CATEGORY THEORY AND THE VAN KAMPEN THEOREM

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Abstract. This paper introduces fundamental concepts in category theory such as categories, functors, limits and colimits, and also briefly reviews the fundamental group, a topological invariant containing information about "holes" in a space. The Seifert-van Kampen Theorem is an important result for calculating the fundamental group. It states that if a space admits an open cover satisfying fairly mild conditions, then its fundamental group is determined by the fundamental groups of the covering sets. The paper aims to demonstrate the practical usage of category theory in algebraic topology by investigating categorical formulations of the Seifert van-Kampen Theorem involving colimits and exploring how it facilitates the statement, proof and extension of the theorem.

Contents

1. Introduction

Category theory is an important tool in modern algebraic topology that allows abstracting away from particular mathematical contexts and objects like sets, groups or topological spaces to define properties and derive results that hold in general, which can turn out easier than working in each specific mathematical context. One class of important general properties are called universal properties, and the statements and proofs of various versions of the Seifert-van Kampen Theorem heavily involve a universal object - an object satisfying a universal property - called the colimit.

Therefore, after presenting basic definitions in category theory, the paper gives an exhaustive treatment of limits and colimits, presenting three definitions: one explicitly invokes the universal property, the second introduces objects called cones and cocones, and the third one ties with the notion of funtor representability. While the three definitions are equivalent, they provide different perspectives for understanding what "universal" means for limits and colimits, and how different universal objects are related.

The rest of the paper builds on the category theory language and applies it to the Seifert-van Kampen Theorem, an important tool for calculating fundamental groups of spaces. After a brief review of the fundamental group, the paper presents the classical formulation of the van Kampen theorem and its proof, followed by a category theoretical version of the theorem and its proof. In comparing and contrasting the different formulations of the van Kampen Theorem and their proofs, we would see that while the main idea behind the proofs remains the same, category theoretical constructions like the colimit simplify the notation and language, and manipulation of these objects also simplifies the proofs themselves.

2. Preliminaries of Category Theory

This section introduces basic category theory concepts that are required for later sections, including categories, functors, natural transformations and notions of equivalence between categories. Like many mathematical constructions such as groups or vector spaces, categories are also collections of things with additional structures. What is special about categories is that it not only contains objects, but also maps between objects, referred to as morphisms, therefore allowing a higher level of abstraction. Functors are maps between categories, and natural transformations transform one functor to another. Notions of equivalence defines what categories can be considered "the same".

Definition 2.1. A *category* consists of a collection $\mathrm{ob}\,\mathcal{C}$ of objects in $\mathcal{C}$, a set $\mathrm{Hom}_{\mathcal{C}}(x,y)$ of morphisms for every pair of objects $x,y\in\mathrm{ob}\,\mathcal{C}$, the identity morphism $\mathrm{id}_x\in\mathrm{Hom}_{\mathcal{C}}(x,x)$ for every object x, and a composition $\circ:\mathrm{Hom}_{\mathcal{C}}(y,z)\times\mathrm{Hom}_{\mathcal{C}}(x,y)\to\mathrm{Hom}_{\mathcal{C}}(x,z)$ for every triple of objects x,y and z satisfying the following axioms:

(1) (Unit.) For any two objects $x,y\in\mathrm{ob}\,\mathcal{C}$ and any morphism $f\in\mathrm{Hom}_{\mathcal{C}}(x,y)$, $f\circ\mathrm{id}_x=f$ and $\mathrm{id}_y\circ f=f$.
(2) (Associativity.) For any four objects $x,y,z,t\in\mathrm{ob}\,\mathcal{C}$ and any three morphisms $f\in\mathrm{Hom}_{\mathcal{C}}(x,y)$, $g\in\mathrm{Hom}_{\mathcal{C}}(y,z)$ and $h\in\mathrm{Hom}_{\mathcal{C}}(z,t)$, $(h\circ g)\circ f=h\circ(g\circ f)$.

Relatedly, the set of *endomorphisms* - morphisms from an object to itself - of an object $x\in\mathrm{ob}\,\mathcal{C}$ is denoted as $\mathrm{End}_{\mathcal{C}}(x)$.

From now on, to simplify the notation, we write $x\in\mathcal{C}$ to mean $x\in\mathrm{ob}\,\mathcal{C}$.

Example 2.2. Below are some examples of categories:

- (Small) sets[1] form a category Set, with the morphisms being set maps.
- Groups form a category Grp, with the morphisms being group homomorphisms. When restricted to Abelian groups this is the category Ab.
- Topological spaces form a category denoted Top in which the morphisms are continuous maps.
- Vector spaces form a category Vect with morphisms being linear maps.
- A discrete category $\mathcal{C}$ consists of a set of objects whose only morphisms are the identity maps from one object to itself.
- The opposite category of a category $\mathcal{C}$, denoted $\mathcal{C}^{\mathrm{op}}$, is one where the objects are the same but all the morphisms are reversed: a morphism $x\to y$ in $\mathcal{C}$ becomes a morphism $y\to x$ in $\mathcal{C}^{\mathrm{op}}$.

[1] Details for set-theoretic issues are omitted in this paper. We mention requirements of "smallness" in the paper for completion, but will not define the terms or elaborate on them.

Definition 2.3. A morphism $f : x \to y$ in a category $\mathcal{C}$ is an *isomorphism* if there exists $f^{-1} \in \mathrm{Hom}_{\mathcal{C}}(y,x)$ such that $f^{-1} \circ f = \mathrm{id}_x$ and $f \circ f^{-1} = \mathrm{id}_y$.

Remark 2.4. When studying different mathematical objects we come across different notions of "sameness", and categorical language provides a simple way of referring to them. For example, a homeomorphism is the isomorphism in the category of topological spaces, a linear isomorphism is the isomorphism in the category of vector spaces, and so on. In other words, the notions of sameness are the same but has different specific manifestations in different categories. While this is a trivial example, the discussion on universal constructions in the next section will present more important implications of this idea.

Next we consider the relations between categories.

Definition 2.5. A *functor* $F : \mathcal{C} \to \mathcal{D}$ is a map between two categories $\mathcal{C}$ and $\mathcal{D}$ that consists of a map $\mathrm{ob}\,\mathcal{C} \to \mathrm{ob}\,\mathcal{D}$ and a map of sets $\mathrm{Hom}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{D}}(F(x),F(y))$ that satisfies the following axioms:

(1) (Unit.) For any object $x \in \mathcal{C}$, $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$.
(2) (Composition.) For any objects $x,y,z \in \mathcal{C}$ and morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(x,y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(y,z)$, $F(g \circ f) = F(g) \circ F(f)$.

A functor can be *covariant* or *contravariant*; a covariant functor F maps a morphism $x \to y$ to a morphism $F(x) \to F(y)$, while a contravariant functor reverses the morphisms and maps a morphism $x \to y$ to a morphism $F(y) \to F(x)$. A functor from $\mathcal{C}$ to $\mathcal{D}$ is called *faithful* if the set map $\mathrm{Hom}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{D}}(F(x),F(y))$ is injective for each pair of $x,y \in \mathcal{C}$ and is called *full* if the set map is surjective for each $x,y \in \mathcal{C}$. A functor that is both full and faithful is called *fully faithful.*

Example 2.6. Below are some examples of functors:

- A forgetful functor $U : \mathcal{C} \to \mathrm{Set}$ can be defined for any category $\mathcal{C}$ whose objects are sets with additional structures, which are forgotten when the functor is applied. For example, $U : \mathrm{Grp} \to \mathrm{Set}$ maps a group to its underlying set and group homomorphisms to set maps. A forgetful functor can be fully faithful, full but not faithful, faithful but not full or neither.
- The abelianization functor from Grp to Ab is a covariant functor that sends every group G to the quotient by its commutator subgroup $G/[G,G]$. It is neither full nor faithful.
- The vector space dual functor from Vect to itself is a contravariant functor that sends a vector space to its dual space and sends a linear map to its transpose. If we restrict the dual functor to finite-dimensional vector spaces, it is fully faithful.
- The Hom functor is from a category $\mathcal{C}$ to Set and is defined for specific objects in $\mathcal{C}$. That is, for an object $x \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(x,-) : \mathcal{C} \to \mathrm{Set}$ is a covariant functor that takes an object $y \in \mathcal{C}$ to the set of morphisms $x \to y$ and take a morphism $f : y \to z$ to a map $\mathrm{Hom}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{C}}(x,z)$ defined by post-composing each morphism in $\mathrm{Hom}_{\mathcal{C}}(x,y)$ with f. The contravariant functor $\mathrm{Hom}_{\mathcal{C}}(-,x)$ is defined similarly.
- A constant functor $\underline{D} : \mathcal{C} \to \mathcal{D}$ takes all objects in $\mathcal{C}$ to a fixed object $D \in \mathcal{D}$ and takes all morphisms to the identity morphism on D.

Given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, their composite $G \circ F$ can be defined objectwise: given $x \in \mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(x,y)$, $(G \circ F)(x) \coloneqq G(F(x))$ and $(G \circ$

$F)(f) \coloneqq G(F(f))$, and the identity functor is defined by taking every object and morphism to itself. So (small) categories form a category Cat with functors being the morphisms. Then we have the usual notion isomorphism for categories: $\mathcal{C}$ and $\mathcal{D}$ are isomorphic if there is a functor $F : \mathcal{C} \to \mathcal{D}$ that has a 2-sided inverse. But this usual notion of isomorphism or "sameness" is rarely used. Instead, a weaker but more useful notion is a category equivalence. Before defining an equivalence, we need to first introduce maps between functors.

Definition 2.7. Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\eta : F \Rightarrow G$ consists of morphisms $\eta_x \in \mathrm{Hom}_{\mathcal{D}}(F(x), G(x))$ for every object $x \in \mathcal{C}$ such that the following diagram commutes for every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$:

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\eta_x"] & F(y) \arrow[d, "\eta_y"] \\ G(x) \arrow[r, "G(f)"] & G(y). \end{tikzcd}$$

A natural transformation between two functors is defined locally at each object, and η_x is called the *component* of η at x. We also say that η is *natural* in x. If the morphism η_x is an isomorphism for every x, then η is a natural isomorphism. If we consider the functors between two categories $\mathcal{C}, \mathcal{D}$ as a functor category $\mathrm{Fun}(\mathcal{C}, D)$, then natural transformations are the morphisms and natural isomorphisms are simply isomorphisms in this category.

Natural transformations can be composed both vertically and horizontally.

Vertical compostion. Let $F, G, H : \mathcal{C} \to \mathcal{D}$ be functors, and let $\eta : F \to G$ and $\varepsilon : G \to H$ be natural transformations. We have the following diagram defining vertical composition:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"] \arrow[r, "G" description] \arrow[r, bend right=50, "H"'] \arrow[r, phantom, shift left=2, "\Downarrow\eta"] \arrow[r, phantom, shift right=2, "\Downarrow\varepsilon"] & \mathcal{D}. \end{tikzcd}$$

Component-wise, the naturality statement says that the diagram

$$\begin{tikzcd} F(x) \arrow[r, "\eta_x"] \arrow[d, "F(f)"] & G(x) \arrow[r, "\varepsilon_x"] \arrow[d, "G(f)"] & H(x) \arrow[d, "H(f)"] \\ F(y) \arrow[r, "\eta_y"] & G(y) \arrow[r, "\varepsilon_y"] & H(y) \end{tikzcd}$$

commutes for any $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$.

Horizontal compostion. Let $F_1, G_1 : \mathcal{C} \to \mathcal{D}$ and $F_2, G_2 : \mathcal{D} \to \mathcal{E}$ be functors. Consider the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D} \arrow[r, bend left, "F_2"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & \mathcal{E}. \end{tikzcd}$$

This horizontal composition $F_2F_1 \Rightarrow G_2G_1$ has components $F_2F_1(x) \overset{F_2(\eta_x)}{\to} F_2G_1(x) \overset{\varepsilon_{G_1(x)}}{\to} G_2G_1(x)$. This is equivalent to $F_2F_1(x) \overset{\varepsilon_{F_1(x)}}{\to} G_2F_1(x) \overset{G_2(\eta_x)}{\to} G_2G_1(x)$.

Equipped with maps between functors, we are now ready to define the equivalence of categories.

Definition 2.8. An equivalence of categories $\mathcal{C}, \mathcal{D}$ consists of a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $e : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$.

Another notion of the "sameness" between categories is the following:

Definition 2.9. An *adjoint equivalence* of categories $\mathcal{C}$ and $\mathcal{D}$ consists of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and natural transformations $e : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and $\varepsilon : FC \Rightarrow \mathrm{id}_{\mathcal{D}}$ such that the composite natural transformation

$$F \cong F \circ \mathrm{id}_{\mathcal{C}} \overset{\mathrm{id}_F \circ e}{\Rightarrow} FGF \overset{\varepsilon \circ \mathrm{id}_F}{\Rightarrow} \mathrm{id}_{\mathcal{D}} \circ F \cong F$$

is the identity natural transformation on F and that the composite natural transformation

$$G \cong \mathrm{id}_{\mathcal{C}} \circ G \overset{e \circ \mathrm{id}_G}{\Rightarrow} GFG \overset{\mathrm{id}_G \circ \varepsilon}{\Rightarrow} G \circ \mathrm{id}_{\mathcal{D}} \cong G$$

is the identity natural transformation on G.

Finally, there is an explicit characterization of a functor that is part of an equivalence of categories that involves the following definition.

Definition 2.10. A functor $F : \mathcal{C} \to \mathcal{D}$ is *essentially surjective* if for any object $d \in \mathcal{D}$, there exists $c \in \mathcal{C}$ such that $F(c) \cong d$.

Theorem 2.11. *The following properties of a functor $F : \mathcal{C} \to \mathcal{D}$ are equivalent:*

(1) F is part of an equivalence of categories;
(2) F is fully faithful and essentially surjective;
(3) F is part of an adjoint equivalence of categories.

Proof. See [6]. □

3. Universal Constructions, Limits and Colimits

Like Remark 2.4 alluded, many mathematical constructions can be abstracted to universal ones, and then each one takes back its concrete meaning in its category. Consider the following example:

Example 3.1. A product of two sets X and Y is $X \times Y \coloneqq \{(x, y) \mid x \in X, y \in Y\}$, known as the *Cartesian product*. Any map $f : Z \to X \times Y$ where Z is another set is uniquely determined by the component maps $f_X : Z \to X$, $f_Y : Z \to Y$ by composing with the projection maps p_X, p_Y where $p_X(x, y) = x$, $p_Y(x, y) = y$.

This abstracts to the following universal property of the *product*: given a category $\mathcal{C}$ and objects $X, Y \in \mathcal{C}$, a collection $(X \times Y, p_X, p_Y)$ where $X \times Y \in \mathcal{C}$, $p_X : X \times Y \to X$, $p_Y : X \times Y \to Y$ is a *product* if for any $W \in \mathcal{C}$ and morphisms $W \to X$ and $W \to Y$, there exists a unique morphism $W \to X \times Y$ such that the following diagram commutes:

$$\begin{tikzcd}
& W \arrow[d, dashed, "\exists!"] \arrow[ddl] \arrow[ddr] & \\
& X \times Y \arrow[dl, "p_X"] \arrow[dr, "p_Y"'] & \\
X & & Y
\end{tikzcd}$$

This generalizes to the product over an index set I and objects $X_\alpha \in \mathcal{C}$ to define the product $\prod_{\alpha\in I} X_\alpha$.

Example 3.2. The following are some examples of the product:

- The product in the category of sets is the Cartesian product of sets: $X \times Y$ has projection maps onto X and Y, and given maps $f_X : W \to X$ and $f_Y : W \to$ Y, there is a unique map $f_X \times f_Y$ sending w to $(f_X(w), f_Y(w))$ making the diagram commute.
- The product in the category of groups is the direct product $G \times H$, whose operation is defined pointwise: $(g_1, h_1)(g_2, h_2) \coloneqq (g_1g_2, h_1h_2)$.
- In the category of abelian group, the product is also the direct product.
- In the category of vector spaces, the product is the direct sum.

In the previous example, the uniqueness of the morphism $W \to X \times Y$ making the relevant diagram commute is the essential characteristic of a universal property. In fact, objects in a category with unique morphisms to or from them are themselves important universal constructions.

Definition 3.3. An object $c \in \mathcal{C}$ is called *initial* if for any $x \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(c, x)$ is an one-element set. An object c is *final* or *terminal* if for any $x \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(x, c)$ is an one-element set.

Note that we can talk about "the" initial or terminal object since they are unique up to unique isomorphism: if c and c' are both initial, then let $f \in \mathrm{Hom}_{\mathcal{C}}(c, c')$ and $g \in \mathrm{Hom}_{\mathcal{C}}(c', c)$, $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(c, c)$ has to be the identity on c.

Example 3.4. The following are some examples of initial and terminal objects:

- In the category of sets, the initial object is the empty set and the terminal object is the one-element set.
- In the category of groups, the trivial group is both the initial and the terminal object.
- In the category of rings with multiplicative identity, the initial object is $\mathbb{Z}$ and the terminal object is the trivial ring.

Products, initial objects and terminal objects are examples of limits and colimits, which are important universal objects in category theory. In what follows, I and $\mathcal{C}$ denote categories and F denotes a functor.

Definition 3.5. (Limit, version 1.) Let $F : I \to \mathcal{C}$ be a functor. A *limit* of F is an object $\lim_I F \in \mathcal{C}$ together with maps $f_i : \lim_I F \to F(i)$ for every $i \in I$ such that for all $g : i \to j$, we have $F(g) \circ f_i = f_j$, and has the following universal property: for any $W \in \mathcal{C}$ with maps $W \to F(i)$ such that the outer triangle in the following diagram commutes, there is a unique compatible morphism $W \to \lim_I F$.

$$\begin{tikzcd}
F(i) \arrow[rr, "F(g)"] & & F(j) \\
& \lim_I F \arrow[ul, "f_i"'] \arrow[ur, "f_j"] & \\
& W \arrow[uul] \arrow[uur] \arrow[u, dashed, "\exists!"] &
\end{tikzcd}.$$

Colimit is dual to a limit: we have $\mathrm{colim}_I F = \lim_{I^{\mathrm{op}}} F$ where $F : I^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is identified with $F : I \to \mathcal{C}$. The fully spelled-out definition is the following:

Definition 3.6. (Colimit, version 1.) Let $F : I \to \mathcal{C}$ be a functor. A *colimit* of F is an object $\mathrm{colim}_I F \in \mathcal{C}$ together with maps $f_i : F(i) \to \mathrm{colim}_I F$ such that for every morphism $g : i \to j$, we have $f_i = f_j \circ F(g)$, and it is the universal such object; that is, for every object $W \in \mathcal{C}$ such that the outer triangle of the following diagram commutes, there is a unique compatible morphism $\mathrm{colim}_I F \to W$:

$$\begin{tikzcd}
F(i) \arrow[rr, "F(g)"] \arrow[dr, "f_i"] \arrow[ddr] & & F(j) \arrow[dl, "f_j"'] \arrow[ddl] \\
& \mathrm{colim}_I F \arrow[d, dashed, "\exists!"] & \\
& W &
\end{tikzcd}$$

Limits and colimits don't always exist. As an example, consider a category $\mathcal{C}$ whose objects are the integers and for $i, j \in \mathbb{Z}$, and there is a morphism $f_{ij} : i \to j$ if and only if $i \leq j$. This satisfies that the morphism is the identity if $i = j$, and $f_{ij} = f_{ik} \circ f_{kj}$ if $i \leq k \leq j$. Then this category has no initial or terminal object, since there is no minimal or maximal element in $\mathbb{Z}$.

We say that the functor $F : I \to \mathcal{C}$ is a *diagram of shape I*. By comparing the two diagrams in Examples 3.1 and 3.5, one can see that product is an example of a limit, and in this case I is a discrete category with its objects being the index set of the product. Its dual concept is a *coproduct*:

Definition 3.7. A *coproduct* of $X, Y \in \mathcal{C}$ is an object $X \sqcup Y \in \mathcal{C}$ together with morphisms $X, Y \to X \sqcup Y$ such that for every object W and morphisms $X, Y \to W$, there is a unique morphism $X \sqcup Y \to W$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[dr] \arrow[ddr] & & Y \arrow[dl] \arrow[ddl] \\
& X \sqcup Y \arrow[d, dashed, "\exists!"] & \\
& W &
\end{tikzcd}$$

Again this can be generalized into a coproduct over any index set I. The following are some examples of the coproduct:

- A coproduct in Set is the disjoint union.
- In the category Ab, the coproduct is the direct sum. For homomorphisms $f : X \to W$ and $g : Y \to W$, define $\phi : X \sqcup Y \to W$ by $\phi(a, b) = f(a) + g(b)$ for $a \in X$ and $b \in Y$.
- In Grp, the coproduct is the free product of groups. A free product of groups G and H, whose intersection consists of the identity element only, is the group generated by all the *reduced words* in the elements of G and H. The free product between two groups is written as $G * H$. Given homomorphisms $f_1 : G \to W$ and $f_2 : H \to W$, define $\phi : G * H \to W$ as the following: for a reduced word $x = x_1 x_2 \ldots x_n$, define $\phi(x) = f_{i_1}(x_1) f_{i_2}(x_2) \ldots f_{i_n}(x_n)$ where f_{i_j} correspond to the group to which x_j belongs.

Different shapes determined by the category I give rise to other limits and colimits. If I is the empty category, then the limit of shape I is the terminal object and the colimit of shape I is the initial object. If I is a category of two objects with the only two non-identity morphisms being $a \rightrightarrows b$, then the limit of shape I is an *equalizer*: let $X, Y \in \mathcal{C}$ together with maps $f, g : X \to Y$. The equalizer E makes the diagram

$$\begin{tikzcd}
E \arrow[r] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\
W \arrow[u, dashed, "\exists!"] \arrow[ur] \arrow[urr]
\end{tikzcd}$$

commute and is the universal such object. That is, for any object $W \in \mathcal{C}$, the compatible morphism $W \to E$ is unique. Dually, a colimit of shape I is the *coequalizer* C associated with the following diagram:

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[drr] & Y \arrow[r] \arrow[dr] & C \arrow[d, dashed, "\exists!"] \\
& & W.
\end{tikzcd}$$

Example 3.8. The following are examples of equalizers and coequalizers:

- In Set, consider two morphisms $f, g : X \to Y$. Then the equalizer is the subset E given by $E := \{x \in X \mid f(x) = g(x)\}$, as any morphism $h : W \to X$ factors through E.
- In Ab, let G, H be abelian groups and let $f, g : G \to H$ be maps between them. Then the equalizer is $\ker(f - g)$. This generalizes to equalizers in the category of R-modules for any ring R.
- The coequalizer in Set is the quotient set $C = Y/\sim$, where $\sim$ is the minimal equivalence relation in Y such that $f(x) \sim g(x)$ for all $x \in X$. The same construction also gives the coequalizer in the categories of abelian groups, vector spaces and modules.
- In the category of groups, if $f, g : X \to Y$ are homomorphisms, the coequalizer is the quotient of Y by the normal closure of $S = \{f(x)g(x)^{-1} \mid x \in X\}$.

If I is a category of three objects with all the non-identity morphisms being $a \to c \leftarrow b$ where a, b and c denote the objects, then the limit of shape I is called the *pullback*. Let $X, Y, Z \in \mathcal{C}$, the pullback $X \times_Z Y$ is such that the composites $X \times_Z Y \to X \to Z$ and $X \times_Z Y \to Y \to Z$ are equal and is the universal such object, meaning for any $W \in \mathcal{C}$ there is a unique morphism $W \to X \times_Z Y$ making the following diagram commute:

$$\begin{tikzcd}
W \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d] \\
& Y \arrow[r] & Z.
\end{tikzcd}$$

Dually, the *pushout* is the colimit of shape I where I is a three-object category with the only non-identity morphisms being $a \leftarrow c \to b$ and is associated with the following diagram:

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr, bend left] & \\
Y \arrow[r] \arrow[drr, bend right] & X \sqcup_Z Y \arrow[dr, dashed, "\exists!"] & \\
 & & W.
\end{tikzcd}$$

Below are some examples for pushouts and pullback:

- In the category of sets, let $f : X \to Z$ and $g : Y \to Z$ be maps, then the pullback $X \times_Z Y$ is $\{(x, y) \in X \times Y \mid f(x) = g(y)\}$. Note that the pullback is a combination of product and equalizer, i.e. $\mathrm{eq}(X \times Y \rightrightarrows Z) \cong X \times_Z Y$, and this is generally true for all categories.
- In Set, the pushout $X \sqcup_Z Y$ is $X \sqcup Y / \sim$, where $\sim$ is the smallest equivalence relation on such that $f(z) \sim g(z)$ for all $z \in Z$. Again note that we have that the pushout is a combination of the coproduct and the coequalizer, i.e. $X \sqcup_Z Y \cong \mathrm{coeq}(Z \rightrightarrows X \sqcup Y)$, and this also holds in general.

That universality means the "unique morphism" condition is another way of expressing that universality is equivalent to being initial or terminal in an appropriate category. In the construction of limits and colimits, this appropriate category is that of the cones.

Definition 3.9. Given a diagram $F : I \to \mathcal{C}$, a *cone* on F consists of an object $c \in \mathcal{C}$ and morphisms $\lambda_i : c \to F(i)$ for each object $i \in I$ such that the following diagram commutes for any morphism $g : i \to j$ in I:

$$\begin{tikzcd}
 & c \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\
F(i) \arrow[rr, "F(g)"] & & F(j).
\end{tikzcd}$$

Dually, a *cocone* on F consists of an object $c \in \mathcal{C}$ and morphisms $\lambda_i : F(i) \to c$ such that the following diagram commutes:

$$\begin{tikzcd}
F(i) \arrow[rr, "F(g)"] \arrow[dr, "\lambda_i"'] & & F(j) \arrow[dl, "\lambda_j"] \\
 & c & .
\end{tikzcd}$$

The cones or cocones over F form a category of cones. We can also view cones as defining a natural transformation $\underline{c} \to F$ where $\underline{c} : I \to \mathcal{C}$ is the constant functor with value $c \in \mathcal{C}$, and similarly for cocones.

By comparing the diagrams associated with cones/cocones and with limits/colimits, we have the following reformulation:

Definition 3.10. (Limit and colimit, version 2.) Given a diagram $F : I \to \mathcal{C}$, a *limit* is a terminal object in the category of cones on F, and a *colimit* is an initial object in the category of cocones on F.

And this formulation leads to an immediate conclusion:

Proposition 3.11. *Limits and colimits are unique up to unique isomorphism.*

Up to this point universality is associated with being initial or terminal in an appropriate category. In fact, another paradigm of universality involves the idea of functor representability.

Definition 3.12. A functor $F : \mathcal{C} \to \text{Set}$ is *representable* if it is naturally isomorphic to the Hom functor for some object $c \in \mathcal{C}$, i.e. $\text{Hom}_{\mathcal{C}}(c, -) \cong F$ or $\text{Hom}_{\mathcal{C}}(-, c) \cong F$ depending on whether F is covariant or contravariant. A *representation* for a functor F is an object $c \in \mathcal{C}$ and a specific natural isomorphism $\text{Hom}_{\mathcal{C}}(c, -) \cong F$ or $\text{Hom}_{\mathcal{C}}(-, c) \cong F$.

A *universal property* of an object $c \in \mathcal{C}$ is then expressed by a representable functor F together with a *universal element* $x \in F(c)$ that defines the natural isomorphism $\psi(x) : \text{Hom}_{\mathcal{C}}(c, -) \cong F$ or $\psi(x) : \text{Hom}_{\mathcal{C}}(-, c) \cong F$ by the components $\psi(x)_d(f) \coloneqq F(f)(x)$. The existence of such an element is a consequence of the Yoneda lemma:

Lemma 3.13. *(Yoneda.) For any functor $F : \mathcal{C} \to Set$ where $\mathcal{C}$ is locally small, and any object $c \in \mathcal{C}$, there is a bijection $Hom(Hom_{\mathcal{C}}(c, -), F) \cong F(c)$ that associates to a natural transformation $\alpha : Hom_{\mathcal{C}}(c, -) \Rightarrow F$ the element $\alpha_c(id_c) \in F(c)$. The contravariant version replaces $\mathcal{C}$ with $\mathcal{C}^{op}$.*

See [6] and [5] for the proof and more details. To see how this definition of universal property is related to being initial or terminal, the appropriate category to consider is that of the elements.

Definition 3.14. The *category of elements* $\int F$ of a covariant functor $F : \mathcal{C} \to \text{Set}$ consists of pairs (c, x) as objects where $c \in \mathcal{C}$, $x \in F(c)$ and as morphisms $(c, x) \to (c', x')$ induced by $f : c \to c'$ such that $F(f)(x) = x'$. That of a contravariant functor $F : \mathcal{C}^{\text{op}} \to \text{Set}$ has the same objects but morphisms $(c, x) \to (c', x')$ induced by $f : c \to c'$ such that $F(f)(x') = x$.

The following result follows easily from the definitions.

Proposition 3.15. *A covariant functor $F : \mathcal{C} \to Set$ is representable if and only if $\int F$ has an initial object. Dually, a contravariant functor $F : \mathcal{C}^{op} \to Set$ is representable if and only if $\int F$ has a terminal object.*

Back to the limit and colimit context, the relevant functors are $\text{Cone}(-, F) : \mathcal{C}^{\text{op}} \to \text{Set}$ and $\text{Cone}(F, -) : \mathcal{C} \to \text{Set}$ respectively, which send $c \in \mathcal{C}$ to the set of cones or the set of cocones in c and send morphisms $c' \to c$ or $c \to c'$ to set maps of cones or cocones induced by pre- or post-composition.

Definition 3.16. (Limit and colimit, version 3.) A *limit* of $F : I \to \mathcal{C}$ is a representation for $\text{Cone}(-, F)$, which consists of an object $\lim_I F \in \mathcal{C}$ and a universal element $\lambda_i : \lim_I F \to F(i)$, called the universal cone or the limit cone, that defines the natural isomorphism $\text{Hom}_{\mathcal{C}}(-, \lim_I F) \simeq \text{Cone}(-, F)$.

Dually, a *colimit* of F is a representation for $\text{Cone}(F, -)$, which consists of an object $\text{colim}_I F \in \mathcal{C}$ and a universal element $\lambda_i : F(i) \to \text{colim}_I F$, called the universal cocone or the colimit cone, that defines the natural isomorphism $\text{Hom}_{\mathcal{C}}(\text{colim}_I F, -) \cong \text{Cone}(F, -)$.

The categories of elements $\int \text{Cone}(-, F)$ and $\int \text{Cone}(F, -)$ are exactly the categories of cones or cocones, so Proposition 3.15 then shows that Definitions 3.10 and 3.16 are equivalent. Indeed, the natural isomorphisms imply the existence and uniqueness of a compatible morphism $W \to \lim_I F$ or $\mathrm{colim}_I F \to W$ for any object $W \in \mathcal{C}$.

4. Review of Fundamental Group

In this section, we switch gears from general category theory context and consider a specific algebraic topological object - the fundamental group. The fundamental group is often the first topological invariant one encounters in algebraic topology. And the Seifert-van Kampen Theorem, the focus of the second half of this paper, provides an important way of calculating them. Before presenting the theorem itself, this section gives a brief review of the definition of the fundamental group and some key properties and results associated with it. For full details and proofs of the following results, see [4].

The fundamental group of a space X, in short, consists of path homotopy classes of loops at a given basepoint and an operation that "concatenates" different loops. To unpack this definition, first recall the definition of a path homotopy.

Definition 4.1. Two paths f and f', mapping the interval $I = [0,1]$ into X, are said to be *path homotopic* if they have the same initial point x_0 and the same final point x_1, and there is a continuous map $F : I \times I \to X$ such that $F(s,0) = f(s)$, $F(s,1) = f'(s)$, $F(0,t) = x_0$, $F(1,t) = x_1$ for each $s, t \in I$. We call F a *path homotopy* between f and f', and write $f \simeq_p f'$.

It is easy to check that $\simeq_p$ is an equivalence relation, so we can talk about path homotopy classes in a space X. Next, we define the product operation.

Definition 4.2. If f is a path in X from x_0 to x_1, and if g is a path from x_1 to x_2, we define the *product* $f \cdot g$ of f and g to be the path h by the equations

$$h(s) = \begin{cases} f(2s) & \text{for } s \in [0, \frac{1}{2}] \\ g(2s-1) & \text{for } s \in [\frac{1}{2}, 1]. \end{cases}$$

This operation on paths induces a well-defined operation on path-homotopy classes defined by

$$[f] \cdot [g] \coloneqq [f \cdot g].$$

Note that $[f] \cdot [g]$ is only defined when $f(1) = g(0)$. It satisfies the following group-like properties that are called the *groupoid properties* of the operation $\cdot$. We will formally introduce the concept of a groupoid in Section 6.

Theorem 4.3. *The operation $\cdot$ satisfies the following properties:*

1. *(Associativity.) If $[f] \cdot ([g] \cdot [h])$ is defined, so is $([f] \cdot [g]) \cdot [h]$ and they are equal.*
2. *(Right and left identities.) Given $x \in X$, let e_x denote the constant path $e_x : I \to X$ carrying all of I to the point x. If f is a path in X from x_0 to x_1, then*
$$[f] \cdot [e_{x_1}] = [f] = [e_{x_0}] \cdot [f].$$
3. *(Inverse.) Given a path f in X from x_0 to x_1, let $\bar{f}$ be the path defined by $\bar{f}(s) = f(1-s)$. Then $[f] \cdot [\bar{f}] = [e_{x_0}]$ and $[\bar{f}] \cdot [f] = [e_{x_1}]$.*

To actually get a group structure, we need to consider a special subset of path homotopy classes for which the operation $\cdot$ is always defined. To do this we pick a basepoint x_0 in X and consider the set of "loops" at x_0, which gives the fundamental group of X.

Definition 4.4. Let X be a space, and let x_0 be a point of X. A path in X that begins and ends at x_0 is called a *loop* at x_0. The set of path homotopy classes of loops based at x_0 with the operation $\cdot$ is called the *fundamental group* of X relative to the *basepoint* x_0. It is denoted by $\pi_1(X, x_0)$.

As an example, a space X is called *simply connected* if it is path-connected and if $\pi_1(X, x_0)$ is the trivial group for some $x_0 \in X$. A natural question that follows is how the fundamental group depends on the basepoint. To consider this, we first define a map that "moves" a loop based at one basepoint to another.

Definition 4.5. Let α be a path in X from x_0 to x_1. We define a map $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by the equation

$$\hat{\alpha}([f]) = [\bar{\alpha}] \cdot [f] \cdot [\alpha]$$

which is a well-defined map given $\cdot$ is well-defined on path homotopy classes.

And we have the following results on the map $\hat{\alpha}$.

Theorem 4.6. *The map $\hat{\alpha}$ is a group isomorphism.*

Corollary 4.7. *If X is path connected and x_0 and x_1 are two points in X, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$.*

However, in general the exact isomorphism depends on the path α we choose, the exception being when the fundamental group is abelian. While the isomorphism result makes it tempting to simply refer to "the" fundamental group of a space, it should be kept in mind that there is no canonical way of identifying it in general.

Continuous maps between topological spaces induce homomorphisms between their fundamental groups. Let $h : (X, x_0) \to (Y, y_0)$ be a continuous map. Then the induced homomorphism is denoted as $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, and is defined by $h_*([f]) = [h \circ f]$. The induced homomorphism has the following properties:

Theorem 4.8. *If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous, then $(k \circ h)_* = k_* \circ h_*$. If $i : (X, X_0) \to (X, x_0)$ is the identity map, then i_* is the identity homomorphism.*

These properties are called the "functorial properties", because it then follows that the fundamental group gives a functor from the category of based topological spaces Top_* to Grp.

We have already invoked the fundamental group as an invariant, which alludes to the following result:

Theorem 4.9. *If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism, then h_* is an isomorphism. More generally, if X and Y are of the same homotopy type, then their fundamental groups are isomorphic.*

Therefore, the fundamental group is an important tool for distinguishing spaces from one another, which motivates finding ways to calculate them.

5. The Classical Seifert-van Kampen Theorem

The Seifert-van Kampen theorem provides a method for calculating the fundamental group of a space by decomposing a space into path-connected open sets. Using categorical language, the van Kampen theorem can be generalized to the fundamental groupoid, which at this point can be thought of as a "fundamental group without basepoint". Before delving into the categorical perspective, we first recall the classical version of the van Kampen theorem.

Theorem 5.1. *Suppose a path-connected space X is the union of path-connected open sets A_α, each of which contains the basepoint $x_0 \in X$ and each intersection $A_\alpha \cap A_\beta$ is path-connected. Let $j_\alpha : \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$ and $i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta, x_0) \to \pi_1(A_\alpha, x_0)$ be the homomorphisms induced by inclusion, and let $\Phi : *_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$ be the extension of j_α to the free product. Then Φ is surjective. If in addition each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of Φ is the normal subgroup N generated by all elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta, x_0)$, and hence Φ induces an isomorphism $\pi_1(X, x_0) \cong *_\alpha \pi_1(A_\alpha, x_0)/N$.*

Proof. We first prove that Φ is surjective. Let $f : I \to X$ be a loop based at x_0. Since f is continuous, for each $s \in I$, we can find a neighborhood V_s such that $f(V_s)$ is contained in a single A_α. In fact we can take V_s to be an interval such that $f(\bar{V_s}) \in A_\alpha$. Since I is compact, finitely many such intervals cover I. The endpoints of these intervals gives a partition $0 = s_0 < s_1 < \cdots < s_m = 1$ of I such that $f([s_{i-1}, s_i])$ is contained in a single A_α.

Let A_i be the A_α containing $f([s_{i-1}, s_i])$, and let f_i be the path obtained by restricting f to $[s_{i-1}, s_i]$, then f is the composition $f_1 \cdots f_m$. Since $A_i \cap A_{i+1}$ is path-connected, we can find a path $g_i(I) \subset A_i \cap A_{i+1}$ from x_0 to $f(s_i) \in A_i \cap A_{i+1}$. Then the loop $(f_1 \cdot \bar{g_1}) \cdot (g_1 \cdot f_2 \cdot \bar{g_2}) \cdots (g_{m-1} \cdot f_m)$ is a composition of loops based at x_0 each contained in a single A_i and is homotopic to f.

Now we prove that the kernel of Φ is indeed N. First, it follows from the definition of N that $N \subset \ker \Phi$ as $j_\alpha \circ i_{\alpha\beta} = j_\beta \circ i_{\beta\alpha}$. To show the reverse inclusion, we need to introduce *factorization* of loops in X. The proof of surjectivity gives a way of factorizing $[f] \in \pi_1(X, x_0)$ into a product of loops f_i, each of which is contained in a single A_α. Call $[f_1] \cdots [f_k]$ a *factorization* of $[f]$ where $[f_i] \in \pi_1(A_\alpha, x_0)$. Surjectivity of Φ ensures that every $[f]$ admits a factorization.

Then we introduce the an equivalence relation for factorizations. Two factorizations of $[f]$ are equivalent if they are related by the following operations or their inverses:

- Combine adjacent terms $[f_i] \cdot [f_{i+1}]$ into $[f_i \cdot f_{i+1}]$ if both terms are in the same group $\pi_1(A_\alpha, x_0)$.
- Regard the term $[f_i] \in \pi_1(A_\alpha, x_0)$ as in $\pi_1(A_\beta, x_0)$ if f_i is a loop in $A_\alpha \cap A_\beta$.

Let $Q = *_\alpha \pi_1(A_\alpha)/N$. Note first that equivalent factorizations give the same element in Q: the first operation does not change the element of $*_\alpha \pi_1(A_\alpha, x_0)$ defined by the factorization; for the second operation, if f_i is in $A_\alpha \cap A_\beta$, $i_{\alpha\beta}([f_i])$ has the same coset in the quotient group as $i_{\beta\alpha}([f_i])$. Now if we show that any two factorizations of $[f]$ are equivalent, the map $Q \to \pi_1(X)$ is injective, then $\ker \Phi \subset N$ and the kernel of Φ is N as desired.

To show this, let $[f_1] \cdots [f_k]$ and $[f_1'] \cdots [f_l']$ be two factorizations of $[f]$. The two composed paths are then homotopic, so let $F : I \times I \to X$ be the homotopy. Then there exist partitions given by $0 = s_0 < s_1 < \cdots < s_m = 1$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ such that each rectangle $R_{ij} = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ is mapped by F to a single A_α which can be labelled as A_{ij}: for each $(a, b) \in I \times I$, $F(a, b)$ belongs to A_α for some α. Then $F^{-1}(A_\alpha)$ is open in $I \times I$, and since rectangles form a basis for $I \times I$ which is regular, there exists a rectangle contained in $F^{-1}(A_\alpha)$ whose closure is also

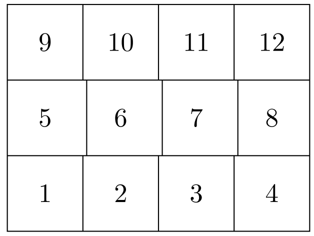

FIGURE 1. A subdivision of $I \times I$

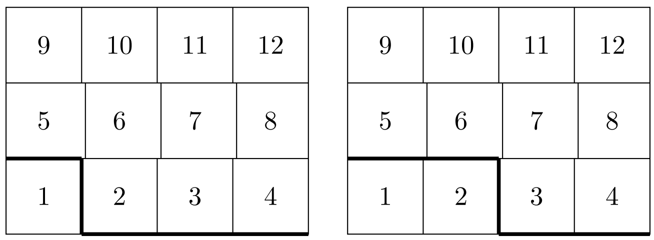

FIGURE 2. The paths γ_1 and γ_2

contained in it. This gives an open cover of $I \times I$. Since $I \times I$ is compact, we can extract a finite subcover, and then take the union of the vertical or horizontal lines containing the edges of the rectangles. We may also assume that this partition is a refinement of the partitions giving the two factorizations. Since F maps a neighborhood of R_{ij} to A_{ij}, we can perturb the edges of the rectangles so that each point of $I \times I$ lies in at most three R_{ij}'s, to ensure that we are always working with a path-connected intersection $A_\alpha \cap A_\beta \cap A_\gamma$. Relabel them as $R_1, R_2, \dots, R_{mn}$ as Figure 3 illustrates.

If γ is a path in $I \times I$ from the left edge to the right edge, then the restriction $F|_\gamma$ is a loop based at x_0. Let γ_r be the path separating the first r rectangles from the rest, then γ_0 is the bottom edge of $I \times I$ and γ_{mn} is the top edge. Figure 2 illustrates how we pass to γ_1 and γ_2, and the remaining R_i's are obtained similarly.

Call the corners of the R_r's vertices. For each vertex v_i with $F(v_i) \neq x_0$, choose a path g_{v_i} from x_0 to $F(v_i)$ that lies in the intersection of the two or three A_{ij}'s corresponding to the R_r's containing v_i. Then we obtain a factorization of $[F|_{\gamma_r}]$ by inserting $\bar{g}_v g_v$ into $F|_{\gamma_r}$ at successive vertices as in the proof of surjectivity of Φ.

This factorization still depends on some choices. If the segment between two vertices lies in two different A_{ij}'s, like the situation in both γ_1 and γ_2 depicted in Figure 2, we need to choose which A_{ij} to regard the resulted loop in. But different choices lead to equivalent factorizations given the loop lies in the intersection. Further, factorizations $[F|_{\gamma_r}]$ and $[F|_{\gamma_{r+1}}]$ are equivalent as they are homotopic, with the homotopy induced by a path homotopy between γ_r and γ_{r+1}, which is a transformation of the left and bottom edge path to the top and right edge path in R_{r+1}, precomposed with a constant path at x_0 for rectangles on the left edge and cutting the constant path for rectangles on the right edge.

We can make sure that the factorization associated to γ_0 is equivalent to $[f_1]\cdots[f_k]$ by choosing the path g_v for each vertex v along the lower edge of $I \times I$ to not only lie in the intersection of the two A_{ij}'s but also in the A_α of the f_i to which $F(v)$ belongs. If v is the common endpoint of the domains of two different f_i's, then $F(v) = x_0$, so such a g_v always exists. Similarly we can ensure that the factorization associated to γ_{mn} is equivalent to $[f'_1]\cdots[f'_l]$. So $[f_1]\cdots[f_k]$ is equivalent to $[f'_1]\cdots[f'_l]$ which completes the proof. □

Before introducing the full categorical version of the van Kampen theorem, we first consider the simple case where X is the union of two open sets U, V whose intersection is path connected and explore what Theorem 5.1 means in categorical terms. If we view the collection $\mathscr{O}$ of open sets that cover X which is closed under finite intersection as a category, in this case $\mathscr{O}$ is a three-item category whose only non-identity morphisms are the inclusions $U \leftarrow U \cap V \rightarrow V$.

Proposition 5.2. *Suppose $X = U \cup V$ where U, V open and $U \cap V$ path connected, $x_0 \in U \cap V$. Then $\pi_1(X, x_0)$ is the pushout:*

$$\begin{tikzcd}
\pi_1(U\cap V, x_0) \arrow[r] \arrow[d] & \pi_1(U, x_0) \arrow[d] \arrow[ddr, bend left] & \\
\pi_1(V, x_0) \arrow[r] \arrow[drr, bend right] & \pi_1(X, x_0) \arrow[dr, dashed, "\exists!"] & \\
 & & W.
\end{tikzcd}$$

Generalizing to an open cover $\mathscr{O}$ containing arbitrary open sets that are closed under finite intersection and $x_0 \in A_\alpha$ for all $A_\alpha \in \mathscr{O}$, we expect the categorical statement of the van Kampen theorem is

$$\pi(X, x_0) \cong \operatorname{colim}_{A_\alpha \in \mathscr{O}} \pi_1(A_\alpha, x_0)$$

which we prove in the next section.

6. Categorical Formulation of the van Kampen Theorem

To prove the generalized version of the van Kampen theorem, we first introduce the notion of fundamental groupoid and prove the van Kampen theorem for the fundamental groupoid. The fundamental group version will follow as a consequence.

Definition 6.1. A category $\mathcal{C}$ is called a *groupoid* if every morphism is invertible. That is, all morphisms are isomorphisms.

The name groupoid comes from the fact that a group can be viewed as a groupoid with a single object, with the group elements being the endomorphisms. Groupoids form a category Grpd by taking morphisms to be functors between groupoids.

Definition 6.2. The *fundamental groupoid* $\Pi(X)$ of a space X is the category whose objects are points of X and whose morphisms $x \to y$ are homotopy equivalence classes of paths from x to y.

Recall that the properties in Theorem 4.3, where we first introduced path homotopy classes in a space, are called "groupoid properties", as the associativity and identity properties ensure that points in the space and the path homotopy classes do form a category, and the inverse property says that this category is a groupoid. The fundamental group of X with base poing x is then the set of endomorphisms of the object x. And Π can be viewed as a functor Top $\to$ Grpd.

Theorem 6.3. *Let $\mathscr{O} = \{U\}$ be a cover of a space X by path-connected open sets such that the intersection of finitely many subsets in $\mathscr{O}$ is again in $\mathscr{O}$. Regard $\mathscr{O}$ as a category whose morphisms are the inclusions of subsets and observe that the functor Π, restricted to spaces and maps in $\mathscr{O}$, gives a diagram $\Pi|_{\mathscr{O}} : \mathscr{O} \to Grpd$ of groupoids. Then $\Pi(X) \cong colim_{U \in \mathscr{O}} \Pi(U)$.*

Proof. To show that $\Pi(X)$ is the colimit, we need to show that it satisfies the universal property. That is, for a groupoid $C \in \text{Grpd}$ and a natural transformation $\eta : \Pi|_{\mathscr{O}} \to \underline{C}$ where $\underline{C}$ is a constant functor $\mathscr{O} \to \text{Grpd}$, we need to construct a map $\tilde{\eta} : \Pi(X) \to C$ that restricts to η_U on $\Pi(U)$ for all $U \in \mathscr{O}$. For objects in $\Pi(X)$, we must have $\tilde{\eta}(x) = \eta_U(x)$ where $x \in U$ (note that morphisms between groupoids, viewed as functors between two categories, are defined on their objects and morphisms). This is independent of the choice of U since for any $V \in \mathscr{O}$ such that $x \in U \cap V$, $\eta_U(x) = \eta_{U\cap V}(x) = \eta_V(x)$. On morphisms, if a path f from x to y lies in a single U, then we must define $\tilde{\eta}([f]) = \eta_U([f])$. Again this does not depend on a particular choice of U since if f lies in $U \cap V$ for another $V \in \mathscr{O}$, $\eta_U([f]) = \eta_{U\cap V}([f]) = \eta_V([f])$. Any paths can be broken down to a composite of finitely many paths f_i, each of which lie in a single U, and we can define $\tilde{\eta}([f])$ to be the composite of $\tilde{\eta}(f_i)$.

Given this definition, $\tilde{\eta}$ clearly restricts to η_U on $\Pi(U)$ if it is indeed well-defined. Suppose $f : x \to y$ is equivalent to g through a path homotopy $F : I \times I \to X$. Divide $I \times I$ to small rectangles such that the resulting subdivision of $I \times 0$ is a refinement of the subdivision of the composites f_i, and similarly for $I \times 1$, as in the proof of the classical version of the van Kampen theorem. And define paths γ_r's in the same way as in that proof. Then $[f]$ is homotopic to $F|_{\gamma_1}$ through a relation in $\Pi(U_1)$ corresponding to the first rectangle, which is then homotopic to $F|_{\gamma_2}$ through a relation in $\Pi(U_2)$, and finally g is homotopic to $F|_{\gamma_{mn}}$ in $\Pi(U_{mn})$. Therefore $\tilde{\eta}([f]) = \tilde{\eta}([g])$. □

The van Kampen theorem for fundamental group then follows from the result above. Before proving it, we need some further observations on the relationship between the fundamental groupoid and the fundamental group.

Definition 6.4. A *full* subcategory $\mathcal{C}'$ of a category $\mathcal{C}$ is such that $\text{ob}\,\mathcal{C}' \subset \text{ob}\,\mathcal{C}'$ and $\text{Hom}_{\mathcal{C}'}(x,y) = \text{Hom}_{\mathcal{C}}(x,y)$ for all $x, y \in \text{ob}\,\mathcal{C}'$. That is, $\mathcal{C}'$ has all morphisms between objects.

Definition 6.5. A *skeleton* of a category $\mathcal{C}$, $\text{sk}\mathcal{C}$, is a full subcategory with one object from each isomorphism class of objects of $\mathcal{C}$.

Proposition 6.6. *The inclusion functor $J : sk\mathcal{C} \to \mathcal{C}$ is part of an equivalence of categories.*

Proof. We can define an inverse functor $F : \mathcal{C} \to \text{sk}\mathcal{C}$ by sending $A \in \mathcal{C}$ to the unique object that is isomorphic to A in $\text{sk}\mathcal{C}$. Choose an isomorphism $\alpha_A : A \to F(A)$, and let α_A be the identity on A if $A \in \text{sk}\mathcal{C}$. For $f : A \to B$ a morphism in $\mathcal{C}$, define $F(f) = \alpha_B \circ f \circ \alpha_A^{-1}$. Clearly JF is isomorphic to $\text{id}_{\mathcal{C}}$ and FJ is the identity on $\text{sk}\mathcal{C}$. □

Further, a category $\mathcal{C}$ is *connected* if any two objects can be connected by a sequence of morphisms. So a groupoid $\mathcal{C}$ is connected if and only if any two objects are isomorphic. In this case, any object $C \in \mathcal{C}$ and its group of endomorphisms is a skeleton of $\mathcal{C}$. Recall that the fundamental group of a path-connected space X with basepoint x is the group of endomorphism of x in $\Pi(X)$, which can be regarded as a single-object groupoid:

Corollary 6.7. *Let X be a path-connected space. Then for each point $x \in X$, the inclusion $\pi_1(X,x) \to \Pi(X)$ is an equivalence of categories.*

Finally, we need an additional result related to colimits in Grp.

Proposition 6.8. *The category of groups has all colimits.*

This results from the fact that categories with coproducts and coequalizers have all colimits. For a complete proof, see [6].

With these results in place, we are ready to prove the full version of the van Kampen Theorem.

Theorem 6.9. *Let X be a path-connected space and choose a basepoint $x \in X$. Let $\mathscr{O}$ be a cover of X by path-connected open subsets such that the intersection of finitely many subsets in $\mathscr{O}$ is again in $\mathscr{O}$, and x is in each $U \in \mathscr{O}$. Regard $\mathscr{O}$ as a category whose morphisms are inclusions of subsets. Note that the functor $\pi_1(-,x)$, restricted to spaces and maps in $\mathscr{O}$, gives an $\mathscr{O}$-shaped diagram $\pi_1|_{\mathscr{O}} : \mathscr{O} \to Grp$ of groups. Then $\pi_1(X,x) \cong colim_{U \in \mathscr{O}} \pi_1(U,x)$.*

Proof. First, we show that the claim holds for a finite $\mathscr{O}$. Again, we need to show that $\pi_1(X,x)$ has the universal property. That is, for a group G, the constant functor $\underline{G}$ that sends everything in $\mathscr{O}$ to the group G and a natural transformation $\eta : \pi_1|_{\mathscr{O}} \to \underline{G}$ between $\mathscr{O}$-shaped diagrams, there is a unique homomorphism $\tilde{\eta} : \pi_1(X,x) \to G$ that restricts to η_U on $\pi_1(U,x)$. That is, recall from the discussion on colimits, the diagram

$$\begin{tikzcd}
\pi_1(X,x) \arrow[d, "\hat{\eta}"] & \pi_1(U,x) \arrow[l] \arrow[dl, "\eta_U"] \\
G &
\end{tikzcd}$$

commutes. Recall also that the inclusion functor $J : \pi_1(X,x) \to \Pi(X)$ is part of an equivalence of categories, with its inverse $F : \Pi(X) \to \pi_1(X,x)$ defined by a choice of path $x \to y$ for $y \in X$. We choose the path to be the constant path c_x when $y = x$ so that FJ is the identity on $\pi_1(X,x)$. Since $\mathscr{O}$ is finite and closed under finite intersections, we can choose a path $x \to y$ that lies entirely in every U that contains y. Then we have a well-defined inverse functor $F_U : \Pi(U) \to \pi_1(U,x)$ to the restricted inclusion $J_U : \pi_1(U,x) \to \Pi(U)$. Then

$$\Pi(U) \xrightarrow{F_U} \pi_1(U,x) \xrightarrow{\eta_U} G$$

defines a natural transformation between $\mathscr{O}$-shaped diagrams of groupoids $\Pi|_{\mathscr{O}} \to \underline{G}$. Then by the fundamental groupoid version, there is a unique map of groupoids $\xi : \Pi(X) \to G$ that restricts to $\eta_U \circ F_U$ on each U. We claim that the composite

$$\pi_1(X,x) \xrightarrow{J} \Pi(X) \xrightarrow{\xi} G$$

is the required homomorphism $\tilde{\eta}$. First, we have that the following diagram

$$\begin{tikzcd}
\Pi(X) \arrow[d, "\xi"] & \Pi(U) \arrow[l] \arrow[r, "F_U"] & \pi_1(U,x) \arrow[dll, "\eta_U"] \\
G & &
\end{tikzcd}$$

commutes given how ξ is chosen. Since $F_U \circ J_U = \mathrm{id}$, the diagram

$$\begin{tikzcd}
\Pi(X) \arrow[d, "\xi"] & \Pi(U) \arrow[l] & \pi_1(U,x) \arrow[l, "J_U"'] \arrow[dll, "\eta_U"] \\
G & &
\end{tikzcd}$$

also commutes. And

$$\begin{tikzcd}
\pi_1(X,x) \arrow[d, "J"'] & \pi_1(U,x) \arrow[l] \arrow[d, "J_U"] \\
\Pi(X) & \Pi(U) \arrow[l]
\end{tikzcd}$$

commutes by definition. So

$$\begin{tikzcd}
\pi_1(X,x) \arrow[d, "J"] & & \\
\Pi(X) \arrow[d, "\xi"] & \Pi(U) \arrow[l] & \pi_1(U,x) \arrow[l, "J_U"'] \arrow[ull] \arrow[dll, "\eta_U"] \\
G & &
\end{tikzcd}$$

commutes.

The homomorphism $\tilde{\eta}$ is unique since ξ is unique. Suppose there is another $\hat{\eta} : \pi_1(X,x) \to G$ that restricts to η_U on each $\pi_1(U,x)$. Then $\hat{\eta} \circ F : \Pi(X) \to G$ restricts to $\eta_U \circ F_U$ on each $\Pi(U)$, so $\xi = \hat{\eta} \circ F$ and $\hat{\eta} = \xi \circ J = \tilde{\eta}$.

Next, we show that the claim also holds for general $\mathscr{O}$. Let $\mathscr{F}$ be the set of finite subsets of $\mathscr{O}$ that are closed under finite intersection. For $\mathscr{S} \in \mathscr{F}$, let $U_{\mathscr{S}}$ be the union of U in $\mathscr{S}$. Then $U_{\mathscr{S}}$ satisfies the assumption for the finite case, so

$$\operatorname{colim}_{U \in \mathscr{S}} \pi_1(U,x) \cong \pi_1(U_{\mathscr{S}}, x).$$

For the next step, regard $\mathscr{F}$ as a category with a morphism $\mathscr{S} \to \mathscr{T}$ when $U_{\mathscr{S}} \subset U_{\mathscr{T}}$. Then we have

$$\operatorname{colim}_{\mathscr{S} \in \mathscr{F}} \pi_1(U_{\mathscr{S}}, x) \cong \pi_1(X,x)$$

by checking the universal property: given any natural transformation $\eta : \pi_1|_{\mathscr{F}} \to \underline{G}$, we can define $\hat{\eta} : \pi_1(X,x) \to G$ in a similar way as before: if a loop f is contained in a single $U_{\mathscr{S}}$, define $\hat{\eta}(f) = \eta_{\mathscr{S}}(f)$, and then use a subdivision argument similar to the one used in proving Theorem 5.1 and 6.3 to show that $\hat{\eta}$ is well-defined, and that $\pi_1(X,x)$ is the colimit of the diagram with shape $\mathscr{F}$.

Next, we claim that $\operatorname{colim}_{U \in \mathscr{O}} \pi_1(U,x) \cong \operatorname{colim}_{\mathscr{S} \in \mathscr{F}} \pi_1(U_{\mathscr{S}}, x)$ and this will complete the proof. Substituting in $\operatorname{colim}_{U \in \mathscr{S}} \pi_1(U,x) \cong \pi_1(U_{\mathscr{S}}, x)$, we have

$$\operatorname{colim}_{\mathscr{S} \in \mathscr{F}} \pi_1(U_{\mathscr{S}}, x) \cong \operatorname{colim}_{\mathscr{S} \in \mathscr{F}} \operatorname{colim}_{U \in \mathscr{S}} \pi_1(U,x)$$

We claim that the iterated limit on the right is isomorphic to a single colimit $\operatorname{colim}_{(U,\mathscr{S}) \in (\mathscr{O},\mathscr{F})} \pi_1(U,x)$, where the category $(\mathscr{O},\mathscr{F})$ has as objects the pairs

$(U, \mathscr{S})$ with $U \in \mathscr{S}$, and morphisms $(U, \mathscr{S}) \to (V, \mathscr{T})$ whenever $U \subset V$ and $U_{\mathscr{S}} \subset U_{\mathscr{T}}$. This is the case since $\operatorname{colim}_{(U,\mathscr{S})\in(\mathscr{O},\mathscr{F})} \pi_1(U, x)$ satisfies the universal property required of the iterated colimit and vice versa.

Since groups have all colimits, $\operatorname{colim}_{U\in\mathscr{O}} \pi_1(U, x)$ exists. We can then show $\operatorname{colim}_{U\in\mathscr{O}} \pi_1(U, x) \cong \operatorname{colim}_{(U,\mathscr{S})\in(\mathscr{O},\mathscr{F})} \pi_1(U, x)$ and complete the proof. We can define a functor $\mathscr{O} \to \mathscr{F}$ that sends U to $(U, \{U\})$. The functor $\pi_1(-, x) : \mathscr{O} \to \mathrm{Set}$ factors through $(\mathscr{O}, \mathscr{F})$ as it ignores the second coordinate. So the universal properties of the two colimits gives rise to maps $\pi_1(U, x) \to \operatorname{colim}_{U\in\mathscr{O}} \pi_1(U, x)$ and $\pi_1(U, x) \to \operatorname{colim}_{(U,\mathscr{S})\in(\mathscr{O},\mathscr{F})} \pi_1(U, x)$ for all U, as well as the unique morphism $\operatorname{colim}_{U\in\mathscr{O}} \pi_1(U, x) \to \operatorname{colim}_{(U,\mathscr{S})\in(\mathscr{O},\mathscr{F})} \pi_1(U, x)$. On the other hand, projection to the first coordinate defines a functor $(\mathscr{O}, \mathscr{F}) \to \mathscr{O}$, and its composite with $\pi_1(-, x)$ gives rise to $\operatorname{colim}_{(U,\mathscr{S})\in(\mathscr{O},\mathscr{F})} \pi_1(U, x)$. Again the universal properties gives the unique morphism $\operatorname{colim}_{(U,\mathscr{S})\in(\mathscr{O},\mathscr{F})} \pi_1(U, x) \to \operatorname{colim}_{U\in\mathscr{O}} \pi_1(U, x)$. These maps are inverse isomorphisms, as both colimits are initial in the category of cocones over $\pi_1(-, x)$.

□

Remark 6.10. One major simplification in the category theoretical formulation of the van Kampen theorem is to cite the universal property of a colimit, instead of describing and proving the relevant universal properties in more specific terms. It also allows working with the fundamental groupoid first and then restricting to the fundamental group, which offers a richer perspective.

The van Kampen Theorem provides a powerful tool for calculating fundamental groups of various spaces. Consider the following example: let X be the space consisting of two circles that intersect on a single point. We can parameterize it as $X = \{(x, y) \in \mathbb{R}^2 \mid d((x, y), (\frac{1}{2}, 0)) = \frac{1}{2}\} \cup \{(x, y) \in \mathbb{R}^2 \mid d((x, y), (-\frac{1}{2}, 0)) = \frac{1}{2}\}$. Let $x_0 = (0, 0)$ be the basepoint, and let $X_1 = \{(x, y) \in \mathbb{R}^2 \mid x < 0.2\} \cap X$, $X_1 = \{(x, y) \in \mathbb{R}^2 \mid x > -0.2\} \cap X$, $X_0 = X_1 \cap X_2$. Then X_0, X_1 and X_2 are all path-connected, both X_1 and X_2 deformation retracts to S^1, and X_0 deformation retracts to a single point. So by the van Kampen Theorem, we have $\pi_1(X, x_0) \cong \mathbb{Z} * \mathbb{Z}$. This generalizes to the space of n circles intersecting on a single point, which is called the *wedge of n circles*, and to a wedge of general path-connected spaces.

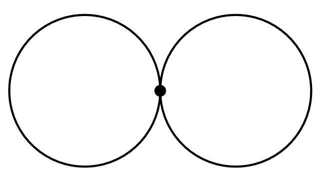

FIGURE 3. A wedge of two circles

However, the van Kampen Theorem does not apply when the open sets in the cover we consider do not have path-connected intersections. An important example is S^1. If we cover it by two open semi-circles, their intersection would be two disjoint open intervals which is not path-connected. This leads to the idea of constructing a "fundamental group with multiple basepoints" and its van Kampen Theorem. This notion is based on the subgroupoids of the fundamental groupoid and deformation retraction between groupoids, which is an analog of the skeleton of a groupoid and the inverse functor to its inclusion. The theorem itself and its proof are both similar to the finite cover part of Theorem 6.9, and the full detail can be found in [1]. This version of the van Kampen Theorem provides another proof for the fact that the fundamental group of S^1 is isomorphic to $\mathbb{Z}$. It also demonstrates how categorical language and manipulation help formulate and derive results related to the fundamental groupoid and its subgroupoids.

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