AN INTRODUCTION TO SHIMURA VARIETIES

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Abstract. A Shimura variety is a higher-dimensional analog of a modular curve, the theory of which is reformulated by Deligne in the 1970s. In this expository paper, we begin with the definition of general Shimura varieties. Afterward, we turn to introduce special cases of Shimura varieties that serve as the moduli space of a family of abelian varieties. We conclude with a brief discussion of canonical models.

Contents

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1. Introduction

The goal of this paper is to introduce the definition of Shimura varieties with some motivations provided and to establish the modular interpretation of Shimura varieties in some specific cases. Shimura varieties are generalizations of modular curves and have played an important role in the research of number theory. As every abelian variety over $\mathbb{C}$ can be associated with a Hodge structure, the moduli space of a family of abelian varieties over $\mathbb{C}$ can be regarded as a parametrizing space of variation of Hodge structures and thus is a finite union of locally symmetric varieties. A locally symmetric variety is of the form $\Gamma\backslash M$ where M is a hermian symmetric domain and Γ is an arithmetic group acting on M. In the case of modular curves $X_i(N)(i = 0, 1)$, M is the complex upper half plane $\mathcal{H}_1$ and Γ is the congruence group $\Gamma_i(N)$. A hermian symmetric domain can be expressed in terms of algebraic groups as $G(\mathbb{R})/K$ where G is a real adjoint algebraic group and $K \subset G(\mathbb{R})$ is some compact subgroup. According to Deligne's definition in his paper [4], a Shimura variety is defined by a Shimura datum including a reductive group G over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class X of homomorphisms $h : \mathbb{C}^\times \to G(\mathbb{R})$ satisfying certain axioms.

This paper is divided into two parts. In Section 2 and 3, we introduce the definition of Shimura varieties, including realizing the hermitian symmetric domain as a parameter space for Hodge structures, constructing a complex variety from a Shimura datum (G, X) and discussing the first properties of Shimura varieties. In the rest of this paper, we attempt to interpret some Shimura varieties as moduli spaces classifying a family of abelian varieties. In Section 4, we organize the necessary knowledge of abelian varieties. Here we describe how to associate a Hodge structure to every abelian variety over $\mathbb{C}$, introduce the structures of abelian varieties to occur in later sections, and explain the representability of the moduli space of abelian varieties. In Section 5, we survey Siegel and PEL-type Shimura varieties and the corresponding moduli problems. In Section 6, we discuss the notion of canonical models which says that the model of a Shimura variety over its canonical field of definition exists and is uniquely determined by the information of some special points.

1.1. Notations for algebraic groups.

Let G be an algebraic group. G^{ad} is defined to be the quotient of G induced by the adjoint representation of G on its Lie algebra $\mathfrak{g}$. There are some important subgroups of G. G° is the Zariski connected component of G containing the identity. The derived group G^{der} is the subgroup generated by the commutators. $Z(G)$ is the center of G. The radical $R(G)$ is the maximal connected normal solvable subgroup of G and the unipotent radical $R_u(G)$ is given by all the unipotent elements in $R(G)$.

Definition 1.1. Let G be an algebraic group. G is called reductive if $R_u(G) = \{e\}$; G is called semisimple if $R(G) = \{e\}$. G is called simple if there is no nontrivial normal connected subgroup of G.

Proposition 1.2. *Let G be an algebraic group. G is reductive if and only if it is the almost direct product of a torus and a semisimple group. These groups can be given by $Z(G)^\circ$ and G^{der}.*

For a reductive group G, there is a diagram (see [4], 1.1)

$$\begin{tikzcd} & G^{\mathrm{der}} \arrow[d] \arrow[dr] & \\ Z \arrow[r] \arrow[dr] & G \arrow[r, "\mathrm{ad}"] \arrow[d, "v"] & G^{\mathrm{ad}} \\ & T & \end{tikzcd} \tag{1.3}$$

where T is the largest commutative quotient of G. The column and the row are short exact sequences, and the diagonal maps are isogenies with the kernel $Z(G^{\mathrm{der}}) = Z \cap G^{\mathrm{der}}$. It gives an exact sequence $1 \to Z \cap G^{\mathrm{der}} \to Z \times G^{\mathrm{der}} \to G \to 1$.

Proposition 1.4. *Let G be a reductive group, then G^{ad} is semisimple.*

Lemma 1.5. *If G is a semisimple connected Lie group with trivial center, then it is isomorphic to a direct product of simple groups with trivial centers.*

Proof. See [3], IV.14.2. □

This lemma can be applied in particular to $G^{\mathrm{ad}}(\mathbb{R})$ when G is reductive since the real points of an algebraic group over $\mathbb{R}$ yield a real Lie group. Some other results concerning the real points of reductive algebraic groups over $\mathbb{R}$ shall be used later in this article.

Proposition 1.6. *For a surjective homomorphism $\phi : G \to H$ of algebraic groups over $\mathbb{R}$, the map $\phi(\mathbb{R}) : G(\mathbb{R}) \to H(\mathbb{R})$ is surjective.*

Proof. See [8], 5.1. □

Theorem 1.7 (Cartan 1927)**.** *Let G be a semisimple algebraic group over $\mathbb{R}$. If G is simply connected group G, then $G(\mathbb{R})$ is connected.*

Proof. See [12], Theorem 7.6. □

Corollary 1.8. *For a reductive group G over $\mathbb{R}$, $G(\mathbb{R})$ has only finitely many connected components (for the real topology).*

Let G be an algebraic group over $\mathbb{Q}$, the cohomology group $H^i(\mathbb{Q}, G)$ is defined to be $H^i(\mathrm{Gal}(\mathbb{Q}^a/\mathbb{Q}), G(\mathbb{Q}^a))$. Similarly, for a finite field extension $K/\mathbb{Q}$, $H^i(K, G_K)$ is defined. We will say that an algebraic group G over $\mathbb{Q}$ satisfies the Hasse principle if $H^i(\mathbb{Q}, H) \to \prod_{l \leq \infty} H^1(\mathbb{Q}_l, H)$ is injective.

Proposition 1.9. *Let G be a simply connected semisimple algebraic group over $\mathbb{Q}$.*
   (1) For every finite prime l, The group $H^1(\mathbb{Q}_l, G)$ is trivial.
   (2) G satisfies Hasse principle.

Proof. See [12], Theorem 6.4, 6.6. □

2. Variation of Hodge Structures

In this section, we explain Deligne's realization of a hermitian symmetric domain as the parameter space for a variation of Hodge structures.

2.1. The definition of Hodge structure.

Definition 2.1. A Hodge structure is a real vector space V together with a Hodge decomposition, we also write it as V. A Hodge decomposition of V is a decomposition of complex vector space

$$V(\mathbb{C}) = \bigoplus_{p,q\in\mathbb{Z}\times\mathbb{Z}} V^{p,q}$$

such that $V^{p,q}=\overline{V^{q,p}}$. The type of the Hodge structure is $\{(p,q)\in\mathbb{Z}\times\mathbb{Z} \mid V^{p,q}\neq 0\}$.

Let V be a Hodge structure. $\bigoplus_{p,q=n} V^{p,q}$ is stable under complex conjugation, thus there exists a real subspace $V_n\subset V$ such that $V_n(\mathbb{C}) = \bigoplus_{p,q=n} V^{p,q}$. $V = \bigoplus_n V_n$ is called the weight decomposition of V. If $V = V_n$, then the Hodge structure is said to have weight n.

The Hodge filtration associated with a Hodge structure of weight n is

$$F^\bullet : \cdots \supset F^p \supset F^{p+1} \supset \ldots, F^p = \bigoplus_{r\geq p} V^{r,s} \subset V(\mathbb{C}).$$

Let V and W be two Hodge structures. A morphism of Hodge structures is a linear map $V\to W$ sending $V^{p,q}$ to $W^{p,q}$.

Let V and W be two Hodge structures of weight m and n, then the tensor product of V and W is defined to be a Hodge structure of weight $m+n$ on $V\otimes W$ as

$$(V\otimes W)^{p,q} = \bigoplus_{\substack{r+r^{'}=p\\ s+s^{'}=q}} V^{r,s}\otimes W^{r^{'},s^{'}}$$

Definition 2.2. An integral (resp. rational) Hodge structure is a free $\mathbb{Z}$ (resp. $\mathbb{Q}$) module together with a Hodge structure of $V(\mathbb{R})$ such that the weight decomposition is defined over $\mathbb{Q}$.

Example 2.3. A Hodge structure arises from a complex structure of a real vector space. Let J be a complex structure on a real vector space V. We define $V^{-1,0}, V^{0,-1}$ to be the $+i, -i$ eigenspaces of J acting on $V(\mathbb{C})$, then $V(\mathbb{C}) = V^{-1,0}\oplus V^{0,-1}$ gives V a Hodge structure associated with the complex structure.

Conversely, giving a Hodge structure of type $\{(-1,0),(0,-1)\}$ amounts to giving a complex structure. Since in this case, the Hodge filtration is $(F^{-1}\supset F^0\supset F^1) = (V(\mathbb{C})\supset V^{0,-1}\supset 0)$, and $\mathbb{R}$-linear isomorphism $V\to V(\mathbb{C})/F^0$ defines the complex structure on V. Moreover, A integral Hodge structure of type $\{(-1,0),(0,-1)\}$ amounts to giving a lattice in a $\mathbb{C}$-vector space.

The Hodge structures can be interpreted in terms of representations of a torus. Before further illustration, we recall a basic fact about the representations of a torus.

Proposition 2.4. *Let T be a torus over a field k and split over a Galois extension K of k. Let V be a k-vector space with a representation ρ of T_K on $K\otimes V$. Then $K\otimes V = \bigoplus_{\chi\in X^*(T)} V_\chi$. ρ is defined over k if and only if $\sigma(V_\chi) = V_{\sigma\chi}$, for all $\sigma\in \mathrm{Gal}(K/k), \chi\in X^*(T)$.*

Proof. See [9], 12.30. □

Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_m$ be a torus over $\mathbb{R}$, which is called Deligne torus. Then, $\mathbb{S}(\mathbb{R}) = \mathbb{C}^*$ and $\mathbb{S}_\mathbb{C}\simeq \mathbb{G}_m\times\mathbb{G}_m$. The character of $\mathbb{S}_\mathbb{C}$ are the homomorphisms

$(z_1, z_2) \to z_1^r z_2^s$ with $(r, s) \in \mathbb{Z} \times \mathbb{Z}$. Let V be a real vector space. Then to give a Hodge structure on V is the same as to give a representation of $\mathbb{S}$ on V, following Proposition 2.4. Also, the weight construction and the filtration associated with a Hodge structure can be interpreted this way. The explicit correspondences are stated as follows.

Proposition 2.5. *A representation $h : \mathbb{S} \to \mathrm{GL}(V)$ corresponds to a Hodge structure on V in which the Hodge decomposition is $V^{p,q} = \{v \in V(\mathbb{C}) \mid h_{\mathbb{C}}(z_1, z_2)v = z_1^{-p} z_2^{-q} v\}$. The Hodge structure is also denoted by (V, h).*

1. *The weight homomorphism $w : \mathbb{G}_m \to \mathbb{S}$ is defined to be the morphism such that $w(\mathbb{R}) : \mathbb{G}_m(\mathbb{R}) \to \mathbb{S}(\mathbb{R})$ is $r \mapsto r^{-1} : \mathbb{R}^\times \to \mathbb{C}^\times$. Then the weight decomposition corresponds to the representation $w_h := w \circ h : \mathbb{G}_m \to \mathrm{GL}(V)$ in the sense of Proposition 2.4. A rational Hodge structure is equivalent to saying that the w_h is over $\mathbb{Q}$.*
2. *Define μ_h to be the cocharacter of $\mathrm{GL}(V)$ over $\mathbb{C}$ such that $\mu_h(z) = h_{\mathbb{C}}(z, 1)$. Then $F_h^p V = \bigoplus_{r \geq p} V_{\mu_h, -r}$*

Definition 2.6. Let (V, h) be a Hodge structure. Weil operator is defined to be the $\mathbb{R}$-linear map $C = h(i)$. Then C acts as i^{q-p} on $V^{p,q}$ and C^2 acts as $(-1)^n$ on V_n. Let $\mathbb{Q}(m)$ be the unique Hodge structure of weight $-2m$ with underlying vector space $(2\pi i)^m \mathbb{Q}$. $\mathbb{Z}(m)$ and $\mathbb{R}(m)$ is defined similarly.

In the following definitions, (V, h) is assumed to be of weight n. Let $R = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$. A Hodge tensor is a multilinear form $t : V^r \to R$ such that the map $V \otimes V \otimes \cdots \otimes V \to R(-nr/2)$ is a morphism of Hodge structures. A polarization of (V, h) is a Hodge tensor $\psi : V \times V \to \mathbb{R}(-n)$ such that $\psi_C(u, v) := (2\pi i)^n \psi(u, Cv)$ is symmetric and positive-definite. Then ψ is symmetric or alternative according to n is even or odd, which can be directly derived from the definition.

More generally, if (V, h) be an R-Hodge structure of weight n, then a polarization of (V, h) is a bilinear form $\psi : V \times V \to R$ such that $\psi_{\mathbb{R}}$ is a polarization.

Example 2.7. Now we rewrite Example 2.3 from this perspective. A complex structure on a real vector space V, by definition, is a homomorphism $h : \mathbb{C}^\times \to \mathrm{GL}_{\mathbb{R}}(V)$ of $\mathbb{R}$-algebra. h is always derived from a morphism between algebraic groups $h : \mathbb{S} \to \mathrm{GL}(V)$. (V, h) gives the same Hodge structure as in Example 2.3, and the Weil operator C coincides with J in the complex structure. Moreover, the functor $(V, (V^{-1,0}, V^{0,-1})) \to (V, C)$ is an equivalence from the category of real Hodge structures of type $\{(-1, 0), (0, -1)\}$ to the category of real vector spaces with complex structures.

In this very case, a polarization is an alternating bilinear form $\psi : V \times V \to 2\pi i R = R(1)$ such that, for every $u, v \in V(\mathbb{R})$,

$$\begin{aligned} &\psi_{\mathbb{R}}(Ju, Jv) = \psi_{\mathbb{R}}(u, v), \text{ and} \\ &\frac{1}{2\pi i}\psi_{\mathbb{R}}(u, Ju) > 0 \text{ if } u \neq 0 \end{aligned} \tag{2.8}$$

Then the form $u, v \mapsto \psi_J(u, v) = \psi_{\mathbb{R}}(u, Jv)$ is symmetric by the first condition and positive-definite by the second condition.

2.2. **Hermitian symmetric domains.**

Definition 2.9. A (Riemannian, hermitian, ...) manifold M is homogeneous if its automorphism group acts transitively. It is symmetric if for every $p \in M$, there is an involution s_p (called the symmetry at p) having p as its isolated fixed point, where s_p is an involution means s_p is an automorphism of M and $s_p^2 = id_M$. A hermitian symmetric space is defined as a connected symmetric hermitian manifold (M, g) where M is a complex manifold and g is a hermitian metric, whose automorphism group consisting of holomorphic isometries is denoted by $\mathrm{Is}(M, g)$.

Example 2.10. A prototypical example of the hermitian symmetric space is the complex upper half plane $\mathcal{H}_1$ endowed with the metric $\dfrac{dxdy}{y^2}$. The action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathcal{H}_1$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz=d}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}), z \in \mathcal{H} \tag{2.11}$$

gives the isomorphism between $\mathrm{SL}_2(\mathbb{R})/\{\pm I\}$ and $\mathrm{Is}(\mathcal{H}_1, g)$. Notice that in this case, the curvature is negative, which is the situation we shall be mainly interested in.

Proposition 2.12. *Let (M, g) be a symmetric space. Then $\mathrm{Is}(M, g)$ can be endowed with compact-open topology and get a natural structure of a Lie group. The compact-open topology is generated by the open subsets of the form $W(C, U) = \{g \in \mathrm{Is}(M, g) \mid g(C) \subset U\}$ where C and U are respectively the compact and open subsets of M.*

Proposition 2.13. *Let (M, g) be a symmetric space. Then the following properties hold:*

(1) $\mathrm{Is}(M, g)$ *is a finite-dimensional Lie group.*
(2) $\mathrm{Is}(M, g)^+$ *denoted the identity component of* $\mathrm{Is}(M, g)$ *acts transitively on M.*
(3) *$p \in M$, the subgroup K_p of* $\mathrm{Is}(M, g)$ *fixing p is compact.*
(4) *The natural map* $\mathrm{Is}(M, g)^+/K_p \to M$ *is an isomorphism of smooth manifolds. On passing to the tangent spaces,* $\mathrm{Lie}(\mathrm{Is}(M, g))/\mathrm{Lie}(K_p) \simeq \mathrm{Tgt}_p D$.

Proof. See [8], 1.5. □

Hermitian symmetric spaces are classified according to the curvature of the metric g.

Proposition 2.14. *There are three families of hermitian symmetric spaces.*

(1) *(M, g) is called noncompact type if it is of negative curvature, which is also called hermitian symmetric domain. Then, M is simply connected and* $\mathrm{Is}(M, g)^+$ *is adjoint and noncompact. An example is $\mathcal{H}$.*
(2) *(M, g) is called compact type if it is of positive curvature. Then, M is simply connected and* $\mathrm{Is}(M, g)^+$ *is adjoint and compact. An example is* $\mathbb{P}^1(\mathbb{C})$.
(3) *(M, g) is called euclidean if it is of zero curvature. M is euclidean if and only if it is a quotient of $\mathbb{C}^g$ by a discrete subgroup of translations. So M is obviously not necessarily simply connected, since an example is $\mathbb{C}/\Lambda$.*

Every hermitian symmetric space decomposes into a product $M^0 \times M^- \times M^+$ where M^0, M^-, M^+ are respectively euclidean, noncompact type, and compact type.

Proof. See [7], VIII. □

Remark 2.15 ([7], VIII)**.** A hermitian symmetric domain is a simply connected complex manifold. What about the converse? It can be shown every symmetric bounded domain in $\mathbb{C}^n$ can be endowed with a structure of symmetric hermitian domain. Here the first "symmetric" means the space is symmetric under holomorphic automorphisms. We begin by introducing a canonical metric on a bounded domain.

Theorem 2.16. *Every bounded domain (a domain means a nonempty open connected subset) has a canonical hermitian metric called the Bergman metric. This metric has negative curvature.*

The construction of the Bergman metric is outlined as follows. Let $D \subset V$ be a domain and $H(D)$ be the Hilbert space consisting of the holomorphic square-integrable functions on D with inner product $(f \mid g) = \int_D f\overline{g}dv$. Then the Bergman kernel function $K : D \times D \to \mathbb{C}$ is defined to be $K(z,\zeta) = \sum_m e_m(z) \cdot \overline{e_m(\zeta)}$ where $(e_m)_{m\in\mathbb{Z}}$ is a complete orthonormal set in $H(D)$. It is the unique function satisfying the following properties: (1) For any $\zeta \in D$, $(z \mapsto K(z,\zeta)) \in H(D)$; (2) $K(z,\zeta) = \overline{K(\zeta,z)}$; (3) $f(z) = \int_D K(z,\zeta)f(\zeta)dv(\zeta)$ for all $f \in H(D)$. Define the hermitian metric to be $h = \sum \dfrac{\partial^2}{\partial z^i \partial \bar{z}^j} \log K(z,z) dz^i d\bar{z}^j$.

Since the Bergman metric is canonical, it is invariant under the action of $\mathrm{Hol}(D)$. Thus, a symmetric bounded domain D equipped with the Bergman metric becomes a hermitian symmetric domain, and $\mathrm{Is}(D,h) = \mathrm{Hol}(D)$. Conversely, it is known that every hermitian symmetric domain has a unique hermitian metric such that every embedding into some V as a bounded symmetric domain (these embeddings exist) maps this metric to the Bergman metric up to a multiple on each irreducible factor.

The rest of this section concentrates on reconstructing hermitian symmetric domains in terms of algebraic groups.

Proposition 2.17. *Let (M,g) be a hermitian symmetric domain. Then,* $\mathrm{Is}(M^\infty,g)^+ = \mathrm{Is}(M,g)^\infty = \mathrm{Hol}(M)^+$. *Hence,* $\mathrm{Hol}(M)^+$ *acts transitively on M, K_p the stabilizer of p in* $\mathrm{Hol}(M)^+$ *is compact, and* $\mathrm{Hol}(M)^+/K_p \simeq K^\infty$.

Proof. See [7], VIII, 4.3. □

Proposition 2.18. *Let (M,g) be a hermitian symmetric domain, and let $\mathfrak{h}$ denote the Lie algebra of* $\mathrm{Hol}(M)^+$. *There exists a unique connected algebraic subgroup G of* $\mathrm{GL}(\mathfrak{h})$ *such that inside* $\mathrm{GL}(\mathfrak{h})$, $G(\mathbb{R})^+ = \mathrm{Hol}(M)^+$. *Also,* $G(\mathbb{R})^+ = \mathrm{Hol}(M) \cap G(\mathbb{R})$

Proof. The finite-dimensional Lie group $\mathrm{Hol}(M)^+$ is adjoint, thus the adjoint representation realizes it as a subgroup of $GL(\mathfrak{h})$. There exists an algebraic group $G \subset \mathrm{GL}(V)$ such that $\mathrm{Lie}(G) = [\mathfrak{h},\mathfrak{h}]$ (see [3], 7.9). Since $\mathrm{Hol}(M)^+$ is semisimple, $[\mathfrak{h},\mathfrak{h}] = \mathfrak{h}$, then $\mathrm{Lie}(G) = \mathfrak{h}$, which proves the first statement. The second follows from [13], 8.5. □

Therefore, the real smooth manifold structure of a hermitian symmetric domain M is $M^\infty = G(\mathbb{R})/K_p$ where G is an algebraic group and K_p is a compact closed subgroup of $G(\mathbb{R})$. The problem left is how to interpret the complex structure of M. This is completed with the observation that there is complex rotation on every point of a hermitian symmetric domain.

Let $U_1 = \{z \in \mathbb{C} \mid |z| = 1\}$ be the unit circle which is viewed as an algebraic group over $\mathbb{R}$.

Theorem 2.19. *Let D be a hermitian symmetric domain. For each $p \in D$, there exists a unique homomorphism $u_p : U_1 \to \mathrm{Hol}(D)$ such that $u_p(z)$ fixes p and acts on $\mathrm{Tgt}_p(D)$ as multiplication by z.*

The proof of this theorem is omitted (see [8], 1.9), which depends on the fact that symmetric spaces are geodesically complete. However, one can easily understand the idea and see how this works on $\mathcal{H}$. Let $p = i \in \mathcal{H}$, then $u_i(z) = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ where $a + bi$ is a square root of z.

Conversely, to depict which representation $u_p : U_1 \to \mathrm{Hol}(D)^+ \simeq G(\mathbb{R})^+$ is derived from the complex structure, the notion of Cartan involutions is introduced.

Definition 2.20. Let G be a connected algebraic group over $\mathbb{R}$, and let $g \mapsto \overline{g}$ denote complex conjugation on $G(\mathbb{C})$. An involution θ of G is said to be Cartan if the Lie group $G^{(\theta)}(\mathbb{R}) := \{g \in G(\mathbb{C}) \mid g = \theta(\overline{g})\}$ is compact.

Theorem 2.21. *Let G be a connected algebraic group over $\mathbb{R}$. There exists a Cartan involution of G if and only if G is reductive, in which case any two are conjugate by an element of $G(\mathbb{R})$.*

Proof. See [13], I 4.3. □

Example 2.22. Let G be a connected algebraic group over $\mathbb{R}$. We say that G is compact if $G(\mathbb{R})$ is compact.

(1) The identity map is a Cartan involution if and only if G is compact.
(2) ([13], I 4.4) If G allows a faithful representation $G \hookrightarrow \mathrm{GL}(V)$ with V a real vector space. Then G is reductive if and only if G is stable under $g \mapsto g^t$ for a suitable choice of a basis for V, in which case all Cartan involution arises from $g \mapsto (g^t)^{-1}$.
(3) Let θ be an involution of G. There is a unique real form $G^{(\theta)}$ of $G_{\mathbb{C}}$ such that complex conjugation on $G^{(\theta)}(\mathbb{C})$ is $g \mapsto \theta(\overline{g})$, and all compact real forms of $G_{\mathbb{C}}$ arise in this way.

If G is a compact real algebraic group, every finite-dimensional real representation $G \to \mathrm{GL}(V)$ carries a G-invariant positive-definite symmetric bilinear form. Choose any positive-definite symmetric bilinear form $f(\cdot,\cdot)$ and an invariant measure dg on G, then the form $f_G(\cdot,\cdot) = \int_G f(\cdot,\cdot)dg$ is a desired one. Conversely, if the faithfully finite-dimensional real representation of G carries such a form, then G is compact. The criterion of Cartan involution in Proposition 2.24 is based on this fact.

Definition 2.23. Let G be a real algebraic group, and C be an element of $G(\mathbb{R})$ such that C^2 is in the center of $G(\mathbb{R})$ (i.e. $\mathrm{ad}(C)$ is an involution). A C-polarization on a real representation V of G is a G-invariant bilinear form ϕ such that the form $\phi_C : (u, v) \mapsto \phi(u, Cv)$ is symmetric and positive-definite.

Proposition 2.24. *If $\mathrm{ad}(C)$ is a Cartan involution of G, then every finite-dimensional real representation carries a C-polarization; conversely, if a faithful finite-dimensional real representation carries a C-polarization, $\mathrm{ad}(C)$ is a Cartan involution.*

Proof. Let $G \to \mathrm{GL}(V)$ be a finite-dimensional real representation of G and ϕ be a G invariant bilinear form on V. ϕ' has a sesquilinear extension of ϕ on $V(\mathbb{C})$ such that $\phi'(u,v) = \phi_{\mathbb{C}}(u,\overline{v})$. Moreover, ϕ' is hermitian (and positive-definite) if and only if ϕ is symmetric (and positive-definite). By the definition of ϕ', $\phi'(gu,\overline{g}v) = \phi'(u,v)$ for all $g \in G(\mathbb{C}), u,v \in V(\mathbb{C})$. Then, $\phi'(gu, C(C^{-1}\overline{g}C)v) = \phi'(u,Cv)$. Thus, ϕ'_C is invariant under $G^{(\mathrm{ad}C)}$. With these in mind, the equivalent relationship in the statement clearly holds. □

Since all the homomorphism $U_1 \to \mathrm{GL}(V)$ of real Lie groups is algebraic, u_p constructed in Theorem 2.19 is algebraic. We finally obtain the following theorem to construct and classify hermitian symmetric domains in terms of real algebraic groups.

Theorem 2.25. *Let D be a hermitian symmetric domain, and let G be the associated real adjoint algebraic group. The homomorphism $u_p : U_1 \to G$ attached to a point p of D has the following properties:*

(a) Only the characters $z, 1, z^{-1}$ occur in the representation of U_1 on $\mathrm{Lie}(G)_{\mathbb{C}}$ defined by $\mathrm{Ad} \circ u_p$.
(b) $ad(u_p(-1))$ is a Cartan involution.
(c) u_p does not project to 1 in any simple factor of G.

Conversely, let G be a real adjoint algebraic group, and let $u : U_1 \to G$ satisfy the above properties. Then the set D of conjugates of u by elements of $G(\mathbb{R})^+$ has a natural structure of a hermitian symmetric domain for which $G(\mathbb{R})^+ = \mathrm{Hol}(D)^+$ and $u(-1)$ is the symmetry at u.

Proof. Let D be a hermitian symmetric domain, G and K_p be defined as before. Then $u_p|_{K_p}$ is trivial and $u_p(z)$ acts on $\mathrm{Lie}(G)/\mathrm{Lie}(K_p) \simeq \mathrm{Tgt}_p D$ as multiplication by z. So (a) holds. The (b) follows from the fact that the symmetry at a point of a symmetric space gives a Cartan involution of G if and only if the space has negative curvature (see [7], V 2). By (b), if u_p projects to the identity map in a simple factor, the factor should be of compact type, contradicting the assumption. This proves (c).

For the converse, let D be the set of $G(\mathbb{R})^+$-conjugates of u. The centralizer K_u of u in $G(\mathbb{R})^+$ is closed subgroup and contained in $\{g \in G(\mathbb{C}) \mid g = u(-1)\cdot\overline{g}\cdot u(-1)^{-1}\}$. So according to the condition (b), it is compact. The expression $D \simeq G(\mathbb{R})^+/K_u$ brings D a structure of a real smooth manifold. Since $\mathrm{Tgt}_u(D) \simeq \mathrm{Lie}(G)/\mathrm{Lie}(K_u) = \mathrm{Lie}(G)/\mathrm{Lie}(G)^o$, (a) gives $\mathrm{Tgt}_u(D)$ the structure of $\mathbb{C}$ vector space. Then the homogeneity of D endows D with an almost-complex structure, which can be proved integrable (see [16]). Then it will make D into a complex manifold. Because K_u acts on the tangent space $\mathrm{Tgt}_u D$ and K_u is compact, there is a K_u-invariant positive-definite bilinear form on $\mathrm{Tgt}_u D$, then it becomes a hermitian metric on $\mathrm{Tgt}_u D$ since $u(i) \in K_p$. Because of the homogeneity of D, D then becomes a hermitian symmetric space, which is a hermitian symmetric domain because each simple factor of its automorphism group is a noncompact semisimple group because of condition (b, c).

□

2.3. **Variation of Hodge structures.** Before our discussion of the variation of Hodge structures, we have to introduce several definitions.

When n is an integer, let $G_d(V(\mathbb{C}))(0 < d < n)$ be the Grassmanian, the $\mathbb{C}$-points on which classifies the set of d-dimensional subspaces of V. When $\mathbf{d} = (d_1, \cdots, d_r)$ with $n > d_1 > \cdots > d_r > 0$, let $G_{\mathbf{d}}(V(\mathbb{C}))$ be the flag variety whose $\mathbb{C}$-points classifies the set of flags $F : V \supset V^1 \supset \cdots \supset V^r \supset 0, \dim V^i = d_i$. There is a natural map $G_{\mathbf{d}}(V(\mathbb{C})) \to \prod_i G_{d_i}(V(\mathbb{C})) : F \mapsto (V^i)$.

Definition 2.26. let S be connected complex manifold and V a real vector space of dimension . For every point $s \in S$. we have a Hodge structure h_s on V of weight n (independent of s). Let $V_s^{p,q} = V_{h_s}^{p,q}$ and $F_s^p = F_s^p V = F_{h_s}^p$.

The family of Hodge structures $(h_s)_{s\in S}$ on V is called continuous if, for fixed p and q, the dimension $d(p, q)$ is constant and the map $S \to G_{d(p,q)}(V(\mathbb{C})) : s \mapsto V_s^{p,q}$ is continuous.

The family is said to be holomorphic if the map $\phi : S \to G_{\mathbf{d}}(V(\mathbb{C}))$ is holomorphic, $\mathbf{d} = (\cdots, d(p), \cdots)$, $d(p) = \dim F_s^p V$. Then the differential of ϕ gives the morphism $d\phi_s : \mathrm{Tgt}_s S \to \mathrm{Tgt}_{F_s^\bullet}(G_{\mathbf{d}}(V(\mathbb{C}))) \subset \bigoplus_p \mathrm{Hom}(F_s^p, V(\mathbb{C})/F_s^p)$.

The family is said to satisfy Griffiths transversality if the image of $d\phi_s$ is contained in $\mathrm{Hom}(F_s^p, F_s^{p-1}/F_s^p)$ for all s. When a family satisfies Griffith transversality, it is called a variation of Hodge structures.

Let V be a real vector space, and let T be a family of tensors on V including a nondegenerate bilinear form t_0. Let $d : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ be a symmetric function such that: (1) for almost all but finite p, q, $d(p, q) = 0$, and (2) $d(p, q) = 0$ unless $p + q = n$. Now we define $S(d, T)$ to be the set of all Hodge structures h on V such that:

(1) $\dim V_h^{p,q} = d(p, q)$ for all p, q;
(2) each $t \in T$ is a Hodge tensor for h;
(3) t_0 is a polarization for h.

The topology of $S(d, T)$ comes from its natural embedding in $\prod_{d(p,q)\neq 0} G_{d(p,q)}(V(\mathbb{C}))$.

Theorem 2.27. *Let S^+ be a connected component of $S(d, T)$. Then the following properties hold.*

(1) The space S^+ has a unique complex structure with which (h_s) is a holomorphic family of Hodge structure.
(2) With this complex structure, S^+ is a hermitian symmetric domain if (h_s) is a variation of Hodge structure.
(3) Every irreducible hermitian symmetric domain is of the form S^+ for a suitable choice of V, d and T.

Proof of Theorem 2.27. (1) Because the Hodge filtration determines the Hodge decomposition, the map ϕ (defined in Definition 2.26) is injective. Let G be the smallest algebraic subgroup of GL_V such that $h(\mathbb{S}) \subset G$ for all $h \in S^+$. Take a $h_0 \in S^+$, then $G(\mathbb{R})^+$ can act on S^+ by conjugation. Following from an argument of Deligne, this action is transitive. Let K_0 be the closed subgroup of $G(\mathbb{R})^+$ fixing h_0, then $S^+ = G(\mathbb{R})^+/K_0$ is a smooth manifold. G is a closed subgroup in $\mathrm{GL}(V)$, which gives the embedding $\mathfrak{g} = \mathrm{Lie}(G) \hookrightarrow \mathrm{End}(V)$ and it is equivariant for the adjoint action of G on both sides. Then h_0(seen as a homomorphism $\mathbb{S} \to G$) makes the injection an inclusion of Hodge structures. Clearly, $\mathfrak{g}^{00} = \mathrm{Lie}(K_0)$. Therefore, $\mathrm{Tgt}_{h_0} S^+ \simeq \mathfrak{g}/\mathfrak{g}^{00} \simeq \mathfrak{g}_{\mathbb{C}}/F^0$. $\mathfrak{g}_{\mathbb{C}}/F^0$ is a complex vector space and by the inclusion of Hodge structures, it is a complex subspace of $\mathrm{End}(V(\mathbb{C}))/F^0 \simeq \mathrm{Tgt}_{h_0}(G_{\mathbf{d}}(V(\mathbb{C})))$. The composition map $\mathfrak{g}/\mathfrak{g}^{00} \hookrightarrow \mathrm{End}(V(\mathbb{C}))/F^0$ is just $(d\phi)_{h_0}$. As this works for all

$h_0 \in S^+$, S^+ acquires an almost complex structure, which can be shown integrable. So, S^+ is equipped with a complex structure, and the above discussion explains that with this complex structure, ϕ is holomorphic. It is clear that this complex structure is the unique one making the conditions true.

(2) As the theorem 2.25 says, we just need to construct $U_1 \to G^{ad}$ satisfying some conditions. Since $h_0(r) \in Z(G)$ for $r \in \mathbb{R}$, $z \mapsto h_0(\sqrt{z})$ is a well defined homomorphism $u_0 : U_1 \to G^{\mathrm{ad}}$. Let $C = u_0(-1) = h_0(i)$. The faithful representation $G \to \mathrm{GL}(V)$ carries a C-polarization, namely t_0. Thus, $\mathrm{ad}C$ is a Cartan involution on G. Griffiths transversality guarantees that the image of $(d\phi)_{h_0} : \mathfrak{g}/\mathfrak{g}^{00} \hookrightarrow \mathrm{End}(V(\mathbb{C}))/F^0$ is in $F^{-1}\mathrm{End}(V(\mathbb{C}))/F^0\mathrm{End}(V(\mathbb{C}))$. Therefore, (G, u_0) satisfies condition (b) in theorem 2.25. As condition (c) is obviously satisfied, it is concluded that S^+ is the hermitian symmetric domain associated to (G, u_0).

(3) Let D be an irreducible hermitian symmetric domain attached to (G, u_0) where G is a real adjoint algebraic group and $u_0 : U_1 \to G$ is such that only the characters $z, 1, z^{-1}$ occur in $\mathrm{Ad} \circ u_0$ and $\mathrm{ad} \circ u_0(-1)$ is a Cartan involution (see Theorem 2.25). Example 2.22 (b) shows that G admits a self-dual faithful representation $G \to \mathrm{GL}(V)$. Thus, there exists a G-invariant nondegenerate bilinear form t_0. Find a set of tensors T containing t_0 as in the following proposition.

Proposition 2.28. *For any faithful self-dual representation $G \to \mathrm{GL}(V)$, there exists a finite set T of tensors of V such that G is the subgroup of $\mathrm{GL}(V)$ fixing the t in T.*

Let h_0 be the composition of the morphisms $\mathbb{S} \xrightarrow{z \mapsto z/\bar{z}} U_1 \xrightarrow{u_0} G \to \mathrm{GL}(V)$. Then it defines a Hodge structure on V such that each $t \in T$ is a Hodge tensor for h_0 and t_0 is a polarization for h. It also can be checked that D is naturally identified with a component of $S(d, T)$ via this composition. ☐

Let S be a complex manifold and F be a local system of $\mathbb{Z}$-modules on S. Suppose a Hodge structure h_s is assigned to $F_s \otimes \mathbb{R}$ for every $s \in S$. Then F and the family of Hodge structures is said to be a variation of integral Hodge structures on S if $(F \otimes \mathbb{R}, (h_s))$ is a variation of Hodge structures on every open subset on which the local system F is trivial. Therefore, when S is simply connected, this definition is simply the previous one. In this more general case, we can consider the universal covering $T \to S$ and lift the variation of Hodge structures to the simply connected space T.

2.4. **Locally symmetric varieties.** Let us now consider locally symmetric varieties, namely the space of the form $\Gamma\backslash D$ where D is a hermitian symmetric domain and Γ is an arithmetic group with a canonical structure of algebraic variety to be introduced.

We demonstrate the following geometric property of such space. The proof is not hard and omitted.

Proposition 2.29. *Let D be a hermitian symmetric domain and Γ be a discrete subgroup of $\mathrm{Hol}(D)^+$. If Γ is torsion-free, then Γ acts freely on D. Also, there is a unique complex structure on $\Gamma\backslash D$ for which the quotient map $\pi : D \to \Gamma\backslash D$ is a local isomorphism.*

To invoke the discussion of arithmetic subgroups, we give the following definitions.

Definition 2.30. Two subgroups S_1 and S_2 of a group H are commensurable if $S_1 \cap S_2$ has finite index in both S_1 and S_2. Commensurability is an equivalence relationship.

Let G be an algebraic group over $\mathbb{Q}$. A subgroup Γ of $G(\mathbb{Q})$ is arithmetic if it is commensurable with $G(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})$ for a embedding $G \hookrightarrow \mathrm{GL}_n$. This definition is independent of the choice of the embedding.

Let H be a real Lie group. A subgroup Γ of H is arithmetic if there exists an algebraic group G over $\mathbb{Q}$, a surjective homomorphism $G(\mathbb{R})^+ \to \mathrm{Hol}(D)^+$ with compact kernel, and an arithmetic subgroup Γ_0 of $G(\mathbb{Q})$ such that $\Gamma_0 \cap G(\mathbb{R})^+$ maps onto Γ.

We proceed to discuss a few properties of arithmetic groups.

Proposition 2.31. *Let $\rho : G \to G'$ be a surjective homomorphism. If $\Gamma \subset G(\mathbb{R})$ is arithmetic then so is $\rho(\Gamma) \subset G'(\mathbb{Q})$.*

Proof. See [12], Theorem 4.1. □

Under the condition that $G \to G'$ is surjective, $G(\mathbb{Q})^+ \to G'(\mathbb{Q})^+$ is far from surjective. An example is $\mathrm{SL}_2(\mathbb{Q})^+ \to \mathrm{PGL}_2(\mathbb{Q})^+$, as an image element can be represented by a matrix with determinant in $\mathbb{Q}^{\times 2}$. However, the arithmetic subgroups are preserved under the isogenies.

Let Γ be an arithmetic subgroup of $G(\mathbb{Q})$, Γ is discrete but does not necessarily have finite covolume in $G(\mathbb{R})$. A counterexample is $G = \mathbb{G}_m/\mathbb{Q}$ and $\Gamma = \{\pm 1\} \subset \mathbb{G}_m(\mathbb{Q})$. Thus, if Γ is of finite covolume, there should not exist nonzero homomorphism $G \to \mathbb{G}_m$. In the case of G reductive, this is sufficient, and with some more conditions, we can even tell if the space $\Gamma\backslash D$ is compact.

Theorem 2.32. *Let G be a reductive group over $\mathbb{Q}$ and Γ be an arithmetic subgroup of $G(\mathbb{Q})$.*

1. *The space $\Gamma\backslash G(\mathbb{R})$ has finite volume if and only if $\mathrm{Hom}(G, \mathbb{G}_m) = 0$. In particular, this condition holds if G is semisimple.*
2. *$\Gamma\backslash G(\mathbb{R})$ is compact if and only if $\mathrm{Hom}(G, \mathbb{G}_m) = 0$ and $G(\mathbb{Q})$ contains no unipotent element.*

Proof. See [12], Theorem 4.12, 4.13. □

Proposition 2.33. *Let H be a semisimple real Lie group that admits a faithful finite-dimensional representation. Every arithmetic subgroup Γ of H is discrete of finite covolume.*

Proof. See [8], 3.6. □

The natural examples of arithmetic subgroups are congruence subgroups.

Definition 2.34. Let G be a reductive algebraic group over $\mathbb{Q}$, and choose one embedding $G \hookrightarrow \mathrm{GL}_n$. Define $\Gamma(N) = G(\mathbb{Q}) \cap \{g \in \mathrm{GL}_n(\mathbb{Z}) \mid g \equiv I_n \mod N\}$. A congruence subgroup of $G(\mathbb{Q})$ is any subgroup containing some $\Gamma(N)$ as a subgroup of finite index.

This definition is actually independent of the choice of embedding. Unlike arithmetic subgroups, the image of a congruence subgroup under an isogeny does not need to be congruence.

Remark 2.35. Are all arithmetic subgroups congruence subgroups? For split simply connected groups other than SL_2, this is true, but SL_2 and all nonsimple connected groups have many noncongruence arithmetic subgroups.

When does a discrete subgroup of finite covolume become an arithmetic group? A Theorem of Margulis states that for a noncompact simple real Lie group except those isogenous to $\mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$, all discrete subgroups of finite covolume are arithmetic. It fails in the case of SL_2 can be shown easily.

So far, we talk about $D(\Gamma) = \Gamma\backslash D$ as a complex manifold. The next two key theorems endow it with the canonical structure of an algebraic variety.

Theorem 2.36 (Barly and Borel 1966, [1])**.** *Let $D(\Gamma) = \Gamma\backslash D$ be the quotient of a hermitian symmetric domain D by a torsion-free arithmetic subgroup Γ of $\mathrm{Hol}(D)^+$. Then $D(\Gamma)$ has a canonical realization as a Zariski-open subset of a projective algebraic variety $D(\Gamma)^*$.*

Definition 2.37. An algebraic variety $D(\Gamma)$ arising as in the theorem is called a locally symmetric variety.

This is called Barly and Borel compactification. Recall the case of modular curves. We set $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$ and extend the action of Γ on $\mathcal{H}$ to $\mathcal{H}^*$. Then $\Gamma\backslash\mathcal{H}^*$ is a compact Riemann surface, and the modular forms of high levels embed it as a projective algebraic variety. $\Gamma\backslash\mathcal{H}$ is this one-dimensional variety with finite points moitted, then is a quasi-projective variety. In the general case, the proof is similar. The variety $D(\Gamma)^*$ is equal to $\mathrm{Proj}(\bigoplus_{n\geq 0} A_n)$ where A_n is the vector space of automorphic forms for the n-th power of the canonical automorphy factor.

Theorem 2.38 (Borel 1972, [2])**.** *Keep the notations of Theorem 2.36. Let V be a nonsingular quasi-projective variety over $\mathbb{C}$. Then every holomorphic map $f : V^{\mathrm{an}} \to D(\Gamma)^{\mathrm{an}}$ is regular.*

Corollary 2.39. *The structure of algebraic variety on $D(\Gamma)$ is unique.*

Remark 2.40. Theorem 2.36 holds when Γ is torsion, but Theorem 2.38 fails then.

Theorem 2.41. *Let $D(\Gamma)$ be the quotient of a hermitian symmetric domain D by a torsion-free arithmetic group Γ of $\mathrm{Hol}(D)^+$. Then $D(\Gamma)$ has only finitely many automorphisms.*

Proof. See [8], 3.21. □

3. Definition of Shimura Varieties

3.1. Definition of connected Shimura varieties.

Definition 3.1. A connected Shimura datum is a pair (G, D) consisting of a semisimple algebraic group G over $\mathbb{Q}$ and a $G^{\mathrm{ad}}(\mathbb{R})$-conjugacy class D of homomorphism $u : U_1 \to G^{\mathrm{ad}}_{\mathbb{R}}$ satisfying the following conditions:

SU1 for all $u \in D$, only the characters $z, 1, z^{-1}$ occur in the representation of U_1 on $\mathrm{Lie}(G^{\mathrm{ad}})_{\mathbb{C}}$ defined by $\mathrm{Ad} \circ u$;
SU2 for all $u \in D$, $\mathrm{ad}(u(-1))$ is a Cartan involution on $G^{\mathrm{ad}}_{\mathbb{R}}$;
SU3 G^{ad} has no $\mathbb{Q}$-factor H such that $H(\mathbb{R})$ is compact.

The conditions (SU1) and (SU2) are the same as the requirement listed in Theorem 2.25. But the condition (SU3) is somehow different from what we have stated before since H is restricted in the $\mathbb{Q}$ factor of G. It says G is of noncompact type. This is an innocent assumption because we can replace G with its quotient by a compact normal subgroup over $\mathbb{Q}$ and it changes things a little.

Recall the exact sequence of algebraic groups over $\mathbb{R}$

$$0 \to \mathbb{G}_m \xrightarrow{w} \mathbb{S} \to U_1 \to 0,$$

where the maps on $\mathbb{R}$-points are $0 \to \mathbb{R}^\times \xrightarrow{r \mapsto r^{-1}} \mathbb{C}^\times \xrightarrow{z \mapsto z/\bar{z}} U_1 \to 0$. Then a homomorphism $u : U_1 \to G^{\mathrm{ad}}_{\mathbb{R}}$ satisfying (SU1) is the same as having a homomorphism $h : \mathbb{S} \to G^{\mathrm{ad}}_{\mathbb{R}}$ such that only characters $z/\bar{z}, 1, \bar{z}/z$ occur in the representation of $\mathbb{S}$ on $\mathrm{Lie}(G^{\mathrm{ad}})_{\mathbb{C}}$ defined by $\mathrm{Ad} \circ h$. In fact, given such a u, h is just the composition of u with the second map in the exact sequence. Conversely, given such a h, as it is of weight 0 it arises from a u that satisfies (SU1) and $u(z) = h(\sqrt{z})$ which is independent of the choice of square root. Therefore, we have the following alternative definition of connected Shimura datum.

Definition 3.2. A connected Shimura datum is a pair (G, D) consisting of a semisimple algebraic group G over $\mathbb{Q}$ and a $G^{\mathrm{ad}}(\mathbb{R})$-conjugacy class D of homomorphism $h : \mathbb{S} \to G^{\mathrm{ad}}_{\mathbb{R}}$ satisfying the following conditions:

SV1 for all $u \in D$, only the characters $z/\bar{z}, 1, \bar{z}/z$ occur in the representation of $\mathbb{S}$ on $\mathrm{Lie}(G^{\mathrm{ad}})_{\mathbb{C}}$ defined by $\mathrm{Ad} \circ h$;
SV2 for all $h \in D$, $\mathrm{ad}(h(i))$ is a Cartan involution on $G^{\mathrm{ad}}_{\mathbb{R}}$;
SV3 G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial.

Then equivalence of (SU3) and (SV3) under the other two conditions comes from the following lemma which just restates what we have discussed in Section 2.

Lemma 3.3. *Let H be an adjoint real Lie group, and let $u : U_1 \to H$ be a homomorphism satisfying SU1,2. Then the following conditions are equivalent:*

(1) $u(-1) = 1$;
(2) u is trivial;
(3) H is compact.

Proposition 3.4. *Giving a connected Shimura datum is the same as giving*

(1) a semisimple algebraic group G over $\mathbb{Q}$ of the noncompact type,
(2) a hermitian symmetric domain D, and
(3) an action of $G(\mathbb{R})^+$ on D defined by a surjective homomorphism $G^{\mathrm{ad}}(\mathbb{R})^+ \to \mathrm{Hol}(D)^+$ with compact kernel.

Proof. Assume (G, D) is a connected Shimura datum. Decompose $G^{\mathrm{ad}}_{\mathbb{R}} = H_1 \times \cdots \times H_s$ where H_i are simple factors. Let u_i be the projective of u into the factor $H_i(\mathbb{R})$. If H_i is compact, then $u_i = 1$ following the above proposition. Otherwise, (H_i, u_i) corresponds to a hermitian symmetric domain D_i and $H_i(\mathbb{R})^+ = \mathrm{Hol}(D_i')^+$. Let D be the product of D_i for all i such that H_i is not compact. Moreover, there is a surjective morphism $G^{\mathrm{ad}}(\mathbb{R})^+ \to \mathrm{Hol}(D)^+$ with compact kernel. The converse is similar. $\square$

Definition 3.5. Let (G, D) be a connected Shimura datum. A connected Shimura variety relative to (G, D) is an algebraic variety of the form $D(\Gamma)$ where Γ an arithmetic subgroup of $G^{\mathrm{ad}}(\mathbb{Q})^+$ containing the image of a congruence subgroup of $G(\mathbb{Q})^+$ such that its image $\overline{\Gamma}$ in $\mathrm{Hol}(D)$ torsion-free. These algebraic varieties form an inverse system with respect to the inclusion of arithmetic subgroups. This inverse system is denoted $\mathrm{Sh}^\circ(G, D)$, called the connected Shimura variety attached to (G, D).

Because the map $G^{\mathrm{ad}}(\mathbb{R})^+ \to \mathrm{Hol}(D)^+$ has compact kernel, then its intersection with the discrete subgroup Γ is finite, which means the kernel of map $\Gamma \to \overline{\Gamma}$ is finite. By definition, $\overline{\Gamma}$ is an arithmetic subgroup in $\mathrm{Hol}(D)^+$. Then Theorem 2.36, 2.38 apply to $D(\Gamma) = \overline{\Gamma}\backslash D$. In particular, $D(\Gamma)$ is endowed with the unique structure of an algebraic variety, and the natural quotient map $D(\Gamma) \to D(\Gamma)$ is regular with $\Gamma' \subset \Gamma$. Thus, this family of algebraic varieties $D(\Gamma)$ forms an inverse system. Moreover, the varieties $\Gamma\backslash D$ with Γ a congruence subgroup of $G(\mathbb{Q})^+$ are cofinal in this inverse system.

In the definition of connected Shimura varieties, we ask Γ to contain the image of a congruence group of $G(\mathbb{Q})^+$. This condition is equivalent to the inverse image of Γ is a congruence group, which can be deduced from Proposition 2.31. Because $\pi : G(\mathbb{Q})^+ \to G^{\mathrm{ad}}(\mathbb{Q})^+$ is usually far from surjective, as shown in the example before Proposition 2.31, the family in this definition is much larger than the family of the images of the congruence subgroups of $G(\mathbb{Q})^+$.

Example 3.6. This example is to show one reason why we do not replace G by G^{ad} in the definition of connected Shimura varieties. let $G = \mathrm{SL}_2$ and $D = \mathcal{H}$. Then $Sh^\circ(G, D)$ is the family of modular curves $\Gamma\backslash\mathcal{H}$ with Γ be a torsion-free arithmetic subgroup containing some image of $\Gamma(N)$. However, if we take $G = \mathrm{SL}_2^{\mathrm{ad}} = \mathrm{PSL}_2$. The congruence subgroup of $\mathrm{PSL}_2(\mathbb{Q})$ is many fewer. Remark 2.35 provides a positive result when the two notions of congruence subgroup and arithmetic subgroup coincide.

Example 3.7. Let B be a quaternion algebra over a totally real field F. Then $B \otimes_{\mathbb{Q}} \mathbb{R} \simeq \prod_{v: F \hookrightarrow \mathbb{R}} B \otimes_{F,v} \mathbb{R}$. The classification of semisimple algebra over $\mathbb{R}$ shows $B \otimes_{F,v} \mathbb{R}$ is isomorphic either to the quaternions $\mathbb{H}$ or to $M_2(\mathbb{R})$. Let G be the semisimple algebraic group over $\mathbb{Q}$ such that

$$G(\mathbb{Q}) = \mathrm{Ker}(\mathrm{Nm} : B^\times \to F^\times).$$

Then,

$$G(\mathbb{R}) \simeq \mathbb{H}^{\times 1} \times \cdots \times \mathbb{H}^{\times 1} \times \mathrm{SL}_2(\mathbb{R}) \times \cdots \times \mathrm{SL}_2(\mathbb{R}), \tag{3.8}$$

where $\mathbb{H}^{\times 1} = \mathrm{Ker}(\mathrm{Nm} : \mathbb{H}^\times \to \mathbb{R}^\times)$ is isomorphic to a sphere topologically. Therefore, G is of noncompact type if at least one $\mathrm{SL}_2(\mathbb{R})$ occurs and D is a product of copies of $\mathcal{H}$, one for each copy of $\mathrm{SL}_2(\mathbb{R})$. Note that the action of $G(\mathbb{R})$ on D depends on the choice of isomorphism in Equation 3.8. It can be checked (G, D) satisfies (SU1, 2, 3) and hence is a connected Shimura datum. The dimension of $D(\Gamma)$ is the same as the number of $M_2(\mathbb{R})$ occurring in $B \otimes_{F,v} \mathbb{R}$. If $B \simeq M_2(F)$, then $G(\mathbb{Q})$ has unipotent elements, and so followed from Theorem 2.32 it is not compact. In this case the varieties $D(\Gamma)$ are called Hilbert modular varieties. On the other hand, if B is a division algebra, then $G(\mathbb{Q})$ has no unipotent elements, thus $D(\Gamma)$ are compact and projective as algebraic varieties.

3.2. **Adelic description.** $\mathbb{A}_f$ is the ring of finite adeles, defined to be the restricted topological product $\mathbb{A}_f = \prod_l (\mathbb{Q}_l, \mathbb{Z}_l)$ where l runs over the finite primes of $\mathbb{Q}$. $\mathbb{A}_f$ is isomorphic to $\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let V be an affine variety over $\mathbb{Q}$, it needs more concern to define $V(\mathbb{A}_f)$ as a topological group. We may choose a $\mathbb{Z}$-model of V to define "$V(\mathbb{Z}_l)$", however, the restricted topological product is independent of the choice of $\mathbb{Z}$-model. A quick explanation is as follows. Let $\alpha : V = \operatorname{Spec} A \to \mathbb{A}^m_{\mathbb{Q}}$ be a closed immersion and let the coordinates be $x_1, \cdots, x_m$. Let $\mathbb{Z}[x_1, \cdots, x_m]/I$ be the $\mathbb{Z}$-subalgebra of A generated by the x_i. Then it defines the Zariski closure V_α of $V \cap \mathbb{A}^m_{\mathbb{Z}}$ in $\mathbb{A}^m_{\mathbb{Z}}$. Let $V(\mathbb{Z}_l)$ be the $\mathbb{Z}_l$ points in V_α. For another embedding β, coordinates $y_1, \cdots, y_n$, and $V_\beta = \operatorname{Spec} \mathbb{Z}[y_1, \cdots, y_n]/J$, there is a $d \in \mathbb{Z}$ such that $\mathbb{Z}[1/d][x_1, \cdots, x_m]/I \simeq \mathbb{Z}[1/d][y_1, \cdots, y_n]/J$, because y_j can be expressed as polynomials of x_i with rational coefficients and vice versa. It follows that when l is coprime to d, these two embeddings give the same $V(\mathbb{Z}_l)$. Thus, the restricted topological product $V(\mathbb{A}_f) = \prod_l (V(\mathbb{Q}_l), V(\mathbb{Z}_l))$ is well-defined this way. Here is an example. When $V = \mathbb{G}_m$, $\mathbb{G}_m(\mathbb{A}_f) = \prod_l (\mathbb{Q}_l^\times, \mathbb{Z}_l^\times) = \mathbb{A}_f^\times$ is the ring of finite ideles.

The appearance of congruence subgroups in the definition of Shimura varieties is natural from the adelic perspective. As a result, Shimura varieties have adelic descriptions.

Proposition 3.9. *let K be a compact open subgroup of $G(\mathbb{A}_f)$. Then $K \cap G(\mathbb{Q})$ is a congruence subgroup of $G(\mathbb{Q})$, and every congruence subgroup arises in this form.*

Proposition 3.10. *Let (G, D) be a connected Shimura datum with G simply connected. Let K be a compact open subgroup of $G(\mathbb{A}_f)$, and let $\Gamma = K \cap G(\mathbb{Q})$ be the corresponding congruence subgroup of $G(\mathbb{Q})$. The map $x \mapsto [x, 1]$ defines a bijection*

$$\Gamma\backslash D \simeq G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K. \tag{3.11}$$

The right side means $G(\mathbb{Q})\backslash(D \times G(\mathbb{A}_f))/K$, where $G(\mathbb{Q})$ acts on both D and $G(\mathbb{A}_f)$ on the left, and K acts only on $G(\mathbb{A}_f)$ on the right. Moreover, this is a homomorphism with D endowed with the usual topology and $G(\mathbb{A}_f)$ endowed with the adelic topology or the discrete topology.

G is simply connected if any isogeny $G' \to G$ with G' connected is an isomorphism. Theorem 1.7 shows that in this case, $G(\mathbb{R})$ is connected, so $G(\mathbb{R})^+ = G(\mathbb{R})$. Then $G(\mathbb{Q}) \subset G(\mathbb{R})$ acts on D via the action of $G(\mathbb{R})^+$.

The proof of the above adelic description exploits the strong approximation theorem.

Theorem 3.12 (Strong Approximation)**.** *let G be an algebraic group over $\mathbb{Q}$. If G is semisimple, simply connected, and of noncompact type, then $G(\mathbb{Q})$ is dense in $G(\mathbb{A}_f)$.*

Proof. See [12], Theorem 7.12. □

Example 3.13. The following examples show that all the conditions in the theorem are necessary.

1. $G = \mathbb{G}_m$ is not semisimple, then $\mathbb{Q}^\times$ is not dense in $\mathbb{A}_f$.
2. $G = \mathrm{PGL}_2$ is not simply connected, then the determinant defines the commutative diagram

$$\begin{tikzcd}
\mathrm{PGL}_2(\mathbb{Q}) \arrow[r] \arrow[d] & \mathbb{Q}^\times/\mathbb{Q}^{\times 2} \arrow[d] \\
\mathrm{PGL}_2(\mathbb{A}_f) \arrow[r] & \mathbb{A}_f^\times/\mathbb{A}_f^{\times 2}
\end{tikzcd}$$

and $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$ is not dense in $\mathbb{A}_f^\times/\mathbb{A}_f^{\times 2}$.

(3) G is of compact type. If $G(\mathbb{Q})$ were dense in $G(\mathbb{A}_f)$, then $G(\mathbb{Z}) = G(\mathbb{Q}) \cap G(\hat{\mathbb{Z}})$ would be dense in $G(\hat{\mathbb{Z}})$. But one fact is that $G(\mathbb{Z})$ is discrete in $G(\mathbb{R})$ so is finite.

Proof of Proposition 3.10. Because K is open, the above strong approximation theorem shows that $G(\mathbb{A}_f) = G(\mathbb{Q}) \cdot K$. Therefore, all elements in $G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K$ can be represented by $[x, 1]$ for some $x \in D$. $[x, 1]$ and $[x', 1]$ represent the same element if and only if there exist $g \in G(\mathbb{Q})$ and $k \in K$ such that $gk = 1, gx = x'$, which is equivalent to say there is a $g \in K \cap G(\mathbb{Q}) = \Gamma$ such that $gx = x'$. Thus, the bijection in the statement holds.

For the second statement, consider the commutative diagram

$$\begin{tikzcd} D \arrow[r, "x \mapsto (x,[1])"] \arrow[d] & D \times (G(\mathbb{A}_f)/K) \arrow[d] \\ \Gamma\backslash D \arrow[r, "{[x] \mapsto [x,1]}"] & G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K. \end{tikzcd}$$

Since K is open, $G(\mathbb{A}_f)/K$ is discrete, then the upper map is a homeomorphism of D onto its image. It follows that the lower map is a homeomorphism. $\square$

As we mentioned in Definition 3.5 that the connected Shimura variety relative to (G, D) form an inverse system, it is natural to concern the inverse image $\varprojlim \Gamma\backslash D$ and its adelic description of this system. The example $\varprojlim \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}$ motivates us to consider the projective limit $\varprojlim \Gamma\backslash D$ as sort of completion of D, and it turns out to be $G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)$.

Proposition 3.14. *Let (G, D) be a connected Shimura datum. Then $\varprojlim_K G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K = G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)$.*

Lemma 3.15. *Let G be a topological group acting continuously on a topological space X and $(G_i)_{i\in I}$ be a directed family of subgroups of G. The following properties hold.*

(a) The canonical map $h : X/\bigcap G_i \to \varprojlim X/G_i$ is continuous.
(b) h is injective if the stabliser of x in G_i is compact for all $i \in I$ and $x \in X$.
(c) h is surjective if the orbit xG_i is compact for all $i \in I$ and $x \in X$.

Proof. (a) is directly from the universal property of inverse limit.

For (b), if $x, x' \in X$ map to the same image in $\varprojlim X/G_i$, then for each $i \in I$, $G_i(x, x') = \{g \in G_i \mid gx = x'\}$ is nonempty. The assumption implies that $G_i(x, x')$ is compact. Since (G_i) is a directed family, $\bigcap_i(\bigcap_i G_i)(x, x') = G_i(x, x')$ is not empty. Therefore, x, x' maps to the same image in $X/\bigcap G_i$, then h is injective.

For (c), let $(x_iG_i) \in \varprojlim X/G_i$, then $\bigcap x_iG_i$ is nonempty since by assumption x_iG_i is compact for each i. Choose $x \in \bigcap x_iG_i$, then the image of $x \cdot \bigcap G_i$ is (x_iG_i). Thus, h is surjective. $\square$

Proof of Proposition 3.14. It can be easily checked that all hypotheses in (a, b, c) in the above lemma hold if every G_i is compact and every orbit xG_i is Hausdorff. In what follows we check this in our case of the directed family of all open compact subgroups K of $G(\mathbb{A}_f)$. It will suffice to show that for each $[x,g] \in G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)$, $[x,g]\cdot K$ is Hausdorff when K is sufficiently small. let $[x,a]$ and $[x,b]$ be two distinct points in $[x,g]\cdot K$. Let $\Gamma = G(\mathbb{Q}) \cap aKa^{-1}$ be a discrete subgroup of $G(\mathbb{Q})$. When K is sufficiently small, Γ is torsion-free, then there exists a neighborhood V of a in D such that $gV \cap V =$. We claim the images of $V\times\{a\}$ and $V\times\{b\}$ do not intersect. This is because if there is $g \in G(\mathbb{Q})$ such that $g(V \times \{a\}) \cap V \times \{b\} \neq$, $ga = b$, thus $gaK = bK$. Since $aK = bK$ by assumption, then $g \in aKa^{-1} \cap G(\mathbb{Q}) = \Gamma$. Then the claim is proved because of the way to choose V. Therefore, every K-orbit is Hausdorff when K is sufficiently small and the lemma applies. □

Remark 3.16. The inverse limit of $\mathrm{Sh}^\circ(G,D)$ exists as an algebraic variety, which is even locally noetherian and regular. Therefore, one can regard $\mathrm{Sh}^\circ \to \Gamma\backslash D$ as an algebraic version of universal covering. To be specific, the $\mathcal{T}$-completion of $G(\mathbb{Q})^+$ acts on Sh° like the action of $G(\mathbb{R})^+$ on D. Also, it is possible to recover the inverse system from Sh° and the action.

3.3. **Definition of Shimura varieties.**

Definition 3.17. A Shimura datum is a pair (G,X) consisting of a reductive group G over $\mathbb{Q}$ and a $G(\mathbb{R})$ conjugacy class X of homomorphisms $h : \mathbb{S} \to G_{\mathbb{R}}$ satisfying the following conditions:

SV1 for all $h \in X$, the Hodge structure on $\mathrm{Lie}(G_{\mathbb{R}})$ defined by $\mathrm{Ad} \circ h$ is of the type $\{(-1,1),(0,0),(1,-1)\}$;
SV2 for all $h \in D$, $\mathrm{ad}(h(i))$ is a Cartan involution on $G^{\mathrm{ad}}_{\mathbb{R}}$;
SV3 G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial.

For every $h \in X$ satisfying (SV1), $\mathrm{ad} \circ h$ defines a Hodge structure of weight 0, so the corresponding weight morphism $w_h = h|_{\mathbb{G}_m}$ factor through the centre $Z = \mathrm{Ker}(G \to G^{\mathrm{ad}})$. Thus, w_h is unchanged by the conjugation of G and does not rely on the choice of $h \in X$. It is denoted by w_X and called the weight homomorphism.

Note that in contrast to a connected Shimura datum, the target of a h is $G_{\mathbb{R}}$ but not $G^{\mathrm{ad}}_{\mathbb{R}}$. The condition (SV1) here is compatible with (SV1) defined earlier because $\mathrm{Lie}(G_{\mathbb{C}}) = \mathrm{Lie}(Z_{\mathbb{C}}) \oplus \mathrm{Lie}(G^{\mathrm{ad}}_{\mathbb{C}})$ and $\mathrm{Ad}\circ h$ acts trivially on $\mathrm{Lie}(Z_{\mathbb{C}})$. The following proposition further explains the relationship between Shimura data and connected Shimura data.

Proposition 3.18. *Let G be a reductive group over $\mathbb{R}$. For a homomorphism $h : \mathbb{S} \to G$, let $\bar{h} : \mathbb{S} \to G^{\mathrm{ad}}$ be the composition of h with $G \to G^{\mathrm{ad}}$. If X is a $G(\mathbb{R})$-conjugacy class of homomorphism $\mathbb{S} \to G$, let $\overline{X}$ be the conjugacy class consisting $\bar{h}$ for every $h \in X$.*

(a) The map $X \to \overline{X} : h \mapsto \bar{h}$ is injective and its image is a union of connected components of $\overline{X}$.
(b) Let X^+ be a connected component of X and $\overline{X}^+$ be its image in $\overline{X}$. If (G,X) satisfies (SV1-3), then $(G^{\mathrm{ad}}, \overline{X}^+)$ satisfies SV1-3. Moreover, the stabiliser of X^+ in $G(\mathbb{R})$ is the inverse image of $G^{\mathrm{ad}}(\mathbb{R})^+$ in $G(\mathbb{R})$. This subgroup of $G(\mathbb{R})$ is denoted by $G(\mathbb{R})_+$, and $\pi_0(X) \simeq G(\mathbb{R})/G(\mathbb{R})_+$.

Proof. (a) Since $\mathrm{Ker}(G \to G^{\mathrm{ad}}) \cap \mathrm{Ker}(G \to T) = Z(G^{\mathrm{der}})$ is a finite discrete group and $\mathbb{S}$ is connected, a homomorphism $h : \mathbb{S} \to G$ is trivial if and only if its projections on G^{ad} and T are trivial. Then it is decided by the two projections.

Because all h lie in one $G(\mathbb{R})$-conjugacy class, their projection to T is the same. Therefore, in X, h is uniquely decided by its projection on G^{ad}, then the injectivity in (a) holds. According to the theory of the hermitian symmetric domain, $G^{\mathrm{ad}}(\mathbb{R})^+$ acts transitively on each component of $\overline{X}$. Because $G(\mathbb{R})^+ \to G^{\mathrm{ad}}(\mathbb{R})^+$ is surjective (Proposition 1.1), the rest of (a) holds.

(b)The first assertion is obvious. The second assertion comes from the result of (a) and the fact that the stabilizer of $\overline{X}^+$ in $G^{\mathrm{ad}}(\mathbb{R})$ is $G^{\mathrm{ad}}(\mathbb{R})^+$.

□

Corollary 3.19. *Let (G,X) be a Shimura datum, and let X^+ be the connected component of X. Then X^+ can be regarded as the $G(\mathbb{R})^+$ conjugacy class of homomorphism $\bar{h} : \mathbb{S} \to G^{\mathrm{ad}}_{\mathbb{R}}$. Thus, X is a finite disjoint union of hermitian symmetric domain and (G^{der}, X^+) is a connected Shimura datum.*

Definition 3.20. Let (G,X) be a Shimura datum. For a compact open subgroup K of $G(\mathbb{A}_f)$, define

$$\mathrm{Sh}_K(G,X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K.$$

A Shimura variety relative to (G,X) is a variety of the form $\mathrm{Sh}_K(G,X)$ for some small compact open subgroup K of $G(\mathbb{A}_f)$. The Shimura variety $\mathrm{Sh}(G,X)$ attached to a Shimura datum (G,X) is the inverse system of varieties equipped with an action of $G(\mathbb{A}_f)$.

In this definition, K is asked to be sufficiently small in consideration of realizing $\mathrm{Sh}_K(G,X)$ as a variety, which shall be explained later. In the following, we attempt to understand $G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$, and we can see when K is sufficiently small, $G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$ is a union of connected Shimura varieties. We shall also develop an understanding of its connected components.

This implies why we consider Shimura varieties regardless of connected Shimura varieties arising naturally from locally symmetric varieties. Every locally symmetric variety is defined over a number field, but one natural question is if there exists a "natural" field of definition of $G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$ which only depends on the datum (G,X) but not on K. It turns out this is true for Shimura varieties $\mathrm{Sh}_K(G,X)$, but it is not true for the connected components of $\mathrm{Sh}_K(G,X)$, the connected Shimura varieties. This is just like the simple case $X = \operatorname{Spec} \mathbb{Q}[x]/(x^2+1)$, where X is defined over $\mathbb{Q}$ and each component is defined over the quadratic extension $\mathbb{Q}(i)$. As we can see soon, the number of connected components of $\mathrm{Sh}_K(G,X)$ increases as K gets much and much smaller. So there is no way to expect the connected Shimura varieties to have such good properties. However, on the other hand, the set of connected components (which is actually a 0-dimensional Shimura variety) is always defined over a "natural" field by class field theory. This crucial property of Shimura varieties will be discussed in Chapter 6.

Lemma 3.21. *For every open subgroup K of $G(\mathbb{A}_f)$, the set $G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K$ is finite.*

The case that G^{der} is simply connected shall be proven in Theorem 3.28

Lemma 3.22. *Let $\mathcal{C}$ be a set of representatives for the double closet space $G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K$. Let X^+ be a connected component of X. Then*

$$G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K \simeq \bigsqcup_{g\in\mathcal{C}} \Gamma_g\backslash X^+ \tag{3.23}$$

where $\Gamma_g = gKg^{-1} \cap G(\mathbb{Q})_+$ *is a subgroup of* $G(\mathbb{Q})_+$*. Moreover, this is a homeomorphism.*

Theorem 3.24 (Real Approximation, [4])**.** *Let* G *be a connected algebraic group over* $\mathbb{Q}$*, then* $G(\mathbb{Q})$ *is dense in* $G(\mathbb{R})$*.*

Lemma 3.25. *For every connected component* X^+ *of* X*, the natural map*

$$G(\mathbb{Q})_+\backslash X^+ \times G(\mathbb{A}_f) \to G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)$$

is a bijection.

Proof. Because $G(\mathbb{R})$ acts transitively on X and $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$ (real approximation theorem), every element in $G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)$ can be represented by $[x, a]$ where $x \in X^+, a \in G(\mathbb{A}_f)$. Then the statement follows from Proposition 3.18 since the stabilizer of X^+ in $G(\mathbb{R})$ is $G(\mathbb{R})_+$. $\square$

Proof of Lemma 3.22. For $g \in \mathcal{C}$, define the map $\Gamma_g\backslash X^+ \to G(\mathbb{Q})_+\backslash X^+ \times G(\mathbb{A}_f)/K :$ $[x] \mapsto [x, g]$. It can be easily checked that this map is injective and that $G(\mathbb{Q})_+\backslash X^+ \times G(\mathbb{A}_f)/K$ is the disjoint union of the images of these maps variating $g \in \mathcal{C}$. The second assertion can be proved similarly to the paralleled assertion in Proposition 3.10. $\square$

We now turn to consider $\Gamma_g\backslash X^+$. To make it an algebraic variety, we want to apply Theorem 2.36 and 2.38, then Γ_g is asked to be torsion-free. In fact, it is known that when K is sufficiently small, Γ_g is neat, then the image of Γ_g in $\mathrm{Aut}(X^+)$ is torsion-free and arithmetic. Hence, $\Gamma_g\backslash X^+$ becomes a locally symmetric variety. Under this condition, $\mathrm{Sh}_K(G, X)$ is a finite disjoint union of locally symmetric varieties. This is the reason why we asked K to be sufficiently small in the definition of Shimura varieties. Moreover, when $K' \subset K$ is an inclusion of sufficiently small open compact subgroups of $G(\mathbb{A}_f)$, the natural map $\mathrm{Sh}_{K'}(G, X) \to \mathrm{Sh}_K(G, X)$ is regular.

There is a natural action of $G(\mathbb{A}_f)$ on the Shimura variety $\mathrm{Sh}(G, X)$, the inverse system. For $a \in G(\mathbb{A}_f)$, there is a map $\mathcal{T}(a) : \mathrm{Sh}_K(G, X) \to \mathrm{Sh}_{a^{-1}Ka}(G, X)$. The action on $\mathbb{C}$-points is $G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K \to G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/a^{-1}Ka : [x, g] \mapsto [x, ga]$. This is a right action because $\mathcal{T}(gh) = \mathcal{T}(h) \circ \mathcal{T}(g)$.

A general philosophy is that we want to relate representations of $G(\mathbb{A}_F)$ and $Gal(\bar{F}/F)$ (F can be either a global field or a local field), so some objects that both can act on nicely are needed. The significance put on Shimura varieties is because they are perfect candidates for this purpose.

Definition 3.26. Let (G, X) and (G', X') be Shimura data.

1. A morphism of Shimura data $(G, X) \to (G', X')$ is a homomorphism $G \to G'$ sending $X \to X'$.
2. A morphism of Shimura varieties $\mathrm{Sh}(G, X) \to \mathrm{Sh}(G', X')$ is an inverse system of regular maps of algebraic varieties compatible with the action of $G(\mathbb{A}_f)$.

Remark 3.27. Obviously, a morphism of Shimura data induces a morphism of Shimura varieties. Deligne also proved that if the homomorphism $G \to G'$ is injective, the morphisms of Shimura varieties is a closed immersion, which means for every sufficiently small compact open subgroup K' of $G'(\mathbb{A}_f)$, there is a compact open subgroup K of $G(\mathbb{A}_f)$ such that the map $Sh_K(G, X) \to \mathrm{Sh}_{K'}(G', X')$ is a closed immersion.

In Lemma 3.22, we show a Shimura variety $\mathrm{Sh}_K(G, X)$ is a union of locally symmetric varieties whose connected components are one-one correspondent to the element in $G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K$ which is finite as stated in Lemma 3.21. Actually, when G^{der} is simply connected, the set of connected components is a zero-dimensional Shimura variety.

Let G be an algebraic group, define $T(\mathbb{R})^\dagger = \mathrm{Im}(Z(\mathbb{R}) \to T(\mathbb{R}))$ and $T(\mathbb{Q})^\dagger = T(\mathbb{Q}) \cap T(\mathbb{R})^\dagger$ (T, Z are defined as in Diagram 1.3). Because $Z \to T$ is surjective and $T(\mathbb{R})^\dagger$ contains $T(\mathbb{R})^+$, therefore $T(\mathbb{R})^\dagger$ and $T(\mathbb{Q})^\dagger$ are of finite index in $T(\mathbb{Q})$ and $T(\mathbb{R})$ respectively.

Theorem 3.28. *Assume G^{der} is simply connected. Let $v : G \to T$ be the natural map. For K sufficiently small, the natural map*

$$G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K \to T(\mathbb{Q})^\dagger\backslash T(\mathbb{A}_f)/v(K) \tag{3.29}$$

defines an isomorphism

$$\pi_0(\mathrm{Sh}_K(G, X)) \simeq T(\mathbb{Q})^\dagger\backslash T(\mathbb{A}_f)/v(K) \tag{3.30}$$

Moreover, $T(\mathbb{Q})^\dagger\backslash T(\mathbb{A}_f)/v(K)$ is finite, and the connected component over [1] *is canonically isomorphism to $\Gamma\backslash X^+$ where Γ is the image of $K \cap G(\mathbb{Q})_+$ in $G^{\mathrm{ad}}(\mathbb{Q})^+$*

Proof. We claim the following results under the conditions of the statement.

(a) $G(\mathbb{R})_+ = G^{\mathrm{der}}(\mathbb{R}) \cdot Z(\mathbb{R})$.
(b) $T(\mathbb{Q})^\dagger$ is the image of $G(\mathbb{Q})_+$ through v.
(c) $v : G(\mathbb{A}_f) \to T(\mathbb{A}_f)$ is surjective and sends compact open subgroups to compact open subgroups.

Assume these claims. Then the morphism (3.29) in the statement is defined by

$$G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K \simeq G(\mathbb{Q})_+\backslash X^+ \times G(\mathbb{A}_f)/K \xrightarrow{[x,g]\mapsto[v(g)]} v(G(\mathbb{Q})_+)\backslash T(\mathbb{A}_f)/v(K) = T(\mathbb{Q})^\dagger\backslash T(\mathbb{A}_f)/v(K).$$

The fiber over [1] is $\Gamma_1\backslash X^+$ where Γ_1 is defined in Lemma 3.22.

We now check the claims. For (a), $G^{\mathrm{der}}(\mathbb{R})$ is connected (Theorem 1.7) so it maps onto $G^{\mathrm{ad}}(\mathbb{R})^+$ (Proposition 1.1). So we have one side inclusion $G(\mathbb{R})_+ \supset G^{\mathrm{der}}(\mathbb{R}) \cdot Z(\mathbb{R})$ on hand. Consider the diagram

$$\begin{tikzcd}
1 \arrow[r] & Z'(\mathbb{R}) \arrow[r, "z\mapsto(z^{-1},z)"] \arrow[d, equal] & Z(\mathbb{R}) \times G^{\mathrm{der}}(\mathbb{R}) \arrow[r, "(z,g)\mapsto zg"] \arrow[d, "(z,g)\mapsto g"] & G(\mathbb{R}) \arrow[r] \arrow[d] & H^1(\mathbb{R}, Z') \arrow[d, equal] \\
1 \arrow[r] & Z'(\mathbb{R}) \arrow[r] & G^{\mathrm{der}}(\mathbb{R}) \arrow[r] & G^{\mathrm{ad}}(\mathbb{R}) \arrow[r] & H^1(\mathbb{R}, Z'),
\end{tikzcd}$$

then for $g \in G(\mathbb{R})$, $g \in G(\mathbb{R})_+ \Leftrightarrow$ image of g in $G^{\mathrm{ad}}(\mathbb{R})$ lies in the image of $G^{\mathrm{der}}(\mathbb{R})$ $\Leftrightarrow$ image of g in $H^1(\mathbb{R}, Z')$ is 0 $\Leftrightarrow g \in Z(\mathbb{R}) \times G^{\mathrm{der}}(\mathbb{R})$, which proves the converse inclusion and the desired equality.

(b) There is an exact sequence $0 \to G^{\mathrm{der}} \to G \to T \to 0$. Since $H^1(\mathbb{Q}, G^{\mathrm{der}}) \to H^1(\mathbb{R}, G^{\mathrm{der}})$ is injective (Hasse principle shown in Proposition 1.9), by definition an element t of $T(\mathbb{Q})^\dagger$ can be lifted to $Z(\mathbb{R}) \subset G(\mathbb{R})$, then it can be lifted to $g \in G(\mathbb{Q})$. Also by definiton, g is in $Z(\mathbb{R})\cdot G^{\mathrm{der}}(\mathbb{R}) = G(\mathbb{R})_+$ (Claim (a)). Therefore, $g \in G(\mathbb{Q}) \bigcap G(\mathbb{R})_+ = G(\mathbb{Q})_+$. The converse is simpler.

(c) The surjection comes from the Hasse principle in Proposition 1.9. The proof of the second statement is omitted.

The finiteness follows Proposition 3.31 below which states that $T(\mathbb{Q})\backslash T(\mathbb{R}) \times T(\mathbb{A}_f)/(T(\mathbb{R}) \times T(\hat{\mathbb{Z}}))$ is finite. When K is sufficiently small, $v(K)$ is contained in

$T(\hat{\mathbb{Z}})$. Since $T(\hat{\mathbb{Z}})$ is compact and $v(K)$ is open, the quotient $T(\hat{\mathbb{Z}})/v(K)$ is finite. So $T(\mathbb{Q})^\dagger\backslash T(\mathbb{A}_f)/v(K)$ is finite because $T(\mathbb{Q})^\dagger$ is of finite index in $T(\mathbb{Q})$. □

Proposition 3.31. *Let T be a torus over $\mathbb{Q}$. Let $T(\mathbb{Z}_l) = \{a \in T(\mathbb{Q}_l) \mid \chi(a)$ is integral for all $\chi \in X^*(T)\}$ and $T(\hat{\mathbb{Z}}) = \prod_l T(\mathbb{Z}_l)$. Then the class group $H(T)$ of T defined as $H(T) = T(\mathbb{Q})\backslash T(\mathbb{R}) \times T(\mathbb{A}_f)/(T(\mathbb{R}) \times T(\hat{\mathbb{Z}}))$ is finite.*

Proof. When $T = \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$ with F a number field, then the class group of T is equal to the class group of F, thus finite. For a proof of general case, one can refer to Ono's paper [11]. □

Remark 3.32. Let $Y = T(\mathbb{R})/T(\mathbb{R})^\dagger$ and endow Y with discrete topology. Y is also isomorphic to $T(\mathbb{Q})/T(\mathbb{Q})^\dagger$ because $T(\mathbb{Q})$ is dense in $T(\mathbb{R})$ (real approximation theorem). Therefore, under condition in Theorem 3.28, $\pi_0(\mathrm{Sh}_K(G,X)) \simeq T(\mathbb{Q})^\dagger\backslash T(\mathbb{A}_f)/v(K) \simeq T(\mathbb{Q})\backslash Y \times T(\mathbb{A}_f)/v(K) = \mathrm{Sh}_{v(K)}(T,Y)$.

Example 3.33. Let $(G,X) = (\mathrm{GL}_2, \mathcal{H}_1^\pm)$ and $K = K(N)$. Then $T = \mathbb{G}_m$ and $Y = \mathbb{R}/\mathbb{R}^+ \simeq \pm 1$ becase $T^\dagger = T^+$. Thus, $\pi_0(\mathrm{Sh}_K(G,X)) = T(\mathbb{Q})\backslash\{\pm 1\} \times \mathbb{A}_f^\times/(1 + N\mathbb{A}_f^\times) \simeq (\mathbb{Z}/N\mathbb{Z})^\times \simeq \mathrm{Gal}(\mathbb{Q}[\zeta_N]/\mathbb{Q})$.

Passage to limit is more subtle in this case compared to the connected case. Let K be a compact open subgroup of $G(\mathbb{A}_f)$ and $Z(\mathbb{Q})^-$ be the closure of $Z(\mathbb{Q})$ in $Z(\mathbb{A}_f)$. Then

$$\begin{aligned}\mathrm{Sh}_K(G,X) &= G(\mathbb{Q})\backslash X \times (G(\mathbb{A}_f)/K)\\ &\simeq \frac{G(\mathbb{Q})}{Z(\mathbb{Q})}\backslash X \times (G(\mathbb{A}_f)/Z(\mathbb{Q}) \cdot K)\\ &\simeq \frac{G(\mathbb{Q})}{Z(\mathbb{Q})}\backslash X \times (G(\mathbb{A}_f)/Z(\mathbb{Q})^- \cdot K)\end{aligned}$$

Theorem 3.34. *Let (G,X) be a Shimura datum. Then*

$$\varprojlim_K \mathrm{Sh}_K(G,X) = \frac{G(\mathbb{Q})}{Z(\mathbb{Q})}\backslash X \times (G(\mathbb{A}_f)/Z(\mathbb{Q})^-) \tag{3.35}$$

The proof can be found in Deligne's paper ([5], 2.1.10), and the main property used is that the action of $G(\mathbb{Q})/Z(\mathbb{Q})$ on $X \times (G(\mathbb{A}_f)/Z(\mathbb{Q})^-)$ is proper. In (3.35), if $Z(\mathbb{Q})$ is discrete in $Z(\mathbb{A}_f)$, then $Z(\mathbb{Q})^- = Z(\mathbb{Q})$ and $\varprojlim_K \mathrm{Sh}_K(G,X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)$.

Thus, some simplifications to the theory of Shimura varieties occur when some additional axioms are satisfied.

- **SV2** For all $h \in X$, $\mathrm{ad}(h(i))$ is a Cartan involution on $G_\mathbb{R}/w_X(\mathbb{G}_m)$.
- **SV4** The weight is rational, which means the weight homomorphism $w_x : \mathbb{G}_m \to G_\mathbb{R}$ is defined over $\mathbb{Q}$.
- **SV5** the group $Z(\mathbb{Q})$ is discrete in $Z(\mathbb{A}_f)$.
- **SV6** The torus Z° splits over a CM-field.

(SV4) corresponds to rational Hodge structures which occur in the cohomology of abelian varieties. Thus, a Shimura variety is a moduli variety for motives when (SV4) holds and a fine moduli variety when additionally (SV5) holds. When (SV6) holds, w is defined over a totally real field, and the reflex field of the Shimura variety is either a totally real or a CM field. We have shown that with SV5, the inverse limit of Shimura varieties has a simple expression. Furthermore, we have some criterion for (SV5) when $Z = T$ is a torus.

Proposition 3.36 ([14], 3.5). *Let T be a torus over $\mathbb{Q}$, every arithmetic subgroup of $T(\mathbb{Q})$ of finite index contains a congruence subgroup.*

Let $T(\mathbb{Z})$ be an arithmetic subgroup of $T(\mathbb{Q})$. For example, $T(\mathbb{Z}) = \mathrm{Hom}(X^*(T), \mathcal{O}_L^\times)^{\mathrm{Gal}(L/\mathbb{Q})}$. This proposition says that the congruence subgroup problem has a positive answer for tori. Then the induced topology on $T(\mathbb{Q})$ from the injection $T(\mathbb{Q}) \subset T(\mathbb{A}_f)$ can be described as $T(\mathbb{Z})$ is open, and the induced topology on $T(\mathbb{Z})$ is the profinite topology. Therefore, $T(\mathbb{Q})$ is discrete $\Leftrightarrow$ $T(\mathbb{Z})$ is discrete $\Leftrightarrow$ $T(\mathbb{Z})$ is finite.

Example 3.37. (1) $T = \mathbb{G}_m$, $T(\mathbb{Z}) = \pm 1$ is finite. So, $T(\mathbb{Q})$ is discrete.
(2) $K = \mathbb{Q}(\sqrt{d})$. Let $T = \{a \in K^\times \mid \mathrm{Nm}(a) = 1\}$, then $T(\mathbb{Z}) = O_K^\times$. Therefore, $T(\mathbb{Q})$ is discrete if $d < 0$, and $T(\mathbb{Q})$ is not discrete if $d > 0$.

A torus T is said to be anisotropic over a field k if there is no character $\chi : T \to \mathbb{G}_m$ defined over k. If T is a torus over $\mathbb{R}$, T is anisotropic if and only if $T(\mathbb{R})$ is compact.

Proposition 3.38. *Let T be a torus over $\mathbb{Q}$, and T^a be the largest anisotropic subtorus of T over $\mathbb{Q}$. Then $T(\mathbb{Q})$ is discrete in $T(\mathbb{A}_f)$ if and only if T^a is anisotropic over $\mathbb{R}$ or equivalently $T^a(\mathbb{R})$ is compact.*

Moreover, if T splits over a CM field L (i.e. $Z = T$ and (SV6) holds), let ι be the complex conjugation on L. Take $T_L^+ = \bigcap_{\iota\chi = -\chi} \mathrm{Ker}(\chi : T_L \to \mathbb{G}_m)$ to be the largest subtorus of T splitting over $\mathbb{R}$. Thus, following from the above proposition, $T(\mathbb{Q})$ is discrete in $T(\mathbb{A}_f)$ if and only if T^+ splits over $\mathbb{Q}$.

4. Abelian Varieties

Definition 4.1. An abelian variety A over a field k is a complete algebraic variety over k with a group law $m : X \times X \to X$ such that m and the inverse map i are both morphisms of varieties.

It can be proven an abelian variety is projective, everywhere non-singular, and commutative as a group scheme. (See [10], §4)

4.1. Complex abelian varieties.

The abelian varieties over $\mathbb{C}$, as complex manifolds, are complex tori. This fact provides them with the corresponding Hodge structures. But when a complex torus can be endowed with a structure of abelian varieties? How is a Hodge structure corresponding to a complex abelian variety distinct from what corresponds to a general complex torus?

We first show complex abelian varieties are complex tori as complex manifolds. The map $\exp : \mathrm{Tgt}_0 A \to A(\mathbb{C})$ is a universal covering map of Lie groups, and the kernel of this map is a discrete group in $\mathrm{Tgt}_0 A \simeq \mathbb{C}^n$ where n is the dimension of A. Therefore, $A(\mathbb{C}) \simeq V/\Lambda$ where Λ is a full lattice in complex vector space $V \simeq \mathbb{C}^n$. A complex manifold of the form V/Λ is called a complex torus.

Proposition 4.2. *Let $M = V/\Lambda$ be a complex torus. $H_1(M, \mathbb{Z}) \simeq \Lambda$, and $H^1(M, \mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$. Moreover, there is a canonical isomorphism $H^d(M, \mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(\bigwedge^d \Lambda, \mathbb{Z})$.*

Then, $H^d(M,\mathbb{C}) \simeq H^d(M,\mathbb{Z})\otimes\mathbb{C} \simeq \bigwedge^d \mathrm{Hom}(\mathrm{Tgt}_0 M,\mathbb{C})$. In fact, for all d-closed forms ω, $[\omega] \in H^d(M,\mathbb{C})$ is represented by a unique translation-invariant n-form ω_α where $\alpha \in \mathrm{Hom}(\bigwedge^d \mathrm{Tgt}_0 M,\mathbb{C})$.

A more general result is stated in the following proposition.

Proposition 4.3. *Let $Y = X/G$, where G is a discrete group acting freely and continuously on a good topological space X. Let $\pi : X \to Y$ be the quotient map. For any sheaf $\mathcal{F}$ over Y, there is a natural map*

$$\phi : H^p(G, \Gamma(X, \pi^*\mathcal{F})) \to H^P(Y, \mathcal{F})$$

which is compatible with the cup product. If $H^i(X, \pi^*\mathcal{F}) = 0, i \geq 1$, ϕ is an isomorphism.

Proof. See [10], Appendix to §2. □

The natural map ϕ can be defined using Cech cochains in a natural approach. The map defined in Proposition 4.2 differs from ϕ by composing with the canonical isomorphism $H^i(\Lambda,\mathbb{Z}) \simeq \bigwedge^i \mathrm{Hom}(\Lambda,\mathbb{Z}) \simeq \mathrm{Hom}(\bigwedge^i \Lambda,\mathbb{Z})$. When $i = 2$, $H^2(\Lambda,\mathbb{Z}) \simeq \bigwedge^2 \mathrm{Hom}(\Lambda,\mathbb{Z})$ is given by $[F] \to [AF]$ in which $F : \Lambda\times\Lambda \to \mathbb{Z}$ and $AF(u,v) = F(u,v) - F(v,u)$.

Since $H^i(V, \phi^*\mathcal{F}) = 0$ for all sheaf $\mathcal{F}$ on M, then there is a commutative diagram induced by the exact sequence $0 \to \mathbb{Z} \to \mathcal{O}_M \to \mathcal{O}_M^\times \to 0$ of sheaves on M:

$$\begin{tikzcd}
 & H^1(\Lambda, H^*) \arrow[r, "\delta"] \arrow[d, "\simeq"] & H^2(\lambda,\mathbb{Z}) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \bigwedge^2 \mathrm{Hom}(\Lambda,\mathbb{Z}) \\
\mathrm{Pic}(M) \arrow[r, "\simeq"] \arrow[rr, bend right, "c_1"'] & H^1(M, \mathcal{O}_M^*) \arrow[r, "\delta"] & H^2(M,\mathbb{Z}) &
\end{tikzcd}$$

where $H = \Gamma(V, \mathcal{O}_V)$, and c_1 maps a line bundle to its Chern class. The Neron-Severi group $NS(M)$ is defined to be the image of c_1. Let $\mathrm{Pic}^0(M) = \mathrm{Ker}(c_1)$ be the subgroup of $\mathrm{Pic}(M)$ consisting of topologically trivial line bundles.

Therefore, the Chern class of a line bundle L of M corresponds to an alternating 2-form Φ on Λ with values in $\mathbb{Z}$. Lefschetz theorem on (1, 1)-classes tells that $c_1(L)$ is of form $(1,1)$, which means $\Phi(iu, iv) = \Phi(u,v)$ under the above isomorphism. An alternating $(1,1)$-form Φ on Λ corresponds to a unique Hermitian form H on V such that $H(x,y) = \Phi(ix,y) + i\Phi(x,y)$ and $\Phi(x,y) = \mathrm{Im}H(x,y)$. Thus, $NS(X)$ can be considered to consist of the hermitian form on V whose image part takes integral values on lattice Λ. Further results are stated in the following explicit way.

Theorem 4.4 (Appell-Humbert)**.** *Let H be a hermitian form on V such that $\mathrm{Im}H(\Lambda,\Lambda) \subset \mathbb{Z}$. Let $\alpha : \Lambda \to U_1$ be the map satisfying $\alpha(u_1+u_2) = e^{i\pi\Phi(u_1,u_2)} \cdot \alpha(u_1)\cdot\alpha(u_2), u_i \in \Lambda$ (such maps exist for every given H).*

If we put $e_u(z) = \alpha(u)e^{\pi H(z,u)+1/2\pi H(u,u)}$ for $u \in \Lambda$, then $u \mapsto e_u$ is 1-cocycle on U with coefficients in H^* called Appell-Humbert cocycle, which determines an element in $H^1(\Lambda, H^*) \simeq \mathrm{Pic}(M)$. The associated line bundle on M denoted by $L(\Phi,\alpha)$ is the quotient of $\mathbb{C}\times V$ by the action of Λ: for $u \in \Lambda$, $\phi_u(x,z) = (e_u(z)\cdot x, z+u)$, the Chern class of which is $\Phi \in H^2(M,\mathbb{Z})$.

The tensor product of these line bundles is given by $L(H_1,\alpha_1)\otimes L(H_2,\alpha_2) \simeq L(H_1+H_2, \alpha_1\alpha_2)$.

Conversely, any line bundle L on the complex torus M is isomorphic to an $L(H,\alpha)$ for a unique (H,α) as above.

Proof. See [10], §2. □

Appell-Humbert theorem provides the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(\Lambda, U_1) & \longrightarrow & \{(H,\alpha) \text{ as above}\} & \longrightarrow & \{H \text{ as above}\} & \longrightarrow & 0 \\ & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\ 0 & \longrightarrow & \mathrm{Pic}^0(M) & \longrightarrow & \mathrm{Pic}(M) & \longrightarrow & NS(M) & \longrightarrow & 0 \end{array}$$

Theorem 4.5 (Lefschetz)**.** *Let M be a complex torus and $L(H,\alpha)$ be a line bundle on M as defined in the above Theorem 4.4. Then, L is ample if and only if H is positive-definite.*

Proof. See [10], §3. □

When a complex torus M becomes algebraic? This asks when M is the complex space associated to an abelian variety over $\mathbb{C}$, which is equivalent to asking if there exists an ample line bundle on M. The above theorem shows the line bundle is ample if and only if the hermitian form H associated to its Chern class is positive-definite.

The isomorphism $\Lambda \otimes \mathbb{R} \simeq V$ defines a complex structure J on $\Lambda \otimes \mathbb{R}$ and therefore a integral Hodge structure of type $\{(-1,0),(0,-1)\}$ on Λ. Furthermore, $M \mapsto (H_1(M,\mathbb{R}), J)$ induces the equivalence between the category of complex tori and integral Hodge structures of type $\{(-1,0),(0,-1)\}$. (A holomorphic morphism between two complex $V/\Lambda \to V/\Lambda'$ is easily checked to be induced by a group homomorphism $\Lambda' \to \Lambda$.)

Definition 4.6. A Riemann form for M is an alternating form $\Phi : \Lambda \times \Lambda \to \mathbb{Z}$ such that

$$\begin{cases} \Phi_{\mathbb{R}}(Ju, Jv) = \Phi_{\mathbb{R}}(u,v) & \text{for all } u, v \in V, \text{ and} \\ \Phi_{\mathbb{R}}(u, Jv) > 0 & \text{for all } u \neq 0, \end{cases} \tag{4.7}$$

or equivalently $2\pi\Phi$ is a Hodge polarization of the Hodge structure on Λ associated to M.

A complex torus is said to be polarizable if there exists a Riemann form. Then Theorem 4.5 can be reformulated in the following way.

Theorem 4.8. *The complex torus is projective if and only if it is polarizable.*

Theorem 4.9 (Riemann's Theorem)**.** *The functor $A \to H_1(A,\mathbb{Z})$ is an equivalence from the category* **AV** *of abelian varieties over $\mathbb{C}$ to the category of polarizable integral Hodge structure of type $\{(-1,0),(0,-1)\}$.*

To understand these results and the notion of polarization, we proceed to discuss dual abelian varieties.

Proposition 4.10. *Denote $Pic^0(M)$ by $\hat{M}$. Then it has a natural structure of complex torus as $\overline{V}^*_{\mathbb{C}}/\Lambda'$, where $\Lambda' = \{v \in \overline{V}^*_{\mathbb{C}} \mid < u, v > \in \mathbb{Z}, \text{ for all } u \in \Lambda\}$. The universal line bundle on $M \times \hat{M}$ parametrizing $Pic^0(M)$ is defined by $H((u_1,v_1),(u_2,v_2)) = < u_1, v_2 > + < u_2, v_1 >$ and $\alpha((u,v)) = \exp(-\pi i \mathrm{Im}(< u, v >))$.*

Proof. First, we construct a group isomorphism $V_{\mathbb{R}}^*/\Lambda^* \to \mathrm{Hom}(\Lambda, U_1)$ where $\Lambda^* = \{v \in V_{\mathbb{R}}^* \mid < u, v > \in \mathbb{Z}, \text{ for all } u \in \Lambda\}$. By the Appell-Humbert theorem, $\mathrm{Pic}^0(M) \simeq \mathrm{Hom}(\Lambda, U_1)$. The map is defined to be $v \mapsto (u \mapsto \exp(2\pi i < u, v >))$ and it is obviously a group homomorphism. Since if $v_1 - v_2 \in \Lambda^*$, for every $u \in \Lambda$, $\exp(2\pi i < u, v_1 >) = \exp(2\pi i < u, v_2 >)$, the map is well-defined. A similar argument shows that it is injective. Given a homomorphism $\alpha : \Lambda \to U_1$ and a $\mathbb{Z}$-basis $e_1, \cdots, e_n$ of Λ, we can choose $a_1, \cdots, a_n \in \mathbb{R}$ such that $\alpha(e_t) = \exp(2\pi i a_t)$, then $v \in V_{\mathbb{R}}^*/\Lambda^* : e_i \mapsto a_i$ is an inverse image of α, explaining the surjection. Therefore, $\mathrm{Pic}^0(M) \simeq V_{\mathbb{R}}^*/\Lambda^*$ as groups.

Because $V_{\mathbb{R}}^* \to \overline{V}_{\mathbb{C}}^* : g \mapsto -g(iv) + ig(v)$ is an isomorphism (the inverse map is taking the imaginary part) and the image of Λ^* under this map is Λ', $V_{\mathbb{R}}^*/\Lambda^*$ can be identified with $\overline{V}_{\mathbb{C}}^*/\Lambda'$ which is a complex torus. Therefore, $\hat{M}$ is viewed as a complex torus whose points are elements in $\mathrm{Pic}^0(M)$. □

Given a hermitian from H on V such that the imaginary part of H takes integral values on the lattice Λ, it defines a map $\phi_H : M \mapsto \hat{M}$ by $u \mapsto H(\cdot, u)$. It is well-defined ensured by the conditions on H. Then, ϕ_H is an isogeny if and only if H is non-degenerate. Moreover, giving a Riemann form Φ is equivalent to giving an isogeny $M \to \hat{M}$ which is called polarization. We also call Φ a polarization.

Definition 4.11. With a suitable choice of a basis of Λ, Φ can be represented by a matrix $E = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$, where D is a diagonal matrix $D = (d_1, ..., d_n)$ for some non-negative integers $d_1, ..., d_n$ such that $d_1 \mid d_2 \mid ... \mid d_n$. The form Φ is non-degenerate if these integers are nonzero. $D = (d_1, ..., d_n)$ is called the type of the polarization Φ. A polarization is called principal if its type is $(1, ..., 1)$.

The type of the polarization Φ is $D = (d_1, ..., d_n)$ means the kernel of the associated isogeny $M \to \hat{M}$ is isomorphic to $\mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_n\mathbb{Z}$.

Proposition 4.12. *$L \in \mathrm{Pic}(M)$ defines an associated hermitian form, thus a map $\phi_L : M \to \hat{M}$. ϕ_L has the following properties:*

1. *The kernel of ϕ_L is $K(L) \simeq \Lambda^\perp/\Lambda$ where $\Lambda^\perp = \{v \in V \mid \text{for all } u \in \Lambda, \Phi_L(u, v) \in \mathbb{Z}\}$;*
2. *ϕ_L is trivial $\Leftrightarrow \Phi_L = 0 \Leftrightarrow L \in \mathrm{Pic}^0(M)$;*
3. *ϕ_L is surjective $\Leftrightarrow K(L)$ is finite $\Leftrightarrow L$ is ample.*

Proof. All these properties can be induced by the Appell-Humbert theorem. □

4.2. **Abelian varieties over any fields.** To survey abelian varieties over any field and define the moduli problems, we have to see the polarization and the Riemann form algebraically. Basically, when X is an abelian variety over k, the functor $\mathrm{Pic}^0_{X/k}$ is representable by an abelian variety $\hat{X}$ over k called the dual abelian variety of X. We use the etale fundamental group to replace $\Lambda = H_1(X, \mathbb{Z})$ in the last section, which can be written explicitly as the Tate module. Every ample $L \in \mathrm{Pic}(X)$ defines an isogeny $X \to \hat{X}$ called a polarization. Then a Riemann form can be associated to each polarization. We start with the construction of dual abelian varieties.

Theorem 4.13 (Theorem of The Cube)**.** *Let X be any variety and Y be an abelian variety, and f, g, h be morphisms from X to Y. Then for every $L \in \mathrm{Pic}(Y)$,*

$$(f+g+h)^*L \simeq (f+g+h)^*L \otimes (f+h)^*L \otimes (g+h)^*L \otimes f^*L^{-1} \otimes g^*L^{-1} \otimes h^*L^{-1}.$$

Proof. See [10], §6. □

Theorem 4.14 (Theorem of Square)**.** *Let X be an abelian variety over k and $x, y \in X(S)$ where S is a scheme over k. Let T_x be the translation of $X \times S$ and its action on sections of $X \times S \to S$ is $t \mapsto t + x$. Then for all line bundles L, $T_{x+y}^* L \otimes L \simeq T_x^* L \otimes T_y^* L$.*

Proof. Take $Y = X$, $f = x, g = y$ the constant morphisms and $h = id$ in (4.13), then this result follows. □

Therefore, $\phi_L : X(k) \to \mathrm{Pic}(X)$, $x \mapsto T_x^* L \otimes L^{-1}$ is a homomorphism from $X(k)$ to $\mathrm{Pic}(X)$.

Definition 4.15. Let X be an abelian variety over k. Then $\mathrm{Pic}^0(X)$ is defined to be the subgroup of $\mathrm{Pic}(X)$ consisting of the line bundles L such that ϕ_L is the trivial morphism.

It can be easily checked that the image of ϕ_L for any $L \in \mathrm{Pic}(X)$ is contained in $\mathrm{Pic}^0(X)$. So in what follows, we consider ϕ_L as a map $X(k)$ to $\mathrm{Pic}^0(X)$. Here is a useful equivalent condition for a line bundle to be in $\mathrm{Pic}^0(X)$. If $L \in \mathrm{Pic}(X)$,

$$\begin{aligned} L \in \mathrm{Pic}^0(X) &\Leftrightarrow T_x^* L \otimes L^{-1} = m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}|_{X \times \{x\}} \text{ is trivial for all } x \in X \\ &\Leftrightarrow m^* L \simeq p_1^* L \otimes p_2^* L \text{ on } X \times X \end{aligned}$$

One result follows is that for any $f, g : Y \to X$ where Y is a variety and X is an abelian variety, then $(f+g)^* L \simeq f^* L \otimes g^* L$ for $L \in \mathrm{Pic}^0(X)$. Thus,

$$[n]^* L \simeq L^n, L \in \mathrm{Pic}^0(X). \tag{4.16}$$

Another result is that if L is a line bundle on $X \times S$ where X is an abelian variety and S is a scheme, $L_s \in \mathrm{Pic}^0(X)$ for every $s \in S$ if and only if this holds for one $s \in S$. To prove this, one can apply the theorem of the cube to $m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$ on $X \times X \times S$.

ϕ_L can be further defined as a morphism between functors. Let S/k be a scheme and $f : S \to X \in X(S)$, denote $p_1 : X_S = X \times S \to X$ the projection to the first factor and $p_2 : X_S \to S$ the projection to the second factor. Define the translation $T_f : X_S \to X_S$ such that the $p_1 \circ T_f$ is the composition $X \times S \xrightarrow{id \times f} X \times X \xrightarrow{m} X$ and $p_2 \circ T_f = p_2$. Let

$$\mathrm{Pic}^0_{X/k}(S) = \{L \in \mathrm{Pic}(X_S) \mid L_s \in \mathrm{Pic}^0(X), \text{ all } s \in S\}/p_2^*(\mathrm{Pic}(S)).$$

Thus, a functor $\phi_L : X \to \mathrm{Pic}_{X/k}$ is constructed as $(f : S \to X) \mapsto T_f^* p_1^* L \otimes p_1^* L^{-1}$. $K(L)$ is defined to be the kernel of ϕ_L.

Proposition 4.17. *$K(L)$ is representable by a subgroup scheme of X. $K(L)$ is finite if and only if L is ample on X.*

Proof. See [10], §6. □

Let $M = m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$ on $X \times X$, then $K(L)$ can be defined as the maximal subscheme of X such that $M|_{K(L) \times X}$ is trivial.

Theorem 4.18. *Let X be an abelian variety over k and L be an ample line bundle on X. Then for any $N \in \mathrm{Pic}^0(X)$, there exists $x \in X$ such that $N \simeq T_x^* L \otimes L^{-1}$. Therefore, $\phi_L : X \to \mathrm{Pic}^0_{X/k}$ is surjective.*

Proof. See [10], §8, 13. ☐

Therefore, we define the dual abelian variety $\hat{X}$ to be the quotient abelian variety $X/K(L)$ and expect $\hat{X}$ to represent $\mathrm{Pic}^0_{X/k}$ with a universal line bundle P on $X\times\hat{X}$. Then the pullback of P along $\pi : X \to \hat{X}$ has to be M, therefore, we define the Poincare line bundle P on $X\times X$ as the quotient of the line bundle M by a suitable action of $K(L)\times\{0\}$ lifting the translation action on $X\times X$. For the details of this construction, refer to Mumford's book.

Theorem 4.19. *$\hat{X}$ represents $\mathrm{Pic}^0_{X/k}$ and the Poincare line bundle P is the universal line bundle.*

Proof. See [10], §8, 13. ☐

Proposition 4.20. *Let X, Y be two abelian varieties and P be a line bundle on $X\times Y$ such that its restrictions on $X\times\{0\}$ and $\{0\}\times Y$ is trivial, then P induced morphism $f : X \to \hat{Y}$ and $g : Y \to \hat{X}$. Then, the following are equivalent:*

(1) f is an isomorphism;
(2) g is an isomorphism;
(3) $|\chi(P)| = 1$.

Proof. See [10], §13. ☐

The proposition indicates that the notion of dual abelian varieties is symmetric, and there is a canonical isomorphism from X to $\hat{\hat{X}}$ defined by the Poincare line bundle.

Proposition 4.21. *If $f : X \to Y$ is a morphism of abelian varieties, then the pullback of line bundles gives the dual morphism $\hat{f} : \hat{Y} \to \hat{X}$. The kernel of f and $\hat{f}$ are naturally dual as finite group schemes.*

Proof. If S is a scheme,

$$\begin{aligned}\mathrm{Ker}(\hat{f})(S) &= \{L \in \mathrm{Pic}(Y\times S) \mid f^*(L) \simeq X\times S\times\mathbb{A}^1\}\\ &= \{\text{Liftings of translation action of } \mathrm{Ker}(f)(S) \text{ on } X\times S \text{ to } X\times S\times\mathbb{A}^1\}\\ &= \mathrm{Hom}_S(\mathrm{Ker}(f), \mathbb{G}_m)\end{aligned}$$

☐

Let X be an abelian variety over $\mathbb{C}$. In the analytic theory of complex abelian varieties, the Riemann form associated to a line bundle is defined to be an alternating form on $\Lambda = H_1(X,\mathbb{Z}) = \pi_1(X)^{ab}$. Here, in order to deal with abelian varieties over an arbitrary field, the replacement of $H_1(X,\mathbb{Z})$ is the etale fundamental group.

Let $[n]$ denote the multiplication by n, then $\{[n] : X \to X\}$ is a cofinal system of etale coverings of X. The reason is for any finite etale $f : Y \to X$, the map $[\deg f] : X \to X$ factors through f. Let X_n be the kernel of $[n]$, then the automorphism group of the covering $X \xrightarrow{prop:[n]} X$ is isomorphic to $X_n(k)$. So we continue to introduce some properties of $[n]$ and X_n, and then define the Riemann form algebraically.

Let X be an abelian variety over k of dimension $2g$, then the following properties of $[n]$ and X_n follow from Theorem 4.13.

Corollary 4.22. $L \in \mathrm{Pic}X, [n]^*L \simeq L^{(n^2+n)/2} \otimes [-1]^*L^{(n^2-n)/2}$.

Proof. Take $f = [n+1], g = id, h = [-1]$ and the equality can be proved by induction. $\square$

Corollary 4.23. $\deg[n] = n^{2g}$.

Proof. Assume L' is an ample line bundle. Then $L = L' \otimes [-1]^* L'$ is again ample and $[n]^* L = L^{n^2}$. Consider the intersaction number $\deg[n] L \cdot \cdots \cdot L = [n]^* L \cdot \cdots \cdot [n]^* L = L^{n^2} \cdot \cdots \cdot L^{n^2} = n^{2g} L \cdot \cdots \cdot L \neq 0$. Thus, $\deg[n] = n^{2g}$. $\square$

Corollary 4.24. *If $n \neq$char k, $X_n(\bar{k}) \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}$.*

Proof. If $n \neq$char k, $[n]$ is separable. Therefore, $\# X_n(\bar{k}) = \deg[n] = n^{2g}$. Note that when $m|n$, X_m is a subgroup of X_n. Then the elementary group theory shows that the only group possible $X_n(\bar{k})$ is $(\mathbb{Z}/n\mathbb{Z})^{2g}$. $\square$

Take $f = [n]$ where n is a integer coprime to char k, then $\hat{f} = [n]$ by (4.16). There is a natural pairing $X_n \times \hat{X}_n \to \mathbb{G}_m$ by Proposition 4.21. This pairing factors through μ_n, then, there is a natural pairing $e_n : X_n \times \hat{X}_n \to \mu_n$. Since this is the pairing arising naturally from dual group schemes, it is non-degenerate.

Definition 4.25. The Riemann form E_L associated to a line bundle L is the map $X_n \times X_n \to \mu_n$ given by $E_L(x, y) = e_n(x, \phi_L(y))$.

Regarding $X_n(\bar{k})$ as a free $\mathbb{Z}/n$ module of rank $2 \dim X$, e_L is a $\mathbb{Z}/n$ bilinear form on $X_n(\bar{k})$.

Definition 4.26. When $l \neq$ char k is a prime, the l-adic Tate group $T_l(X)$ is defined to be $\varprojlim X_{l^n}$, where the inverse system is $\cdots \to X_{l^{n+1}} \xrightarrow{\cdot n} X_{l^n} \xrightarrow{\cdot n} X_{l^{n-1}} \to \cdots$.

Let $\mu_{l^\infty} = T_l(\mathbb{G}_m)$. Then the pairing e_n extends to $e_n : T_l(X) \times T_l(\hat{X}) \to \mu_{l^\infty}$. Also, the Riemann form can be extended to be defined as a pairing on $T_l(X)$, like $E_L : T_l(X) \times T_l(X) \to \mu_{l^\infty}$.

Proposition 4.27. *Let X be an abelian variety and L be a line bundle on X. Let $l \neq$ char k be a prime, then the Riemann form E_L on $T_l(X)$ is skew-symmetric.*

Proof. See [10], §20. $\square$

Proposition 4.28 ([10], §20)**.** *Let X be an abelian variety of dimension g and L be a line bundle on X, and $l \neq$ char k be a prime. Then there is a generator $v \in \mathrm{Hom}_{\mathbb{Z}_l}(\bigwedge^{2g} T_l(X), \mu_{l^\infty}^{\otimes g})$ such that for any g divisors $D_1, \cdots, D_g$ on X, let L_i is the corresponding line bundles, then $E^{L_1} \wedge \cdots \wedge E^{L_g} = (D_1 \cdot \cdots \cdot D_g) \cdot v$.*

This proposition shows how the Riemann form relates to the intersection of the line bundles, which is compatible with the Riemann form in the last section derived from the Chern class of line bundles. The next proposition finally connects the algebraic construction with the former analytic construction when $k = \mathbb{C}$.

Proposition 4.29. *X is an abelian variety over $\mathbb{C}$, If $L = L(H, a)$ is a line bundle on $X = V/\Lambda$, $E = \mathrm{Im} H$, and $\pi : V \to X$ is the natural map. Let π_l be the natural morphism $\Lambda \to T_l(X) : u \mapsto (\pi(u/l^n))_n$ and $\zeta = (e^{2\pi i/l^n})_n \in \mu_{l^\infty}$ be a canonical basis element. Then*

$$E^L(\pi_l u, \pi_l v) = -E(u, v) \cdot \zeta$$

Proof. See [10], §24. $\square$

A polarization of abelian varieties is an isogeny $\phi_L : X \to \hat{X}$ where L is an ample line bundle on X. The type of the polarization is $D = (d_1, \cdots, d_n)$ if $\mathrm{Ker}(\phi_L)(\bar{k}) \simeq \mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_n\mathbb{Z}$.

4.3. **Endormorphism ring.** Define $\mathrm{Hom}^0(X, Y) = \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Hom}(X, Y)$ and $\mathrm{End}^0(X) = \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{End}(X)$. Then we can define the category $\mathbf{AV}^0(k)$ of abelian varieties up to isogenies to be the category whose objects are abelian varieties over k and morphisms from X to Y is $\mathrm{Hom}^0(X, Y)$ and the composition of two morphisms is the obvious one. It is actually the localization of the additive category $\mathbf{AV}(k)$ of abelian varieties over k with respect to isogeny. Since $\{[n] : X \to X\}$ is a cofinal system of isogenies with target of X, then $\varprojlim_{X' \to X \text{ isogeny}} \mathrm{Hom}(X', Y) \simeq \mathrm{Hom}^0(X, Y)$, which verifies the two definition is the same. We proceed to introduce some ingredients needed in the further discussion of endomorphism rings of abelian varieties. This section is necessary in understanding the PEL moduli problem since PEL stands for polarization, endomorphism, and level structure.

Theorem 4.30 (Poincare Complete Reducibility Theorem)**.** *If X is an abelian variety and Y is an abelian subvariety, there is a subvariety Z such that $Z \cap Y$ is finite and $Y + Z = X$.*

Proof. See [10], §19. □

This theorem says $\mathbf{AV}^0(k)$ is a semi-simple abelian category. An abelian variety is called simple if it has no nontrivial abelian subvariety, and any abelian variety is isogeneous to a unique product of simple abelian varieties.

Proposition 4.31. *If X is a simple abelian variety,* $\mathrm{End}^0(X)$ *is a division algebra.*

Proof. Every element $f \in \mathrm{End}X \backslash \{0\}$ is an isogeny by the definition of simple abelian variety, thus is invertible in $\mathrm{End}^0 X$. □

Proposition 4.32. *Let X be an abelian variety.* $\mathrm{End}^0(X)$ *is a semisimple algebra.*

Proof. By Theorem 4.30, X is isogeneous to $Y_1^{d_1} \times \cdots \times Y_n^{d_n}$ where Y_i are distinct simple abelian varieties. Therefore, $\mathrm{End}^0 X \simeq M_{d_1}(\mathrm{End}^0 Y_1) \times \cdots \times M_{d_n}(\mathrm{End}^0 Y_n)$ is a semisimple algebra. □

Theorem 4.33. *Let X, Y be two abelian varieties over k and l be a prime$\neq$char k. $f \in \mathrm{Hom}(X, Y)$ restricts to maps $f : X_{l^n} \to Y_{l^n}$ and induces a map $T_l(f) : T_l(X) \to T_l(Y)$ called the l-adic representation which can be extended to $T_l : \mathbb{Z}_l \otimes_{\mathbb{Z}} \mathrm{Hom}(X, Y) \to \mathrm{Hom}_{\mathbb{Z}_l}(T_l(X), T_l(Y))$. Then it is injective.*

Proof. See [10], §19. □

Since $T_l(X)$ is a free $\mathbb{Z}_l$-module of rank $2g$ where $g = \dim X$, this theorem presents the finiteness of the rank of $\mathrm{Hom}(X, Y)$.

Corollary 4.34. $\mathrm{Hom}(X, Y) \simeq \mathbb{Z}^\rho$*, where $\rho \leq 4 \dim X \cdot \dim Y$.*

Proof. Note that $\mathrm{Hom}(X, Y)$ is torsion-free, since $\mathrm{Ker}[n]$ is finite and $[n] \circ f = 0 \Leftrightarrow f = 0$. Then this statement follows Theorem 4.33 □

Corollary 4.35. $\mathrm{End}^0(X)$ *is a finite-dimensional semisimple algebra over $\mathbb{Q}$.*

$f \mapsto \deg f$ acts as a natural norm on $\mathrm{End}^0 X$, and we can also construct a trace on $\mathrm{End}^0 X$ using the degree map.

Theorem 4.36. *The function $\phi \mapsto \deg\phi$ on $\mathrm{End}X$ extends to a homogeneous polynomial function of degree $2g$ on End^0X.*

Proof. We have shown $\deg[n] = n^{2g}$ and $\deg(nf) = \deg[n]\deg f = n^{2g}\deg f$. So, it will suffice to show that for $\phi, \psi \in \mathrm{End}X$, $\deg(n\phi+\psi)$ is a polynomial on n. Since $\deg(n\phi+\psi) = \chi((n\phi+\psi)^*L)/\chi(L)$ for an ample line bundle L. It will suffice to show that $\chi((n\phi+\psi)^*L)$ is a polynomial on n. This follows the induction and an application of Theorem 4.13. □

Theorem 4.37. *Let $f \in \mathrm{End}X$ where X is an abelian variety over k, and l be a prime $\neq$char k. Then $\deg f = \det T_l(f)$. Hence, $\deg([n]-f) = P(n)$ where $P(t)$ is the characteristic polynomial $\det(t - T_l(f))$ of $T_l(f)$ acting on T_lX. P has coefficients in $\mathbb{Z}$ and $P(f) = 0$.*

Proof. See [10], §19. □

Definition 4.38. The above polynomial $P(t) \in \mathbb{Z}[t]$ (independent of l) is called the characteristic polynomial of f. Its constant term and the negative of the coefficient of t^{2g-1} are called the norm $\mathrm{Nm}f$ and trace $\mathrm{Tr}f$ of f respectively.

Let A be a simple abelian variety, then $\mathrm{End}^0(A)$ is a division algebra over $\mathbb{Q}$. Let K be the center of $\mathrm{End}^0(A)$, then $\mathrm{End}^0(A)\times_K \bar{K} \simeq M_d(\bar{K})$ for some $d \in \mathbb{Z}$ and $[\mathrm{End}^0(A) : K] = d^2$. Recall the following well-known facts before some further discussion.

Lemma 4.39. *Let A be a finite-dimensional associative simple algebra over a field Γ with center Λ separable over Γ. There is a canonical norm form N^0 and a canonical trace form Tr^0 of A over Λ such that any norm form (resp. trace form) of A over Γ is of the type $(\mathrm{Nm}_{\Lambda/\Gamma}\circ N^0)^k$ with some integer k (resp. $\phi\circ\mathrm{Tr}^0$ where $\phi : \Lambda \to \Gamma$ is a Γ-linear form). If $[A : \Lambda] = d^2$, N^0 is homogeneous of degree d. $\mathrm{Nm}_{\Lambda/\Gamma}\circ N^0$ is called the reduced norm of A over Γ and $\mathrm{Tr}_{\Lambda/\Gamma}\circ\mathrm{Tr}^0$ is called the reduced trace of A over Γ.*

Proposition 4.40. *Let A be a simple abelian variety and K be the center of $\mathrm{End}^0(A)$. $[K : \mathbb{Q}] = e$, $[\mathrm{End}^0(A) : K] = d^2$, then $de|2g$.*

Proof. This proposition follows Lemma 4.39 and Theorem 4.37. □

Definition 4.41. Let A be a simple abelian variety, and keep the same notation. A is of CM type if $de = 2g$.

There is another structure called Rosati involution on End^0X to be defined.

Definition 4.42. Fix an ample line L on an abelian variety X over k, then $\phi_L : X \to \hat{X}$ is an isogeny, and then an isomorphism in $M(k)$. The Rosati involution on $\mathrm{End}^0(X)$ associated with L is defined to be the involution $f \mapsto f' = \phi_L^{-1}\circ f\circ\phi_L$, for every $f \in \mathrm{End}^0(X)$.

Proposition 4.43. *The Rosati involution on $\mathrm{End}^0(X)$ satisfies the following properties:*

1. *$f'' = f$, $f \mapsto f'$ is $\mathbb{Q}$-linear and $(\phi\psi)' = \psi'\phi'$.*
2. *Let $l \neq$ char k, and let E^L be the Riemann form of L on Tate module T_lX, then $E^L(\phi x, y) = E^L(x, \phi' y)$.*

(3) $\mathrm{Tr}(\phi\phi^*) = \dfrac{2g}{L^g}(L^{g-1}\cdot\phi^*L)$. *Therefore,* $\phi \mapsto \mathrm{Tr}(\phi\phi^*)$ *is a positive-definite quadiatic form on* $\mathrm{End}^0 X$.

Proof. See [10], §20, 21. □

The first two properties directly follow the definition. They show $f \mapsto f'$ is really an involution and f' is the transpose of f with respect to E^L. The third property indicates the essential positivity of the Rosati involution, which becomes the key point to classify the structure of $\mathrm{End}^0 A$ for simple A. This property can be induced by Theorem 4.28. The only thing to check is $\dfrac{(E^L)^{\wedge(g-1)}\wedge(E^L\circ(\phi\times\phi))}{(E^L)^{\wedge g}} = \dfrac{1}{g}\mathrm{Tr}(\phi\phi')$, which is purely a linear algebra problem.

The classification of $\mathrm{End}^0 A$ for simple abelian variety A is due to Albert. We just cite the results that will be used in this article. As usual, K is the centre of $\mathrm{End}^0 A$. Let K_0 be the subfield of K fixed by the Rosati involution. Then, $K = K_0$ is totally real, or K is a CM field and the involution is the conjugation due to the following lemma.

Lemma 4.44. *Let K be a number field with a positive involution $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$, which means $\sigma^2 = id$ and $\mathrm{Tr}(a\sigma(a)) > 0$. $K_0 \subset K$ is the fixed field of σ. Then, it lies in one of the two cases:*

(1) $K = K_0$ *is a totally real field.*
(2) $K = K_0(\sqrt{\alpha})$, *where* $\alpha \in K_0$ *and* K_0 *is totally real.* α *satisfies* $\sqrt{\alpha} \notin K$, $\sigma(\sqrt{\alpha}) = -\sqrt{\alpha}$ *and for every homomorphism* $\rho : K_0 \to \mathbb{R}$, $\rho(\alpha) < 0$. *Thus,* K *is totally imaginary over* K_0 *and a CM field.*

Proposition 4.45. *Let A be a simple abelian variety of CM type. Then* $\mathrm{End}^0 A$ *is a CM field.*

Proof. See [10], §21. □

4.4. **Abelian schemes and moduli problems.**

Definition 4.46. An abelian scheme over a scheme S is a smooth proper group scheme with connected geometric fibers. Being a group scheme, X is equipped with the following structures:

(1) a unit section $e : S \to X$,
(2) a multiplication morphism $m : X \times_S X \to X$, and
(3) an inverse morphism $i : X \to X$,

such that the usual axioms for abstract groups hold.

Let X be an abelian scheme over S, the functor $\mathrm{Pic}_{X/S} : \mathrm{Sch/S} \to \mathrm{Grp}$ is defined as

$$\begin{aligned}\mathrm{Pic}_{X/S}(T) &= \{L \in \mathrm{Pic}(X\times_S T)\}/p_2^*(\mathrm{Pic}T)\} \\ &= \{(L,\iota) \mid L \in \mathrm{Pic}(X\times_S T), \iota : L|_{e\times T} \simeq \mathcal{O}_T\}\end{aligned}$$

Theorem 4.47. *Let X be a projective abelian scheme over S. Then the functor* $\mathrm{Pic}_{X/S}$ *is representable by a smooth separated S-scheme which is locally of finite presentation over S.*

The smooth scheme $\mathrm{Pic}_{X/S}$ equipped with the unit section corresponding to the trivial line bundle $\mathcal{O}_X$ admits a neutral component $\mathrm{Pic}^0_{X/S}$ which is an abelian scheme over S.

Definition 4.48. Let X/S be a projective abelian scheme. The dual abelian scheme$\hat{X}/S$ is the neutral component $\mathrm{Pic}^0_{X/S}$ of the Picard functor $\mathrm{Pic}_{X/S}$. The Poincare sheaf P is the restriction of the universal invertible sheaf on $X \times_S \mathrm{Pic}X/S$ to $X \times_S \hat{X}$.

Definition 4.49. Let X/S be an abelian scheme. A polarization of X/S is a symmetric isogeny$\phi : X \to \hat{X}$ which locally for the etale topology of S, is of the form ϕ_L for some ample line bundle L of X/S. The polarization is of type $D = (d_1, \cdots, d_n)$ $(d_1 \mid \cdots \mid d_n)$ if for every geometric point $\bar{s} \in S$, $\phi_{\bar{s}}$ is of type D.

Fix positive integers n, N and a type $D = (d_1, \cdots, d_n)$ with $d_1 \mid \cdots \mid d_n$ such that d_n is prime to N. Let $\mathcal{A}$ be the following moduli problem: for every scheme S over $\mathbb{Z}[(Nd_n)^{-1}]$, $\mathcal{A}(S)$ is the set of isomorphy classes of triples (X, λ, η):

(1) X is an abelian scheme over S,
(2) $\lambda : X \to \hat{X}$ is a polarization of type D, and
(3) η is an isomorphism between two symplectic spaces $(\mathbb{Z}/N\mathbb{Z})^2 \to X[N]$,

where the symplectic structure on $(\mathbb{Z}/N\mathbb{Z})^2$ is the standard one, namely the one presented by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and the symplectic structure on $X[N]$ is given by the polarization λ and the natural pairing $X_n \times \hat{X}_N \to \mathbb{G}_m$.

Theorem 4.50 (See [6], 2.3.1)**.** *When N is large enough (with respect to D), the functor $\mathcal{A}$ defined above is representable by a smooth quasi-projective scheme over $\mathbb{Z}[(Nd_n)^{-1}]$.*

A sketch proof is given below to show roughly how this moduli space is constructed. Two properties of the ample bundles on abelian varieties are used in the construction.

Proposition 4.51 (See [10], §17)**.** *Let $L = L(D)$ be an ample line bundle on an abelian variety A. Then, $|2D|$ is base point free and L^n is very ample when $n \geq 3$.*

Theorem 4.52 (Vanishing, see [10], §16)**.** *Let L be a line bundle on X. If $K(L)$ is finite, there is a unique integer $i = i(L), 0 \leq i(L) \leq g$, such that $H^j(X, L) = 0$ for $j \neq i$ and $H^i(X, L) \neq 0$. Moreover, L is ample if and only if $i(L) = 0$.*

Sketchy proof of Theorem 4.50. Let X be an abelian scheme over S and $\hat{X}$ be its dual abelian scheme, and let $\lambda : X \to \hat{X}$ be a polarization of type $D = (d_1, \cdots, d_n)$. P is the Poincare line bundle on $X \times_S \hat{X}$. Define $L^\Delta(\lambda) = (id_X, \lambda)^* P$. We claim that $\lambda_{L^\Delta(\lambda)} = 2\lambda$. In fact, locally for etale topology, we can assume $\lambda = \lambda_L$ for some line bundle over X which is relatively ample. Then

$$
\begin{aligned}
L^\Delta(\lambda) &= \Delta^*(id_X \times \lambda)^* P \\
&= \Delta^*(m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}) \\
&= [2]^* L \otimes L^{-2}
\end{aligned}
\tag{4.53}
$$

$\lambda_{[2]^*L} = 4\lambda_L$, deduced from Proposition 4.22. Thus, $\lambda_{[2]^*L \otimes L^{-2}} = 2\lambda_L$.

(4.53) also shows that $L^\Delta(\lambda)$ is a line bundle relatively ample locally over S, then $L^\Delta(\lambda)$ is a relatively ample line bundle over S. Then $L^\Delta(\lambda)^3$ is very ample by Proposition 4.51. Together with Theorem 4.52, $\mathrm{R}^i \pi_* L^\Delta(\lambda) = 0, i > 0$ and $M = \pi_* L^\Delta(\lambda)$ is a vector bundle on S of rank $m + 1 = 6^n d$, where $d = \prod_i d_i$.

An isomorphism $\alpha : \mathbb{P}^m_S \to \mathbb{P}_S(M)$ is called a linear rigidification of a polarized abelian scheme (X, λ). Define functor $\mathcal{H}$ which assigns every S to isomorphy classes of quadruples $(A, \lambda, \eta, \alpha)$ where α is a linear rigidification of (X, λ). There is a natural forgetting functorial morphism $\mathcal{H} \to \mathcal{A} : (A, \lambda, \eta, \alpha) \mapsto (A, \lambda, \eta)$, which is a $\mathrm{PGL}(m+1)$-torsor. Each linear rigidification provides an embedding $X \hookrightarrow \mathbb{P}^m_S$. With Theorem 4.52 and since the rank of vector bundle M is m, we obtain a functorial morphism $f : \mathcal{H} \to \mathrm{Hilb}^{Q(t),1}(\mathbb{P}^m)$, where $\mathrm{Hilb}^{Q(t),1}(\mathbb{P}^m)$ is the Hilbert scheme of 1-pointed subschemes of $\mathbb{P}^m$ with Hilbert polynomial $Q(t) = 6^n dt^n$. We cite the following result.

Proposition 4.54. *The morphism f identifies $\mathcal{H}$ with an open subfunctor of $\mathrm{Hilb}^{Q(t),1}(\mathbb{P}^m)$ which consist of pointed smooth subschemes of of $\mathbb{P}^m$.*

This can be explained by a theorem of Grothendieck, which shows any smooth projective morphisme $f : X \to S$ over a geometrically connected base S with a section $e : S \to X$ has an abelian scheme structure if and only if one geometric fiber X_s does.

Since a polarized abelian varieties with principal N-level structure has no trivial automorphisms, $\mathrm{PGL}(m+1)$ acts freely on $\mathcal{H}$. Thus, take $\mathcal{A}$ to be the quotient of $\mathcal{H}$ by this free action. This construction depends on the GIT theory. $\square$

5. Examples of Shimura Varieties and Moduli Intepretations

In this section, we survey some important cases of Shimura varieties by specifying the reductive group G in Definition 3.17, and introduce the modular interpretations of these Shimura varieties. To be more specific, $\mathbb{C}$-points on a Shimura variety parameterize a family of abelian varieties, which can be attained immediately when we combine the adelic description of the Shimura variety and the knowledge of abelian varieties mentioned in the last section. Furthermore, the moduli space constructed in Theorem 4.50 provides an integral model for a Shimura variety, and it turns out to be canonical in the sense of what is to be defined in the next section.

5.1. Siegel modular varieties.

Definition 5.1. Let k be a field of characteristic $\neq 2$. (V, ψ) is a symplectic space of dimension $2n$ over k if V is a $2n$-dimensional k vector space and ψ is a nondegenerate alternating form. A subspace W of V is totally isotropic if $\psi(W, W) = 0$. A symplectic basis of a symplectic space V is a basis $(e_{\pm i})_{1 \le i \le n}$ such that

$$\psi(e_i, e_{-i}) = 1, \text{ for } 1 \le i \le n$$
$$\psi(e_i, e_j) = 0, \text{ for } j \neq \pm i.$$

With some basic linear algebra knowledge, one can see every symplectic space has symplectic bases. Then every maximal totally isotropic subspace of V is of dimension n, called a lagrangian.

Definition 5.2. Let (V, ψ) be a nonzero symplectic space. The group of symplectic similitude $\mathrm{GSp}(\psi)$ is defined to be the group of automorphisms of V preserving ψ up to a scalar. Thus

$$\mathrm{GSp}(\psi)(k) = \{g \in \mathrm{GL}(V) \mid \psi(gu, gv) = v(g) \cdot \psi(u, v) \text{ for some } v(g) \in k^\times\}$$

where $v : \mathrm{GSp} \to \mathbb{G}_m$ is a homomorphism. The symplectic group $\mathrm{Sp}(\psi)$ is defined to be the kernel of v.

$\mathrm{Sp}(\psi)$ is actually the derived group of $\mathrm{GSp}(\psi)$ and we have a diagram like (1.3)

$$
\begin{tikzcd}
 & \mathrm{Sp}(\psi) \arrow[d] \arrow[dr] & \\
\mathbb{G}_m \arrow[r] \arrow[dr] & \mathrm{GSp}(\psi) \arrow[r, "\mathrm{ad}"] \arrow[d, "v"] & \mathrm{GSp}(\psi)^{\mathrm{ad}} \\
 & \mathbb{G}_m &
\end{tikzcd}
\tag{5.3}
$$

The kernel of the diagonal maps is the centre of $\mathrm{Sp}(\psi) = \mathbb{G}_m \cap \mathrm{Sp}(\psi) = \mu_2$.

When $\dim V = 2$, $\dim_k \mathrm{Hom}(\bigwedge^2 V, k) = 1$, then $\mathrm{GSp}(\psi) = \mathrm{GL}_2$ and $\mathrm{Sp}(\psi) = \mathrm{SL}_2$ (because there is a unique symplectic structure on the vector space). This is the case of classical modular curves, where SL_2 acts on the complex upper surface $\mathcal{H}_1$. Now we define the Siegel upper space $\mathcal{H}_n$ as a generalization of $\mathcal{H}_1$ with an action of GSp.

Definition 5.4 (Siegel upper half space)**.** The Siegel upper half space $\mathcal{H}_g$ of degree g consists of the symmetric complex $g \times g$ matrices $Z = X + iY$ with positive-definite imaginary part Y. It is a complex manifold by identifying the set of symmetric complex matrices with $\mathbb{C}^{g(g+1)/2}$ with the map $(z_{ij}) \mapsto (z_{ij})_{j \geq i}$. The symplectic group $\mathrm{Sp}_{2g}(\mathbb{R})$ is the group fixing the alternating form $\sum_{i=1}^g x_i y_{-i} - \sum_{i=1}^g x_{-i} y_i$ on $\mathbb{R}^{2g}$ where $\{e_1, \cdots, e_g, e_{-1}, \cdots, e_{-g}\}$ is a basis. Then

$$\mathrm{Sp}_{2g}(\mathbb{R}) = \left\{ \left(\begin{array}{cc} A & B \\ C & D \end{array} \right) \middle| \begin{array}{ll} A^t C = C^t A & A^t D - C^t B = I_g \\ D^t A - B^t C = I_g & B^t D = D^t B \end{array} \right\}$$

The group $\mathrm{Sp}_{2g}(\mathbb{R})$ acts transitively on $\mathcal{H}_g$ by

$$\left(\begin{array}{cc} A & B \\ C & D \end{array} \right) Z = (AZ + B)(CZ + D)^{-1}.$$

The matrix $\left(\begin{array}{cc} 0 & -I_g \\ I_g & 0 \end{array} \right)$ acts as an involution on $\mathcal{H}_g$, the only fixed point of which is iI_g. Therefore, $\mathcal{H}_g$ is a homogeneous and symmetric complex manifold. Identifying the set of symmetric complex matrices with $\mathbb{C}^{g(g+1)/2}$, the map $Z \mapsto (Z - iI_g)(Z + iI_g)^{-1}$ maps $\mathcal{H}_g$ to a bounded open set of $\mathbb{C}^{g(g+1)/2}$. Actually, the image of this map is D_g consisting of the symmetric complex matrices Z such that $I_g - \bar{z}^t Z$ is positive-definite. Using the Bergman metric for the bounded domain, we can endow an invariant hermitian metric on $\mathcal{H}_g$. Thus, $\mathcal{H}_g$ is a hermitian symmetric domain with this metric.

Fix a symplectic space (V, ψ) of dimension $2g$ over $\mathbb{Q}$, and let $\mathrm{GSp} = \mathrm{GSp}(\psi)$ and $\mathrm{Sp} = \mathrm{Sp}(\psi)$. In the spirit of Theorem 2.25 and 2.27, we construct X^+ as a conjugacy set of $h : \mathbb{S} \to \mathrm{Sp}$, and it can be identified with $\mathcal{H}_g$ after a choice of a symplectic basis for V.

Let X^+ denote the set of complex structures J on $V(\mathbb{R})$ such that ψ is a polarization of Hodge structure (V, h_J). (2.8) shows that this is equivalent to

$\psi(Ju, Jv) = \psi(u, v)$ and ψ_J is positive-definite. Moreover, a direct calculation indicates

$$\psi(Ju, Jv) = \psi(u, v) \Leftrightarrow \psi(zu, zv) = |z|^2\psi(u, v) \text{ , for all } z \in \mathbb{C}^\times \tag{5.5}$$

Thus, If $J \in X^+$, $J \in \mathrm{Sp}(\mathbb{R})$, $h_J(z) \in \mathrm{GSp}(\mathbb{R})$ for all $z \in \mathbb{C}^\times$, and $h_J(z) \in \mathrm{GSp}(\mathbb{R})$ for all $|z| = 1$. For a symplectic basis $(e_{\pm i})$ of V, a complex structure $J \in X^+$ can be defined as $Je_{\pm i} = \pm e_{\mp i}$. Conversely, if $J \in X^+$, an orthonormal basis of ψ_J gives a symplectic basis defining J in this way. Therefore, with a choice of a symplectic basis for V, X^+ is identified with $\mathcal{H}_g$. Let X^- denote the set of complex structures J on $V(\mathbb{R})$ such that $-J \in X^+$ and let $X = X^+ \bigsqcup X^-$.

GSp acts on X by conjugation $(g, J) \mapsto gJg^{-1}$. The action on corresponding morphism $\mathbb{C}^\times \to \mathrm{GSp}(\mathbb{R})$ is given by $h_{gJg^{-1}}(z) = gh_J(z)g^{-1}$. Since $\psi_{gJg^{-1}}(u, v) = v(g) \cdot \psi(g^{-1}u, g^{-1}v)$, the stabilizer in $\mathrm{GSp}(\mathbb{R})$ of X^+ is $\mathrm{GSp}(\mathbb{R})^+ = \{g \in \mathrm{GSp}(\mathbb{R}) \mid v(g) > 0\}$. Because every $J \in X$ corresponds to some symplectic basis, $\mathrm{Sp}(\mathbb{R})$ acts transitively on X^+. $\mathrm{GSp}(\mathbb{R})$ acts transitively on X since $\mathrm{GSp}(\mathbb{R}) \neq \mathrm{GSp}(\mathbb{R})^+$. Actually, $g : e_{\pm i} \to e_{\mp i}$ interchanges X^+ and X^- because $v(g) = -1$. In addition, the identification of X^+ and $\mathcal{H}_g$ with a choice of symplectic basis is compatible with the action of GSp on both sets.

Denote $X(\psi)$ by X and $X(\psi)^+$ by X^+.

Proposition 5.6. *The pair* $(\mathrm{GSp}(\psi), X(\psi))$ *is a Shimura datum and satisfies the axioms (SV1)-(SV6). This is called a Siegel Shimura datum.*

Proof. **SV1** : $\mathrm{GSp}(\psi)$ is a subgroup of $\mathrm{GL}(V)$, and this induces an injection $\mathrm{Lie}(\mathrm{GSp}(\psi)) \hookrightarrow \mathrm{Lie}(\mathrm{GL}(V)) \simeq \mathrm{Hom}(V, V)$. $\mathrm{Lie}(\mathrm{GL}(V)) \simeq \mathrm{Hom}(V, V)$, and the action ad of $\mathrm{GL}(V)$ on $\mathrm{Hom}(V, V)$ under this isomorphism is $(\alpha f)(v) = \alpha \circ f \circ \alpha^{-1}v$, for $\alpha \in \mathrm{GL}(V)$, $f \in \mathrm{Hom}(V, V)$, $v \in V$.

For $h = h_J$ where J is a complex structure on V, let $V^+ = V^{-1,0}$ and $V^- = V^{0,-1}$, so $V(\mathbb{C}) = V^+ \oplus V^-$. Therefore, $h(z)$ acts on V^+, V^-, $\mathrm{Hom}(V^+, V^+) \oplus \mathrm{Hom}(V^-, V^-)$, $\mathrm{Hom}(V^+, V^-)$ and $\mathrm{Hom}(V^-, V^+)$ as z, $\bar{z}$, 1, $\bar{z}/z$ and $z/\bar{z}$ respectively. Therefore, (SV1) holds.

SV2 : $J^2 = -1$ lies in the centre of $\mathrm{Sp}(\mathbb{R})$, and ψ as a $\mathrm{Sp}_{\mathbb{R}}$-invariant form is a J-polarization. By Proposition 2.24, J is a Cartan involution on $\mathrm{GSp}^{\mathrm{ad}}$.

SV3 : The symplectic group is simple over every algebraically closed field because its root system is indecomposable. Therefore, G^{ad} is $\mathbb{Q}$-simple. As $G^{\mathrm{ad}}(\mathbb{R})$ is not compact, (SV3) holds.

SV4 : $w_{h_J}(r)$ acts on V as multiplication by r when $r \in \mathbb{R}^\times$ (consider a symplectic basis corresponding to J). Then, $w_X : \mathbb{G}_{m\mathbb{R}} \to \mathrm{GL}(V(\mathbb{R}))$ is $r \mapsto rI, r \in \mathbb{R}^\times$ defined over $\mathbb{Q}$.

SV5 and **SV6** : The centre of GSp is $\mathbb{G}_m$, then (SV5) and (SV6) holds following the definition and Example 3.37. □

The Siegel modular variety attached to (V, ψ) is defined to be the Shimura variety $\mathrm{Sh}(\mathrm{GSp}(\psi), X(\psi))$.

Let K be a compact open subgroup of $G(\mathbb{A}_f)$. Define $\mathcal{H}_K$ to be the groupoid consisting of the following triples $((W, h), s, \eta K)$, where

- (W, h) is a rational Hodge structure of type $\{(-1, 0), (0, -1)\}$,
- s or $-s$ is a polarization for (W, h), and
- ηK is K-orbit of $\mathbb{A}_f$-linear isomorphism $V(\mathbb{A}_f) \to W(\mathbb{A}_f)$ under which ψ corresponds to an $\mathbb{A}_f^\times$-multiple of s.

A morphism $((W,h),s,\eta K) \to ((W',h'),s',\eta' K)$ is an isomorphism of rational Hodge structures $b : (W,h) \to (W',h')$ sending s to cs' where $c \in \mathbb{Q}$ such that $b \circ \eta = \eta' \bmod K$.

Proposition 5.7. *The set $Sh_K(\mathbb{C})$ classifies the isomorphism classes of $\mathcal{H}_K$. In particular, W appearing in a triple in $\mathcal{H}_K$ is of the same dimension of V, then there is an isomorphism $a : W \to V$ sending s to $\mathbb{Q}^\times$-multiple of ψ. Then the map $\mathcal{H}_K/\sim \to \mathrm{GSp}(\mathbb{Q})\backslash X \times \mathrm{GSp}(\mathbb{A}_f)/K : ((W,h),s,\eta K) \mapsto [ah, a\circ\eta]K$ is well-defined and is a bijection.*

Proof. We first check this map is well-defined. It will suffice to show the image of a triple is defined independent of the choice of a. If $((W,h),s,\eta K) \to ((W',h'),s',\eta' K)$ is an isomorphism induced by isomorphic $b : (W,h) \to (W',h')$ and choose a for $((W,h),s,\eta K)$ and a' for $((W',h'),s',\eta' K)$, one can apply the above independence to $a' \circ b$ and a, then the map sends isomorphic triples to the same class. The independence of a is explained as follows. If we choose another a' instead of a, $q = a' \circ a^{-1} \in \mathrm{GSp}$ and the map defined in the statement is unchanged.

We then explain two triples are isomorphic if they map the same class. In every isomorphism class of $\mathcal{H}_K$, we can take a representative triple of the form $((V,h),\psi,\eta K)$, where $h \in X$, $\eta \in G(\mathbb{A}_f)$, and a can be chosen to be id. Two triples $((V,h),\psi,\eta K)$ and $((V,h'),\psi,\eta' K)$ are isomorphic if and only if there exists $g \in \mathrm{GL}(V)(\mathbb{Q})$, such that $g \circ h = h'$, g sends ψ to $c\psi$ for some $c \in \mathbb{Q}^\times$, and $g \circ \eta = \eta'$. g sends ψ to $c\psi$, then $g \in \mathrm{GSp}(V)(\mathbb{Q})$. Thus (h,η) and (h',η') are in the same class. Then the map is injective. It is surjective because $((V,h),\psi,gK)$ lies in the inverse image of $[h,g]$ for any $[h,g] \in \mathrm{GSp}(\mathbb{Q})\backslash X \times \mathrm{GSp}(\mathbb{A}_f)/K$. □

Now we introduce the modular description of the $\mathbb{C}$-points of Siegel varieties. Let (V,ψ) be a symplectic space over $\mathbb{Q}$. Let $\mathcal{M}_K$ denote the groupoid whose objects are triples $(A,s,\eta K)$, where

- A is an abelian variety over $\mathbb{C}$,
- s or $-s$ is a polarization on $H_1(A,\mathbb{Q})$, and
- η is an isomorphism $V(\mathbb{A}_f) \to V_f(A)$ under which ψ corresponds to an $\mathbb{A}_f^\times$-multiple of s.

An isomorphism map $(A,s,\eta K) \to (A',s',\eta' K)$ is an isomorphism $A \to A'$ in $M(\mathbb{C})$ sending s to cs' where $c \in \mathbb{Q}$ such that $b \circ \eta = \eta' \bmod K$.

Proposition 5.8. *There is a canonical bijection $\mathcal{M}_K/\sim \to G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$ Thus, the set $Sh_K(\mathbb{C})$ classifies the isomorphism classes of $\mathcal{M}_K$.*

Proof. From Proposition 5.7 and Theorem 4.9. □

Example 5.9. In the general setting of Siegel modular varieties, K is taken to be any compact open subgroup of $G(\mathbb{A}_f)$. But as the case of familiar modular curves $Y_i(N) = \Gamma_i(N)\backslash\mathcal{H}$, there are some more concrete level structures together with certain choices of K as congruence groups, which links the moduli interpretation described here with moduli spaces introduced in Theorem 4.50.

Fix N, n, D as in Theorem 4.50. Let Λ be a free $\mathbb{Z}$-module of rank $2n$ and E be an alternating form on Λ with values in $\mathbb{Z}$. Assume the type of E is D. Then $\mathrm{GSp}(E \otimes \mathbb{Q})$ has a natural $\mathbb{Z}$-model G such that for any ring R, $G(R) = \{(g,c) \in \mathrm{GL}(\Lambda \otimes R) \times R^\times \mid E(gx,gy) = cE(x,y)$, for any $x, y \in \Lambda \otimes R\}$. G is a group scheme over $\mathbb{Z}$ and reductive over $\mathbb{Z}[1/d]$.

Fix a prime p not dividing N or d_n. For a prime $l \neq p$, let $K_{N,l}$ be the compact open subgroup of $G(\mathbb{Q}_l)$ defined as:

- if $l \not| N$, then $K_{N,l} = G(\mathbb{Z}_l)$;
- if $l|N$, then $K_{N,l} = \mathrm{Ker}(G(\mathbb{Z}_l) \to G(\mathbb{Z}_l/N\mathbb{Z}_l)$.

To relate moduli problems $\mathcal{A}$ and $\mathcal{M}_K$, another $\mathcal{A}'$ is introduced here to make this relationship more clear. The functor $\mathcal{A}'$ is defined as: for every scheme S whose residual characteristics are 0 or p, the objects of the groupoid $\mathcal{A}'(S)$ are triples $(X, \lambda, \tilde{\eta})$, where

- X is an abelian scheme over S,
- $\lambda : X \to \hat{X}$ is a $Z_{(p)}$-multiple of a polarization of degree prime to p, such that for every prime l and for every $s \in S$, the symplectic form induced by λ on $H_1(X_s, \mathbb{Q}_l)$ is similar to $U \otimes \mathbb{Q}_l$, and
- for every prime $l \neq p$, $\tilde{\eta}_l$ is a $K_{N,l}$-orbit of symplectic similitude from $H_1(X_s, \mathbb{Q}_l)$ to $\Lambda \otimes \mathbb{Q}_l$ which is invariant under $\pi_1(S, s)$. Assume for almost all prime l, this orbit corresponds to the auto-dual lattice $H_1(X_s, \mathbb{Z})$.

A morphism between two triples $(X, \Lambda, \tilde{\eta})$ and $(X', \Lambda', \tilde{\eta}')$ is a quasi-isogeny $\alpha : X \to X'$ of degree prime to p such that $\alpha^*(\lambda')$ is a $\mathbb{Z}_(p)^\times$-multiple of λ and $\alpha^*(\tilde{\eta}') = \tilde{\eta}$. There is an obvious functor $\mathcal{A} \to \mathcal{A}'$ assigning (X, λ, η) to $(X, \lambda, \tilde{\eta})$. $\tilde{\eta}$ is defined as follows. When $l \not| N$, a $K_{N,l}$-orbit $\tilde{\eta}_l$ in a triple is equivalent to an auto-dual $\mathbb{Z}_l$-lattice of $H_1(X, \mathbb{Q}_l)$. When $l|N$, a $K_{N,l}$-orbit $\tilde{\eta}_l$ in a triple is equivalent to an auto-dual $\mathbb{Z}_l$-lattice of $H_1(X, \mathbb{Q}_l)$ together with a triviality of rigidification of the pro-l-part of N-torsion points of X_s. In either case, $H_1(X, \mathbb{Z}_l)$ and the level structure provides the desired $\tilde{\eta}_l$. The key point is that this functor actually defines an equivalence of categories.

Proposition 5.10. *The functor $\mathcal{A} \to \mathcal{A}'$ defines an equivalence of categories.*

Proof. Only the definition of the functor needed in order to check the functor is fully faithful. To see it is surjective, notice that for any triple $(X, \Lambda, \tilde{\eta}) \in \mathcal{A}'$, there is a (unique) quasi-isogeny $\alpha : X' \to X$ such that $\alpha^*(\tilde{\eta})$ identifies $\Lambda \otimes \mathbb{Z}_l$ with $H_1(X', \mathbb{Z}_l)$. There is a unique way to pick a rigidification η' of $X'[N]$ compatible with $\tilde{\eta}_l$ for $l|N$. Because the polarization $\alpha^*\lambda$ on X' is of the same type D as the symplectic form E on U, $(X', \alpha^*\lambda, \eta') \in \mathcal{A}(S)$. □

Take $K_N = \prod_{l \neq p} K_{N,l} \times G(\mathbb{Z}_p)$. It is now simple to identify the description of $\mathcal{A}'$ and $\mathcal{M}_K$, so $\mathcal{A}'(\mathbb{C})$, and thus $\mathcal{A}(\mathbb{C})$, is classified by the points on the Siegel Shimura variety $\mathrm{Sh}_K(\mathbb{C}) = G(\mathbb{Q}) \backslash X^\pm \times G(\mathbb{A}_f)/K$. Since $\mathcal{A}$ is representable by a scheme over $\mathbb{Z}[1/Nd_n]$, it becomes a model of the Shimura variety Sh_K.

5.2. **Shimura varieties of Hodge type.**

Definition 5.11. A Shimura datum (G, X) is of Hodge type if there exists a symplectic space (V, ψ) over $\mathbb{Q}$ and an injection $\rho : G \hookrightarrow \mathrm{GSp}(\psi)$ carrying X to $X(\psi)$. Then the Shimura variety $\mathrm{Sh}(G, X)$ is said to be of Hodge type.

Lemma 5.12. *There exist multilinear maps of Hodge structures $t_i : V \times \cdots \times V \to \mathbb{Q}(r_i), 1 \leq i \leq n$, such that G is the subgroup of $\mathrm{GSp}(\psi)$ fixing the $t_i, 1 \leq i \leq n$.*

Let $\mathcal{H}_K$ denote the groupoid whose objects are the triples $((W, h), (S_i)_{0 \leq i \leq n}, \eta K)$, where

- (W, h) is a rational Hodge structure of type $\{(-1, 0), (0, -1)\}$,

- s_0 or $-s_0$ is a polarization for (W, h),
- $S_1, \cdots, s_n$ are multilinear maps $s_i : W \times \cdots \times W \to \mathbb{Q}(r_i), 1 \leq i \leq n$, and
- ηK is K-orbit of $\mathbb{A}_f$-linear isomorphism $V(\mathbb{A}_f) \to W(\mathbb{A}_f)$ under which ψ corresponds to an $\mathbb{A}_f^\times$-multiple of s_0 and t_i corresponds to s_i for $i = 1, \cdots, n$.

A morphism $((W, h), s, \eta K) \to ((W', h'), s', \eta' K)$ is an isomorphism of rational Hodge structures $b : (W, h) \to (W', h')$ sending s to a $\mathbb{Q}^\times$ multiple of s' and sending s_i to s'_i for $i = 1, \cdots, n$ such that $b \circ \eta = \eta'$ mod K.

Proposition 5.13. *$Sh_K(\mathbb{C})$ classifies the isomorphism classes of $\mathcal{H}_K$. The map can be defined in the same way as in Proposition 5.7.*

Now we introduce a modular description of the points on Shimura varieties of Hodge type as in the Siegel case. Let $\mathcal{M}_K$ denote the isomorphy classes of triples $(A, (s_i), \eta K)$, where

- A is an abelian variety over $\mathbb{C}$,
- s_0 or $-s_0$ is a polarization on $H_1(A, \mathbb{Q})$,
- $s_1, \cdots, s_n$ are Hodge tensors for A or its powers, and
- η is a K-orbit of isomorphisms $V(\mathbb{A}_f) \to V_f(A)$ under which ψ corresponds to an $\mathbb{A}_f^\times$-multiple of s and each t_i to s_i,

such that there exists an isomorphism $a : H_1(A, \mathbb{Q}) \to V$ sending s_0 to a $\mathbb{Q}^\times$-multiple of ψ, s_i to t_i each $i \geq 1$, and h to an element of X. $(*)$

An isomorphism map $(A, s, \eta K) \to (A', s', \eta' K)$ is an isomorphism $A \to A'$ in $\mathbf{AV^0}(\mathbb{C})$ sending s to cs' with $c \in \mathbb{Q}^\times$, s_i to s'_i, and η to η' modulo K.

Theorem 5.14. *The set $Sh_K(\mathbb{C})$ classifies the isomorphy classes of $\mathcal{M}_K$.*

One problem is that the condition $(*)$ is hard to check. In some PEL cases, a trace condition serves as an alternative.

5.3. **PEL Shimura varieties.** In this section, we first introduce the definition of PEL Shimura data and the relevant properties, then we describe the PEL moduli problem and connect it to the Shimura varieties associated with some PEL Shimura data. Basically, PEL type Shimura varieties can be interpreted as moduli spaces of polarized abelian schemes with multiplication by the ring of integers of some number field and some level structure, as "P" means "polarization", "E" means "endomorphisms" and "L" means "level structure".

Definition 5.15. Let B be a semisimple $\mathbb{Q}$-algebra with centre F. An involution of a B is a $\mathbb{Q}$-linear bijective map $B \to B : b \mapsto b^*$ such that $(ab)^* = b^* a^*$ and $a^{**} = a$ for every $a, b \in B$. An involution is said to be positive if $\mathrm{Tr}_{B/\mathbb{Q}}(b^* b) > 0$ for all $b \in B \backslash \{0\}$.

Let B be a k-algebra with involution $*$, a symplectic $(B, *)$-module is a B-module V equipped with a skew-symmetric k-bilinear form $\psi : V \times V \to k$ such that

$$\psi(b^* u, v) = \psi(u, bv) \text{ for all } u, v \in V$$

Proposition 5.16. *Let $(B, *)$ be a semisimple k-algebra with involution. If the field k is algebraically closed, then $(B, *)$ is isomorphic to a product of pairs of the following types:*

- $(\mathbf{A})$ $M_n(k) \times M_n(k), (a, b)^* = (b^t, a^t)$;
- $(\mathbf{C})$ $M_n(k), a^* = a^t$;

- (**BD**) $M_n(k), a^* = J \cdot a^t \cdot J^{-1}$ *where* $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$;

Definition 5.17. Let $(B,*)$ be a semisimple k-algebra with involution. The centre of B is F and F_0 is the subalgebra of invariants of $*$ in F. We say $(B,*)$ has type (A),(C) or (BD) if for all k homomorphism $\rho : F_0 \to k^a$, $(B \otimes_{F_0,\rho} k^a, *)$ has this type.

Let $(B,*)$ be a semisimple $\mathbb{Q}$-algebra with involution and (V,ψ) be a faithful $(B,*)$-symplectic module. Define G as a algebraic subgroup of $\mathrm{GL}_B(V)$ such that for every $\mathbb{Q}$-algebra R,

$$G(R) = \{(g, \mu(g)) \in \mathrm{GL}_B(V)(R) \times R^\times \mid \psi(gx, gy) = \psi(\mu(g)x, y) \text{ for any } x, y \in V(R)\}$$

Proposition 5.18. *G is reductive, and it is connected if $(B,*)$ is of the case (A) or (C).*

Proof. See [8], 8.7. □

Therefore, throughout this section, we assume $(B,*)$ is of the case (A) or (C). This assumption also provides us with other conveniences.

Assume there exists a homomorphism $h : \mathbb{S} \to G_{\mathbb{R}}$ such that (V,h) is of type $\{(-1,0),(0,-1)\}$ and the form $\psi(u, h(i)v)$ is symmetric and positive-definite. Let X denote its $G(\mathbb{R})$-conjugacy class. A quintuple $(B,*,V,\psi,h)$ is called a rational PE datum. The action of G on V defines an embedding $G \hookrightarrow \mathrm{GSp}(\psi)$ which sends X to $X(\psi)$. In fact, for $b \in B$, denote the tensor $(x,y) \mapsto \psi(x, by)$ by t_b, then an element $g \in \mathrm{GSp}(\psi)$ commutes with b if and only if it fixes t_b. Hence, (G,X) is the Shimura datum of Hodget type associated with the system $(V, \{\psi, t_{b_1}, \cdots, t_{b_s})$ where $\{b_1, \cdots, b_s\}$ are taken to be a set of generators of B (as a $\mathbb{Q}$-algebra). The pair (G,X) satisfies the axioms (SV1-4), called a PEL Shimura datum.

Definition 5.19. Given a PE datum, let V_1 be the complex vector space $V(\mathbb{C})/F_h^0(V(\mathbb{C})$. Define the trace map $t : B \to \mathbb{C}$ to be $t(b) = \mathrm{Tr}_{\mathbb{C}}(b|V_1)$. Let $x_1, ..., x_r$ be indeterminates and $V_1 = F^0V/F$. Define the determinant polynomial $det_{V_1} = det(x_1b_1 + \cdots + x_rb_r, V_1 \times \mathbb{C}[x_1, \cdots, x_r])$ which is a homogenous polynomial of degree $\dim_{\mathbb{C}} V_1$. The reflex field E of the PEL datum is the field of definition of the isomorphism class of V_1 as an $\mathfrak{B} \otimes_{\mathbb{Q}} \mathbb{C}$-module. That is the subfield of $\mathbb{C}$ generated by the elements $t(b)$, for$b \in B$, or equivalently, the subfield of $\mathbb{C}$ generated by the coefficients of the polynomial $\det_{V_1}$. All that is defined above only depends on X instead of depending on h.

As usual, the moduli problem attached to a PEL Shimura variety is illustrated.

Theorem 5.20. *Let K be a open compact subgroup of $G(\mathbb{A}_f)$, then the $\mathbb{C}$-points on the corresponding Shimura variety $\mathrm{Sh}_K(G,X)$ classifies the isomorphy classes of quadruples $((A,i), s, \eta K)$, where*

- *A is an abelian variety over $\mathbb{C}$,*
- *s or $-s$ is a polarization on $H_1(A,\mathbb{Q})$,*
- *i is a homomorphism $B \to End^0(A)$, and*
- *ηK is a K-orbit of isomorphisms $V(\mathbb{A}_f) \to V_f(A)$ under which ψ corresponds to an $\mathbb{A}_f^\times$-multiple of s and each t_i to s_i,*

such that there exists a B-linear isomorphism $a : H_1(A,\mathbb{Q}) \to V$ sending s to a $\mathbb{Q}^\times$-multiple of ψ, and h_A to an element of X. $(*)$

The following proposition presents that if B is a simple algebra, the pair (G, X) only depends on G and X is nonempty. In this case, the PEL Shimura data is called simple PEL data.

Proposition 5.21. *There exists a homomorphism $h : \mathbb{S} \to G_{\mathbb{R}}$ such that (V, h) has type $\{(-1, 0), (0, -1)\}$ and $2\pi i\psi$ is a polarization of (V, h). Moreover, h is unique up to conjugation by an element of $G(\mathbb{R})$.*

Another problem left is when the condition $(*)$ in the above theorem holds. If it holds, we have the following condition $(**)$:

(1) $s(bu, v) = s(u, b^*v)$, and
(2) $\mathrm{Tr}(i(b)|\mathrm{Tgt}_0 A) = \mathrm{Tr}_{\mathbb{C}}(b|V(\mathbb{C})/F_h^0(V(\mathbb{C})))$, $h \in X$. Notice the right-hand side depends only on X.

However, it is not obvious if the condition $(*)$ can be derived from the condition $(**)$. Let $W = H_1(A, \mathbb{Q})$. One difficulty owes to the existence of a B-linear isomorphism $\alpha : W \to V$ sending s to a $\mathbb{Q}^{\times}$-multiple of Ψ. The existence of η shows that V and W has the same dimension, then there is a $B \otimes \mathbb{Q}^a$-linear isomorphism $\alpha : V(\mathbb{Q}^a) \to W(\mathbb{Q}^a)$ sending s to a $\mathbb{Q}^{a\times}$-multiple of Ψ. Thus, it defines a cycle $a = (a_\sigma = \alpha^{-1} \circ \sigma\alpha)_{\alpha \in \mathrm{Gal}(\mathbb{Q})} \in H^1(\mathbb{Q}, G)$. The existence of η shows that the image of this class is trivial in $H^1(\mathbb{Q}_l, G)$ for all prime l. Therefore, it determines a cycle $a \in \mathrm{Ker}^1(\mathbb{Q}, G) = \mathrm{Ker}(H^1(\mathbb{Q}, G) \to \prod_l H^1(\mathbb{Q}_l, G))$. The following lemma implies that, under the assumption $(B, *)$ is of type (A) and (C), $(*)$ holds if and only if $(**)$ holds and a is trivial. It also shows, under the same assumption, that the image of a in $H^1(\mathbb{R}, G)$ is trivial.

Lemma 5.22 ([4], 5.7)**.** *(1) The $G^{\mathrm{der}}(\mathbb{R})$-conjugacy class of h is uniquely determined by the map t.*
(2) In case (A), the isomorphism class of (V, ψ) is determined by t.
(3) In case (C), the isomorphism class of (V, ψ) is determined by $\dim V$.

Proposition 5.23. *Let $(B, *)$ be of type (Aeven) or (C), and keep the notations in Theorem 5.20. The condition $(*)$ in Theorem 5.20 is implied by the condition $(**)$.*

Proof. See [8], 8.19. □

As in the Siegel case, we want to give some open compact subgroups K which correspond to more concrete moduli problems. We can define some integral models of G with the aid of integral PE datum.

Definition 5.24. An integral PE datum is a quintuple $(O, *, \Lambda, \psi, h)$, where

(1) O is an order in a finite-dimensional semisimple $\mathbb{Q}$-algebra B (that is, O is a subring of B that is a free $\mathbb{Z}$-module and spans the $\mathbb{Q}$-vector space B),
(2) $*$ is a positive involution of O, i.e. it extends to a positive involution on B,
(3) Λ is an O-module that is finitely generated and free as a $\mathbb{Z}$-module,
(4) $\psi(\cdot, \cdot) : \Lambda \times \Lambda \to \mathbb{Z}$ is a $\mathbb{Z}$-bilinear alternating map such that $\psi(bx, y) = \psi(x, b^*y)$ for $x, y \in \Lambda$, and
(5) h is a homomorphism $\mathbb{S} \to G_{\mathbb{R}}$, where G is defined to be an algebraic group over $\mathbb{Z}$ such that for every commutative ring R,

$$G(R) = \{(g, \mu(g)) \in \mathrm{End}_{O \otimes_{\mathbb{Z}} R}(\Lambda \otimes_{\mathbb{Z}} R) \times R^{\times} \mid \psi(gx, gy) = \psi(\mu(g)x, y) \text{ for any } x, y \in \Lambda \otimes_{\mathbb{Z}} R\},$$

and such that (V, h) has type $\{(-1, 0), (0, -1)\}$ and the form $\psi(u, h(i)v)$ is symmetric and positive-definite.

Fix a prime p such that the PEL datum is unramified at p. PEL moduli problem over a discrete valuation ring with residual characteristic p is to be described. Fix an integral $N \geq 3$, consider the moduli problem $\mathfrak{B}$ of abelian schemes with a PE-structure and principal N-level structures. Let E be the reflex field of the PE datum. The contravariant functor $\mathfrak{B}$ assigns every $O_{E,p} = O_E \otimes \mathbb{Z}_{(p)}$-scheme S a groupoid $\mathfrak{B}(S)$ whose objects are $(A, \lambda, \iota, \eta)$, where

(1) A is an abelian scheme over S,
(2) $\eta : A \to \hat{A}$ is a polarization,
(3) $\iota : O \to \mathrm{End}_S(A)$ is a homomorphism such that the Rosati involution induced by η restricts to the involution $*$ of O

$$\det(\beta_1 x_1 + \cdots + \beta_n x_n, \mathrm{Lie} A) = \det_{V_1},$$

and for every prime $l \neq p$ and every geometric point s of S, the Tate module $T_l(A_s)$, equipped with the action of O and with the alternating form induced by η, is similar to $\Lambda \otimes \mathbb{Z}_l$, and
(4) η is a similitude from $A[N]$ equipped with the symplectic form from the polarization and the action of O to $\Lambda/N\Lambda$ that can be lifted to an isomorphism $H_1(A_s, \mathbb{A}_f^p)$ with $\Lambda \otimes_{\mathbb{Z}} \mathbb{A}_f^p$, for every geometric point s.

Theorem 5.25. *The functor which assigns to each $O_{E,p}$-scheme S the set of isomorphy classes $\mathfrak{B}(S)$ is smooth and representable by a quasi-projective scheme over $O_{E,p}$.*

Proof. There is a natural forgetting functor $\mathfrak{B} \to \mathcal{A}$, the fiber of which is the isomorphy set of ι. After choosing a $\mathbb{Z}$-basis $\{b_1, \cdots, b_n\}$ of O, an ι is equivalent to actions of $b_1, \cdots, b_n$ on A satisfying certain equations. Since $\mathcal{A}$ is representable by a quasi-projective scheme shown in Theorem 4.50, it will suffice to use the next lemma. □

Lemma 5.26 (See [6], 3.4.4)**.** *Let A be a projective abelian scheme over a locally noetherian scheme S. Then the functor that assigns to every S-scheme T the set $End(A_T)$ is representable by a disjoint union of projective schemes over S.*

For every prime $l \neq p$, let $K_{N,l} = \begin{cases} K_{N,l} = G(\mathbb{Z}_l) & ,l \not| N \\ \mathrm{Ker}(G(\mathbb{Z}_l) \to G(\mathbb{Z}_l/N\mathbb{Z}_l)) & ,l | N \end{cases}$ and $K_{N,p} = G(\mathbb{Z}_p)$ be a compact open subgroup of $G(\mathbb{Q}_l)$. Let $K = \prod K_{N,l}$ be a compact open subgroup of $G(\mathbb{A}_f)$. It can be shown when the PEL datum is of type A and C, $\mathfrak{B}(\mathbb{C})$ is classified by $G(\mathbb{Q})\backslash X \times G(\mathbb{A})/K$.

The way to prove this is similar to the Siegel case. We can define another moduli problem $\mathfrak{B}'$ as follows: an object of $\mathfrak{B}'(S)$ where S is a scheme over $O_{E,p}$ is a quintuple $(A, \lambda, \iota, \tilde{\eta})$, where

(1) A, ι are the same as in $\mathfrak{B}(S)$,
(2) $\lambda : A \to \hat{A}$ is a $\mathbb{Z}_{(p)}$-multiple of a polarization,
(3) fixing a geometric point s of S, for every prime $l \neq p$, $\tilde{\eta}$ is a $K_{N,l}$-orbit of isomorphisms from $V_l(A_s)$ to $\Lambda \otimes \mathbb{Q}_l$ compatible with symplectic forms and action of O and stable under the action of $\pi_1(S, s)$.

A morphism from $(A, \lambda, \iota, \tilde{\eta})$ to $(A', \lambda', \iota', \tilde{\eta'})$ is a quasi-isogeny $\alpha : A \to A'$ of degree prime to p carrying λ to a $\mathbb{Q}^\times$-multiple of λ' and carrying $\tilde{\eta}$ to $\tilde{\eta'}$.

There is a natural functor $\mathfrak{B} \to \mathfrak{B}'$ defines the equivalence of categories, similar to Proposition 5.10. It is also easy to check that $\mathfrak{B}'$ describes the same moduli problem as M_K defined earlier in this section. Therefore, Theorem 5.20 shows that $\mathfrak{B}(\mathbb{C})$ is classified by $\mathrm{Sh}_K(G, X)(\mathbb{C})$.

6. Canonical Models

The notion of canonical models is natural either when we survey models of the Shimura varieties associated with tori or when we consider the moduli problems of Shimura varieties. One of the most prominent properties of Shimura varieties is that the canonical models exist for any Shimura variety. In this section, we first introduce the basic theory of complex multiplication of abelian varieties. Next, the tori case is discussed, which motivates the definition of canonical models. We conclude with attempts to show the uniqueness and existence of canonical models, where the existence is accomplished by combining the moduli interpretation and complex multiplication theory.

6.1. **Complex multiplication.**

Definition 6.1. A number field E is a CM field if it is a quadratic totally imaginary extension of a totally real field F. Then each embedding of $F \hookrightarrow \mathbb{R}$ will extend to two conjugate embeddings of $E \hookrightarrow \mathbb{C}$. A CM-type Φ of E is a subset $\Phi \subset \mathrm{Hom}(E, \mathbb{C})$ such that $\mathrm{Hom}(E, \mathbb{C}) = \Phi \sqcup \bar{\Phi}$ where $\bar{\Phi}$ consists of the conjugate of the embeddings in Φ.

The involution $a \mapsto a^*$ on E is defined by the nontrivial element in $\mathrm{Gal}(E/F)$, and the fixed field of it is F. It is a positive involution in the sense that for every $\alpha : F \to \mathbb{R}$, $\alpha(\mathrm{Tr}_{E/F}(b^* b)) > 0$ for any $b \in E$.

Definition 6.2. Let E be a CM-field of degree $2g$ over $\mathbb{Q}$. An abelian variety A of dimension g is of CM type if there exists a CM field E and a homomorphism $i : E \to \mathrm{End}^0 A$. A pair (A, i) of CM type (E, Φ) if as $E \otimes_{\mathbb{Q}} \mathbb{C}$-module, $\mathrm{Tgt}_0(A) \simeq \mathbb{C}^{\Phi}$.

Actually, if $A = V/\Lambda$ is an abelian variety over $\mathbb{C}$ of CM type, where $V \simeq \mathbb{C}^g$ and Λ is a full lattice in V, with the homomorphism $i : E \to \mathrm{End}^0 A$,then $\Lambda \otimes \mathbb{Q}$ is a one-dimensional E-module. Therefore, as E-module (via i), $\prod_{\sigma: E \hookrightarrow \mathbb{C}} \mathbb{C} \simeq E \otimes_{\mathbb{Q}} \mathbb{C} \simeq \Lambda \otimes \mathbb{C} \simeq H_1(A, \mathbb{C}) \simeq H^{-1,0} \oplus H^{0,-1} = \mathrm{Tgt}_0 A \oplus \overline{\mathrm{Tgt}_0 A}$. Thus, (A, i) is always of CM type (E, Φ) with a unique Φ.

Proposition 6.3. *Let E be a CM field and Φ be a CM type of E. The complex torus $A_\Phi = \mathbb{C}^\Phi / \Phi(O_E)$ is an abelian variety over $\mathbb{C}$ of CM type (E, Φ) with the natural homomorphism $i_\Phi : E \to \mathrm{End}^0 A_\Phi$. Moreover, every abelian variety A over $\mathbb{C}$ of CM type (E, Φ) is E-isogenous to (A_Φ, i_Φ).*

Proof. To prove A_Φ is an abelian variety, it will suffice to construct a Riemann form on A_Φ. Take α to be a totally imaginary element of E, which means $\sigma\alpha = -\alpha$ for the nontrivial $\sigma \in \mathrm{Gal}(E/F)$. α is further asked to be an algebraic integer and satisfy for all $\phi \in \Phi$, $\mathrm{Im}(\phi\alpha) > 0$. Such α can be obtained by weak approximation theorem. Define $\Psi(u, v) = \mathrm{Tr}_{E/\mathbb{Q}}(\alpha u v^*)$, for $u, v \in O_E$. Since α, u, v are all algebraic integers, the form Ψ takes values in $\mathbb{Z}$. To show this form is a Riemann form, observe that $\Psi = \sum_{\phi \in \Phi} \Psi_\phi$, where $\Psi_\phi(u, v) = \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(\phi(\alpha) \cdot u \cdot \bar{v})$, $u, v \in \mathbb{C}$. Because α is totally imaginary, $\Psi_\phi(u, v) = \phi(\alpha)(u\bar{v} - \bar{u}v) \in \mathbb{R}$ is alternating, and satisfies $\Psi_\phi(u, iu) > 0$ and $\Psi_\phi(iu, iv) = \Psi_\phi(u, v)$.

Assume (A, i) is an abelian variety over $\mathbb{C}$ of CM type (E, Φ). Let $A = V/\Lambda$ and $O = E \cap \mathrm{End}(A)$. Choose any $u \in \Lambda$. As $\Lambda \otimes \mathbb{Q} \simeq H_1(A, \mathbb{Q})$ is a one-dimensional E-module, $[\Lambda : Ou]$ is finite. In addition, there exists an integer N such that $NO_E \subset O$ of finite index. Thus, $A \to A' = V/Ou \to V/O_E(Nu)$ are isogenies between abelian varieties of CM type (E, Φ). Since $V/O_E(Nu) \simeq A_\Phi$, the proposition follows. $\square$

Before starting the main theorem in this section, a few more definitions and preparations are needed.

Proposition 6.4. *Let (A, i) be an abelian variety of CM-type over $\mathbb{C}$. Then (A, i) has a model over $\mathbb{Q}^a$, uniquely determined up to isomorphism.*

Definition 6.5. Let (E, Φ) be a CM-type. The reflex field E^* of (E, Ψ) is the subfield of $\mathbb{Q}^a$ generated by $\sum_{\phi \in \Phi} \phi(a)$,$a \in E$. It is characterized by the following two equivalent properties.

(1) $\sigma \in \mathrm{Gal}(\mathbb{Q}^a/\mathbb{Q})$ fixes E^* if and only if $\sigma\Phi = \Phi$.
(2) E^* is the smallest subfield K of $\mathbb{Q}^a$ such that there exists a K-vector field V with an action of E for which $\mathrm{Tr}_K(a|V) = \sum_{\phi \in \Phi} \phi(a)$.

If an abelian variety A over $\mathbb{C}$ is of CM-type (E, Φ), Proposition 6.4 shows that A has a model over $K \subset \mathbb{Q}^a$, which should contain E^* by (2) in the above definition. Take a E^*-vector space V as in (2), and it can be viewed as a $E^* \otimes_{\mathbb{Q}} E$-vector space. It is unique up to isomorphism in this view. Define the reflex norm $N_{\Phi^*} : \mathrm{Res}_{E^*/\mathbb{Q}} \mathbb{G}_m \to \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m$ as $N_{\Phi^*}(a) = \det_E(a|V)$, for $a \in E^{*\times}$.

There is a connection between the Galois action on abelian varieties of CM type and the Artin reciprocity map which will be stated in the main theorem. Let E be a number field. According to class field theory, there exists a continuous surjective homomorphism called Artin map $\mathrm{rec}_E : \mathbb{A}_E^\times \to \mathrm{Gal}(E^{\mathrm{ab}}/E)$ such that for every finite extension L/K, the restriction map $\mathrm{rec}_{L/E} : \mathbb{A}_E^\times \to \mathrm{Gal}(L/E)$ is surjective with kernel $\mathrm{Nm}_{L/K}(\mathbb{A}_L^\times)$. $\mathrm{rec}_{L/E}$ maps the idele $\alpha = (1, \cdots, 1, \pi, 1, \cdots, 1)$ (π at place v is a prime element in $\mathcal{O}_{E_v}$) to the Frobenius element $(v, L/E) \in \mathrm{Gal}(L/E)$. For the sake of simplicity, we shall use the reciprocal map $\mathrm{art}_E(\alpha) = \mathrm{rec}_E(\alpha)^{-1}$ instead in the rest of this article.

Now we define the Galois action on abelian varieties of CM type. Let k be an algebraically closed field and $\sigma : k \to k$. Then σ induces a map $A(k) \to \sigma A(k)$ and further a map $V_f(A) \to V_f(\sigma A)$. An endomorphism of A can base change to an endomorphism of σA, giving the homomorphism $\sigma : \mathrm{End}^0(A) \to \mathrm{End}^0(\sigma A)$. Now let (A, i) be an abelian variety over $\mathbb{C}$ of CM type (E, Φ). Take $k = \mathbb{C}$ and $\sigma \in \mathrm{Aut}(\mathbb{C}/E^*)$. we can define ${}^\sigma i : E \to \mathrm{End}^0(\sigma A)$ by the composition of i and $\sigma : \mathrm{End}^0(A) \to \mathrm{End}^0(\sigma A)$. Then $(\sigma A, {}^\sigma i)$ is of CM type $(E, \sigma\Phi) = (E, \Phi)$ by the definition of E^*.

Theorem 6.6. *Let (A, i) be an abelian variety over $\mathbb{C}$ of CM type (E, Φ) and $\sigma \in \mathrm{Aut}(\mathbb{C}/E^*)$. For any $s \in \mathbb{A}_{E^*,f}^\times$ with $\mathrm{art}_{E^*}(s) = \sigma|E^{*\mathrm{ab}}$, there exists a unique E-linear isogeny $\alpha : A \to \sigma A$ such that $\alpha(N_{\Phi^*}(s) \cdot x) = \sigma x$ for all $x \in V_f A$ where $V_f A = \prod_{l\ prime} V_l A$*

The proof relies on the Shimura-Taniyama formula. To provide some insight into this theorem, we shall introduce the Shimura-Taniyama formula for which a lot more preparation work is needed, and then give a proof of the theorem assuming the Shimura-Taniyama formula.

Let A be an abelian variety over a number field K. Let $\mathfrak{B}$ be a prime of K, $\mathcal{O}_{K,\mathfrak{B}}$ be the localization of $\mathcal{O}_K$ at $\mathfrak{B}$ and $k = \mathcal{O}/\mathfrak{B}$ be the residual field. A has good reduction at $\mathfrak{B}$ if it extends to an abelian scheme $\mathcal{A}$ over $\mathcal{O}_{K,\mathfrak{B}}$. The reduction of A at $\mathfrak{B}$ is an abelian variety $\bar{A}$ over k defined to be the special fiber $\mathcal{A} \times_{\mathcal{O}_{K,\mathfrak{B}}} k$. The specialization map $A(K) \to \bar{A}(k)$ is constructed as follows. Let x be a K point of A, it extends to the unique $\mathcal{O}_{K,\mathfrak{B}}$ point $x' : \operatorname{Spec} \mathcal{O}_{K,\mathfrak{B}} \to A$, thus we get a k point $\bar{x}$ by the composition $\operatorname{Spec} k \hookrightarrow \mathcal{O}_{K,\mathfrak{B}} \xrightarrow{x'} A$.

For $l \neq \text{char } k$, the specialization map restricts on $A[l^n] \to \bar{A}[l^n]$ is an isomorphism, then it induced an isomorphism $V_l(A) \to V_l(\bar{A})$.Similarly, there is a homomorphism $\operatorname{End}(A) \to \operatorname{End}(\bar{A})$. This is an injective map since it is compatible with $V_l(A) \simeq V_l(\bar{A})$. The injection $\operatorname{End}(A) \to \operatorname{End}(\bar{A})$ extends to an injection $\operatorname{End}^0(A) \to \operatorname{End}^0(\bar{A})$. If (A, i) is an abelian variety of CM type, composing i with this injection gives $\bar{A}$ a CM structure $(\bar{A}, \bar{i})$.

Proposition 6.7. *Let (A, i) be an abelian variety of CM-type over a number field $K \subset \mathbb{C}$, then A has everywhere potential good reduction. This means for every prime $\mathfrak{B} \subset O_K$, after possibly replacing K by a finite extension, A will have good reduction at $\mathfrak{B}$*

An important observation is that the Frobenius map on the reduction of an abelian variety over a number field can be lifted to the original abelian variety.

Lemma 6.8. *Let (A, i) be an abelian variety of CM-type (E, Φ) over a number field $K \subset \mathbb{C}$ and $\mathfrak{B} \subset O_K$ be a prime. Assume A has god reduction at $\mathfrak{B}$, and the reduction of A at $\mathfrak{B}$ is $(\bar{A}, \bar{i})$ over $\mathbb{F}_q = \mathcal{O}_K/\mathfrak{B}$. Then the Frobenius map $\pi_{\bar{A}}$ of $\bar{A}$ lies in $\bar{i}(E)$.*

Theorem 6.9 (Weil, [15])**.** *For an abelian variety over $\mathbb{F}_q$, the Frobenius map π_A is in the centre of* $\operatorname{End}^0(A)$. *Regard π_A as an algebraic number, it is a Weil q-number, that is, for any embedding $\mathbb{Q}[\pi_A] \to \mathbb{C}$, $|\rho(\pi_A)| = \sqrt{q}$.*

In order to determine a Weil q-integer $\pi \in E$, it suffices to know its value at every place v of E above $p = \text{char } q$, which is accomplished by the Shimura-Taniyama formula.

Theorem 6.10 (Shimura -Taniyama)**.** *Preserve the notations in Lemma 6.8. Further, assume K is Galois over $\mathbb{Q}$ and contains all the conjugates of E. For a prime v of E dividing $q = p^n$, set $H_v = \{\rho : E \to K \mid \rho^{-1}(\mathfrak{B}) = \mathfrak{p}_v$. Then,*

$$\frac{\operatorname{ord}_v(\pi)}{\operatorname{ord}_v(q)} = \frac{|\Phi \cap H_v|}{|H_v|}. \tag{6.11}$$

Proof. See [8], 10.10. □

Proof of Theorem 6.6. Since (A, i) and $(\sigma A, {}^\sigma i)$ are both of CM type, there is an E-linear isogeny $\alpha : A \to \sigma A$ due to Proposition 6.3. The composition map $V_f(A) \xrightarrow{\sigma} V_f(\sigma A) \xrightarrow{V_f(\alpha)^{-1}} V_f(A)$ is $\mathbb{A}_{E,f} = E \otimes_{\mathbb{Q}} \mathbb{A}_f$-linear and $V_f(A)$ is free of rank one as $\mathbb{A}_{E,f}$-module. Thus, this map is the multiplication of a, for some $a \in \mathbb{A}_f$. Varying α, a changes by multiplying an element of $E^\times$, then we attain the well-defined map $\gamma : \operatorname{Gal}(\mathbb{Q}^a/E^*) \to \mathbb{A}_{E,f}/E^\times : \sigma \mapsto [a]$. This map factors through $\operatorname{Gal}(E^{*\text{ab}}/E^*)$. Composing it with the reciprocity map art_{E^*}, we get a homomorphism $\eta : \mathbb{A}_{E^*,f}/E^{*,\times} \to \mathbb{A}_{E,f}/E^\times$. The only remaining task is to show this homomorphism is induced by Nm_{Φ^*}, which follows from the Shimura-Taniyama formula. Because of the density theorem, one just needs to check this in the case $\sigma \in \mathrm{Gal}(L/E^*)$ where L is Golais over E^* and contains the field of definition of A and E, and σ is a Frobenius element $(\mathfrak{B}, L/E)$ such that A has good reduction at the prime $\mathfrak{B}$. In this situation, $\gamma(\sigma)$ is calculated by Shimura-Taniyama formula. □

6.2. Definition of canonical models.

Definition 6.12. Let (G, X) be a Shimura datum. A model of $\mathrm{Sh}(G, X)$ over a subfield K of $\mathbb{C}$ is an inverse system $M(G, X) = (M_K(G, X))_K$ of varieties over K endowes with a right action of $G(\mathbb{A}_f)$ such that $M(G, X)_{\mathbb{C}} = \mathrm{Sh}(G, X)$.

We are going to introduce the "natural" field of definition of a Shimura variety, the reflex field of a Shimura variety. Indeed, the reflex field of the Shimura datum is, whenever the Shimura variety is a moduli variety in some natural way, the field of definition of the moduli problem. Therefore, it is natural to ask the canonical model of a Shimura variety be defined over the reflex field.

Definition 6.13. For a reductive group G over $\mathbb{Q}$ and a subfield $k \subset \mathbb{C}$, we write $C(k)$ for the set of $G(k)$-conjugacy classes of cocharacters of G_k defined over k. That is, $C(k) = G(k)\backslash\mathrm{Hom}(\mathbb{G}_m, G_k)$. A homomorphism $k \to k'$ induces a map $C(k) \to C(k')$. In particular, $\mathrm{Aut}(k'/k)$ acts on $C(k')$.

Assume G splits over K. Then G_K contains a split maximal torus T. The Weyl group $W = W(G_K, T) = N/N^\circ$ is a constant etale algebraic group where N is the normalizer of T in G_K.

Lemma 6.14. *Let T be a split maximal torus in G_K. Then the map*

$$W\backslash\mathrm{Hom}(\mathbb{G}_m, T_K) \to G(k)\backslash\mathrm{Hom}(\mathbb{G}_m, G_K)$$

is bijective.

Proof. See [9], 17.105. □

This lemma implies all elements in $W\backslash\mathrm{Hom}(\mathbb{G}_m, T_{\mathbb{C}})$ are defined over $\mathbb{Q}^a$ as the same is true for W and $\mathrm{Hom}(\mathbb{G}_m, T_{\mathbb{C}})$, indicating the map $C(\mathbb{Q}^a) \to C(\mathbb{C})$ is bijective.

Let (G, X) be a Shimura datum. Each $h \in X : \mathbb{S} \to G_{\mathbb{R}}$ induces a cocharacter μ_h of $G_{\mathbb{C}}$ as $\mu_h = h_{\mathbb{C}} \circ r$ where $r : \mathbb{G}_{m,\mathbb{C}} \to \mathbb{S}_{\mathbb{C}}, z \mapsto (z, 1)$. Altering h with another one conjugates μ_h, then X defines a class $c(X)$ of $C(\mathbb{C})$. Moreover, $c(X)$ can be seen as an element in $C(\mathbb{Q}^a)$ with the bijection $C(\mathbb{Q}^a) \to C(\mathbb{C})$.

Definition 6.15. The reflex field $E(G, X)$ is the field of definition of $c(X)$, which is the fixed field of the subgroup of $\mathrm{Gal}(\mathbb{Q}^a/\mathbb{Q})$ fixing $c(X)$.

Example 6.16 (The Case of Tori)**.** Let $G = T$ be a torus and $h : \mathbb{S} \to T_{\mathbb{R}}$ be a morphism. Then we get a Shimura datum (T, h) since (SV1-3) are trivially satisfied in this situation. For every open compact subgroup K of $T(\mathbb{A}_f)$, $T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K$ is finite. Thus, a model $M_K(T, h)$ of $\mathrm{Sh}_K(T, h)$ over $F \subset \mathbb{C}$ is equivalent to an action of $\mathrm{Gal}(\bar{F}/F)$ on $T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K$. By passage to limit shown in (3.35), there should be an action of $\mathrm{Gal}(\bar{F}/F)$ on $\varprojlim_K T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K = \overline{T(\mathbb{Q})}\backslash T(\mathbb{A}_f)$ commuting with the action of $T(\mathbb{A}_f)$. $\overline{T(\mathbb{Q})}\backslash T(\mathbb{A}_f)$ is an abelian profinite group, and it is actually $\pi_0(T(\mathbb{Q})\backslash T(\mathbb{A}_f))$ with the obvious bijective map. Moreover, the action of $\mathrm{Gal}(\bar{F}/F)$

factors through $\mathrm{Gal}(F^{\mathrm{ab}}/F)$. Hence, after composing with the Artin map, with the aim of constructing such an action, it is necessary to construct a homomorphism of groups $\eta : F^\times\backslash\mathbb{A}_F^\times \to \overline{T(\mathbb{Q})}\backslash T(\mathbb{A}_f)$.

It will suffice to construct a morphism between algebraic groups $\mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_m \to T$. Given h, we have the corresponding cocharacter $\mu_h : \mathbb{G}_{m,\mathbb{C}} \to T(\mathbb{C})$ defined over a finite extension of $\mathbb{Q}$, and the field of definition of μ_h is compatible with the above definition of the reflex field $E(T,h)$. If F contains $E(T,h)$, μ_h induces a F-homomorphism $\mathbb{G}_m \to T_F$, and therefore a map $\mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_m \to \mathrm{Res}_{F/\mathbb{Q}}\,T$. Composing it with the norm map $\mathrm{Nm}_{F/\mathbb{Q}} : N_{F/\mathbb{Q}}T_F \to T$, we have the desired $\mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_m \to T$. When $F = E(T,h)$ is taken to be the field of definition of μ_h, this homomorphism $r(h) : \mathrm{Res}_{F/\mathbb{Q}}\,\mathbb{G}_m \to T$ is called the reciprocity morphism for (T,h). This can be written in an explicit way: for $P \in F^\times$, $r(h)(P) = \sum_{\rho:F\to\mathbb{Q}^a} \rho(\mu_h(P))$ where $\sum_{\rho:F\to\mathbb{Q}^a} \rho(\mu_h(P)) \in T(\mathbb{Q}^a)$ is stable under the action of $\mathrm{Gal}(\mathbb{Q}^a/\mathbb{Q})$ and hence lies in $T(\mathbb{Q})$.

Example 6.17. Let (E,Φ) be a CM-type, and let $T = \mathrm{Res}_{E/\mathbb{Q}}\,\mathbb{G}_m$. $h_\Phi : \mathbb{S} \to T_\mathbb{R}$ is defined by the $\mathbb{R}$-isomorphism $T(\mathbb{R}) = (E \times_\mathbb{Q} \mathbb{R})^\times \simeq (\mathbb{C}^\Phi)^\times$. The map $h_\Phi : \mathbb{S}(\mathbb{C}) \to T(\mathbb{C})$ is

$$\mathbb{C}^\times \times \mathbb{C}^\times \to (\mathbb{C}^\Phi)^\times \times (\mathbb{C}^{\bar\Phi})^\times : (z_1,z_2) \mapsto (z_1,\cdots,z_1,z_2,\cdots,z_2),$$

and the map $\mu_{h_\Phi} : \mathbb{C}^\times \to (\mathbb{C}^\Phi)^\times \times (\mathbb{C}^{\bar\Phi})^\times : z \mapsto (z,\cdots,z,1,\cdots,1)$. Hence, the reflex field of Shimura datum (T,h) is just the reflex field of CM-type (E,Φ).

Example 6.18. When a PEL Shimura datum (G,X) is of type (A) and (C), the reflex field $E(G,X)$ defined here is compatible with the reflex field of the corresponding PEL datum defined in Theorem 5.20, because $E(G,X)/\mathbb{Q}$ is generated by $t_h(b)$, $b \in B$.

Definition 6.19. A point $x \in X$ is called special if there exists a torus $T \subset G$ such that $h_x(\mathbb{C}^\times) \subset T(\mathbb{R})$. (T,x) is called a special pair in (G,X). When (SV4) and (SV6) hold, the special points and special pairs are called CM points and CM fields.

Remark 6.20. Notice that a point $x \in X$ is special if and only if h_x is fixed by $T(\mathbb{R})$ ($\mathrm{ad}(t) \circ h_x = h_x$ for all $t \in T(\mathbb{R})$) in which T is a maximal torus of G.

Let $(T, h = h_x) \subset (G,X)$ be a special pair and let $E(x)$ be the field of definition fo μ_x. We define $r_x : \mathbb{A}_{E(x)}^\times \to T(\mathbb{A}_f)$ to be $\mathbb{A}_{E(x)}^\times \xrightarrow{r(h)} T(\mathbb{A}_\mathbb{Q}) \xrightarrow{\text{project}} T(\mathbb{A}_f)$ where $r(h)$ is the homomorphism defined in Example 6.16.

Definition 6.21. Let (G,X) be a Shimura datum, and let K be a compact open subgroup of $G(\mathbb{A}_f)$. A model $M_K(G,X)$ of $\mathrm{Sh}_K(G,X)$ over $E(G,X)$ is canonical if, for every special pair $(T,x) \subset (G,X)$, and $a \in G(\mathbb{A}_f)$,

- $[x,a]_K$ has coordinates in $E(x)^{\mathrm{ab}}$,
- $\sigma[x,a]_K = [x, r_x(s)a]_K$, for all $s \in \mathbb{A}_{E(x)}^\times$ and $\sigma = \mathrm{art}_{E(x)}(s) \in \mathrm{Gal}(E(x)^{\mathrm{ab}}/E(x))$.

A model $M(G,X)$ of $\mathrm{Sh}(G,X)$ over $E(G,X)$ is canonical if each $M_K(G,X)$ is canonical.

Example 6.22 (The Case of CM Tori)**.** Let (E,Φ) be a CM-type, and let (T,h_Φ) be the same as in Example 6.17. Then, $E(T,h_\Phi) = E^*$. Moreover, $r(h_\Phi) : \mathrm{Res}_{E^*/\mathbb{Q}}\,\mathbb{G}_m \to \mathrm{Res}_{E/\mathbb{Q}}\,\mathbb{G}_m$ is the reflex norm N_{Φ^*} by direct computation.

Let K be a compact open subgroup of $T(\mathbb{A}_f)$, and let V be a one-dimensional E vector space seen as a $\mathbb{Q}$-vector space with the action of $T(\mathbb{Q})$, which gives an embedding $V \hookrightarrow \mathrm{GL}(V)$. We assert that the Shimura variety $\mathrm{Sh}_K(T, h_\Phi)$ classifies isomorphy classes of triples $(A, i, \eta K)$ in which (A, i) is an abelian variety over $\mathbb{C}$ of CM type (E, Φ) and η is an $E \otimes \mathbb{A}_f$-linear isomorphism $V(\mathbb{A}_f) \to V_f(A)$. An isomorphism $(A, i, \eta K) \to (A', i', \eta' K)$ is an E-linear isogeny sending ηK to $\eta' K$. The isomorphism set is called $\mathcal{M}_{\mathcal{K}}$. There is a E isomorphism $a : H_1(A, \mathbb{Q}) \to V$ sending h_A to h_Φ. The composition $V(\mathbb{A}_f) \xrightarrow{\eta} V_f(A) \xrightarrow{a} V(\mathbb{A}_f)$ defines an element $g \in T(\mathbb{A}_f)$, since it is a $(E \otimes \mathbb{A}_f)^\times$-linear map, and therefore an element $[g]$ in $T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K$. This map $M_K \to Sh_K(T, h_\Phi)(\mathbb{C})$ can be easily checked to be a bijection.

There is a natural Galois action on $\mathcal{M}_K$. Let $\sigma \in \mathrm{Aut}(\mathbb{C}/E^*)$, define $\sigma(A, i, \eta K) = (\sigma A, {}^\sigma i, {}^\sigma \eta K)$ where $(\sigma A, {}^\sigma i)$ is defined in last section and ${}^\sigma\eta$ is defined to be the composite $V(\mathbb{A}_f) \xrightarrow{\eta} V_f(A) \xrightarrow{\sigma} V_f(\sigma A)$. As all abelian varieties of CM type (E, Φ) and the morphisms between them are defined over $\mathbb{Q}^a$ as shown in Proposition 6.4, this action factors through $\mathrm{Gal}(\mathbb{Q}^a/\mathbb{Q})$ and defines a model of $Sh_K(T, h_\Phi)$ over E^* via the bijection map above. Then the main result of complex multiplication in Theorem 6.6 shows exactly this model is canonical.

Example 6.23 (The Galois Action on Connected Components)**.** Assume $\mathrm{Sh}_K(G, X)$ has a canonical model. Then it will define an action of $\mathrm{Aut}(\mathbb{C}/E(G, X))$ on the set $\pi_0(\mathrm{Sh}_K(G, X))$. When G^{der} is additionally simply connected, $\pi_0(\mathrm{Sh}_K(G, X)) \simeq T(\mathbb{Q})\backslash Y \times T(\mathbb{A}_f)/v(K)$ is a zero-dimensional Shimura variety, where Y is the quotient of $T(\mathbb{R})$ by $T^(\mathbb{R})\dagger$ (Theorem 3.28). Let $h = v \circ h_x$ for any $x \in X$. Then μ_h is a cocharacter of T defined over $E(G, X)$. As shown in Example 6.16, it defines a homomorphism $r = r(h) : \mathbb{A}^\times_{E(G,X)} \to T(\mathbb{A}_{\mathbb{Q}})$ and then determines a $E(G, X)$-model of $\pi_0(\mathrm{Sh}_K(G, X))$. This map r can be described explicitly as follows. $\sigma \in \mathrm{Aut}(\mathbb{C}/E(G, X))$. Let s be an element of $\mathbb{A}^\times_{E(G,X)}$ satisfying $\mathrm{art}_{E(G,X)}(s) = \sigma|E(G, X)^{\mathrm{ab}}$ and let $r(s) = (r(s)_\infty, r(s)_f) \in T(\mathbb{R}) \times T(\mathbb{A}_f)$. Then

$$\sigma[y, a]_K = [r(s)_\infty y, r(s)_f a] \text{ ,for all } [y, a] \in T(\mathbb{Q})\backslash Y \times T(\mathbb{A}_f)/v(K) \tag{6.24}$$

If (6.24) is used to define the canonical model of the zero-dimensional Shimura varieties. We get the result that the π_0 of a canonical model of $\mathrm{Sh}(G, X)$ is a canonical model of $\mathrm{Sh}(T, Y)$.

6.3. **Uniqueness and existence of canonical models.** Before the proof of uniqueness, a general criterion is displayed below.

Proposition 6.25. *Let k be a subfield of an algebraically closed field Ω, then the functor $V \to V_\Omega +$ action of $\mathrm{Aut}(\Omega/k)$ on $V(\Omega)$. A variety V over k is uniquely determined by V_Ω and the action of $\mathrm{Aut}(\Omega/k)$ on $V(\Omega)$.*

Theorem 6.26. *(1) If $\mathrm{Sh}_K(G, X)$ and $\mathrm{Sh}_{K'}(G, X)$ have canonical models over $E(G, X)$, then $\mathcal{T}(g)$ is defined over $E(G, X)$.*
(2) A canonical model of $\mathrm{Sh}_K(G, X)$ (if it exists) is unique up to a unique isomorphism.
(3) If, for all compact open subgroups K of $G(\mathbb{A}_f)$, $\mathrm{Sh}_K(G, X)$ has a canonical model, then also does $\mathrm{Sh}(G, X)$, and it is unique up to a unique isomorphism.

The proof of this theorem relies on two key lemmas.

Lemma 6.27 ([4], 5.1)**.** *For every finite extension L of $E(G,X)$ in C, there exists a special point x_0 such that $E(x_0)$ is linearly disjoint from L.*

Lemma 6.28. *For any $x \in X$, $\{[x,a]K \mid a \in G(\mathbb{A}_f)\}$ is Zariski dense in $\mathrm{Sh}_K(G,X)$.*

Proof. See [8], 13.5. Mainly use the real approximation theorem. □

Proof of Theorem 6.26. Let (G,X) be a Shimura datum. We first show that there exists a special point in X. Let $x \in X$ and T be a maximal torus in $G_{\mathbb{R}}$ containing $h_x(\mathbb{S})$. Then T is the centralizer of a regular element in $\mathrm{Lie}(G_{\mathbb{R}})$. The set of regular elements is open in $\mathrm{Lie}(G_{\mathbb{R}})$. So if $\lambda_0 \in \mathrm{Lie}(G)$ is chosen sufficiently closed to λ, λ_0 will be regular. Let T_0 be the centraliser of λ_0, then it is a maximal torus in G. A fact is that two maximal tori of $G_{\mathbb{R}}$ in the same component conjugate to each other. So there are some $g \in G(\mathbb{R})$ such that $T_0 = gTg^{-1}$. Thus, gx is a special point since $h_{gx}(\mathbb{S}) = gh_x(\mathbb{S})g^{-1} \subset T_0(\mathbb{R})$.

proof of (1): To prove $\mathcal{T}(g)$ is defined over $E(G,X)$, by Proposition 6.25, it will suffice to show that $\mathcal{T}(g)$ is fixed by the action of $\mathrm{Gal}(\mathbb{C}/E(G,X))$. Let x_0 be a special point in X, we first that $\mathcal{T}(g)$ is fixed by the action of $\mathrm{Gal}(\mathbb{C}/E(x_0))$. Choose an $s \in \mathbb{A}_{E_0}^{\times}$ such that $\mathrm{art}(s) = \sigma|E(x_0)^{\mathrm{ab}}$. Consider the following commutative diagram holding for all $a \in G(\mathbb{A}_f)$:

$$\begin{tikzcd}
{[x_0,a]_K} \arrow[r, "\mathcal{T}(g)"] \arrow[d, "\sigma"] & {[x_0,ag]_K} \arrow[d, "\sigma"] \\
{[x_0,r_{x_0}(s)a]_K} \arrow[r, "\mathcal{T}(g)"] & {[x_0,r_{x_0}(s)ag]_K}.
\end{tikzcd}$$

By Lemma 6.28, this is enough to show $\mathcal{T}(g)$ is fixed by $\mathrm{Gal}(\mathbb{C}/E(x_0))$. However, Lemma 6.27 shows that $\mathrm{Aut}(\mathbb{C}/E(G,X))$ is generated by $\sigma \in \mathrm{Gal}(\mathbb{C}/E(x_0))$ for some x_0 being special points in X.

(2) is immediately from (1), when we take $K = K'$, $g = 1$. (3) is from (1) and the definition. □

We have shown that, in many examples, the reflex field of Shimura varieties is the natural field of definition of the moduli problems. One may further ask if the moduli spaces we constructed in Theorem 5.9, 5.25 are canonical and if there is a general approach to prove the existence of canonical models. We attempt to provide a rough idea to these questions. Take Siegel Shimura varieties Sh_K for example. Let (V,ψ) be a symplectic space over $\mathbb{Q}$, and let $(G,X) = (\mathrm{GSp}(\psi), X(\psi))$ be the associated Shimura datum. We first find out the reflex field and special points in this case.

Proposition 6.29. *The reflex field $E(G,X) = \mathbb{Q}$.*

Proof. Our previous discussion on symplectic space shows that $h : \mathbb{S} \to G$ is always of the form $(z,z') \in \mathbb{S}(\mathbb{C})$ acts as z on L and $\bar{z}'$ on L' where L, L' are complementary lagrangians in $V(\mathbb{C})$. Then given such a pair (L,L'), the corresponding $\mu_{(L,L')} : \mathbb{G}_m \to G(\mathbb{C})$ satisfies $\mu(z)$ acts as z on L and as 1 on L' runs through $c(X)$. $G(\mathbb{C})$ acts transitively on the pairs (L,L'), so on $c(X)$. There exist pairs of complementary lagrangians (L,L') in V instead of $V(\mathbb{C})$, then $\mu_{(L(\mathbb{C}),L'(\mathbb{C}))}$ is defined over $\mathbb{Q}$. Therefore, $E(G,X) = \mathbb{Q}$. □

Proposition 6.30. *An abelian variety A over $\mathbb{C}$ is CM if and only if there exists a torus $T \subset \mathrm{GL}(H_1(A, \mathbb{Q}))$ such that $h_A(\mathbb{C}^\times) \subset T(\mathbb{Q})$*

Proof. The statements depend solely on A up to isogeny, then we can assume A is simple because of Theorem 4.30. Assume A is simple. Then, $\mathrm{End}^0 A$ is a CM field E and $H_1(A, \mathbb{Q})$ is a one-dimensional E-module. Since $h_A(\mathbb{C}^\times)$ commutes with the action of $E \otimes_{\mathbb{Q}} \mathbb{R}$ on $H_1(A, \mathbb{R})$, $h_A(\mathbb{C}^\times) \subset (E \otimes \mathbb{R})^\times = \mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_m)(\mathbb{R})$. Take $T = \mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_m) \subset \mathrm{GL}(H_1(A, \mathbb{Q}))$, the "only if" part follows.

Conversely, assume $h_A(\mathbb{C}^\times) \subset T(\mathbb{Q})$ for a torus $T \subset \mathrm{GL}(H_1(A, \mathbb{Q}))$. As $\mathrm{End}^0 A$ is the subalgebra of $\mathrm{End}(H_1(A, \mathbb{Q}))$ preserving the Hodge structure, we have

$$\mathrm{End}^0 A \supset \{\alpha \in \mathrm{End}(H_1) \mid \alpha \text{ commutes with the action of } T\}.$$

Because T is a torus, $H_1(A, \mathbb{C}) = \bigoplus_{\chi \in X^*(T)} H_\chi$, thus $\mathrm{End}_T(H_1(A, \mathbb{C}))$ contains a $\mathbb{C}$-algebra of degree $2 \dim A$. It follows that $\mathrm{End}_0(A)$ contains a $\mathbb{Q}$-algebra of degree $2 \dim A$. Proposition 4.40 shows $\mathrm{End}^0 A$ is of $2 \dim A$ dimension over $\mathbb{Q}$, thus is CM. $\square$

Corollary 6.31. *$\mathcal{M}_K \to \mathrm{Sh}_K : (A, \cdots) \mapsto [x, \cdot]$ is defined as before, then A is CM if and only if x is special.*

We first define an action of $\mathrm{Aut}(\mathbb{C})$ on $\mathcal{M}_K$. Let $(A, s, \eta K) \in \mathcal{M}_K$. $\sigma(A, s, \eta)$ is defined to be $(\sigma A, {}^\sigma \eta, {}^\sigma s)$, where $\sigma A, {}^\sigma \eta$ are defined the same way as in Example 6.22. Recall s corresponds to an ample line bundle on A, and thus a rational divisor D on A. ${}^\sigma s$ is taken to be the Hodge tensor corresponding to σD on σA.

Proposition 6.32. *Suppose that Sh_K has a model M_K over $\mathbb{Q}$ for which the map $\mathcal{M}_K \to M_K(\mathbb{C})$ commutes with the action of $\mathrm{Aut}(\mathbb{C})$. Then M_K is canonical.*

Combining Proposition 6.32, Corollary 6.31 and Example 6.22, we immediately obtain that the models constructed by Theorem 5.9 (when they base change to the reflex field) are canonical.

In general, the action of $\mathrm{Aut}(\mathbb{C})$ on $\mathcal{M}_K$ preserves the isomorphism classes and induces an action of $\mathrm{Aut}(\mathbb{C})$ on $\mathrm{Sh}_K(\mathbb{C})$, denoted by $\cdot$. Keeping the notations on Proposition 6.25, we have to show $(Sh_K, \cdot)$ is in the essential image of this functor. To complete this, one can make use of the following criterion. See [8], page 125.

Proposition 6.33. *The pair $(V, \cdot)$ arises from a variety over k if*

(a) V is quasiprojective,
(b) $\cdot$ is regular, and
(c) $\cdot$ is continuous.

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