IMAGES OF MORPHISMS: CHEVALLEY'S THEOREM AND APPLICATIONS

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Abstract. We consider the natural question: under what conditions on the coefficients is a system of polynomial equations solvable? This may be interpreted geometrically as asking for conditions on the images of morphisms of schemes. To this end, we prove the Main Theorem of Elimination Theory and Chevalley's Theorem, using these theorems to determine conditions for when a system of polynomial equations has a solution. We assume familiarity with the definitions of schemes, and morphisms between them, as well as some basic theory of affine schemes.

Contents

1. A Motivating Example

It is quite a natural question to wonder: under what conditions on the coefficients do systems of polynomial equations have a solution? The case of linear equations spawned the subject of linear algebra. In some sense, linear equations constitute a very simple case, as all polynomials are degree 1. We begin by considering another simple case, where we only have two polynomials in one variable.

Let $p(X), q(X) \in \mathbb{C}[x]$, and wish to determine if they have a common root. The determinant of the classical Sylvester matrix gives a method. Write $p(X) = a_nX^n + \cdots + a_0$ and $q(X) = b_mX^m + \cdots + b_0$, with $a_i, b_j \in \mathbb{C}$, and define the Sylvester matrix S to be the $n+m \times n+m$ matrix with rows given as follows: Let the first row be

$$(a_n, a_{n-1}, \ldots, a_1, a_0, 0, \ldots, 0),$$

the second row be

$$(0, a_n, a_{n-1}, \ldots, a_1, a_0, 0, \ldots, 0)$$

with the kth row the coefficients of $p(X)$ shifted $k-1$ places to the right (and the rest of the entries 0) for $k \leq n$.

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Define the $m+1$th through $n+m$th rows similary, using the coefficients of $q(x)$ instead of $p(x)$.

Example 1.1. Let $n=4, m=2$. Then, the Sylvester matrix is given by

$$S = \begin{pmatrix} a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & a_4 & a_3 & a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 & 0 & 0 & 0 \\ 0 & b_2 & b_1 & b_0 & 0 & 0 \\ 0 & 0 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & 0 & b_2 & b_1 & b_0 \end{pmatrix}.$$

It is a remarkable fact that the determinant of S equals 0 if and only if $p(x)$ and $q(x)$ have a common root. To see why, the reader may check that S has nontrivial kernel if and only if there exist $a(X), b(X) \in \mathbb{C}[X]$ such that $\deg a < m$ and $\deg b < n$ and $a(X)p(X) + b(X)q(X) = 0$, which occurs if and only if $p(X)$ and $q(X)$ have a common root. In particular, there is a polynomial $\det(S)$ in the coefficients of p and q such that $\det(S) = 0$ if and only if p and q have a shared root.

We can phrase this phenomenon geometrically through the language of schemes. First, observe we may view p, q as elements $A_n x^n + \cdots + A_0$ and $B_m X^m + \cdots + B_0$ of $R[X]$, where

$$R := k[A_0, \ldots, A_n, B_0, \ldots, B_m].$$

Note that p, q both vanish at a point $(x_0, a_0, \ldots, a_n, b_0, \ldots, b_m)$ if and only if

$$p, q \in (X - x_0, A_0 - a_0, \ldots, A_n - a_n, B_0 - b_0, \ldots, B_m - b_m).$$

Thus, p and q have a common root when $A_i = a_i$ and $B_j = b_j$ for all i, j if and only if there exists some closed point $\mathfrak{m} \in \text{Spec } R[x]$ such that

$$\mathfrak{m} \cap R = (A_0 - a_0, \ldots, A_n - a_n, B_0 - b_0, \ldots, B_m - b_m),$$

i.e. p and q have a common root when $A_i = a_i$ and $B_j = b_j$ for all i, j if and only if $(A_0 - a_0, \ldots, A_n - a_n, B_0 - b_0, \ldots, B_m - b_m)$ is in the image of $V(p,q)$ under the map $\text{Spec } R[x] \to \text{Spec } R$. This argument shows that, at least on closed points, the image of $V(p,q)$ under the natural map $\text{Spec } R[x] \to \text{Spec } R$ is $V(\det(S))$.

Thus, understanding when systems of polynomial equations have a solution leads us to a natural question:

Question 1.1. Let $\pi : X \to Y$ be a morphism of schemes. What is the image of a closed subset of X under π?

The Main Theorem of Elimination Theory gives an answer to this question when $\pi : \mathbb{P}^n_A \to \text{Spec } A$ is the natural map. This will provide insight into the conditions under which systems of homogeneous polynomial equations have a natural solution. We will then develop some commutative algebra and state Chevalley's Theorem, which gives an answer in the more general case where π, X and Y are sufficiently nice, i.e. π is a morphism of finite type, and X, Y are Noetherian schemes. We conclude by giving two applications of Chevalley's Theorem: a condition for when systems of polynomial equations over Noetherian rings are solvable, and a simple proof of Zariski's Lemma, the main obstacle in proving the Nullstellensatz.

2. The Main Theorem of Elimination Theory

A natural case to understand when studying whether systems of polynomial equations are solvable is the homogeneous case: given homogeneous polynomials $h_1, \dots, h_n \in R[X_1, \dots, X_p]$ where $R = k[Y_1, \dots, Y_q]$ is the coefficient ring, for what values of $(Y_1, \dots, Y_q)$ can we guarantee that $h_1, \dots, h_n$ have a common nonzero solution? (Note: throughout the paper, when we refer to a homogeneous polynomial in $R[X_1, \dots, X_p]$, we mean homogeneous with respect to the variables $X_1, \dots, X_p$ and not with respect to $X_1, \dots, X_p, Y_1, \dots, Y_q$).

The Main Theorem of Elimination Theory provides a satisfying answer to this question. It guarantees a Zariski-closed condition on $Y_1, \dots, Y_q$. In other words, there exist polynomials $g_1, \dots g_m$ in $k[Y_1, \dots Y_q]$ such that $g_i(y_1, \dots, y_q) = 0$ for all i if and only if $h_1, \dots, h_n$ have a common root $(x_1, \dots, x_p)$ when $(Y_1, \dots, Y_q) = (y_1, \dots, y_q)$.

Example 2.1. For $i \in [n]$, let $h_i(X_1, \dots, X_n) = A_{i,1}X_1 + \cdots A_{i,n}X_n$ be a system of linear equations. Then, there exists a nonzero solution $(X_1, \dots, X_n) = (x_1, \dots, x_n)$ if and only if $\det(A_{i,j}) \neq 0$.

This example can be interpreted geometrically. Let $B = k[A_{i,j} : i, j \in [n]]$, and observe that $h_1, \dots, h_n$ have a solution for fixed values of the $A_{i,j}$, $A_{i,j} = a_{i,j}$, if and only if there exists some point $\mathfrak{p} \in V(h_1, \dots, h_n)$ such that $A_{i,j} - a_{i,j} \in \mathfrak{p} \cap B$ for all $i, j \in [n]$. This occurs if and only if $(A_{i,j} - a_{i,j} : i, j \in [n])$ is in the image of $V(h_1, \dots, h_n)$. Thus, at least on closed points, the image of $V(h_1, \dots, h_n)$ is Zariski-closed, it is the vanishing set of $\det((A_{ij}))$.

The Main Theorem of Elimination Theory shows that in general, the map from $\mathbb{P}^n_A$ to Spec A always maps closed sets to closed sets.

Theorem 2.2. *(Main Theorem of Elimination Theory) Let A be any commutative ring, and let $\pi : \mathbb{P}^n_A \to$ Spec A be the natural map. Then, π is closed.*

Proof. Let $V_+(I)$ be a closed subset of $\mathbb{P}^n_A \to$ Spec A, where I is some homogeneous ideal in $A[x_0, \dots, x_n]$. Note that the points $\mathfrak{p} \in$ Spec A in the image of $V_+(I)$ are precisely the points where all $f' \in I$ have a common zero in Proj $\kappa(\mathfrak{p})[x_0, \dots, x_n]$, where $\kappa(\mathfrak{p})$ is the residue field, and for $f \in I$, f' denotes the image of f in $\kappa(p)[x_0, \dots, x_n]$. This holds due to the natural bijection between points in Proj $\kappa(\mathfrak{p})[x_0, \dots, x_n]$ and points in $\mathbb{P}^n_A$ whose image in Spec A is $\mathfrak{p}$.

Note that all $f' \in I$ have a common zero in $\kappa(\mathfrak{p})[x_0, \dots, x_n]$ if and only if in Spec $\kappa(\mathfrak{p})[x_0, \dots, x_n]$, $\{f' \in I\}$ vanishes outside the origin, i.e. we do not have $V(I) \subset V(x_0, \dots, x_n)$, where $V(I)$ is the vanishing set of I in Spec $\kappa(\mathfrak{p})[x_0, \dots, x_n]$. This holds if and only if

$$\sqrt{(x_0, \dots, x_n)} \not\subset \sqrt{I},$$

and since $\sqrt{(x_0, \dots, x_n)}$ is radical, this holds if and only if $(x_0, \dots, x_n)^N \not\subset I$ for all $N \in \mathbb{N}$. Fix $N \in \mathbb{N}$ and let S_N be the set of degree N elements in $A[x_0, \dots, x_n]$. Notice that $(x_0, \dots, x_n)^N \not\subset I$ if and only if the $\kappa(p)$-linear map

$$(*) \qquad \bigoplus_{f \in I}(S_{N-\deg f'}) \to S_N$$

given by $S_{N-\deg f'} \to f' S_{N-\deg f'}$ is not surjective. This occurs if and only if for every $\dim S_N$ vectors of $\bigoplus_{f \in I} f'(S_{N-\deg f'})$, the map (*) is not surjective, which occurs if and only if for every $\dim S_N$ basis vectors of $\bigoplus_{f\in I}(S_{N-\deg f'})$, the determinant of the corresponding matrix is 0.

Now, consider the map

$$\bigoplus_{f\in I}(S_{N-\deg f'}) \to S_N$$

given by $S_{N-\deg f} \to fS_{N-\deg f}$, and observe that the determinants of the $\dim S_n \times \dim S_N$ matrices given by considering the action of the A-linear map (*) on each set of $\dim S_N$ basis vectors in $\bigoplus_{f\in I}(S_{N-\deg f'})$ are all elements of A. Let J be the collection of all such determinants ranging over all values of N. Then, observe that (*) not being surjective as a $\kappa(\mathfrak{p})$-linear map is equivalent to, for all $a \in J$, the condition that $a \in \mathfrak{p}$. In particular, it follows that the set of all $\mathfrak{p}$ in the image of π is precisely $V(J)$, so π is closed, as desired.

□

The Main Theorem of Elimination Theory enables us to reduce determining if a system of homogeneous polynomial equations has a nonzero solution to a polynomial condition on the coefficients of the system. The following corollary gives a precise statement of this fact.

Corollary 2.3. *Let k be algebraically closed, and suppose that $f_1, \ldots, f_p$ are polynomials in $k[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$ that are homogeneous with respect to variables $Y_1, \ldots, Y_n$, and with $k[X_1, \ldots, X_m]$ as the coefficient ring. Then, there exist polynomials $g_1, \ldots, g_q \in k[X_1, \ldots, X_m]$ such that for each $(x_1, \ldots, x_m) \in k^m$ the system*

$$f_i(x_1, \ldots, x_m, Y_1, \ldots, Y_n) = 0$$

for all $i \in [p]$ has a nonzero solution $(Y_1, \ldots, Y_n) = (y_1, \ldots, y_n) \in k^n$ if and only if

$$g_i(x_1, \ldots, x_m) = 0$$

for all $i \in [q]$.

Proof. Let $A = k[X_1, \ldots, X_m]$. By the Main Theorem of Elimination Theory, the image of $V_+(f_1, \ldots, f_p)$ is closed under the map $\mathbb{P}^n_A \to \operatorname{Spec} A$. Let $V(g_1, \ldots, g_q)$ be the image. Note that $f_1, \ldots, f_p$ have a common nonzero solution for $(X_1, \ldots, X_m) = (x_1, \ldots, x_m)$ if and only if $(X_1 - x_1, \ldots, X_m - x_m)$ is in the image, which occurs if and only if $g_i(x_1, \ldots, x_m) = 0$ for all $i \in [q]$, as desired. □

3. Necessary Definitions

Since the Main Theorem of Elimination Theory gives a simple characterization of the image of closed sets, one may naively hope that in general, the images of morphisms of schemes are reasonably nice. This turns out to be false, as the following example shows.

Example 3.1. Let X be a scheme and $S \subset X$ be any subset. Consider the natural map

$$\pi : \coprod_{\mathfrak{p}\in S} \operatorname{Spec} \kappa(\mathfrak{p}) \to X.$$

Note the image of π is S. Thus, any set in S can be the image of a morphism of schemes.

Thus, we need to impose some natural conditions in order to achieve a reasonable characterization of the images of morphisms of schemes. We now give a potpourri of useful definitions that will help us give a precise statement of Chevalley's Theorem.

The following gives a useful finiteness condition on schemes.

Definition 3.2. A scheme X is Noetherian if it is quasicompact and for all affine open sets Spec $A \subset X$, A is Noetherian.

For virtually every instance one wishes to check if a scheme is Noetherian, they use the following fact:

Proposition 3.3. *A scheme X is Noetherian if and only if X is the union of finitely many Spec A_i where each A_i is Noetherian.*

Proof. This is a consequence of Noetherianness satisfying the hypotheses of what [5] calls the Affine Communication Lemma. For details, see [5], Ch 5.3. □

Assuming the above proposition, we can give some examples of Noetherian schemes.

Example 3.4. Let A be a Noetherian ring (e.g. a field). Then, Spec A is Noetherian. Moreover, $A[x_1, \ldots, x_n]$ is Noetherian by Hilbert's Basis Theorem, so Spec $A[x_1, \ldots, x_n]$ is Noetherian.

Example 3.5. Note that if A is Noetherian, $A[x_0, \ldots, x_n]_{x_i}$ is Noetherian for each $x_i \in \{0, 1, \ldots, n\}$. In particular, it follows by the above proposition that $\mathbb{P}^n_A$ is Noetherian.

One particularly useful fact about Noetherian schemes is that they form Noetherian topological spaces.

Definition 3.6. A topological space X is Noetherian if its closed sets satisfy the descending chain condition.

Another useful notion for us will be finite morphisms and morphisms of finite type.

Definition 3.7. A morphism of schemes $\pi : X \to Y$ is finite if for all affine open sets Spec $B \subset Y$, $\pi^{-1}(\text{Spec } B) \cong$ Spec A for some ring A, where A is a finitely generated B-module.

We will later use the fact that finite morphisms are closed.

Proposition 3.8. *If $\pi : X \to Y$ be a finite morphism, then π is closed.*

Proof. This requires some results about integrality. We will not prove it here, but instead refer the reader to sections 8.2 and 8.3 of [5]. □

Definition 3.9. A morphism of schemes $\pi : X \to Y$ is of finite type if for all affine open sets Spec $B \subset Y$ and all affine open sets Spec $A \subset \pi^{-1}(\text{Spec } B)$, A is a finitely generated B-algebra.

Once again, for practical purposes, it suffices to check the "finite" and "of finite type" condition on an affine open cover of the target

Proposition 3.10. *Let $\pi : X \to Y$ be a morphism of schemes. Then, π is of finite type (resp. finite) if and only if there exists an affine open cover $\bigcup \text{Spec } A_i$ of Y such that for each Spec A_i, there exists an affine open cover Spec B_{ij} of $\pi^{-1}(\text{Spec } A_i)$ such that for each Spec B_{ij}, B_{ij} is a finitely generated A_i-algebra (resp. $\pi^{-1}(\text{Spec } A_i) = \text{Spec } B_i$ and B_i is a finitely generated A_i-module).*

Proof. Once again, we will not prove this, but this is a consequence of the Affine Communication Lemma, see Ch 5.3 and Ch 8.3 of [5] for more details. □

Assuming the above proposition enables us to give some concrete examples.

Example 3.11. Let A be a ring. Then, the map Spec $A[x]/(x^2) \to$ Spec A is finite, as $A[x]/(x^2)$ is a finitely generated A-module.

Example 3.12. Let A be a ring. Then, the map Spec $A[x, y] \to$ Spec A is of finite type, as $A[x, y]$ is generated as a A-module by x, y over A. However, it is not finite.

Example 3.13. Let A be a ring. Then, the natural morphism $\mathbb{P}^n_A \to$ Spec A is of finite type. To see this, recall we may cover $\mathbb{P}^n_A$ by affine open sets of the form Spec $A[x_0, \ldots, x_n]_{x_i}$, and the induced morphism Spec $A[x_0, \ldots, x_n]_{x_i} \to$ Spec A is of finite type, as $A[x_0, \ldots, x_n]_{x_i}$ is a finitely generated A-algebra with generators $x_0, \ldots, x_n, 1/x_i$.

Now that we have defined them, we hope that placing nice conditions such as Noetherianness and finite type on X, Y and π will give us a nice characterization of $\pi(X)$ like it did in the Main Theorem of Elimination Theory. However, if $\pi : X \to Y$ is a morphism of finite type, where X and Y are Noetherian schemes, then $\pi(X)$ may not necessarily be closed.

The correct notion to describe the image is the notion of constructible sets. To define constructible sets, we must first define locally closed sets.

Definition 3.14. Let X be a topological space. A set $S \subset X$ is locally closed if $S = U \cap V$, where U is open and V is closed.

We can now give a definition of constructible sets in Noetherian topological spaces.

Definition 3.15. Let X be a Noetherian topological space. A subset $S \subset X$ is constructible if it is the finite union of locally closed sets.

Example 3.16. The reader may wonder why the notion of constructible is not equivalent to the notion of locally closed. For an example of a constructible set that is not locally closed, consider 2-dimensional affine space with the x-axis removed, but the origin put back. More formally, let $S \subset$ Spec $k[x, y]$, be the set

$$S = D(y) \cup V(x, y)\}.$$

We claim that S is not locally closed. Suppose for the sake of contradiction that $S = U \cap V$, where V is closed and U is open. But note that S contains the generic point (0), and hence $V =$ Spec $k[x, y]$, as the only closed set containing the generic point of Spec $k[x, y]$ is Spec $k[x, y]$. Thus, S is open, a contradiction, as desired.

Chevalley's theorem will tell us that, for morphisms of finite type of Noetherian schemes, the image of constructible sets is constructible.

4. A Proof of Chevalley's Theorem

We now prove Chevalley's Theorem.

Theorem 4.1. *(Chevalley) Let X and Y be Noetherian schemes, and $\pi : X \to Y$ be a morphism of finite type. Then, if $C \subset X$ is constructible, its image under π is also constructible.*

Remark 4.2. With the proper alterations to the definitions, Chevalley's Theorem holds in the non-Noetherian setting as well, but we will not consider this here.

Example 4.3. The hypothesis of constructible is necessary. We now give an example of a morphism of finite type of Noetherian schemes whose image is constructible, but not locally closed.

Consider the map of schemes $\pi : \text{Spec } \mathbb{C}[x, y] \to \text{Spec } \mathbb{C}[x, y]$ induced by the ring map ϕ given by $x \mapsto xy$ and $y \mapsto y$. We claim that $\pi(\text{Spec } \mathbb{C}[x, y]) = D(y) \cup V(x, y)$.

First, note that if $y \in \mathfrak{p}$, and $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$ for some $\mathfrak{q} \in \text{Spec } \mathbb{C}[x, y]$, then we have $y \in \mathfrak{q}$, so $xy \in \mathfrak{q}$, and thus $x \in \mathfrak{p}$. In particular, if $\mathfrak{p} \notin D(y)$, then $\mathfrak{p} \in V(x, y)$, so $\pi(\text{Spec } \mathbb{C}[x, y]) \subset D(y) \cup V(x, y)$.

We now show equality. First, note that since $xy, y \in (x, y)$, we have that $x, y \in \phi^{-1}((x, y))$, and thus $\phi^{-1}((x, y)) = (x, y)$. Thus, it suffices to show that $D(y) \subset \pi(\text{Spec } \mathbb{C}[x, y])$. We claim that $\pi(D(y)) = D(y)$. Observe that $\pi(D(y)) \subset D(y)$, so we may consider $\tilde{\pi} = \pi|_{D(y)} : \text{Spec } \mathbb{C}[x, y]_y \to \text{Spec } \mathbb{C}[x, y]_y$, and it suffices to show that $\tilde{\pi}$ is surjective.

Fix $\mathfrak{p} \in \text{Spec } \mathbb{C}[x, y]_y$, and consider the prime ideal $\tilde{\mathfrak{p}} \in \text{Spec } \mathbb{C}[x, y]_y$ given by

$$\tilde{\mathfrak{p}} = \{f(x, y) \in \mathbb{C}[x, y]_y : f(x/y, y) \in \mathfrak{p}\}.$$

Observe that $\tilde{\pi}(\tilde{\mathfrak{p}})$, so $\tilde{\pi}$ is surjective, as desired.

In particular, by Example 3.13, we have given an example of a morphism of finite type of Noetherian schemes where the image is constructible, but not locally closed.

The following lemma uses a sequence of technical tricks to reduce proving Chevalley's Theorem to the special case where $\pi : \text{Spec } A[t] \to \text{Spec } A$ is the natural map and A is an integral domain. The uninterested reader may safely skip it without losing the main ideas of the proof of Chevalley's Theorem.

Lemma 4.4. *Suppose that Chevalley's Theorem holds for the special case of τ : Spec $A[t] \to$ Spec A, where A is an integral domain and τ is the morphism induced by the map of rings $A \hookrightarrow A[t]$. Then, the theorem holds in general.*

Proof. Let $\pi : X \to Y$ be a finite type morphism, where X and Y are Noetherian schemes, and let $Z \subset X$ be a constructible set.

Since Y is quasicompact,

$$Y = \bigcup_{i=1}^{n} \text{Spec } A_i$$

for Noetherian rings A_i. Since constructible sets are closed under finite unions and intersections, it suffices to prove that the image of $Z \cap \pi^{-1}(\text{Spec } A_i)$ under the map $\pi|_{\pi^{-1}(\text{Spec } A_i)} : \pi^{-1}(\text{Spec } A_i) \to \text{Spec } A_i$ is constructible for each i. Thus, we may assume that $Y = \text{Spec } A$, for some Noetherian ring A.

Since X is Noetherian, open subsets of X are quasicompact, and in particular, $\pi^{-1}(\text{Spec } A_i)$ is the union of finitely many affine opens,

$$\pi^{-1}(\text{Spec } A) = \bigcup_{j=1}^{m} \text{Spec } B_j,$$

where, since π is of finite type, each B_j is a finitely generated A-algebra (with respect to the ring map $A \to B_j$ induced by π). As each Spec B_j is open in X, it suffices to prove that the image of $Z \cap \text{Spec } B_j$ under $\pi|_{\text{Spec } B_j} : \text{Spec } B_j \to \text{Spec } A$ is constructible for each j. Thus, by replacing Z with $Z \cap \text{Spec } B_j$ and X with Spec B_j for a fixed B_j, we may write $X = \text{Spec } B$, where B is a finitely generated A-algebra, and $\pi : \text{Spec } B \to \text{Spec } A$ with $Z \subset \text{Spec } B$.

Note that since B is a finitely generated A-algebra with respect to the ring map induced by π, we may write $B = A[t_1, \ldots, t_k]/I$ where $\pi : \text{Spec } A[t_1, \ldots, t_k]/I \to \text{Spec } A$ is the morphism of schemes induced by the natural map $A \to A[t_1, \ldots, t_k]/I$. But note that π factors through Spec $A[t_1, \ldots, t_k]$ via the following commutative diagram:

$$\begin{array}{ccc} & \text{Spec } A[t_1, \ldots, t_k] & \\ \overset{\iota}{\nearrow} & & \overset{\pi'}{\searrow} \\ \text{Spec } A[t_1, \ldots, t_k]/I & \xrightarrow[\pi]{} & \text{Spec } A \end{array}$$

where ι is the natural inclusion of points and π' is induced from the natural map $A \to A[t_1, \ldots, t_k]$. If Z is constructible in Spec $A[t_1, \ldots, t_k]/I$, then $\iota(Z)$ is constructible in Spec $A[t_1, \ldots, t_k]$, and hence it suffices to show the result when π is the natural map Spec $A[t_1, \ldots, t_k] \to \text{Spec } A$.

Now, if Chevalley's Theorem holds for the natural map Spec $R[x] \to \text{Spec } R$ for arbitrary Noetherian rings R, observe that the inclusion of rings

$$A \hookrightarrow A[t_1] \hookrightarrow A[t_1][t_2] \hookrightarrow \cdots \hookrightarrow A[t_1, \ldots, t_{k-1}][t_k]$$

induces a corresponding map of spectra

$$\text{Spec } A[t_1, \ldots, t_k] \to \text{Spec } A[t_1, \ldots, t_{k-1}] \to \cdots \to \text{Spec } A[t_1] \to \text{Spec } A$$

that implies Chevalley's theorem holds when π is the natural map Spec $A[t_1, \ldots, t_k] \to$ Spec A.

Thus, it remains to show that given an arbitrary Noetherian ring R, and a constructible subset $Z \subset \text{Spec } R[x]$, the image of Z under the natural map $\pi :$ Spec $R[x] \to \text{Spec } R$ is constructible. Since R is Noetherian, it has finitely many minimal primes $p_1, \ldots, p_l$. Observe that for each i, we may identify Spec $R/p_i[x]$ with $\pi^{-1}(V(p_i))$, and, with the natural maps, the following diagram commutes.

$$\begin{array}{ccc} \text{Spec } R[x] & \xrightarrow{\pi} & \text{Spec } R \\ \uparrow & & \uparrow \\ \text{Spec } R/p_i[x] & \longrightarrow & \text{Spec } R/p_i \end{array}$$

Note that if Z is constructible, we have that $Z \cap \pi^{-1}(V(p_i))$ is constructible, Since the maps Spec $R/p_i[x] \to \text{Spec } R[x]$ and Spec $R/p_i \to \text{Spec } R$ are inclusions, and our assumption that Chevalley's theorem holds for Spec $A[x] \to \text{Spec } A$ when

A is an integral domain, we have that $\pi(Z \cap \pi^{-1}(V(p_i)))$ is constructible. Since R is the union of the $V(p_i)$, we have that

$$\pi(Z) = \bigcup_{i=1}^{l} \pi(Z \cap \pi^{-1}(V(p_i))),$$

and thus $\pi(Z)$ is constructible, as desired. □

We first prove Chevalley's Theorem over an open set of Spec A.

Lemma 4.5. *Let A be a Noetherian integral domain and π : Spec $A[t] \to$ Spec A be the natural map. Let $Z \subset$ Spec $A[t]$ be locally closed. Then, there exists a nonempty open set $U \subset$ Spec A such that $U \cap f(Z)$ is constructible.*

Proof. First, note by taking irreducible components of $\overline{Z}$, we may assume that $\overline{Z}$ is irreducible, and thus write $\overline{Z} = V(\mathfrak{p})$ for some $\mathfrak{p} \in$ Spec A. Now, let η be the generic point of Spec A. We consider $\overline{Z} \cap \pi^{-1}(\eta)$. Note that $\mathfrak{q} \in$ Spec $A[t]$ satisfies $\pi(\mathfrak{q}) = \eta$ if and only if $\mathfrak{q} \cap A = (0)$, so it follows that $\pi^{-1}(\eta) \cong$ Spec $K(A)[t]$, where $K(A)$ is the fraction field of A.

Now, observe that either $\mathfrak{p} \cap (A \setminus \{0\}) \neq \emptyset$, in which case $V(\mathfrak{p}) \cap \pi^{-1}(\eta) = \emptyset$, or $\mathfrak{p} \cap A = (0)$, in which case $\mathfrak{p} \in \pi^{-1}(\mathfrak{q})$ and thus $\overline{Z} \cap \pi^{-1}(\eta)$ corresponds to an irreducible closed set of $\pi^{-1}(\eta) \cong$ Spec $K(A)[t]$. Observe that the only irreducible closed sets of Spec $K(A)[t]$ are closed points and Spec $K(A)[t]$.

In summary, we have three cases:

(i) $\overline{Z} \cap \pi^{-1}(\eta) = \emptyset$.
(ii) $\overline{Z} \cap \pi^{-1}(\eta) = \pi^{-1}(\eta)$.
(iii) $\overline{Z} \cap \pi^{-1}(\eta) = \{\mathfrak{m}\}$ for some closed point $\mathfrak{m} \in$ Spec $K(A)[t]$.

For case (i), note that since $\mathfrak{p}$ is the generic point of $\overline{Z}$, we have that $\pi(\overline{Z}) \subset \overline{\{\pi(\mathfrak{p})\}}$. Since $\overline{Z} \cap \pi^{-1}(\eta)$ is empty, we in particular have that $\pi(\mathfrak{p}) \neq \eta$, so $\overline{\{\pi(\mathfrak{p})\}}$ is a proper subset of Spec A. In particular, let $U :=$ Spec $A \setminus \overline{\{\pi(\mathfrak{p})\}}$. Then, U is an open set disjoint from $\pi(\overline{Z})$ and thus from $\pi(Z)$. In particular, $U \cap \pi(Z)$ is constructible.

For case (ii), observe that Z is open in $\overline{Z}$, so in particular, $Z \cap \pi^{-1}(\eta)$ can be written as the union of finitely many $D(f) \subset$ Spec $K(A)[t]$, where $f \in K(A)[t]$. In particular, it suffices to show the result for where $Z \cap$ Spec $K(A)[t] = D(f)$ for $f \in K(A)[t]$. By clearing denominators, we may assume that $f \in A[t]$. Write

$$f = a_n t^n + \cdots + a_0.$$

Then, let $U =$ Spec $A_{a_n} = D(a_n)$. We claim that $U \subset \pi(Z)$. Let $\mathfrak{a} \in U$. Note that $\pi^{-1}(\mathfrak{a}) \cong$ Spec $\kappa(\mathfrak{a})[t]$. Let $\mathfrak{y}$ be the generic point of Spec $\kappa(\mathfrak{a})[t]$. Then, note that since $a_n \notin \mathfrak{a}$, we have that $f \notin \mathfrak{y}$. In particular, $\mathfrak{y} \in Z$. Since there is a point $\mathfrak{y}$ in Z and in $\pi^{-1}(\mathfrak{a})$, it follows that $\mathfrak{p} \in \pi(Z)$, so $U \subset \pi(Z)$. Since $U \cap \pi(Z) = U$, it is constructible.

For case (iii), observe if $\mathfrak{m} \in V(\mathfrak{p})$, and $\mathfrak{p} \cap A \neq (0)$, then $\mathfrak{m} \cap A \neq (0)$, a contradiction, so it follows that $\mathfrak{p} \in \pi^{-1}(\eta)$, and thus that $\mathfrak{p} = \mathfrak{m}$. Since $K(A)[t]$ is a PID, there exists some $g \in K(A)[t]$ such that $\mathfrak{p} \cdot K(A)[t] = g \cdot K(A)[t]$. By clearing denominators, we may assume WLOG that $g \in \mathfrak{p}$.

Now, write $g(t) = a_n t^n + \cdots + a_0$ and let $V =$ Spec A_{a_n}. Then, observe the morphism of schemes $V(g) \cap \pi^{-1}(V) \to V$ given by restricting the domain and range of π is a finite morphism. To see this, note that $V(g) \cap \pi^{-1}(V) =$ Spec $A_{a_n}[t]/gA_{a_n}[t]$, and since the leading coefficient of g is invertible in $A_{a_n}[t]$, $1, t, \dots, t^{n-1}$ generate $A_{a_n}[t]/gA_{a_n}[t]$ as an A_{a_n}-module.

Now, Z is locally closed, so we have that Z is open in $\overline{Z}$, and thus that $\overline{Z} \setminus Z$ is closed in Spec $A[t]$. Since finite morphisms are closed, we thus have that $\pi(\overline{Z} \setminus Z)) \cap V$ and $\pi(\overline{Z}) \cap V$ are both closed in V. If $\pi(\overline{Z}) \cap V \neq V$, then $V \setminus (\pi(\overline{Z}) \cap V)$ is an open set disjoint from $\pi(V)$, so we may let $U := V \setminus (\pi(\overline{Z}) \cap V)$. Thus, we may assume that $\pi(\overline{Z}) \cap V = V$, i.e. $V \subset \pi(\overline{Z})$.

Since Z is open in $\overline{Z}$, Z must contain the generic point of $\overline{Z}$, $\mathfrak{p} = \mathfrak{m}$, and in particular, $\eta \notin \pi(\overline{Z} \setminus Z) \cap V$, so $V \setminus (\pi(\overline{Z} \setminus Z) \cap V)$ is nonempty, and since $\pi(\overline{Z})$ contains V, it follows that $U := V \setminus (\pi(\overline{Z} \setminus Z) \cap V)$ is an open set contained entirely in $\pi(V)$, as desired. □

Given Lemma 4.5, we can now prove Chevalley's Theorem. We proceed by Noetherian induction. Let $\pi :$ Spec $A[t] \to$ Spec A be the natural map, and suppose that the image of Z is not constructible. Since Spec A is Noetherian, there exists a closed set $V \subset$ Spec A minimal with respect to the property that $\pi(Z) \cap V$ is not constructible. Note that V must be irreducible, as if $V(I) = V_1 \cup V_2$, where V_1, V_2 are closed proper subsets of V, then since $\pi(Z) \cap V_1, \pi(Z) \cap V_2$ are constructible by minimality, we have that $\pi(Z)$ is so as well.

Thus, we may write $V = V(\mathfrak{p})$, where $\mathfrak{p}$ is a prime ideal of A. Now, observe that $\pi^{-1}(V(\mathfrak{p})) \cong$ Spec $A/\mathfrak{p}[t]$, and so π induces a map $\pi' :$ Spec $A/\mathfrak{p}[t] \to$ Spec $A/\mathfrak{p}$. Let $Z' = Z \cap \pi^{-1}(V(\mathfrak{p}))$. Then, note that $\pi'(Z')$ is not constructible, by our definition of $V(\mathfrak{p})$. Since $A/\mathfrak{p}$ is an integral domain, we may apply Lemma 4.5, so there exists a nonempty open set $U \subset$ Spec $A/\mathfrak{p}[t]$ such that the $\pi'(Z') \cap U$ is constructible. In particular, $\pi'(Z') \cap U^c$ is not constructible. Since $V(\mathfrak{p})$ is closed, U^c is a closed subset of Spec A, and since $Z' = Z \cap \pi^{-1}(V(\mathfrak{p}))$, $\pi(Z) \cap U^c$ is not constructible. But this implies that U^c is a proper closed subset of V such that $\pi(Z) \cap U^c$ is not constructible, contradicting our choice of V. It follows that $\pi(Z)$ is constructible, as desired. □

5. Applications of Chevalley's Theorem

Chevalley's Theorem also gives a clean proof of Zariski's Lemma, which is the main technical obstacle in proving the Nullstellensatz.

Theorem 5.1. *(Zariski's Lemma) Let K/k be a field extension such that K is a finitely generated k-algebra. Then, K/k is a finite extension.*

Proof. Let $x_1, \dots, x_n$ generate K as a k-algebra. It suffices to show each x_i is algebraic over k. If x_i is not, then note the map of k-algebras $k[x] \to K$ given by $x \mapsto x_i$ is an injection, and in particular, we have a map Spec $K \to$ Spec $k[x]$. Since $k[x] \to K$ is an injection and Spec K is one point, the image of Spec K must be dense and thus must be the generic point. If we show that the generic point is not constructible, this contradicts Chevalley's Theorem, and thus x_i must be algebraic over k.

To see why the generic point is not constructible, suppose that $\{(0)\} = U \cap V$ for U open and V closed, note that since $\{0\} \in V$, we must have $V =$ Spec $k[x]$. Thus, $\{(0)\}$ is open, and by taking a distinguished open set, we may assume that $\{(0)\} = D(f)$ for some $f \in k[x]$. But then note that $k[x]_f$ is an integral domain but not a field, and thus $|\text{Spec } k[x]_f| > 1$. In particular, the generic point is not constructible, as desired. □

From Zariski's Lemma, it is easy to recover a proof of the weak Nullstellensatz.

Corollary 5.2. *(Weak Nullstellensatz) Let k be algebraically closed. Then, the only maximal ideals of $k[x_1, \dots, x_n]$ are of the form $(x_1 - a_1, \dots, x_n - a_n)$ where $a_1, \dots, a_n \in k$.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of $R := k[x_1, \dots, x_n]$. Then, observe $R/\mathfrak{m}$ is a finitely generated k-algebra, so by Zariski's Lemma, R/m is a finite extension of k. Since k is algebraically closed, we thus have that $R/\mathfrak{m} = k$. In particular, note that for each $i \in [n]$, there exists some $a_i \in k$ such that $x_i \equiv a_i \mod \mathfrak{m}$. Thus, $x_i - a_i \in \mathfrak{m}$ for all $i \in [n]$, and so it follows that $\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n)$, as desired. □

Remark 5.3. For a full proof of the Nullstellensatz, which gives a bijection between algebraic sets in F^n and radical ideals of $F[x_1, \dots, x_n]$ for an algebraically closed F, one applies the Rabinowitsch trick. For details, see [3].

Finally, as promised, Chevalley's Theorem gives us our desired characterization of when systems of polynomial equations are solvable in the spirit of Corollary 2.3, stating that the solvability of a system of polynomial equations can be checked via polynomial relations of the coefficients.

Theorem 5.4. *(Elimination of Quantifiers) Let k be algebraically closed, and $f_1, \dots, f_p \in k[X_1, \dots, X_m, Y_1, \dots, Y_n]$. Then, there is a Zariski-constructible subset Y of k^m such that the system of polynomial equations*

$$f_i(y_1, \dots, y_m, X_1, \dots, X_n) = 0$$

for all $i \in [p]$ has a solution if and only if $(y_1, \dots, y_m) \in Y$.

Proof. Let $A = k[X_1, \dots, X_m]$, and consider the natural map $\pi : \text{Spec } A[Y_1, \dots, Y_n] \to \text{Spec } A$. Observe that this is a morphism of finite type between Noetherian schemes, so Chevalley's Theorem applies. In particular, the image of $V(f_1, \dots, f_p)$ is constructible. By the weak Nullstellensatz, we can identify closed points in Spec A with points in k^m. We show that a closed point $(y_1, \dots, y_m)$ is in the image of $V(f_1, \dots, f_p)$ if and only if the system of polynomial equations

$$f_i(y_1, \dots, y_m, X_1, \dots, X_n)$$

for all $i \in [q]$ has a solution.

If $(y_1, \dots, y_m)$ is in the image of $V(f_1, \dots, f_p)$, then there exists a point $\mathfrak{a} \in V(f_1, \dots, f_p)$ such that $\mathfrak{a} \cap A = (Y_1 - y_1, \dots, Y_m - y_m)$. Let $\mathfrak{m}$ be a maximal ideal in $A[Y_1, \dots, Y_m]$ containing $\mathfrak{a}$. Again applying the weak Nullstellensatz, $\mathfrak{m} = (X_1 - x_1, \dots, X - x_n, Y_1 - y_1, \dots, Y_m - y_m)$ for some $(x_1, \dots, x_n) \in k^n$. Thus, $(x_1, \dots, x_n)$ is a solution to the system of equations

$$f_i(y_1, \dots, y_m, X_1, \dots, X_n)$$

for all $i \in [q]$.

Now, let $(y_1, \dots, y_m)$ be a m-tuple in k^m such that the system of polynomial equations:

$$f_i(y_1, \dots, y_m, X_1, \dots, X_n) = 0$$

for all $i \in [q]$, has a solution. Then, let $(x_1, \dots, x_n)$ be a solution to the system of equations

$$f_i(y_1, \dots, y_m, X_1, \dots, X_n) = 0$$

, for all $i \in [q]$, and note that $(X_1 - x_1, \ldots, X_n - x_n, Y_1 - y_1, \ldots, Y_m - y_m)$ is in $V(f_1, \ldots, f_p)$, and

$$\pi((X_1 - x_1, \ldots, X_n - x_n, Y_1 - y_1, \ldots, Y_n - y_n)) = (Y_1 - y_1, \ldots, Y_n - y_n),$$

so $(y_1, \ldots, y_n)$ is in the image, as desired.

$\square$

Example 5.5. Consider the system of equations over $\mathbb{C}$ in x, y with coefficients in a, b, c, d, e, f

$$ax + by - e = 0$$
$$cx + dy - f = 0.$$

This has a solution if and only if the matrices

$$M = \begin{pmatrix} a & b & e \\ c & d & f \end{pmatrix}$$

and

$$N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

have the same rank. Since the rank of a matrix M is the largest k such that M has a $k \times k$ submatrix with nonzero determinant, the solution to this equation is a Zariski-constructible condition on a, b, c, d, e, f. In particular, if rank $M = 2$, then we must have $ad - bc \neq 0$. If rank $M = 1$, then we must have $ad - bc = af - ec = bf - ed = 0$, with at least one of a, b, c, d nonzero (i.e. $abcd \neq 0$). If rank $M = 0$, we must have $a, b, c, d, e, f = 0$. Thus, the set of a, b, c, d, e, f where the system has a solution is

$$V(a, b, c, d, e, f) \cup (D(abcd) \cap V(ad - bc, af - ec, bf - ed)) \cup D(ad - bc).$$

In general, by identifying $M_{m \times n}(\mathbb{C}) \times \mathbb{C}^n$ with $\mathbb{C}^{mn+n}$, a similar argument shows that the set of $(A, v) \in M_{m \times n}(\mathbb{C}) \times \mathbb{C}^n$ such that $v \in \mathbb{C}^m$ is in the image of A is Zariski-constructible.

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