REGULARITY OF ELLIPTIC PDES

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Abstract. We present the most important techniques to establish regularity of energy minimizers in smooth calculus of variations, explain their connection to elliptic partial differential equations, and show how they are applied to some classical problems. We prove the theorems of De Giorgi and Schauder to verify that minimizers are smooth, resolving Hilbert's 19th problem. Finally, we tackle the obstacle problem, a canonical example of nonsmooth calculus of variations using Calderon-Zygmund estimates.

Contents

1. Introduction

In Partial Differential Equations (PDE), the minimization of energy is an important class of problems, where we seek to minimize some $I[\cdot]$ that takes functions and maps them to $\mathbb{R}$. The study of these energy functionals is called *calculus of variations*. We will limit ourselves to studying second-order energy functionals.

Definition 1.1. The *energy functional* I has the explicit form

$$I[w] = \int_U L(Dw(x), w(x), x)dx$$

where w satisfies some requisite conditions, and for $U \subset \mathbb{R}^d$, $L : \mathbb{R}^d \times \mathbb{R} \times \overline{U}$ is a smooth function. We call L the *Lagrangian*.

These requisite conditions are necessary in order for the functional to make sense, such as requiring regularity on w, so that Dw exists, or an integrability condition on Dw and w so that $I[\cdot]$ makes sense, or a boundary condition $w = g$ on ∂U.

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Remark 1.2. Suppose some particular function u satisfies the requisite conditions (we say u is *admissible*), and is the minimizer among all such functions satisfiying the requisite conditions. Then for some smooth function $\eta \in C_c^\infty(U)$, $u+\varepsilon\eta$ is admissible. If we set $i(\varepsilon) = I[u+\varepsilon\eta]$ i is a real-valued function, so classical calculus techniques apply. Because u is a minimizer of I, $i'(0) = 0$. Expanding,

$$\begin{aligned} i'(0) &= \int_U \sum_{i=1}^n L_{p_i}(Du,u,x)\eta_{x_i} + L_z(Du,u,x)\eta dx \\ &= \int_U -\sum_{i=1}^n [(L_{p_i}(Du,u,x))_{x_i} + L_z(Du,u,x)]\eta = 0 \end{aligned}$$

for all test functions $\eta \in C_c^\infty(U)$, which implies that u solves the PDE

$$-\sum_{i=1}^n (L_{p_i}(Du,u,x))_{x_i} + L_z(Du,u,x) = 0 \tag{1.3}$$

in the weak sense. We call (1.3) the *Euler-Lagrange equation*.

This is a powerful characterization of the minimizers of energy functionals. But immediately, there are three questions to answer. First: what is the right space for these functions to exist in? Choosing the right space will make the PDE well posed. Second: does the function that minimizes $I[\cdot]$ even exist? Without it, our techniques would not give us any information. Third: If the minimzer exists, how smooth is it?

For the first question, the answer is $W^{1,q}$. An integrability condition on u and ∇u and selecting the correct q allows I to be well defined on the whole space. Then we call $\mathcal{A} = \{u \in W^{1,q} : u = g \text{ on } \partial U\}$ the *admissible set*, and functions belonging to $\mathcal{A}$ *admissible*. For the second question, we can show existence for an extremely large class of functionals with only some light assumptions.

Finally, the question of regularity.

Example 1.4. One of the most famous variational problems is to minimize

$$I[u] = \int_U \frac{1}{2}|\nabla u|^2,$$

for which the Euler-Lagrange equation is $\Delta u = 0$, that is, u is harmonic[1] and therefore smooth. This result, known as *Dirichlet's principle*, implies that a harmonic function on U is a minimizer of $I[\cdot]$ among all functions that equal u on ∂U.

For this Lagrangian $L(p,z,x) = F(p) = |p|^2/2$, F is smooth, uniformly convex, has quadratic growth. In fact, minimizers to *any* energy functionals which have smooth, uniformly convex, and quadratic growth Lagrangians are indeed smooth. But for many years, this question of regularity remained unanswered. It is not possible to find an explicit form for a minimizer to an arbitrary Lagrangian, so trying to take derivatives directly isn't possible. Thus we are limited to studying the minimizers in terms of calculus of variations processes, such as looking at the

[1]Specifically, u solves $\Delta u = 0$ in the weak sense. Looking at mollifiers of u, noting that they satisfy the mean-value property and are therefore harmonic, then since $u_\varepsilon \to u$ uniformly, proves that u also satisfies the mean-value property. Therefore u is harmonic in the regular sense. We will use the fact that weakly harmonic functions are harmonic multiple times.

Euler-Lagrange equations that minimizers solve. We will specifically study energy minimizers of the form

$$I[\cdot] = \int_U a_{ij}\partial_i u \partial_j u + fu = \int_U \langle A\nabla u, \nabla u\rangle + fu = \int_U F(\nabla u) + fu. \tag{1.5}$$

for a_{ij} uniformly elliptic, which solves the equation $\operatorname{div}(\nabla F(\nabla u)) = \operatorname{div}(f)$, or $\partial_i(a_{ij}\partial_j u) = \partial_i f_i$.[2]

Theorem 1.6 (Existence of minimizer)**.** *Let*

$$I[w] = \int_U a_{ij}\partial_i w \partial_j w + gw,$$

for a_{ij} uniformly elliptic and g smooth. Then $I[\cdot]$ attains its minimum on the admissible set $\mathcal{A} = \{w : w \in H^1, w = f \text{ on } \partial U\}$ for $f \in H^1$.

Proof. First, we want to show that $I[\cdot]$ is convex. $I[\cdot]$ is linear in the $g(x)u(x)$ term, so we only need to worry about the first term. Let u, v be in the admissible set. We wish to show that for $t \in [0,1]$, $0 \leq tI(u) + (1-t)I(v) - I(tu + (1-t)v)$, which expanding, is

$$t(1-t)\int_U \langle A\nabla u, \nabla u\rangle - \langle A\nabla u, \nabla v\rangle - \langle A\nabla v, \nabla u\rangle + \langle A\nabla v, \nabla v\rangle =$$
$$t(1-t)\int_U \langle A(\nabla u - \nabla v), \nabla u - \nabla v\rangle \geq t(1-t)\lambda \int_U |\nabla(u-v)|^2 \geq 0$$

as desired. I is also continuous, since the H^1 includes a norm on ∇u. For coercivity, we need a poincare inequality for fixed boundary data. Since $u - f$ has zero boundary, we have the Poincare-type inequality

$$\|u - f\|_{L^2} \leq C\|\nabla u\|_{L^2} + C\|\nabla f\|_{L^2} \implies \|u\|_{L^2} \leq C\|\nabla u\|_{L^2} + C_1\|f\|_{H^1}$$

and since the second term is fixed, we know that $\|u\|$ is controlled by $\|\nabla u\|$. So

$$I[u] \geq \lambda \int_U |\nabla u|^2 - \|g\|_{L^2}\|u\|_{L^2} \geq \lambda\|\nabla u\|_{L^2}^2 - \|g\|_{L^2}(C\|\nabla u\|_{L^2} + C_1\|f\|_{H^1})$$

Since the quadratic term dominates the linear term, as $\|u\|_{H^1}$ (which is comparable to $\|\nabla u\|_{L^2}$) grows large, $I[u] \to \infty$, and we have coercivity. Now, by Evans [1] Theorem 8.2.2, we conclude that the minimizer exists. □

To see that minimizers are smooth, we will show three important regularity theorems:

Theorem 1.7 (Interior H^2 regularity)**.** *Let $u \in H^1(U)$ be a weak solution to $\partial_i(a_{ij}\partial_j u)) = \partial_i f_i$. Then $u \in H^2_{loc}$.*

Theorem 1.8 (De Giorgi)**.** *Let $u \in H^1(B_2)$ be a weak solution to $\partial_i(a_{ij}\partial_j(u)) = \partial_i f_i$, and suppose that the coefficient matrix a_{ij} is measurable and uniformly elliptic with constants $\lambda I \leq a_{ij} \leq \Lambda I$. Then $u \in C^\alpha(B_1)$ for some α.*

Theorem 1.9 (Schauder)**.** *Let $a_{ij} \in C^{k,\alpha}(B_1)$ be a uniformly elliptic matrix. Let $f : B_1 \to \mathbb{R}$ be such that $f_i \in C^{k,\alpha}(B_1)$. Then if $\partial_i(a_{ij}\partial_j u) = \partial_i f_i$, then $u \in C^{k+1,\alpha}$.*

Using these regularity theorems, we will conclude that minimizers of such energy functionals are smooth, an important result illuminated by the following example:

[2] Here we are using summation notation. We are minimizing $\int_U \langle A\nabla w, \nabla w\rangle$ which corresponds to the Euler-Lagrange equation $\sum_{i=1}^d \partial_i f_i = \operatorname{div}(f) = \operatorname{div}(A\nabla u) = \sum_{i=1}^d \sum_{j=1}^d \partial_i(a_{ij}\partial_j u)$.

Example 1.10 (Hilbert's 19th Problem)**.** Take

$$I[u] = \int_U F(\nabla u)$$

so that D^2F is bounded and uniformly elliptic, i.e. $\lambda I \leq D^2F \leq \Lambda I$ in the sense of symmetric matrices. Note that the boundedness and ellipticity imply quadratic growth. The Euler-Lagrange equation to this is

$$\operatorname{div}(\nabla F(\nabla u)) = 0 \implies \partial_i(F_{ij}(\nabla u)\partial_j(\partial_k u)) = 0, \tag{1.11}$$

after taking a derivative. This is the general case to Example 1.4. By using De Giorgi and Schauder, we can conclude that the solution is indeed smooth.

The first step to Hilbert's 19th Problem is the H^2 estimate. Given that u is H^2, then we may take an arbitrary partial derivative on the Euler-Lagrange equation, say in the e_j direction, to find that $v = \partial_k u$ solves a uniformly elliptic equation $\partial_i(\partial_{ij}F(\nabla u)\partial_j v) = 0$, with $\partial_{ij}F(\nabla u) = a_{ij}$. De Giorgi then implies that $v \in C^\alpha$, so that $u \in C^{1,\alpha} \implies D^2F(\nabla u) \in C^\alpha$. Then by Schauder, v solves an equation with C^α coefficients, so v is $C^{1,\alpha}$, so u is $C^{2,\alpha}$. Then by Schauder again, v solves an equation with $C^{1,\alpha}$ coefficients, so v is $C^{2,\alpha}$. Reapplying Schauder estimates concludes that u is smooth on the interior. For decades, De Giorgi's theorem was the missing step needed to start the infinite iteration that Schauder estimates imply.

It is indeed true that minimizers to (1.5) are smooth under the conditions that all of the coefficients and f are smooth, which is the general form to Hilbert's 19th problem. However, it is more complicated to show regularity theorems for $\partial_i(a_{ij}\partial_j u) = \partial_i f_i$, which corresponds to the full energy functional (1.5), than for $\partial_i(a_{ij}\partial_j u) = 0$, which corresponds to (1.11). For Hilbert's 19th problem, it is enough to show the theorems with zero right hand side, that is, $f \equiv 0$. However, extensions of the aforementioned theorems with right hand side and lower order terms do exist. For the sake of brevity, we will show the H^2 and De Giorgi theorems with zero right hand side, and briefly discuss their full versions, and show Schauder with right hand side.

2. H^2 Theory

The first theorem necessary in our journey to show smoothness is to prove that minimizers have weak second derivatives, that is, minimizers are in H^2.

Theorem 2.1. *Suppose $u \in H^1(B_1)$ is a solution to the equation $\operatorname{div}(\nabla F(\nabla u)) = 0$ for F smooth, uniformly convex, and such that $\lambda I \leq D^2F(p) \leq \Lambda I$ for all $p \in B_1$. Then $u \in H^2_{loc}(B_1)$.*

Proof. Fix $0 \leq |h| \leq 1/4$ and choose an arbitrary unit vector e. Then, set

$$v = \frac{u(x+he) - u(x)}{h}.$$

Since $u \in H^1(B_1)$, $v \in H^1(B_{3/4})$. Showing that $u \in H^2_{loc}$ is equivalent to showing that ∇v is bounded in L^2 indepdendently of choice of e and h.

Now, since u is a solution on B_1,

$$0 = \frac{1}{h}\left(\partial_{x_i}\frac{\partial F}{\partial p_i}(\nabla u(x+he)) - \partial_{x_i}\frac{\partial F}{\partial p_i}(\nabla u(x))\right) \tag{2.2}$$

in $B_{3/4}$. Then because we can rewrite

$$\partial_{p_i} F(z) - \partial_{p_i} F(w) = \int_0^1 \frac{\partial^2 F}{\partial_{p_i p_j}}(tz + (1-t)w(z_j - w_j)dt,$$

we conclude from (2.2) that

$$0 = \partial_{x_i}\left(\frac{1}{h}\int_0^1 \frac{\partial^2 F}{\partial_{p_i p_j}}(t\nabla u(x+he) + (1-t)\nabla u(x))(\partial_{x_j} u(x+he) - \partial_{x_j} u(x))dt\right).$$

Now set

$$\tilde{a}_{ij}(x) = \int_0^1 \frac{\partial^2 F}{\partial_{p_i p_j}}(t\nabla u(x+he) + (1-t)\nabla u(x))dt,$$

so that $\tilde{a}_{ij}$ is an average of the a_{ij} that correspond to u. Therefore, $\tilde{a}_{ij}$ inherits the same ellipticity as a_{ij}. Moving the $1/h$ inside the integral, it follows that $0 = \partial_{x_i}(a_{ij}\partial_{x_j} v)$ inside $B_{3/4}$. Now fix the cutoff function $\varphi \in C_c^\infty(B_{3/4})$ such that $\varphi = 1$ on $B_{1/2}$. Testing the above equation with $v\varphi^2$, we obtain

$$0 = \int_{B_3/4} \tilde{a}_{ij}\partial_j v \partial_i(v\varphi^2) = \int_{B_{3/4}} \varphi^2 a_{ij}\partial_j v \partial_i v + 2\int_{B_{3/4}} v\varphi\tilde{a}_{ij}\partial_j v\partial_i\varphi.$$

Then subtracting the first term and applying Hölder's inequality, we find

$$\lambda\int_{B_{3/4}} \varphi^2|\nabla v|^2 \le \int_{B_{3/4}} |\varphi^2 a_{ij}\partial_i v\partial_j v| \le 2\int_{B_{3/4}} |v\varphi\tilde{a}_{ij}\partial_i\varphi\partial_j v| \le$$
$$2\Lambda\left(\int_{B_{3/4}} \varphi^2|\nabla v|^2\right)^{1/2}\left(\int_{B_{3/4}} v^2|\nabla\varphi|^2\right)^{1/2}.$$

Finally, we can conclude that

$$\int_{B_{1/2}} |\nabla v|^2 \le \int_{B_{3/4}} \varphi^2|\nabla v|^2 \le 4\frac{\Lambda^2}{\lambda^2}\|\nabla\varphi\|_{L^\infty}\int_{B_{3/4}} v^2.$$

By Evans [1] Theorem 5.8.3, we know that $\|v\|_{L^2(B_{3/4})} \le C\|\nabla u\|_{L^2(B_1)}$, from which we conclude that $\|\nabla v\|_{L^2(B_{1/2})}$ is uniformly bounded. From this, we can conclude that u is H^2_{loc} after rescaling and applying the above calculation. For the case with a righthand side, look at Δv, then test with the same equation, and use Young's inequality with ε to cancel extra terms. □

3. De Giorgi's Theorem

The next step to showing regularity of minimizers is De Giorgi's theorem. We will prove the theorem for a specific type of energy functional, which solves the PDE $\partial_i(a_{ij}\partial_j u) = 0$ for a_{ij} uniformly elliptic. Critically, there are no other assumptions on a_{ij}, so it could be very discontinuous—zooming in will not necessarily improve the equation. The proof of Schauder estimates rely on having some control on the a_{ij}, so zooming in flattens out the equation. Instead of scaling, like in the proof of Schauder's estimates, De Giorgi's brilliant argument studied the extrema of solutions. We will use the following lemma to achieve regularity:

Lemma 3.1 (Improvement of oscillation)**.** *Let $f : B_1 \to \mathbb{R}$ be continuous. If there exists some $0 < \theta < 1$ such that for every $B_r(x) \in B_1$,*

$$\operatorname{osc}_{B_{r/2}} f \le (1-\theta)\operatorname{osc}_{B_r} f,$$

then $f \in C^\alpha(B_{1/2})$ for some α depending on θ.

Proof. Recall that $\operatorname{osc}_X f = \sup_X f - \inf_X f$. Let $M = \operatorname{osc}_{B_1} f$, fix $x, y \in B_{1/2}$, let $z = (x+y)/2$, and $r = |x-y|$. Then

$$|f(x) - f(y)| \leq \operatorname{osc}_{B_r(z)} f.$$

Even though $x, y \notin B_r(z)$, continuity allows us to take this inequality. Then for some n, we know that $2^{-(n+2)} \leq |x, z| = |y, z| \leq 2^{-(n+1)}$, and equivalently $d(x, y) \leq 2^{-n}$. This means we can double the radius of $B_r(z)$ a total of n times until we are outside of B_1. Reapplying the conditions of the lemma, we obtain

$$\operatorname{osc}_{B_r(z)} f \leq (1-\theta)^n \operatorname{osc}_{B_{(2^n)r}} f \leq (1-\theta)^n M \leq |x-y|^{-\log_2(1-\theta)} M$$

Since $0 < (1-\theta) < 1$, $\alpha = -\log_2(1-\theta) \geq 0$ is our desired Hölder exponent. We conclude that $[f]_{C^\alpha} \leq \operatorname{osc}_{B_1} f$. $\square$

We now have a new way of looking at regularity. Critically, we only need to prove the improvment of oscillation lemma at scale $1/2$, that is, show that

$$\sup_{B_{1/2}} u \leq (1-\theta) \sup_{B_1} u$$

By invariance of the Euler-Lagrange equation under scaling, this is sufficient to meet the conditions of the Lemma 3.1, as long as θ is based on ellipticity constants and $\|u\|_{L^\infty}$. For some solution u, we scale with $u_r(x) = u(rx)$ with $r < 1$, to obtain that $\sup_{B_{1/2}} u_r(x) \leq (1-\theta) \sup_{B_1} u_r(x)$, which is equivalent to $\sup_{B_{r/2}} u \leq (1-\theta) \sup_{B_r} u$. Note that $u_r(x)$ solves $\partial_i(a^r_{ij} \partial_j u_r(x)) = 0$, where $a^r_{ij}(x) = a_{ij}(rx)$, an equation with the same uniform ellipticity that u solves[3] and has same or less L^∞ norm than u. So the half-ball estimate holds for u_r, which is what we wanted.

To use it, we will prove the following steps. First, we will show an L^∞ bound from an L^2 bound, which rules out the minimizer having interior spikes and so that we can make sense of $\sup_{B_r} u$. Second, we will control the maximum and minimum of u on smaller and smaller balls. Finally, we can use the improvement of oscillation lemma to take the L^∞ bound to an C^α bound.

Notation 3.2. We write the specific energy functional J to be

$$J[u] = \int_U a_{ij} \partial_i u \partial_j u$$

for a_{ij} uniformly elliptic and symmetric. This is because the Euler-Lagrange equation associated with the energy functional $J[\cdot]$ is the divergence form PDE $-2\sum_{i,j=1}^n \partial_i(a_{ij}\partial_j u) = 0$, or equivalently $\partial_i(a_{ij}\partial_j u) = 0$. The existence of a minimizer to $J[\cdot]$ is guaranteed by Theorem 1.6. For the first step, we will study subsolutions and supersolutions to obtain the L^∞ bound.

Definition 3.3. $u : U \to \mathbb{R}$ is a *subsolution* to the equation $\partial_i(a_{ij}\partial_j u) \geq 0$ if for any H^1_0 function $\varphi : U \to \mathbb{R}$ with $\varphi \geq 0$, $\varphi = 0$ on ∂U, we have

$$\int_U a_{ij} \partial_i u \partial_j \varphi \leq 0.$$

We call u a *supersolution* if $-u$ is a subsolution.

[3] or $\partial_i(a^r_{ij}\partial_j u_r(x)) = \partial_i f^r_i(x)$, in which case the regularity on f stays the same, and the right hand side even gets smaller in norm. The ellipticity stays the same as well.

Remark 3.4. Importantly, u is a subsolution if and only if $J[u] \leq J[u-v]$ for all $v \in H_0^1(U)$ with $v \geq 0$. That is, u is a subsolution if and only if any downward perturbation increases the energy. First, we note that since $a_{ij}\partial_i u \partial_j u = \langle A\nabla u, \nabla u\rangle \geq \lambda|\nabla u|^2$, $J[\cdot]$ is nonnegative. For the forwards direction, we expand and find

$$J[u-v] = \int_U a_{ij}(\partial_i u - \partial_i v)(\partial_j u - \partial_j v) = J[u] - 2\int_U a_{ij}\partial_i u \partial_j v + J[v].$$

Since u is a subsolution, the middle term is nonnegative, and since $J[v]$ is positive, we conclude that $J[u] \leq J[u-v]$. For the converse, let u be such that $J[u] \leq J[u-v]$ for all $v \in C_c^\infty, v \geq 0$. Fix v. For any $\varepsilon > 0$, we have that $J[u] \leq J[u - \varepsilon v]$. Since

$$J[u-\varepsilon v] = \int_U a_{ij}(\partial_i u - \varepsilon\partial_i v)(\partial_j u - \varepsilon\partial_j v) = J[u] + \varepsilon^2 J[v] - 2\varepsilon\int_U a_{ij}\partial_i u\partial_j v$$

we know that

$$0 \leq J[u-\varepsilon v] - J[u] = \varepsilon^2 J[v] - 2\varepsilon\int_U a_{ij}\partial_i u \partial_j v.$$

and finally

$$2\varepsilon\int_U a_{ij}\partial_i u\partial_j v \leq \varepsilon^2 J[v] \implies 2\int_U a_{ij}\partial_i u \partial_j v \leq \varepsilon J[v].$$

Sending $\varepsilon \to 0$, we conclude that $\int_U a_{ij}\partial_i u\partial_j v \leq 0$, and therefore u is a subsolution.

Additionally, if u is a subsolution, then u_+ is a subsolution as well. If we write $u = u_+ - u_-$, then expanding $J[u] = J[u_+ - u_-]$ and seeing that $\partial_i u_+$ is nonnegative where $\partial_i u_-$ is zero and vice versa gives $J[u] = J[u_+] + J[u_-]$. Moreover, if we let $w = (u_+ - v)_+ - u_- < u$ for some v positive, the same calculation yields $J[w] = J[(u_+ - v)_+] + J[u_-]$. Therefore $J[u_+] = J[u] - J[w] + J[(u_+ - v)_+]$. Since u is a subsolution and $w \leq u$, $J[u] \leq J[w]$, which implies that $J[u_+] \leq J[(u_+ - v)_+] \leq J[u_+ - v]$. Therefore, u_+ is a subsolution.

Using these properties, we will show the L^∞ bound after the following inequality:

Lemma 3.5 (Cacioppoli)**.** *Let $u \geq 0$ be a solution in $B_{r+\delta}$. Then there exists some $C > 0$ such that*

$$\|\nabla u\|_{L^2(B_r)} \leq C\delta^{-1}\|u\|_{L^2(B_{r+\delta})}$$

Proof. Set $\varphi = u\eta^2$ as a smooth cutoff function where $\eta = 1$ on B_r and $\eta = 0$ on $\partial B_{r+\delta}$ such that $|\nabla\eta| \leq C\delta^{-1}$. Then

$$\int_{B_{r+\delta}} a_{ij}\partial_i u\partial_j u\eta^2 \leq 2\int_{B_{r+\delta}} a_{ij}|u\partial_i u\eta\partial_i\eta|$$

We can use the fact that from symmetry

$$2a_{ij}p_iq_j \leq a_{ij}p_ip_j + a_{ij}q_iq_j$$

with $p_i = \sqrt{2}^{-1}\partial_i u\eta$ and $q_i = \sqrt{2}\partial_j\eta u$ which gives

$$2\int_{B_{r+\delta}} a_{ij}|u\partial_i u\eta\partial_i\eta| \leq \frac{1}{2}\int_{B_{r+\delta}} a_{ij}|\partial_i u\partial_j u\eta^2| + 2\int_{B_{r+\delta}} a_{ij}|u^2\partial_i\eta\partial_j\eta|$$

so subtracting off the first term from both sides and multiplying by 2 gives

$$\int_{B_{r+\delta}} a_{ij}\partial_i u\partial_j u\eta^2 \leq 4\int_{B_{r+\delta}} a_{ij}|u^2\partial_i\eta\partial_j\eta|.$$

The right sides becomes

$$4\int_{B_{r+\delta}} a_{ij}|u^2\partial_i\eta\partial_j\eta| \le 4\Lambda\delta^{-2}\int_{B_{r+\delta}}|u|^2$$

and the left is

$$\int_{B_{r+\delta}} a_{ij}\partial_i u\partial_j u\eta^2 \ge \lambda\int_{B_r}|\nabla u|^2,$$

so dividing and taking square roots we find

$$\|\nabla u\|_{L^2(B_r)} \le 2\sqrt{\frac{\Lambda}{\lambda}}\delta^{-1}\|u\|_{L^2(B_{r+\delta})},$$

as desired. $\square$

Now we can finish the desired bound. The proof is based on a nonlinear iteration, and uses the fact that combining the Sobolev and Cacioppoli inequalities gives a reverse Hölder inequality after stepping in. We will show that we can bound the higher norm ($L^{2^*}(B_r)$) by a lower norm, but on a slightly smaller ball ($L^2(B_{r-\varepsilon})$). The scaling factors are chosen to allow for a limit to be taken.[4]

Lemma 3.6. *Let $u : B_2 \to \mathbb{R}$ be a nonnegative subsolution to $\partial_i(a_{ij}\partial_j u) \ge 0$, then*

$$\sup_{B_1} u \le C_{d,\lambda,\Lambda}\|u\|_{L^2(B_2)}.$$

The interpretation is that subsolutions cannot have upward spikes, and equivalently, supersolutions cannot have downward spikes.

Proof. It is equivalent to show that there exists some δ based on d, λ, Λ such that if $\|u\|_{L^2(B_2)} < \delta$, then $\|u\|_{L^\infty(B_1)} \le 1$. Now we consider the following program: Let

$$l_k = 1 - 2^{-k}, \quad r_k = 1 + 2^{-k}, \quad u_k = (u - l_k)_+, \quad a_k = \|u_k\|_{L^2(B_{r_k})}.$$

Because the r_k are decreasing and $u_k \ge u_{k+1} \ge \ldots$, we know the a_k are decreasing as well. Additionally, if we are able to show that the $a_k \to 0$, then we have shown the L^∞ bound $u \le 1$, since $a_k \to \|(u-1)_+\|_{L^2}$. Applying Hölder's inequality with $p = 2^*/2$, we find

$$\begin{aligned}
a_{k+1} &= \left[\int_{B_{r_{k+1}}} |u_{k+1}|^2 \cdot \chi_{\{u_{k+1}>0\}}\right]^{1/2} \\
&\le \left[\left(\int_{B_{r_{k+1}}} |u_{k+1}|^{2^*}\right)^{2/2^*}\left(\int_{B_{r_{k+1}}} \chi_{\{u_{k+1}>0\}}\right)^{(2^*-2)/2^*}\right]^{1/2} \\
&= \|u_{k+1}\|_{L^{2^*}(B_{r_{k+1}})}|\{u_{k+1} > 0 \ \cap B_{r_{k+1}}\}|^{1/d}.
\end{aligned}$$

Observing that $u_{k+1} > 0 \iff u_k > 2^{-k-1}$ and applying Chebyshev's inequality,

$$|\{u_{k+1} > 0\} \cap B_{r_{k+1}}|^{1/d} = |\{u_k > 2^{-k-1}\} \cap B_{r_{k+1}}|^{1/d} =$$

$$|\{u_k^2 > 2^{2(-k-1)}\} \cap B_{r_{k+1}}|^{1/d} \le \left[2^{2(k+1)}\int_{B_{r_{k+1}}} u_k^2\right]^{1/d} \le 2^{2(k+1)/d}a_k^{1/d}$$

[4]Importantly, when we use sup and inf from here on, we mean the essential supremum and essential infimum. If we can change u to equal infinity on a measure zero set, it is still a solution.

Now define η_k to equal 1 on B_{r_k} and 0 on $\partial B_{r_{k-1}}$. The Sobolev inequality yields

$$\|u_{k+1}\|_{L^{2^*}(B_{r_{k+1}})} \leq \|u_{k+1}\eta_{k+1}\|_{L^{2^*}(\mathbb{R}^d)} \leq C\|\nabla(u_{k+1}\eta_{k+1})\|_{L^2(B_{r_{k-1}})},$$

and Cacioppoli's inequality (Lemma 3.5) yields

$$C\|\nabla(u_{k+1}\eta_{k+1})\|_{L^2(B_{r_{k-1}})} \leq C2^k\|u_{k+1}\eta_{k+1}\|_{L^2(B_{r_k})} =$$
$$C\|\nabla u_{k+1}\|_{L^2(B_{r_k})} \leq C2^k\|u_k\|_{L^2(B_{r_k})}.$$

Finally, we conclude

$$a_{k+1} \leq C2^{2/d}2^{k(1+2/d)}a_k^{1+1/d} = Ch^k a_k^{1+1/d}.$$

For a new C in the last inequality, and $h = 2^{1+2/d} > 2$. But then, if $a_0 \leq [C^{-1}h^{-d}]^d$,

$$a_1 \leq Ca_0^{1+1/d} = Ca_0[C^{-1}h^{-d}] = a_0h^{-d}.$$

Now we work by induction. We claim that for all k, $a_k \leq a_{k-1}h^{-d}$, which holds for the base case $k = 1$. To show the statement for $k+1$, we need to show that $a_{k+1} \leq a_kh^{-d}$. Since

$$a_k^{1/d} \leq a_{k-1}^{1/d}h^{-1} \leq a_{k-2}^{1/d}h^{-2} \leq \cdots \leq a_0^{1/d}h^{-k} \implies$$
$$a_{k+1} \leq Ch^k a_k a_0^{1/d}h^{-k} = Ca_ka_0^{1/d} \leq a_kh^{-d}$$

so the statement holds for $k+1$, and we have shown the inductive step. We can conclude that

$$a_k \leq a_0(h^{-kd})$$

which approaches zero as $k \to \infty$. Thus $\|(u-1)_+\|_{L^2(B_1)} = 0 \implies \sup_{B_1} u \leq 1$ as desired. Analyzing our constants, we know that $a_0 = \|u\|_{L^2(B_2)}$ needs to be less than $[C^{-1}h^{-d}]^d$ for C based on the Cacioppoli and Sobolev constants, which are based on ellipticity and dimension, and h dependent on dimension, which concludes the estimate. □

Now, we use the improvement of oscillation lemma.

Proposition 3.7. *Let $u : B_2 \to \mathbb{R}$ be a nonnegative supersolution. Then there is a constant ε such that if*

$$|\{x \in B_2 : u(x) \geq 1\}| \geq (1-\varepsilon)|B_2|,$$

then $u(x) \geq 1/2$ almost everywhere on B_1.

Proof. Set $v = (1-u)_+$. Since v is a subsolution, Lemma 3.6 gives $\sup_{B_1} v \leq C||v||_{L^2(B_2)} \leq C(|\{u \leq 1\}|) = C|B_2|\varepsilon$. If $\varepsilon = \frac{1}{2C|B_2|}$, $\sup_{B_1} v \leq 1/2$ which implies $1/2 \leq \inf_{B_1} u$ as desired. Since C is dependent on ellipticity constants and dimension, ε is also dependent on ellipticity constants and dimension. □

We use this to obtain a bound on the oscillation.

Proposition 3.8. *Let $u : B_2 \to \mathbb{R}$ be a subsolution with $-1 \leq u \leq 1$ and $|\{u \leq 0\}| \geq \delta_0$. Define*

$$\delta_k = |\{x \in B_1 : u(x) > 1 - 2^{-k}\}.$$

Then $\delta_k \to 0$.

Proof. Let $w_k = 2^k(u - (1-2^{-k}))_+$, so that $\delta_k = |\{w_k > 0\}|$. Note that w_k is a subsolution, $0 \le w_k \le 1$, and the w_k have bounded $H^1(B_1)$ norms. First, the fact that $0 \le w_k \le 1$ gives bounds on $\|w_k\|_{L^2(B_1)}$. Second, since w_k is also a solution to $\partial_i(a_{ij}\partial_j w_k) = 0$, Lemma 3.5 implies that $\|\nabla w_k\|_{L^2(B_1)} \le C\|w_k\|_{L^2(B_2)} \le C|B_2|$, so we have a uniform bound on $\|\nabla w_k\|_{L^2(B_1)}$ and thus a uniform bound on $\|w_k\|_{H^1(B_1)}$.

Moreover, if $w_k(x) < 1/2$, then

$$(u(x) - (1-2^{-k}))_+ < 2^{-(k+1)} \implies u(x) - (1-2^{-(k+1)}) < 0 \implies w_{k+1}(x) = 0.$$

The sets where $0 < w_k < 1/2$ are therefore disjoint, and the δ_k are decreasing. Now, if the proposition were false, then $|\{w_k = 1\}| > \delta'$ for all k and therefore $|\{u(x) = 1\}| > \delta'$ for some $\delta' > 0$. If we consider $v_k(x) := \min(2w_k, 1)$, then $|\{v_k = 0\}| > \delta_0$ and $|\{v_k = 1\}| > \delta'$. We claim that $|\{0 < w_k < 1/2\}| = |\{0 < v_k < 1\}| > \varepsilon$ for all k, that is, the δ_k decrease by at least ε every iteration.

Assume to the contrary that there does not exist such a bound. Then there exists some sequence $f_n \in H^1$ uniformly bounded but with $|\{f_k = 0\}| > \delta_0$ and $|\{f_k = 1\}| > \delta'$ under the condition that $|\{0 < f_n < 1\}| \to 0$. By Rellich-Kondrachov this sequence embeds compactly into L^2 and converges to an indicator function of a non measure zero set, which cannot be in H^1, reaching the desired contradiction.

But now $\delta_k < \delta_{k-1} - \varepsilon < \cdots < (1-\delta_0) - k\varepsilon$, so taking k large we see this cannot be true. Therefore we must have that $\delta_k \to 0$. $\square$

Corollary 3.9. *Let $v : B_2 \to \mathbb{R}$ be a nonnegative supersolution. Assume that $|\{x \in B_2 : v(x) \ge 1\}| \ge \delta$. Let $\delta_k = |\{x \in B_1 : v(x) < 2^k\}|$. Then $\delta_k \to 0$.*

If we take $v = 1 - u$ and observe that it is a supersolution, the statements are equivalent.

Proposition 3.10 (Weak Harnack inequality)**.** *Let $u : B_2 \to \mathbb{R}$ be a nonnegative supersolution. Assume that $|\{x \in B_2 : u(x) \ge 1\}| \ge \delta$. Then $\inf_{B_1} u \ge \theta$ for some $\theta > 0$ depending on δ, ellipticity constants, and dimension.*

Proof. Let ε be the one from Proposition 3.7, which depends on ellipticity constants, and dimensions. Choose k large so that $\delta_k < \varepsilon|B_2|$. Then the supersolution $2^k u$ has the property that $\{2^k u(x) \ge 1\} \ge (1-\varepsilon)|B_2|$. Therefore $2^k u(x) \ge 1/2$ on B_1, that is, $2^{-k-1} \le u$ on B_1 as desired. $\square$

Again, this proposition has a subsolution form:

Corollary 3.11. *Let $u : B_2 \to \mathbb{R}$ be a subsolution such that $u \le 1$. Assume that $|\{x \in B_2 : u(x) \le 0\}| \ge \delta$. Then $\sup_{B_1} u \le 1 - \theta$.*

Lemma 3.12. *Let $u : B_2 \to [0,1]$ be a solution. Then*

$$\operatorname{osc}_{B_1} u \le (1-\theta)$$

for $\theta > 0$ depending on dimension and ellipticity constants.

Proof. In the first case, let $|\{x \in B_2 : u(x) \ge 1/2\}| \ge |B_2|/2$. Then applying a scaled version of the Weak Harnack Inequality (Proposition 3.10) implies that $u \ge \theta$ on B_1. Since $2u$ is also a solution, and

$$|\{x \in B_2 : 2u(x) \ge 1\}| \ge |B_2|/2,$$

we know that $\inf_{B_1} 2u \geq \theta_1 \implies \inf_{B_1} u \geq \theta$ for $\theta = \theta_1/2$. We conclude that because $\sup u \leq 1$, $\operatorname{osc}_{B_1} u \leq 1 - \theta$. Because the set that $2u \geq 1$ is fixed in size, θ is based only on ellipticity and dimension.

In the second case, let $|\{x \in B_2 : u(x) \leq 1/2\} \geq |B_2|/2|$. Applying a scaled version of the subsolution form of the Weak Harnack inequality (Corollary 3.11), we know that $u \leq 1 - \theta$ on B_1, so that $\operatorname{osc}_{B_1} u \leq 1 - \theta$. Similarly, θ is based only on ellipticity and dimension. □

Now we arrive at the theorem. We need to use an iteration.

Theorem 3.13 (De Giorgi)**.** *Let $u \in H^1(B_2)$ be a weak solution to $\partial_i(a_{ij}\partial_j(u)) = 0$, and suppose that the coefficient matrix a_{ij} is measurable and uniformly elliptic with constants $\lambda I \leq a_{ij} \leq \Lambda I$. Then $u \in C^\alpha(B_1)$ for some α.*

Proof. By the first step, we have an L^∞ bound for u. Set $\tilde{u} = u/\|u\|_\infty$ so that $\tilde{u} : B_2 \to [0,1]$. Note that $\tilde{u}$ is a solution as well, but to $\tilde{a}_{ij}$. Now choose x_0, r such that $B_r(x_0) \subset B_1$. Now, write

$$u_{x_0}(x) = \Big(\tilde{u}\big(\frac{r}{2}(x - x_0)\big) - \inf_{B_r(x_0)} \tilde{u}\Big)\frac{1}{\sup_{B_r(x_0)} \tilde{u}}$$

So that $u_{x_0} : B_2 \to [0,1]$ solves $\partial_i(\tilde{a}_{ij}(r/2(x - x_0))u_{x_0}(x)) = 0$ in B_2. Since u_{x_0} solves an equation with the same ellipticity constants as $\tilde{u}$ or better, we apply Lemma 3.12 and conclude that $\operatorname{osc}_{B_1} u_{x_0} \leq (1-\theta) = (1-\theta)\operatorname{osc}_{B_2} u_{x_0}$. But since

$$\big(\sup_{B_r(x_0)} \tilde{u}\,\big)\operatorname{osc}_{B_2} u_{x_0}(x) = \operatorname{osc}_{B_2} \tilde{u}\big(\frac{r}{2}(x - x_0)\big) = \operatorname{osc}_{B_r(x_0)} \tilde{u},$$

and similarly $(\sup_{B_r(x_0)} \tilde{u})\operatorname{osc}_{B_1} u_{x_0}(x) = \operatorname{osc}_{B_{r/2}(x_0)}$, so we can conclude that

$$\operatorname{osc}_{B_{r/2}(x_0)} \tilde{u} \leq (1-\theta)\operatorname{osc}_{B_r(x_0)} \tilde{u}$$

for θ absolute, meeting the conditions of Lemma 3.1. Since $\tilde{u}$ is $C^\alpha(B_1)$, so is u. We conclude the estimate

$$\|u\|_{C^\alpha(B_1)} \leq C\|u\|_{L^\infty(B_1)} \leq C\|u\|_{L^2(B_2)}.$$

□

Remark 3.14. What about the case $\partial_i(a_{ij}\partial_j u) = \partial_i f_i$? De Giorgi theory also applies in this scenario, with the estimate

$$\|u\|_{C^\alpha(B_1)} \leq C(\|u\|_{L^2(B_2)} + \|f\|_{L^q(B_2)}) \tag{3.15}$$

for $q > d$. We can define a subsolution as satisfying $\partial_i(a_{ij}\partial_j u) \geq \partial_i f_i$ weakly, that is, for some test function $\varphi \in C_c^\infty$ we have that

$$\int_U a_{ij}\partial_j u \partial_i \varphi \leq \int_U f_i \partial_i \varphi.$$

Taking u^+ as a subsolution, the L^∞ bound we obtain is

$$\sup_{B_{1/2}} u_+ \leq C(\|u\|_{L^2} + \|f\|_q) \tag{3.16}$$

via a Cacciopoli-type inequality testing with $u_+\eta^2$. We can simply consider the case $q = \infty$. Using (3.16) similarly as in the proof of Lemma 3.12 and Theorem 3.13 by scaling vertically and iterating concludes the estimate. For a full proof including even more lower order terms, see [5], Theorem 4.1.

The reason we treat $q > d$ comes from scaling; set $u_r(x) = u(rx)$ so that u_r solves $\partial_i(a^r_{ij}\partial_j u_r) = r\partial_i f^r_i$, where $a^r_{ij}(x) = a_{ij}(rx)$ and $f^r_i(x) = f_i(rx)$. Thus the ellipticity of the equation stays constant when we zoom in. But the righthand side has the L^q norm

$$r\left(\int_{B_2} |f(rx)|^q\right)^{1/q} = r^{1-d/q}\|f\|_{L^q(B_{2r})}.$$

so we need $1 - d/q > 0 \iff q > d$ in order for the righthand side not to blow up. So, for generality, we can take $q = \infty$.

4. Harnack Inequality for Elliptic PDEs

By using De Giorgi's theorem, we are able to establish a Harnack inequality for uniformly elliptic equations with no regularity on a_{ij}.

Proposition 4.1. *There are universal constants $C, q > 0$ such that the following holds: for any $r < R$ such that $B_{4R}(x_0) \subset B_2$, we have*

$$\inf_{B_R(x_0)} u \geq C(r/R)^q \inf_{B_r(x_0)} u.$$

Proof. We may assume that $u > 0$ by looking at the solution $u + \varepsilon$. For the sake of notation, all balls are centered at x_0. Set $a = \inf_{B_r} u$ so that $B_r \subset \{u/a \geq 1\}$. Then applying the Weak Harnack inequality (Proposition 3.10), we know that $\inf_{B_{2r}} u/a \geq \theta$. Then $u/(a\theta) \geq 1$ on B_{2r}, so applying again, $\inf_{B_{4r}} u/(a\theta) \geq \theta$ for the same θ. Iterating until $B_R \subset B_{2^n r}$, we obtain

$$\inf_{B_R} \frac{u}{a\theta^n} \geq \inf_{B_{2^n r}} \frac{u}{a\theta^n} \geq 1.$$

But since $2^{-n} \leq (r/R)^n$, and $0 < \theta < 1$ will be very small,

$$\theta^n = 2^{\log_2 \theta^n} = (2^{-n})^{\log_2(1/\theta)} \geq (1/2)^{\log_2(1/\theta)}(r/R)^{\log_2(1/\theta)}.$$

Thus

$$\inf_{B_R} u \geq a(1/2)^{\log_2(1/\theta)}(r/R)^{\log_2(1/\theta)}$$

and we are done. Notice that we can conclude $u(0) \geq cr^q \inf_{B_r} u$ $\square$

Now we are ready to state the theorem. The strategy of the proof is to work by contradiction. If the Harnack inequality is not true, then we can construct a converging sequence $x_k \in B_{1/2}$ such that $u(x_k) \to \infty$, contradicting the L^∞ bound of De Giorgi.

Theorem 4.2. *Let $u : B_2 \to \mathbb{R}$ be a nonnegative solution. Then there exists a universal constant C such that $\sup_{B_{1/8}} u \leq C \inf_{B_{1/8}} u$.*

Proof. Contruct a sequence $x_k \in B_{1/2}$ by picking $x_0 \in B_{1/2}$ such that $u(x_0) = \sup_{B_{1/2}} u$. Then choose x_{k+1} in the following way - for some r_{k+1}, set x_{k+1} to be the point where $u(x_{k+1}) = \sup_{B_{r_k}(x_k)} u$. Then since $u(0)c^{-1}r_k^{-q} \geq \inf_{B_{r_k}(x_k)} u$, we have

$$u(x_k) - u(0)c^{-1}r_k^{-q} \leq u(x_k) - \inf_{B_{r_k}(x_k)} u = \sup_{B_{r_{k-1}}(x_{k-1})} u - \inf_{B_{r_k}(x_k)} u \leq$$

$$\sup_{B_{r_k}(x_k)} u - \inf_{B_{r_k}(x_k)} u \leq (1-\theta) \sup_{B_{r_k}(x_k)} u = (1-\theta)u(x_{k+1})$$

and finally we can conclude that

$$u(x_{k+1}) \geq \frac{u(x_k) - c^{-1}r_k^{-q}u(0)}{1-\theta} \tag{4.3}$$

where $1-\theta$ comes from Lemma 3.12. Note that $1-\theta$ is fixed, based on the same argument as in Theorem 3.13,

Then, set

$$M = \frac{\sup_{B_a} u}{u(0)}.$$

for some a small to be determined later. We claim that if M is sufficiently large, then we can choose x_k such that $x_k \to \infty$. To see this, take θ from Lemma 3.12 and select $\beta > 0$ such that $(1-\theta)(1+\beta) \leq 1-\beta < 1$. Suppose that we can choose $u(x_k) \geq M(1+\beta)^{k-1}u(0)$. Choose $r_k = a\delta^k$, with $a > 0$ small and $\delta \in (0,1)$ yet to be chosen. We know that

$$u(x_{k+1}) \geq u(0)\frac{M(1+\beta)^{k-1} - c^{-1}a^{-q}\delta^{-kq}}{1-\theta} \geq u(0)\frac{M(1+\beta)^{k-1} - c^{-1}a^{-q}\delta^{-kq}}{1-\theta} \tag{4.4}$$

So for the iteration to hold, we need that $u(x_{k+1}) \geq M(1+\beta)^k u(0)$, which is equivalent to showing that

$$\frac{M(1+\beta)^{k-1} - c^{-1}a^{-q}\delta^{-kq}}{1-\theta} \geq M(1+\beta)^k \iff$$
$$\frac{1}{(1+\beta)(1-\theta)} - \frac{1}{ca^q\delta^{kq}M(1+\beta)^k(1-\theta)} \geq 1.$$

Since $1 < (1+\beta)(1-\theta)$, we know that the statement makes sense, and we just need the second term to be small enough. Now, choose δ so that $\delta^q(1+\beta) > 1$. Then choose a so that we do not escape $B_{1/2}$, i.e. $a\sum \delta^k < 1/2$. Since the rest of the terms in the denominator are not based on k, we know that the inequality is true for k large, and that the second term decreases as k increases. So if the inequality is true for $k = K$, it is true for all $k \geq K$. Now, simply choose M to be large enough so that the statement is true for $k = 1$, and the iteration holds.

The base case is just our assumption on M. So if M is too large, then we can select x_k such that $u(x_k) \to \infty$. However, we also know that $u \in C^\alpha_{loc}$ so is bounded on B_1 from De Giorgi (Theorem 3.13), which is a contradiction.

Therefore, $\sup_{B_{1/4}} u \leq Mu(0)$ for M depending only on absolute constants. Finally, if we consider $\tilde{u} = u(x - x_0)$ for $x_0 \in B_{1/4}$, then select δ so that the iteration on $\tilde{u}$ does not exceed B_1 and reapply the above, we can conclude that

$$\sup_{B_{1/8}} u \leq \sup_{B_{1/4}(x_0)} u \leq \sup_{B_{1/4}} \tilde{u} \leq M\tilde{u}(0) = Mu(x_0)$$

for all $x_0 \in B_{1/8}$. Therefore $\sup_{B_{1/8}} u \leq M \inf_{B_{1/8}} u$ as desired. □

5. Schauder Estimates

De Giorgi's proof allows for a cleaner proof of Schauder's estimate. Unlike De Giorgi's proof, Schauder's proof relies on the classical techniques of zooming in over and over to achieve some sort of flatness. Given C^α regularity, scaling makes the equation flatter and flatter. The strategy of the proof will be to zoom in, then perturb the solution by smaller and smaller amounts until we have reached a perturbation that squeezes the solution into a very thin hyperplane. For each scale, we find a plane near which we can trap the solution with appropriate control on the error. From this, we can conclude $C^{1,\alpha}$ regularity.

We will show a gain in regularity on minimizers of energy that look like

$$I[u] = \int_U a_{ij}(x)\partial_i u \partial_j u + f(x)u(x)dx. \tag{5.1}$$

Recall that the minimizer to (5.1) exists via Theorem 1.6. The Euler-Lagrange equation corresponding to $I[\cdot]$ is of the form

$$\partial_i(a_{ij}\partial_j u) = \partial_i f_i.$$

Much like Di Giorgi, Schauder's proof uses a new perspective on regularity:

Lemma 5.2 (Improvement of flatness)**.** *Let $f \in C^0(U)$. For any x_0 in the interior of U, there exists a linear function $l(x) = a \cdot (x - x_0) + b$ so that for any ball $B_r(x_0) \in U$,*

$$\sup_{B_r(x_0)} |f(x) - l(x)| \leq Cr^{1+\alpha}$$

if and only if $f \in C^{1,\alpha}(U)$.

Proof. We will prove that f is $C^{1,\alpha}$ with the same Hölder constant C at every point. Moreover, we can take $b = f(x_0)$. This is because at x_0, we have

$$|f(x_0) - b| \leq Cr^{1+\alpha} \implies |f(x) - l(x) - f(x_0) + b| \leq 2Cr^{1+\alpha}$$

So if we take $l_1(x) = l(x) + f(x_0) - b$, l_1 is a linear function that meets the conditions of the lemma but with $2C$ instead of C. Then at x_0, $l_1(x_0) = f(x_0)$. We also know that f is differentiable at x_0, since

$$\sup_{B_r(x_0)} \frac{|f(x) - f(x_0) - a \cdot (x - x_0)|}{r} \leq Cr^\alpha \to 0$$

as $r \to 0$, so that $\nabla f(x_0) = a$. Now fix r so that $B_r(x_0) \subset U$. The inequality now reads for $|x - x_0| = h < r$,

$$|f(x) - f(x_0) - \nabla f(x_0) \cdot (x - x_0)| \leq Ch^{1+\alpha}.$$

Now fix some x such that $B_{r_0}(x) \in U$, and then choose some $y \in B_{r_0}(x)$. Set $r = |x - y|$. For any $z \in B_{r_0(x)}$, we know that

$$f(z) = f(x) + (z - x) \cdot \nabla f(x) + O(|z - x|^{1+\alpha}),$$
$$f(z) = f(y) + (z - y) \cdot \nabla f(y) + O(|z - y|^{1+\alpha}).$$

So taking the first line with $z = y$, we conclude that

$$f(z) = f(x) + (y - x) \cdot \nabla f(x) + (z - y) \cdot \nabla f(y) + O(|z - y|^{1+\alpha}) + O(|x - y|^{1+\alpha}).$$

Now equating this with the first line, we conclude

$$|(z - y) \cdot (\nabla f(x) - \nabla f(y))| \leq |O(|z - x|^{1+\alpha}) + O(|z - y|^{1+\alpha}) + Cr^{1+\alpha}|.$$

Therefore if we set $z = y + r\frac{\nabla f(x) - \nabla f(y)}{|\nabla f(x) - \nabla f(y)|}$, the inequality yields

$$r|\nabla f(x) - \nabla f(y)| \leq 4Cr^{1+\alpha}$$

so f is $C^{1,\alpha}$ with absolute Hölder seminorm. The converse is straightforward and follows from Taylor expansion. $\square$

To use this approach for establishing regularity, we will need the following lemma:

Lemma 5.3. *Suppose u is a solution so that $\operatorname{osc}_{B_1} u \leq 1$. Then for some $r_0 \in (0,1)$, there exists ε_0 such that if*

$$\operatorname{osc}_{B_1} a_{ij} < \varepsilon_0,\ \operatorname{osc}_{B_1} f < \varepsilon_0,$$

then there exists $b \in \mathbb{R}^d$ with $|b| \leq C_0$ where

$$osc_{B_{r_0}}\big(u(x) - b \cdot x\big) \leq r_0^{1+\alpha}. \tag{5.4}$$

The proof of the lemma is a perturbation compactness argument based on regularity of the Laplacian.

Proof. We prove by contradiction. Suppose the lemma does not hold. Then there exists some decreasing sequence $\varepsilon_k \to 0$, solutions u_k with coefficients a_{ij}^k for which $\lambda I \leq a_{ij}^k \leq \Lambda I$ and $f^k \in C^\alpha$ such that $\operatorname{osc}_{B_1} a_{ij}^k, \operatorname{osc}_{B_1} f_i^k < \varepsilon_k$ such that there is no vector $|b|$ where (5.4) holds. We can consider the f^k to be uniformly bounded, just like we consider the a_{ij}^k to have the same ellipticity constants. The statement we will prove will then have the condition that $|f| \leq M$ for some arbitrary M and $\lambda I \leq a_{ij} \leq \Lambda I$ for arbitrary $0 < \lambda \leq \Lambda$, so there is no loss of generality. Then look at $f^k(0)$, uniform boundedness allows us to pass to a subsequence f^{k_j} so that $f^{k_j}(0) \to f^{k_1}(0)$, and since the oscillations of $f^{k_j} \to 0$, we conclude that $f^{k_j} \to f^\infty$ a constant vector uniformly. Let us rename f^{k_j} as f^k and A^{k_j} as A^k.

Now, write $w_k = u_k - (u_k)_{B_1}$, which solves the same equation, and has the same oscillation. Using Poincare's inequality, we know that

$$\|w^k\|_{L^2(B_1)} = \|u^k - (u^k)_{B_1}\|_{L^2(B_1)} \leq C\|\nabla u^k\|_{L^2(B_1)} = C\|\nabla w^k\|_{L^2(B_1)}.$$

which amounts to saying that we only need to control $\|\nabla w^k\|_{L^2}$ to control $\|w\|_{H^1}$. But then

$$\lambda \int_U |\nabla w^k|^2 \leq \int_U a_{ij}^k \partial_i w^k \partial_j w^k = -\int_U \partial_i(a_{ij}\partial_j w^k) w^k = -\int_U \partial_i f_i w^k = \int_U f_i^k \partial_i w^k$$

so that

$$\lambda\|\nabla w^k\|_{L^2}^2 \leq \int_U f^k \cdot \nabla w^k \leq \|\nabla w^k\|_{L^2}\|f^k\|_{L^2}.$$

Since f^k converges to f^∞ a constant bounded function, we conclude that $\|\nabla w^k\|_{L^2}$ stays bounded as well. Then because the u^k are bounded in H^1, we can pass to another weakly convergent subsequence. Furthermore, $\{u_k\}$ is compact in C^0: since $\|w^k\|_{C^\alpha(B_{1/2})} \leq C(\|w^k\|_{L^2(B_1)} + \|f^k\|_{L^\infty(B_1)})$ via De Giorgi's estimate (3.15), and since $C^\alpha \subset\subset C^0$, we can extract the uniform limit $w^{k_j} \to w^\infty$. Moreover, by Arzela-Ascoli, we can also extract a uniform limit on the $a_{ij}^{k_j} \to a_{ij}^\infty$, which is a constant. Now for test functions v,

$$\int_U a_{ij}^\infty \partial_i u^\infty \partial_j v = \int_U (a_{ij}^\infty - a_{ij}^k)\partial_i u^\infty \partial_j v + a_k^{ij}(\partial_i u^\infty - \partial_i u^k)\partial_j v + a_{ij}^k \partial_i u^k \partial_j v.$$

The first term goes to zero since $a_{ij}^k \to a_{ij}^\infty$ uniformly, the second term goes to zero because $u^k \to u^\infty$ in H_1 and therefore the partials converge weakly, and the last term goes to zero since

$$\int_U a_{ij}^k \partial_i u^k \partial_j v = -\int_U \partial_i(a_{ij}^k \partial_i u^k) v = -\int_U \partial_i f_i v = \int_U f^k \cdot \nabla v = \int_U f^\infty \cdot \nabla v = 0$$

because f^∞ is constant and v is compactly supported. But then u^∞ is harmonic after a linear change of variables[5]. Therefore we have the quadratic approximation by Taylor expansion[6],

$$|u^\infty(x) - \nabla u^\infty(0)\cdot x - u^\infty(0)| \le C|x|^2.$$

for C based on $\|u^\infty\|_{L^\infty}$. But then the lemma holds for u^∞. If $b = \nabla u^\infty(0)$, we fulfill the conditions of the lemma. Since the $u^k \to u^\infty$ uniformly, the lemma also holds for $\|u^k - u\|_{L^\infty} < \varepsilon$ for ε small enough, contradicting our assumption. □

Now we prove the iteration necessary to move to smaller and smaller scales.

Lemma 5.5. *Assume $u : B_1 \to \mathbb{R}$ is a solution with $\operatorname{osc}_{B_1} u \le 1$. Let C_0 and r_0 be as in Lemma 5.3. Then there exists $\varepsilon_1 > 0$ such that if $[a_{ij}]_\alpha, [f]_\alpha < \varepsilon_1$, then*

$$\operatorname{osc}_{B_{r_0^k}} u - b^k\cdot x \le r_0^{k(1+\alpha)}$$

for some $b^k \in \mathbb{R}^d$. Furthermore, $|b^k - b^{k-1}| \le C_0 r_0^{\alpha k}$.

Proof. For $k = 0$, the statement is just $\operatorname{osc}_{B_1} u \le 1$, and for $k = 1$, the statement is the previous lemma (b^k is an indexing term). Set

$$v(x) = r_0^{-k(1+\alpha)}[u(r_0^k x) - b^k \cdot r_0^k x]$$

so that

$$\partial_i(v) = r_0^{-k\alpha}[\partial_i u(r_0^k x) - b_i^k]$$

Set $\tilde{a}_{ij}(x) = a_{ij}(r_0^k x)$ and $\tilde{f} = f(r_0^k x)$. But then

$$\partial_i(\tilde{a}_{ij}\partial_j v) = \partial_i(a_{ij}(r_0^k x) r_0^{-k\alpha}[\partial_j u(r_0^k x) - b_i^k]) = \partial_i g_i$$

for some g_i where $\operatorname{osc}_{B_1} g_i \le \varepsilon_1 + \varepsilon_1|b^k|$. This is because

$$\sup \frac{|a_{ij}(r_0^k x) - a_{ij}(r_0^k y)|}{(r_0^k|x-y|)^\alpha} \le [a_{ij}]_\alpha \le \varepsilon_1 \implies \operatorname{osc}\tilde{a}_{ij}(x) r_0^{-k\alpha}|b_i^k| \le \varepsilon_1 |b_i^k|,$$

and since

$$r_0^{-k\alpha} a_{ij}(r_0^k(x)\partial_j u(r_0^k x)) = r_0^{-k\alpha}\tilde{f}_i,$$

a similar calculation gives

$$\operatorname{osc}_{B_1} r_0^{-k\alpha}\tilde{f}_i = \operatorname{osc}_{B_1} f_i = \varepsilon_1.$$

By subadditivity of oscillation we obtain $\operatorname{osc}_{B_1} g \le \varepsilon_1 + \varepsilon_1|b^k|$. Now if we assume the lemma holds up to k, then

$$|b^k| \le \sum_{j=0}^{k-1}|b^{j+1} - b^j| \le \sum_{j=0}^{k-1} C_0 r^{j\alpha} \le \frac{C_0}{1 - r_0^\alpha}$$

for all k (we let $b^0 = 0$). Therefore taking ε_1 small such that $\varepsilon_1(1+C_0)/(1-r_0^\alpha) < \varepsilon_0$, we can continue to smaller and smaller scales. By assumption, $\operatorname{osc}_{B_1} v \le 1$, and

[5] Let $A = a_{ij}^\infty = UDU^{-1}$ by spectral theorem. Then change variables with $x = UD^{-1}$ on the integral $0 = \int_U a_{ij}^\infty \partial_i u^\infty \partial_j v$.

[6] Via Gilbarg-Trudinger [4] Theorem 2.10, for harmonic functions w, we have the estimate $\sup_{B_{1/2}}|\nabla w| \le C_1 \sup_{B_1} w$ and $\sup_{B_{1/2}}|D^2 w| \le C_2 \sup_{B_1} w$, so that C is absolute and controlled by $\|w\|_{L^\infty}$. Then since $u^k \to u^\infty$ uniformly, and the u^k have controlled L^∞ norms, so does u^∞, so our C is absolute.

$\operatorname{osc}_{B_1} g < \varepsilon_0$, and $\operatorname{osc}_{B_1} a_{ij}(r_0^k(x)) = r^{k\alpha} \operatorname{osc}_{B_r} a_{ij}(x) \le \varepsilon_1$. Therefore we may apply Lemma 5.3 to v to conclude that there exists $\tilde{b}$ such that

$$\operatorname{osc}_{B_{r_0}} v - \tilde{b} \cdot x \le r_0^{1+\alpha}.$$

Then expand

$$v - \tilde{b}\cdot x = r_0^{-k}(1+\alpha)[u(r_0^k x) - b^k \cdot r_0^k x] - \tilde{b}\cdot x = r_0^{-k(1+\alpha)}[u(r_0^k x) - (b^k + r^{k\alpha}\tilde{b})\cdot r_0^k x]$$

So if we let $b^{k+1} = b^k + r^{k\alpha}\tilde{b}$, we verify $|b^{k+1} - b^k| = |r^{k\alpha}\tilde{b}| \le Cr^{k\alpha}$. Then multiplying over we find

$$\operatorname{osc}_{B_{r_0^k}} u - b^{k+1}\cdot x = \operatorname{osc}_{B_{r_0}} u(r_0^k x) - b^{k+1}\cdot r_0^k x \le r^{(k+1)(1+\alpha)}$$

as desired. $\square$

Since $|b^k - b^{k+1}| \le Cr^{k\alpha}$, $b_k \to b_\infty$, and $|u(x) - b^\infty \cdot x - u(0)| \le Cr^{k(1+\alpha)}$ on $B_{r_0^k}$. Then $r^{k(1+\alpha)} \ge \operatorname{osc}_{B_{r^k}} \big(u - b^k\cdot x\big) = \operatorname{osc}_{B_{r^k}} \big(u - b^k \cdot x - u(0)\big)$, testing at $x=0$ yields that $|u - b^k\cdot x - u(0)| \le r^{k(1+\alpha)}$. Therefore on $B_{r_0^k}$,

$$|u - b^\infty \cdot x - u(0)| \le |u - b^k - u(0)| + |\sum_{j=k}^{\infty}(b^{j+1} - b^j)\cdot x| \le$$

$$r^{k(1+\alpha)} + r^k \sum_{j=k}^{\infty} C_0 r^{\alpha(j+1)} \le r^{k(1+\alpha)} + r^k \frac{C_0 r^{\alpha(1+k)}}{1-r^\alpha} \le r^{k(1+\alpha)}\left(1 + \frac{C_0 r^\alpha}{1-r^\alpha}\right)$$

as desired, and we meet the conditions of Lemma 5.2.

Theorem 5.6 (Schauder)**.** *Let $a_{ij} \in C^\alpha(B_1)$ be a uniformly elliptic matrix with ellipticity constants λ, Λ. Let $f : B_1 \to \mathbb{R}$ be such that $f_i \in C^\alpha(B_1)$. Then if $\partial_i(a_{ij}\partial_j u) = \partial_i f_i$, then $u \in C^{1,\alpha}$.*

Note that without loss of generality we can assume the conditions of Lemma 5.5 since u is $C^{1,\alpha}$ if $\tilde{u}$ is $C^{1,\alpha}$. To get into the scheme of the lemma, we apply

$$\tilde{u} = \frac{u}{\operatorname{osc}_{B_1} u + [f]_\alpha/\varepsilon_1}, \quad \tilde{f} = \frac{f}{\operatorname{osc}_{B_1} u + [f]_\alpha/\varepsilon_1}.$$

Proof. Assume that u meets the conditions of Lemma 5.5. For $x_0 \in B_{1/2}$ set $u^r(x) = u(r(x - x_0))$ and likewise for a_{ij}^r and f^r. Then

$$\partial_i(a_{ij}^r \partial_j u^r) = r\partial_i f_i^r$$

and $[a_{ij}^r]_{C^\alpha(B_1)} = r^\alpha [a_{ij}]_{C^\alpha(B_r)} < [a_{ij}]_{C^\alpha(B_1)}$. Therefore u^r meets the conditions of the lemma, which allows us to conclude via Lemma 5.2 that u^r is $C^{1,\alpha}$ at 0 so that u is $C^{1,\alpha}$ at x_0. Therefore, $u \in C^{1,\alpha}(B_{1/2})$. $\square$

Schauder also give higher regularities. If the equation $\partial_i(a_{ij}\partial_j u) = \partial_i f_i$ where a_{ij}, f have higher regularity, we expect u to also have higher regularity.

Corollary 5.7. *Let the conditions of Theorem 3.4 hold except with $a_{ij}, f \in C^{k,\alpha}$. Then $u \in C^{(k+1),\alpha}$.*

The proof is roughly the same. Instead of subtracting a linear function l, we subtract a $k+1$-degree polynomial, and show that control of the oscillation corresponds to $C^{(k+1),\alpha}$ regularity. Then we can approximate the harmonic solution by a $(k+1)$ degree polynomial using the fact that all derivatives of harmonic functions are bounded by their L^∞ norm. Finally, we iterate as before.

Using Di Giorgi and Schauder, we are now ready to resolve Hilbert's 19th Problem (Example 1.10).

Proof. First, we need that u is H^2_{loc}, which follows from Theorem 2.1.

If we commute a ∂_j on the equation, we get

$$\partial_i(\partial_{ij}F(\nabla u)\partial_j(\partial_k u)) = 0$$

so letting $\partial_k u = v \in H^1$, this implies

$$\partial_i(\partial_{ij}F(\nabla u)\partial_j v) = 0.$$

Since $\partial_{ij}F$ is uniformly elliptic, we satisfy the the conditions of Di Giorgi's theorem, and therefore v is C^α, i.e. u is $C^{1,\alpha}$. This means $\partial_{ij}F(\nabla u)$ is C^α, so by Schauder, v is $C^{1,\alpha}$, so u is $C^{2,\alpha}$. Then $\partial_{ij}F(\nabla u)$ is $C^{1,\alpha}$, so v is $C^{2,\alpha}$, so u is $C^{3,\alpha}$, and so on. This process, called *bootstrapping*, implies that u is smooth. $\square$

Using De Giorgi and the H^2 bound for the $\partial_i(a_{ij}\partial_j u) = \partial_i f_i$, we can obtain smoothness for minimizers of the full energy function (1.5), finally answering the question: minimizers of energy functinonals with uniformly convex, quadratic growth and smooth Lagrangian are smooth.

6. Calderon-Zygmund Estimates

Schauder estimates are very powerful, as long as we have some sort of $C^{k,\alpha}$ regularity. It is natural to ask if Schauder estimates hold at the endpoints, that is, if $f \in C^0$, it is the case that $u \in C^2$? We can interpret this as Schauder for the endpoint $\alpha = 0$. Unfortunately, this is not the case. Let $u : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$u(x, y) = (x^2 - y^2)\log|\log(x^2 + y^2)|.$$

Here, we can check that Δu is continuous, but second derivatives are not. However, the Calderon-Zygmund estimates allow us to show that if $\Delta u = f \in L^\infty(B_1)$, then $u \in C^{1,1-\varepsilon}_{loc}(B_1)$ for any $\varepsilon > 0$. Roughly, if all second derivatives of u are bounded, then $u \in C^{1,1}$, but only having boundedness on Δu can almost get us there. We will use a fact about scaling. Let $u_r(x) = u(rx)$. Then

$$\Delta u_r(x) = r^2 f(rx). \tag{6.1}$$

Since f is uniformly bounded, the Laplacian of $u_r(x)$ gets smaller and smaller as we zoom in and $r \to 0$, so as we zoom in, $u_r(x)$ looks more and more harmonic and try to gain regularity like that. To do this, we first need to decompose u into harmonic and zero-boundary parts. Set

$$I[w] = \int_{B_1} \frac{1}{2}|\nabla w|^2 dx.$$

With fixed boundary condition $w = u$ on ∂B_1, Theorem 1.6 gives that the minimizer exists. But from the Euler-Lagrange equation, we find that the minimizer w solves $\Delta w = 0$ in B_1 with $w = u$ on ∂B_1. Then for $v = u - w$, $\Delta v = \Delta u$ in B_1, and $v = 0$ on ∂B_1. We call w the *harmonic replacement* of u and v the *zero boundary replacement*.

Proposition 6.2. *Let u solve*

$$\begin{cases} \Delta u = f & \text{in } B_r, \\ u = g & \text{in } \partial B_r. \end{cases}$$

Then $\|u\|_{L^\infty(B_r)} \le C(\|f\|_{L^\infty(B_r)} + \|g\|_{L^\infty(\partial B_r)})$.

Proof. Define $\tilde u = u/\big(\|f\|_{L^\infty(B_r)} + \|g\|_{L^\infty(\partial B_r)}\big)$, so that $\tilde u$ solves

$$\begin{cases} \Delta \tilde u = \tilde f & \text{in } B_r, \\ \tilde u = \tilde g & \text{in } \partial B_r \end{cases}$$

with $|\tilde f| \le 1$, $|\tilde g| \le 1$. Now we let $h(x) = \frac{r^2 - |x|^2}{2d} + 1$, then $\Delta h = -1$ in B_r, and $h \ge 1$ on ∂B_r. By comparison principle, we know that $\tilde u \le h$ in B_r so the estimate becomes

$$u \le \frac{r^2 + 2d}{2d}(\|f\|_{L^\infty(B_r)} + \|g\|_{L^\infty(\partial B_r)}).$$

Repeating the same argument with $-\tilde u$ gives the lower bound and concludes the estimate. In the special case of the zero-boundary replacement v, we have the estimate $\|v\|_{L^\infty(B_r)} \le \frac{r^2}{2d}\|f\|_{L^\infty(B_r)}$. Since $v = 0$ on the boundary, we can take $h(x) = \frac{r^2 - |x|^2}{2d}$ and apply the same argument. □

Now, we mimic the improvement of flatness lemma in the Schauder inequality. Importantly, the contradiction argument is not necessary, and we can provide a constructive proof due to the fact that every time we zoom in, the Laplacian gets smaller and smaller, and therefore "more harmonic".

Lemma 6.3. *For all $\varepsilon > 0$, there exists $\delta > 0$ and $r_0 \in (0,1)$ such that if $\operatorname{osc}_{B_1} u \le 1$, and $\|f\|_{L^\infty}(B_1) < \delta$, then there exists $b \in \mathbb{R}^d$ such that*

$$\operatorname{osc}_{B_{r_0}} u - b \cdot x \le r_0^{2-\varepsilon}.$$

Observe that by Proposition 6.2, there is no loss of generality in assuming the conditions of the lemma because we can replace any solution u by

$$\tilde u = \frac{u}{\operatorname{osc}_{B_1} u + \frac{\|f\|_{L^\infty(B_1)}}{\delta}}.$$

Alternatively, by (6.1), zooming in reduces the Laplacian and the oscillation, so applying the lemma on $u_r(x)$ for $x \in B_1$ for r small concludes the full estimate.

Proof. First, since $|\sup_{B_r} f| \le \|f\|_{L^\infty(B_r)}$ and $|\inf_{B_r} f| \le \|f\|_{L^\infty(B_r)}$,

$$\operatorname{osc}_{B_r} v \le 2\|v\|_{L^\infty(B_r)}$$

Write $u = v + w$ with w as the harmonic replacement. Since w is harmonic and therefore smooth, we can estimate $|w(x) - w(0) - \nabla w(0)\cdot x| \le Cr^2$. Since $w = u - v$, we write

$$|u(x) - v(x) - w(0) + x \cdot \nabla w(0)| \le Cr^2 \implies$$
$$\operatorname{osc}_{B_r}\big(u(x) - x \cdot \nabla w(0)\big) \le 2Cr^2 + 2\operatorname{osc}_{B_r} v(x)$$

Crucially, C is absolute based on $\operatorname{osc}_{B_1} u$ because it is dependent on first and second derivatives of w, and we can bound $|D^2 w|$ and $|\nabla w|$ by $\operatorname{osc} u$[7]. Choose r_0 small so that $r_0^\varepsilon \le \frac{1}{4C}$, which gives that $2Cr_0^2 \le r_0^{2-\varepsilon}/2$. In view of the bound on v by Proposition 6.2,

$$\|v\|_{L^\infty(B_{r_0})} \le \|v\|_{L^\infty(B_1)} \le \frac{\|f\|_{L^\infty(B_1)}}{2d}$$

[7] see the proof of Lemma 5.3. Let $\tilde u = u - \inf u$, so that $\nabla w = \nabla \tilde w$. But then $\sup_{B_{1/2}} |\nabla w| \le C \sup_{B_1} \tilde w = \sup_{B_1} \tilde u = \operatorname{osc}_{B_1} u$, and similarly for $|D^2 w|$.

Therefore as long as $\|f\|_{L^\infty(B_1)} \leq dr_0^{2-\varepsilon}$, $\operatorname{osc}_{B_{r_0}} v(x)+\operatorname{osc}_{B_{r_0}} o(x) \leq r_0^{2-\varepsilon}$ as desired. □

We arrive at the theorem:

Theorem 6.4 (Calderon-Zygmund)**.** *Let $u \in H^1(B_1)$ solve $\Delta u = f \in L^\infty(B_1)$. Then $u \in C^{1,1-\varepsilon}_{\text{loc}}(B_1)$ for any $\varepsilon > 0$.*

Proof. Again, there is no loss of generality assuming $\operatorname{osc}_{B_1} u \leq 1$ and $\|f\|_{L^\infty(B_1)} \leq \delta$ for δ given by Lemma 6.3 by simply dividing. The first step is given by the lemma. We have that

$$\operatorname{osc}_{B_{r_0}} u - b^1 \cdot x \leq r_0^{2-\varepsilon},$$

where $b_1 = \nabla w(0)$. Then we zoom in. Set $u_1(x) = r_0^{-(2-\varepsilon)}[u(r_0x) - b^1 \cdot r_0x]$ so $\Delta u_1(x) = r_0^\varepsilon f(x)$. Since the Laplacian of u_1 is smaller than the Laplacian of u, and of course the oscillation of u_1 is smaller than 1, we may continue reapplying the lemma. We then have the estimate

$$\operatorname{osc}_{B_r} u_1 - \tilde{b}_2 \cdot x \leq r^{2-\varepsilon} \implies \operatorname{osc}_{B_{r^2}} u(x) - [b + r^{1-\varepsilon}\tilde{b}_2] \cdot x \leq r^{2(2-\varepsilon)}$$

to conclude $b_2 = b + r^{1-\varepsilon}\tilde{b}_2$. Continuing, $u_k(x) = r^{k(-2+\varepsilon)}[u(r^kx) - b^k \cdot r^kx]$ implies

$$\begin{aligned} r^{2-\varepsilon} &\geq \operatorname{osc}_{B_r} u_k - \tilde{b}_{k+1} \cdot x \implies \\ r^{(k+1)(2-\varepsilon)} &\geq \operatorname{osc}_{B_r} u(r^kx) - b_k \cdot r^{kx} - r^{k(1-\varepsilon)}\tilde{b}_{k+1} \cdot r^kx \\ &= \operatorname{osc}_{B_{r^{k+1}}} u(x) - [b_k + r^{k(1-\varepsilon)}\tilde{b}_{k+1}] \cdot x. \end{aligned}$$

Since $\operatorname{osc}_{B_1} u_k = 1$, by Lemma 6.3, $\tilde{b}_{k+1} = \nabla w_k(0)$ is uniformly bounded. Since $|b_{k-1} - b_k| = (r^{1-\varepsilon})^k|\tilde{b}^k| \leq Cr^{k(1-\varepsilon)}$, b_k converges to some b_∞ such that

$$\operatorname{osc}_{B_1} |u_k(x) - b_\infty| \leq Cr^{2-\varepsilon}.$$

Using Lemma 5.2, we conclude that u is $C^{1,1-\varepsilon}$ at 0. Rescaling to different points gives the regularity on u for the interior of B_1. We have the estimate $\|u\|_{C^{1,1-\varepsilon}(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)})$. □

7. The Obstacle Problem

Many interesting problems lie outside the realm of "smooth calculus of variations", where De Giorgi and Schauder estimates may not apply. In Definition 1.1 we assumed that the Lagrangian was smooth, yet there are many such problems where this is not the case. One canonical example is the obstacle problem. Given a smooth function φ, called the *obstacle*, the solution to the *obstacle problem* is the function that minimizes the energy functional

$$J[v] = \int_U \frac{1}{2}|\nabla v|^2 dx$$

subject to the conditions $v \geq \varphi$ in U, and boundary condition $v \geq g$ on ∂U. The interpretation is that admissible functions must lie above the obstacle, and the minimizer is the function with the least amount of "energy". The existence of minimizers is guaranteed by Theorem 1.6. Again, we are interested in the regularity that solutions have.

Remark 7.1. We can consider the obstacle problem to minimize the energy of $u = v - \varphi$, and require $u \geq 0$, and $u = g - \varphi$ on the boundary. Then, we are minimizing

$$\int_U \frac{1}{2}|\nabla(u+\varphi)|^2 = \int_U \frac{1}{2}\left[|\nabla u|^2 + |\nabla\varphi|^2 + 2\nabla u\nabla\varphi\right] = \int_U \frac{1}{2}|\nabla u|^2 - u\Delta\varphi + \frac{1}{2}|\nabla\varphi|^2$$

which amounts to minimizing

$$\int_U \frac{1}{2}|\nabla u|^2 + fu\,dx$$

for $f = -\Delta\varphi$ among all functions $u \geq 0$, because $\int_U |\nabla\varphi|^2$ is a fixed value.

Here, we have the stricter boundary condition $u = \tilde{g} = g - \varphi$. We call this interpretation the *zero obstacle problem*, or the *variational formulation* of the obstacle problem. The problems are identical (so long as $\varphi \in C^{1,1}$), and to move from one to another, we need only to add/subtract the obstacle. At first, it appears that this problem is the same as a smooth calculus of variations problem. However, we will soon notice that the same perturbation methods do not work, or must be modified to accomodate the obstacle, so that the problem does not fit the scheme of smooth calculus of variations. We can consider this as

$$\int_U \frac{1}{2}|\nabla u|^2 + fu_+ = \int_U \frac{1}{2}|\nabla u|^2 + f\chi_{\{u>0\}}u,$$

because we require $u \geq 0$, causing the minimization problem to be nonsmooth. Our calculus of variations techniques may not apply.

We note a nice interpretation in the zero obstacle problem. The first term, which increases as $|\nabla u|$ increases, is the elastic potential of the system, and the second term, which increases as u increases, is the gravitational potential of the system, leading to the gravitational interpretation of the obstacle problem

$$\int_U \frac{1}{2}|\nabla u|^2 + u_+ dx \tag{7.2}$$

solving $\Delta u = \chi_{\{u>0\}}$. Here, being above the obstacle at any point is the same potential energy as being above the obstacle at any other point. We can view, for example, a membrane descending over an obstacle, as an obstacle problem. Critically, since we require the obsctacle to be smooth, we can zoom in to a point until f is constant, then normalize so that the problem is in the same form as (7.2).

Now we are ready to show some properties of solutions to the obstacle problem.

Proposition 7.3. *The minimizer u is weakly superharmonic, that is, $\Delta u \leq 0$ in U.*

Proof. Let $\eta \in C_c^\infty(U), \eta \geq 0$, so that $u+t\eta$ is an admissible function, and therefore $J[u] \leq J[u+t\eta]$ because u is the minimizer. Then for all t,

$$0 \leq J[u+t\eta] - J[u] = \int_U t\partial_i u\partial_i\eta + \frac{t^2}{2}\partial_i\eta^2 = -t\int_U u\partial_{ii}\eta + \frac{t^2}{2}\int_U \partial_i\eta^2.$$

Sending $t \to 0$ we conclude that $\int_U u\Delta\eta \leq 0$, so u is weakly superharmonic. □

Finally, we can fully characterize solutions to the obstacle problem.

Proposition 7.4. *The solution to the obstacle problem is the least supersolution above the obstacle and boundary, i.e. the least superharmonic* ($\Delta u \leq 0$) *function such that*

$$\begin{cases} u \geq \varphi & \text{on} B_1, \\ u \geq g & \text{on } \partial B_1. \end{cases}$$

Moreover, u is harmonic where it does not touch the obstacle, i.e. $\Delta u = 0 \quad \text{in } \{u > \varphi\}$*. We denote $\Lambda := \{u = \varphi\}$ the contact set, and $\Omega := \{u > \varphi\}$ the harmonic set, and $\Gamma := \partial\Lambda = \partial\{u = \varphi\}$ the free boundary.*

Proof. First we show that Ω is open. Since u is weakly superharmonic, u is lower semicontinuous[8]. Fix x_0, then $u(x_0) - \varphi(x_0) > \varepsilon$ for some ε. Since u is lower semicontinuous and φ is continuous, there exists some $\delta > 0$ such that $u - \varphi > \varepsilon/2$ on $B_\delta(x_0)$. Becuase x_0 was arbitrary, we can write Ω as the union of open sets. Now, because we know Ω is open, pick $x_0 \in \Omega$ with $B_\delta(x_0) \subset \Omega$ and $\inf_{B_\delta(x_0)}(u - \varphi) > \varepsilon_0$. Then choose some $\eta \in C_c^\infty(B_\delta(x_0))$. Since $u + \varepsilon\eta$ is an admissible function, we set

$$i(\varepsilon) = J(u + \varepsilon\eta) \implies 0 = i'(0^+) = \lim_{\varepsilon \to 0} \frac{J(u + \varepsilon\eta) - J(u)}{\varepsilon},$$

limiting $i(\cdot) : (\varepsilon_0/\|\eta\|_{L^\infty}, \infty) \to \mathbb{R}$, so that i is well defined in a neighborhood around 0. Since u is a minimizer, the last expression is equal to

$$\lim_{\varepsilon \to 0} \int_{B_\delta(x_0)} \partial_i u \partial_i \eta + \varepsilon \partial_i \eta^2 = 0 \implies \int_{B_{\delta(x_0)}} u \Delta \eta = 0,$$

i.e. u is weakly harmonic on $B_\delta(x_0)$ because η is any $C_c^\infty(B_\delta(x_0))$ function. Since x_0 was arbitrary, we are done.

Then let w be such that $\Delta w \leq 0$, $w \geq \varphi$ on B_1, and $w \geq u$ on ∂B_1, meeting the same conditions as v. We want to show that $w \geq v$, so that v is the *least* supersolution that meets the requirements. On Λ, we know that w cannot be any lower than $f = u$. Then on Ω, we can use the maximum principle. Consider $w - v$. Since $w \geq v$ on the boundary, and $\Delta w \leq 0$, $\Delta(w - v) = \Delta w - \Delta v = \Delta w \leq 0$ is superharmonic and attains its minimum on the boundary. Therefore $w - v \geq 0$. $\square$

Now, we try to establish that $u \in C^{1,1}$. The Calderon-Zygmund estimate (Theorem 6.4) gives that $u \in C^{1,1-\varepsilon}_{loc}(B_1)$ for any $\varepsilon > 0$. But the absolute best we can do is $C^{1,1}$ since $\Delta u = f\chi_{u>0}$ may be discontinuous. Indeed, solutions are $C^{1,1}$.

There is some intuition that indicates that u is $C^{1,1}$. Since u is harmonic and therefore smooth on Ω, we focus our attention on Ω^C. Critically, semiconvex and superhamonic functions are $C^{1,1}$. If u is convex and $\Delta u \leq C$, then eigenvalues of $D^2 u$ are nonnegative, and since $\sum \lambda_i \leq C$, each $\lambda_i \leq C$ and $|D^2 u| \lesssim C$. So if u is semiconvex, then $D^2 u \geq -C_1 I$, so looking at $v = u + C_1|x|^2$ gives that $\Delta v = \Delta u + 2dC_1$ is bounded and convex, so v is $C^{1,1}$, which implies that u is $C^{1,1}$. Notice $u - \varphi$ is convex at the free boundary because it is a local minimum, and has bounded Laplacian, so u is semiconvex, and by Proposition 7.4, u is superharmonic. Therefore, u is $C^{1,1}$ at free boundary points. However, this proof is not complete. Our bound on $|D^2 u|$ on the set Ω, from harmonic interior estimates, becomes worse and worse as we approach the free boundary, and we have not yet stitched together the regularity of Ω with the regularity of the free boundary. Instead of this approach, we will use compactness to achieve the optimal regularity.

[8]See Fernández-Real, Ros-Oton [2] Lemma 1.17.

Lemma 7.5. *Let u solve the obstacle problem*

$$\begin{cases} \Delta u = f\chi_{\{u>0\}} & \text{in } B_1 \\ u \geq 0 & \text{in } B_1. \end{cases}$$

Suppose that $0 \in \partial\{u>0\}$. Then there exists some $C>0$ such that at least one of the following holds for each $0 \leq r \leq 1/2$:

1) $\sup_{B_r} u \leq Cr^2$

2) $\sup_{B_r} u \leq 4^{-j} \sup_{B_{2^j r}}$ *for* ***some*** $2^j r \leq 1$.

Proof. We work by contradiction. Suppose that there are infinitely many r_n such that $\sup_{B_{r_n}} u > nr_n^2$, and $\sup_{B_{r_n}} u > 4^{-j} \sup_{B_{2^j r_n}} u$ for *all* $2^j r_n < 1$. Choose r_{n_k} a decreasing subsequence and rename them to r_n. Then fix n, and let $a_n = \sup_{B_{r_n}} u$. Since

$$\frac{1}{n} \sup_{B_{r_n}} u > \frac{1}{n+k} \sup_{B_{r_n}} u \geq \frac{1}{n+k} \sup_{B_{r_{n+k}}} u \implies a_n > \frac{n+k}{n} a_{n+k}$$

so

$$a_n > \frac{(n+k)^2}{n} \frac{a_{n+k}}{n+k} > (n+k) r_{n+k}^2$$

so as $n \to \infty$, r_n must approach zero. Set $u_n(x) = u(r_n x)/\|u\|_{L^\infty(B_{r_n})}$, so that

$$\Delta u_n(x) = \frac{r_n^2 \Delta u(r_n x)}{\|u\|_{L^\infty(B_{r_n})}} = \frac{r_n^2}{\|u\|_{L^\infty(B_{r_n})}} f(r_n x)\chi_{\{u_n>0\}}.$$

But since $1/\|u\|_{L^\infty(B_{r_n})} \leq 1/(nr_n^2)$, we conclude that

$$|\Delta u_n| \leq \frac{r_n^2}{nr_n^2} |f(r_n x)\chi_{u_n>0}| \leq \frac{1}{n}\|f\|_{L^\infty(B_1)} \to 0.$$

Since $0 \in \partial\{u>0\}$, there exists some $x_0 \in B_{r_n}(x)$ such that $x_0 \in \{u>0\}$ so there is no issue dividing by $\|u\|_{L^\infty(B_{r_n})}$. Fix some arbitrary n. Then by the second assumption we contradicted, for all k with $2^n r_k \leq 1$,

$$4^n \sup_{B_1} u(r_k x) = 4^n \sup_{B_{r_k}} u > \sup_{B_{2^n r_n}} u = \sup_{B_{2^n}} u(r_n x),$$

then dividing by $\|u\|_{L^\infty(B_{r_k})}$, we obtain $4^n = 4^n \sup_{B_1} u_k(x) > \sup_{B_{2^n}} u_k(x)$. Therefore if we pick some B_r with $r < 2^n$, then find K such that $2^n r_k \leq 1$, then for all $k \geq K$, $\sup_{B_r} u_k(x) \leq \sup_{B_{2^n}} u_k \leq 4^n$. With this uniform bound for $k \geq K$, and the fact that $\Delta u_k \leq \|f\|_{L^\infty(B_1)}/K$, Theorem 6.4 allows us to conclude that $\|u_k\|_{C^{1,1-\varepsilon_1}} \leq C_1$. Then since $C^{1,1-\varepsilon_1} \subset\subset C^{1,1-\varepsilon}$ for $1 > \varepsilon > \varepsilon_1$, passing to a subsequence $u_k \to u_\infty$ strongly in $C^{1,1-\varepsilon}$. Now, fix any test function $\psi \in C_c^\infty$, and n large so that $\operatorname{supp}\psi \subset B_{2^n}$. Now for $k \geq K$,

$$\lim_{k\to\infty} \int_{B_{2^n}} |\nabla\psi \cdot \nabla u_k| \leq \lim_{k\to\infty} \frac{1}{k} \|\nabla\psi\|_{L^2(B_{2^n})} \|f\|_{L^\infty(B_1)} = 0,$$

so that $\int_{B_{2^n}} \nabla\psi \cdot \nabla u_k \to 0$. But also, since $\nabla u_k \to \nabla u_\infty$ in $C^{0,1-\varepsilon}$, by dominated convergence theorem,

$$\int_{B_{2^n}} \nabla\psi \cdot \nabla u_k \to \int_{B_{2^n}} \nabla\psi \cdot \nabla u_\infty = 0.$$

Therefore, $u_\infty \geq 0$ is harmonic, and $u_\infty(0) = 0$. But then by Liouville's theorem, u_∞ must be zero everywhere, contradicting that $\sup u_\infty = 1$. $\square$

Theorem 7.6 (Optimal regularity)**.** *Let u solve the obstacle problem*

$$\begin{cases} \Delta u = \chi_{\{u>0\}} & \text{in } B_1 \\ u \geq 0 & \text{in } B_1. \end{cases}$$

Then u is $C^{1,1}$.

Proof. First, when $u \equiv 0$ which happens on Λ , D^2u is controlled by $D^2\varphi$, and harmonic estimates can control D^2u on Ω at points away from Γ. Therefore we only need to look at the points on and near the free boundary. Now we use the lemma. Pick some point $x_0 \in \Gamma \cap B_{1/2}$, and zoom in, that is, let $u_{x_0} = u(rx_0)$ for $r = 1/2$, which also solves the obstacle problem, with $\Delta u_{x_0} = \chi_{\{u_{x_0}>0\}}/2$. Take $r_k = 2^{-k}$. Then, if condition fails at J, for some $j < J$, $\sup_{B_{r_J}} u_{x_0} \leq 4^{-j} \sup_{B_{2^j r_J}} u_{x_0}$. However,

$$4^{-j} \sup_{B_{2^j r_J}} u_{x_0} = 4^{-j} \sup_{B_{2^{j-J}}} u_{x_0} \leq 4^{-j} C(4^{j-J}) = C4^J = Cr_J^2,$$

so the iteration never fails at any point. After upgrading C, $\sup_{B_r} u_{x_0} \leq Cr^2$.[9] Since u_{x_0} has quadratic growth near zero, $D^2u(0)$ stays bounded, which implies that u is $C^{1,1}$ at x_0, with a bound $|D^2u(x_0)| \leq C$ for all $x_0 \in B_{1/2} \cap \Gamma$.

Now for points very close to the free boundary, choose $x_1 \in \{u > 0\}$, and set $r = \text{dist}(x_1, \Gamma)$. Find $x_0 \in \Gamma$ such that $r = |x_1 - x_0|$, so $B_r(x_1) \subset B_{2r}(x_0)$. Then set $v = u(x) - |x - x_1|/2d$. Because $\Delta u = 1$ on $B_r(x_1)$, $\Delta v = 0$ on $B_r(x_1)$, we can bound $\sup_{B_{r/2}(x_1)} |D^2v| \leq \frac{C_1}{r^2} \sup_{B_r(x_1)} |v|$. Therefore $\sup_{B_r(x_1)} v \leq r^2/2d + \sup_{B_{2r}(x_0)} \leq 4Cr^2$, so that $\sup_{B_{r/2}(x_1)} |D^2v| \leq 4C_1C$ is bounded. Finally, because $D^2v = D^2u + I/d$, so our bound on D^2v implies a bound on D^2u. □

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References

[1] Lawrence C. Evans. *Partial Differential Equations.* American Mathematical Society. 2010.

[2] Xavier Fernández-Real, Xavier Ros-Oton. *Regularity Theory for Elliptic PDE.* Zurich Lectures in Advanced Mathematics, EMS Press. 2022. Available here.

[3] Alessio Figalli. *Free boundary regularity in obstacle problems.* arXiv, 1807.01193. 2018. Available here.

[4] David Gilbarg, Neil S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag Berlin Heidelberg. 2001.

[5] Qing Han, Fanghua Lin. *Elliptic Partial Differential Equations 2nd ed.* American Mathematical Society, Courant Institute of Mathematical Sciences.

[9] There are multiple ways to do this critical Cr^2 bound. The author has chosen the above method because it is fun and interesting. For an alternative treatment, see Figalli's very detailed proof [3], Lemma 4.2, or Fernández-Real and Ros-Oton's approach [2], Lemma 5.14.