AN INTRODUCTION TO KNOT POLYNOMIALS

CHAD BERKICH

Abstract. This paper is devoted to the exploration of different knot invariants, particularly polynomials. Three polynomials will be explored: the Jones, Alexander-Conway, and HOMFLY polynomials, with three other non-polynomial knot invariants considered. We prove their invariance and demonstrate any important properties of these invariants, and conclude with an explicit calculation. The goal is to provide the reader with an understanding of the basics of knot invariants and the importance of polynomials within this field.

Contents

1. Introduction

Before beginning the paper, an important question must be addressed: why knot theory? It would be easy to dismiss it as simply a topological novelty without much importance. However, knot theory has ties to algebra, algebraic topology, and graph theory, among other mathematical fields. It also has applications outside of math, such as knotted DNA. But, even without external motivation, it is an incredibly interesting field with unexpected results and great visualizations. This paper is designed to be accessible, and little mathematical background will be assumed, so anyone with interest in the topic can use this as a starting point.

Intuitively, a mathematical knot is a piece of string with its ends tied together. Note the difference from its everyday use; a knotted shoelace, for example, does not have the ends tied together, but separated such that the knot can be "unknotted."

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If mathematical knots could be unknotted in this way, they would all be equivalent and there would be nothing interesting we could say about them.

Intuitively, some knots can be "untied" in the way we would think of an everyday knot. An everyday knot would be considered untied when the string becomes a straight line with no overlapping points. With a mathematical knot, the knot is untied when it becomes a simple circle. In knot theory, this is achieved through a continuous deformation. The broad goal of knot theory is to tabulate all knots, or find every possible distinct knot. In this case, distinct means that two knots cannot be deformed into each other.

However, it is often unclear whether two knots are equal. If they are not, it is simply a waste of time trying to deform one into the other. If they are, it may not be clear why and will require extensive deformations to achieve. How does one even work on achieving this goal then?

This problem is approached through knot invariants, or a quantity that may be derived from a knot that is consistent across all manipulations. For example, there is a special way to color knots, called tricolorability, that will be explained later in the paper in Section 3.1. If one knot is tricolorable and another is not, then the knots are distinct.

The problem that arises is the intuitive invariants, such as tricolorability, are difficult to calculate and work with, or don't offer much information. The latter is the case for tricolorability. This motivates the study of less intuitive, but more useful, invariants, which are the main subject of this paper.

In particular, we will focus on polynomials that can be defined from knots. These invariants sacrifice some of the intuition that invariants like tricolorability provide; in exchange, they can be calculated easily and used to differentiate between knots. Three polynomials will be shown, with an emphasis on the more computational aspects. The overall goal of the paper is to introduce these invariants, show why they are invariants, and demonstrate how they may be used.

It is recommended that the reader have a piece of string, or something equivalent, that may be used as a physical representation while reading the paper, since being able to manipulate an actual knot will help with intuition and understanding.

2. Knot Basics

Definition 2.1. A **knot** is an embedding of S^1 into $\mathbb{R}^3$.

Often $\mathbb{R}^3$ is replaced by its one point compactification, S^3. The simplest knot is called the unknot, which is simply a circle in $\mathbb{R}^3$ with no crossings.

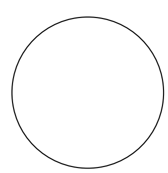


Figure 1. The Trivial Knot or Unknot

The next simplest knot is the trefoil knot.

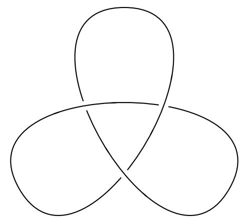


FIGURE 2. The Trefoil Knot

The core goal of knot theory is differentiating between knots. For example, the trefoil and unknot appear to be different, and in fact this turns out to be true. There is no way to manipulate one into the other without "cutting" the string and reforming the knot. With this in mind, we define knot equivalence.

Definition 2.2. Given knots $K, K' \subset \mathbb{R}^3$, K is **equivalent** to K' if there exists $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that h is an orientation-preserving homeomorphism and $h(K) = K'$. This is denoted by $K \approx K'$

It is easy to show that this defines an equivalence relation on knots. Therefore, when speaking about a given knot, we are generally considering the class of knots that are equivalent to each other.

Knots that are equivalent to the unknot are called **trivial** knots. Likewise, knots that are distinct from the unknot are called **nontrivial**.

The definition of a knot can be generalized to include multiple circles.

Definition 2.3. A **link** is an embedding of the disjoint union of finitely many copies of S^1 in $\mathbb{R}^3$.

The simplest link that is not a knot is the Hopf link, or two linked circles.

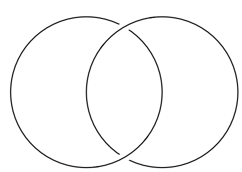


FIGURE 3. The Hopf Link.

As most results that hold for knots also hold for links, for simplicity "knot" will be used going forward to refer to both, unless a distinction need be made.

A convenient way to look at knots is through knot diagrams. You have already seen examples of these in the knots as shown before. These are projections of three-dimensional knots into a two-dimensional space. The precise definition is given as follows:

Definition 2.4. Let $p : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $p(x, y, z) = (x, y)$, or alternatively $p : \mathbb{R}^3 \to \mathbb{R}^3$ where $p(x, y, z) = (x, y, 0)$. If K is a knot, then let $p(K) = \hat{K}$ be the projection of K. $\hat{K}$ is called a **regular projection** if it satisfies the following conditions:

(1) $\hat{K}$ has at most a finite number of intersection points.
(2) If $q \in \hat{K}$ is an intersection point, then $p^{-1}(q) \cap K$ is composed of two points. q is called a **double point** of $\hat{K}$.

Going forward, regular projections will simply be referred to as projections. When drawing a projection, a cut will be made at each double point to one of the strands crossing. To symbolize the over/under information at a double point, the understrand is drawn cut, as shown in the previous figures.

Given a knot projection, there are three operations that may be made on the diagram that affect the knot's crossings. These are called the **Reidemeister moves**. Diagrams of the moves are shown below.

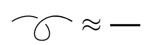


Figure 4. A Type I Reidemeister move.

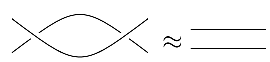


Figure 5. A Type II Reidemeister move.

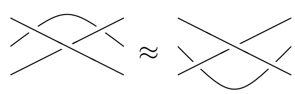


Figure 6. A Type III Reidemeister move.

Note Reidemeister moves hold the same way whether they are over- or under-crossings.

These moves are important because of their effects on crossings. Type I and II moves change the number of crossings, and Type III affects the position of crossings. With this in mind, we define a new type of equivalence.

Definition 2.5. If D and D' are two knot diagrams that can be changed into each other by a finite amount of Reidemeister moves, then D is **equivalent** to D'. This is denoted by $D \approx D'$.

Likewise, this can be shown to be an equivalence relation. Thus, following our intuition, we consider two diagrams to be equivalent under these moves in the same way two knots are equivalent. This gives way for an important theorem.

Theorem 2.6. *If D is the projection of a knot K, and D' is the projection of a K', then $K \approx K'$ if and only if $D \approx D'$.*

The proof of the theorem can be found in [Mur96], Theorem 4.1.1, and is omitted due to its length.

This is a vital result, and will be used implicitly for the rest of the paper. What this result says is that if we can manipulate diagrams into one another, particularly through the use of Reidemeister moves, then the two knots the diagrams represent are equivalent. This is a significantly easier way to show that two knots are equivalent than constructing a map between them. With this background established, we move onto knot invariants.

3. Knot Invariants

One of the problems presented with trying to classify all knots is the sheer amount of knots there are. While there is only one knot with three and four crossings, respectively, there are 2,176 knots with 12 crossings, and 253,293 knots with 15 crossings[1].

Additionally, in the case of more complicated knots, it is often not immediately clear whether the knot is unique or equivalent to some other knot we already found. For example, consider the following knot. Against intuition, this knot is actually equivalent to the unknot.

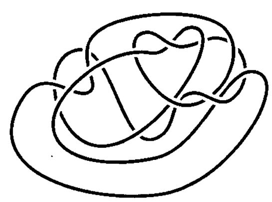


Figure 7. A complicated unknot. Image credit to [Ada94], Exercise 1.3.

Therefore, invariants are a key area of knot theory. Invariants distinguish between knots, no matter how complicated they are. However, many distinct knots share the same value for a given invariant. Invariants can only distinguish between knots; they cannot tell us when two knots are the same.

Definition 3.1. If K is a knot, then let $\rho(K)$ denote some quantity. If for any two equivalent knots the quantity is always the same, then $\rho(K)$ is called a **knot invariant**. This likewise holds for knot diagrams.

Therefore, if we can find quantities that are unaffected by Reidemeister moves, we will have found knot invariants by Theorem 2.6.

Several invariants have been proposed in order to address this problem, which I will spend the rest of the paper discussing. In this paper, I divide the invariants we will study into two types: intuitive invariants and knot polynomials. This section will focus on intuitive invariants.

[1] Stated in Table 1.2 in [Lic97].

3.1. **Tricolorability.** To begin, we will consider one of the simplest knot invariants: tricolorability. To define tricolorability, we first must define a strand.

Definition 3.2. A **strand** in a knot projection is a section of the knot that starts and ends at an undercrossing and only has overcrossings in between.

For example, the blue colored part of Figure 8 is a strand.

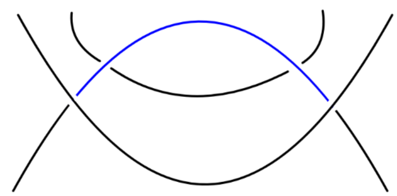


Figure 8. An example of a strand.

Note that at each crossing, there are three strands: the overstrand and two understrands. Now we can define tricolorability.

Definition 3.3. A knot is called **tricolorable** if each strand of a knot is colored one of three colors such that the following conditions are met:

(1) At least two of the colors are used.
(2) At each crossing, either the strands meeting there are all the same color or each strand is a different color.

The reason at least two colors must be used is to ensure that the entire diagram is not one color, in which case no useful information would be available since all knots would satisfy the criteria.

It can be easily demonstrated to be an invariant under Reidemeister moves. To show its usefulness, consider the unknot. It only has one strand, so it may only use one color. Therefore, it is not tricolorable. However, the trefoil knot *is* tricolorable, as shown by the diagram below:

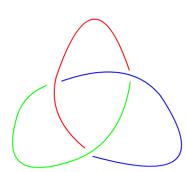


Figure 9. A tricolored trefoil knot.

Therefore, the trefoil knot and the unknot are not the same knot. This is a simple proof that there is actually more than one knot, something we had not shown up until this point.

However, tricolorability is a relatively weak invariant. We can show, for example, that the knot with four crossings (called the figure-eight knot) is not tricolorable, and therefore not equivalent to the trefoil knot. But, with just this information, we have no way of showing that it is nontrivial. For that, we will need stronger invariants.

3.2. **Crossing Number.** Crossing number is arguably the most intuitive of the knot invariants. It is simply the minimum amount of crossings a knot diagram may have. Its formal definition is as follows:

Definition 3.4. Let K be a knot and $\mathcal{D}$ be the collection of regular diagrams, D, of K. Let $c(D)$ denote the number of crossings in D. Then the **crossing number** of K, $c(K)$, is defined as follows:

$$c(K) = \min_{D \in \mathcal{D}} c(D)$$

If $c(D) = c(K)$, then D is said to be a **minimal regular diagram** of K.

For the purposes of this paper, we will only consider knots with a finite number of crossings.

Theorem 3.5. *$c(K)$ is a knot invariant.*

Proof. Suppose D_0 is the minimal regular diagram of K, and D_0' is the minimal regular diagram of K'. Additionally, assume $K \approx K'$. D_0' is therefore also a regular diagram for K, so $c(D_0) \leq c(D_0')$. Likewise, $c(D_0) \geq c(D_0')$. Therefore, $c(K) = c(D_0) = c(D_0') = c(K')$. Thus, $c(K)$ is a knot invariant. $\square$

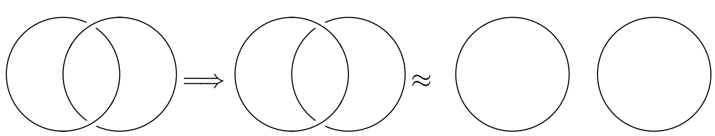
The crossing number of the unknot is 0 and the crossing number of the trefoil is 3, as is likely expected. In fact, crossing number is how knots are typically labeled in knot tables. This also allows us to differentiate between the trefoil and figure-eight, something tricolorability did not allow.

However, there is no method of determining $c(K)$ for an arbitrary knot K. There are for certain types of knots, specifically alternating knots, but not in general. Thus, while crossing number is a useful invariant, it is not simply computable. This is what polynomials will solve at the cost of intuition.

3.3. **Unknotting Number.** First, we will define a new type of move on a knot diagram.

Definition 3.6. Consider some knot diagram D of a knot K. An **unknotting operation** is a move performed on D where the over- and under-crossing at an intersection are switched.

To see a visual representation of this, consider the Hopf link. The top crossing will be unknotted.



Notice that, once the unknotting operation has been performed, using Reidemeister moves the Hopf link can be turned into two unknots. This leads us to the next proposition.

Proposition 3.7. *Consider some knot diagram D of a knot K. Using a finite number of Reidemeister moves and unknotting operations, D can be changed into a diagram of the unknot.*

Proof. Proof by induction. In the case where K is already the unknot, then D is clearly already a diagram of an unknot. Note this is when $c(D) = 0$.

Now suppose that the proposition holds when $c(D) < m$. Suppose $c(D) = m$. Choose an orientation on K which then transfers to D and a point p on D. Move along the knot as directed by the orientation, starting at p until you come upon an intersection. If you are on the overstrand, continue along. If you are on the understrand, then perform an unknotting operation on that intersection such that you are now on the overstrand. Continue repeating this for every new intersection point.

Eventually, you will arrive at an intersection you have been at before. Note that you will arrive on the understrand. Call this intersection point A. As can be seen in the diagram below, this is a loop that includes A and can be removed using a Type I Reidemeister move. Call the new diagram created D'. D' will have fewer intersection points then D, or $c(D') < c(D) = m$ so applying the induction hypothesis to D' concludes the proof.

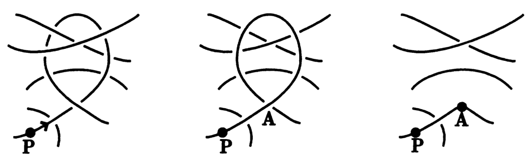


FIGURE 10. A demonstration of what happens when an intersection has been reached again. Image credit to [Mur96], Figure 4.4.3.

□

Using what we have gained from this proposition, we can define a new quantity.

Definition 3.8. Let K be a knot and $\mathcal{D}$ be the collection of regular diagrams, D, of K. The **unknotting number** of a diagram D, denoted $u(D)$ is the minimum number of unknotting operations required to change D to a diagram of the unknot. Likewise,

$$u(K) = \min_{D \in \mathcal{D}} u(D)$$

Theorem 3.9. *$u(K)$ is a knot invariant.*

Proof. The proof follows from the same argument as Theorem 3.5. □

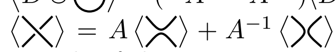
As we showed earlier, the Hopf link has unknotting number 1. The trefoil knot also has unknotting number 1.

As can be more clearly seen now, crossing and unknotting numbers are hard to compute, and tricolorability is very weak as an invariant. However, their intuitive nature is useful to introduce knot invariants. With a firmer grasp on invariants, we move onto knot polynomials.

4. Bracket and Jones Polynomials

The Jones polynomial is an easily computable knot invariant that can help distinguish between knots, something that the other invariants we have looked at have not provided. However, before defining the Jones polynomial, we need to look at another knot polynomial, the bracket polynomial.

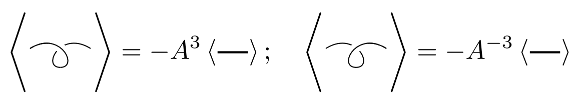
Definition 4.1. The **Kauffman bracket polynomial**, or the bracket polynomial, is a function from knot diagrams in $\mathbb{R}^2$ or S^2 to Laurent polynomials in A, or polynomials in $\mathbb{Z}[A, A^{-1}]$, where A is an indeterminate. If D is a diagram, then $\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$ is characterized by

(1) $\langle \bigcirc \rangle = 1$;
(2) $\langle D \cup \bigcirc \rangle = (-A^2 - A^{-2})\langle D \rangle$;
(3)  where the knots are assumed to be identical outside of the crossings considered here.

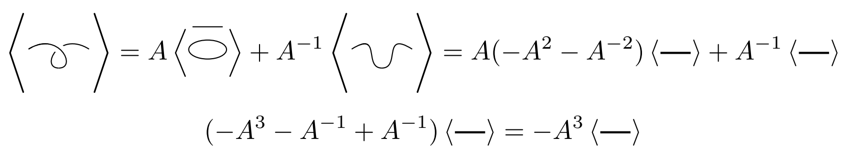
$\bigcirc$ is simply the unknot, and $D \cup \bigcirc$ means the original knot with an extra unknot that does not cross itself nor D. Additionally, note that .

Now we will discuss the affect of Reidemeister moves on the polynomial.

Lemma 4.2. *When a Type I Reidemeister move is applied to a diagram, the bracket polynomial changes in the following way:*



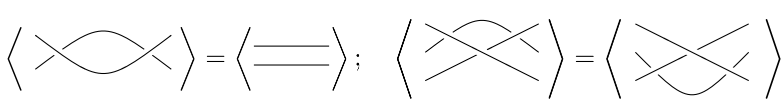
Proof.



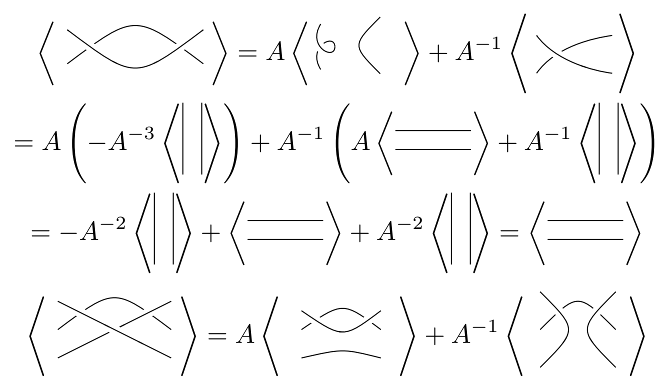
The second equation follows in the same fashion. □

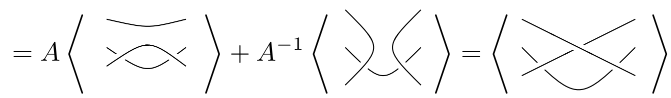
Thus, it appears we have a problem. The first Reidemeister move prevents the bracket polynomial from being invariant. As you probably expect, this will be resolved; however, we first must introduce new machinery. That will come after showing the other Reidemeister moves effect on the bracket polynomial.

Lemma 4.3. *The bracket polynomial is invariant under Type II and III Reidemeister moves, or*



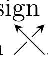
Proof.





Note that the transition from the first to second line is achieved by applying a Type II Reidemeister move twice. $\square$

We will now define writhe, which will solve the problem with Type I moves.

Definition 4.4. Consider some oriented knot diagram D of an oriented knot K. The **writhe** of a diagram, denoted $w(D)$, of an oriented knot is the sum of the crossings of D. We assign numbers to the crossings in the following fashion: if the crossing is of the form , then we assign it -1. If it is of the form , then we assign it $+1$. Note that the type of crossing is determined by the orientation of the knot.

It can be shown that writhe is invariant under Type II and III Reidemeister moves. Performing a Type I Reidemeister move either adds or subtracts 1 from the writhe. Using writhe, we can create a knot invariant with the Jones polynomial.

Theorem 4.5. *Let D be a diagram of an oriented knot K. Then*

$$(-A)^{-3w(D)}\langle D\rangle$$

is an invariant of K.

Proof. The core idea of the proof will be that the writhe's sign is counterbalanced against the sign that the Type I Reidemeister move introduces, and multiplying it by 3 causes them to cancel.

By Lemma 4.3, the expression is invariant under Type II and III Reidemeister moves. It remains to show that it is invariant under Type I. Suppose the writhe increases by one, creating a diagram D'. Then

$$(-A)^{-3w(D')}\langle D'\rangle = (-A)^{-3w(D)-3}(-A^3\langle D\rangle)$$
$$= -A^{-3}(-A)^3(-A)^{-3w(D)}\langle D\rangle = (-A)^{-3w(D)}\langle D\rangle$$

Now suppose the writhe decreases by one, creating a diagram D''. Then

$$(-A)^{-3w(D'')}\langle D''\rangle = (-A)^{-3w(D)+3}(-A^{-3}\langle D\rangle)$$
$$= -A^{3}(-A)^{-3}(-A)^{-3w(D)}\langle D\rangle = (-A)^{-3w(D)}\langle D\rangle$$

Therefore, $(-A)^{-3w(D)}\langle D\rangle$ is invariant under Reidemeister moves, so it is a knot invariant. $\square$

Definition 4.6. The **Jones polynomial** $V(K)$ of an oriented knot K is defined as

$$V(K) = (-A)^{-3w(D)}\langle D\rangle \in \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$$

where $t^{\frac{1}{2}} = A^{-2}$ and D is an oriented diagram for K.

Note that this still gives $V\left(\bigcirc\right) = 1$. Additionally, it can be seen that the Jones polynomial for knots, not links, is independent of the knot's orientation since if every component's orientation changes, then each crossing's sign remains unchanged. This works for knots since knots only have one component. Another characterization of the Jones polynomial is given below.

Theorem 4.7. *The Jones polynomial invariant is a function*

$$V : \{\textit{Oriented knots in } S^3\} \to \mathbb{Z}[t^{-\frac{1}{2}}, t^{\frac{1}{2}}]$$

such that $V(\bigcirc) = 1$ *and when given three oriented links* L_+, L_-, L_0 *that are identical except in the neighborhood of a crossing as shown below, then the following relation holds:* $t^{-1}V(L_+) - tV(L_-) + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})V(L_0) = 0$ *or*

$$t^{-1}V\left(\times\right) - tV\left(\times\right) + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})\Delta\left(\uparrow\uparrow\right) = 0$$

where the knots are assumed to be identical outside of the crossings considered here. This type of relation is called a ***skein relation.*** *Additionally, the Jones polynomial is uniquely determined by these two conditions.*

Proof. $\langle \times \rangle = A\langle \asymp \rangle + A^{-1}\langle)(\rangle$ and $\langle \times \rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle$. Multiplying the first equation by A^{-1} and the second by A, then subtracting the first from the second gives

$$A\langle \times \rangle - A\langle \times \rangle = (A^2 - A^{-2})\langle)(\rangle$$

Since these are oriented links, $w(L_+) - 1 = w(L_0) = w(L_-) + 1$ then

$$-A^4V(L_+) + A^{-4}V(L_-) - (A^2 - A^{-2})V(L_0) = 0$$

since

$$-A^4V(L_+) + A^{-4}V(L_-) - (A^2 - A^{-2})V(L_0)$$
$$= -A^4(-A)^{-3w(L_+)}\langle L_+\rangle + A^{-4}(-A)^{-3w(L_-)}\langle L_-\rangle - (A^2 - A^{-2})(-A)^{-3w(L_0)}\langle L_0\rangle$$
$$= -A^4(-A)^{-3w(L)-3}(-A^{-3})\langle L_0\rangle + A^{-4}(-A)^{-3w(L)+3}(-A^3)\langle L_0\rangle - (A^2 - A^{-2})(-A)^{-3w(L_0)}\langle L_0\rangle$$
$$= -A^4(-A^{-3})(-A^{-3})(-A)^{-3w(L)}\langle L_0\rangle + A^{-4}(-A)^3(-A^3)(-A)^{-3w(L)}\langle L_0\rangle - (A^2 - A^{-2})(-A)^{-3w(L_0)}\langle L_0\rangle$$
$$= -A^{-2}(-A)^{-3w(L)}\langle L_0\rangle + A^2(-A)^{-3w(L)}\langle L_0\rangle - (A^2 - A^{-2})(-A)^{-3w(L_0)}\langle L_0\rangle$$
$$(-A^{-2} + A^2 - A^2 + A^{-2})(-A)^{-3w(L_0)}\langle L_0\rangle = 0$$

Substituting in $A^{-2} = t^{\frac{1}{2}}$ gives us the desired result:

$$-A^4V(L_+) + A^{-4}V(L_-) - (A^2 - A^{-2})V(L_0) = -t^{-1}V(L_+) + tV(L_-) - (t^{-\frac{1}{2}} - t^{\frac{1}{2}})V(L_0)$$
$$= t^{-1}V(L_+) - tV(L_-) + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})V(L_0) = 0$$

Since the Jones polynomial, as we have originally defined it, satisfies this relation, uniqueness implies that the two are equal. Therefore, it suffices to prove uniqueness, which we will now do.

Proof by induction. Suppose a diagram D has no crossings. If D has m trivial components, then $V(D) = (t^{1/2} + t^{-1/2})^{m-1}$ which can be shown by inducting on m and using the skein relation. Thus, the base case holds.

Now assume that for any diagram D with crossing number less than $n-1$, $V(D)$ exists and is unique. Consider some diagram D with n crossings, and choose one of these crossings. When this crossing is positive denote it D_+, when it is negative denote it D_-, and when the crossing is removed denote it D_0. By the skein relation,

$$t^{-1}V(D_+) - tV(D_-) + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})V(D_0) = 0$$

$V(D_0)$ is known by the induction assumption. Therefore, if we can show either of the other two terms is uniquely determined by these axioms, then the other is as well.

To do this requires another induction. Thus, fix n. Let L be a diagram of n crossings with unknotting number $k \leq n$. We will induct on k to show $V(L)$

is uniquely determined. Suppose $k = 0$. This is same base case as previously considered, so it holds.

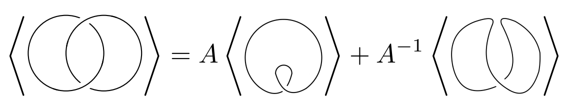
Now assume it holds true for $k-1$. Consider some diagram with n crossings and unknotting number k. There is a crossing where changing said crossing creates a knot diagram with unknotting number $k-1$. Then when this crossing is positive denote it L_+, when it is negative denote it L_-, and when the crossing is removed denote it L_0. By the skein relation,

$$t^{-1}V(L_+) - tV(L_-) + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})V(L_0) = 0$$

$V(L_0)$ is known by the induction assumption. Therefore, if we can show either of the other two terms is uniquely determined by these axioms, then the other is as well. However, either L_+ or L_- must have unknotting number $k-1$. Therefore, by the induction assumption, whichever has unknotting number $k-1$ is determined uniquely by V. Thus, this concludes the induction on k. Therefore, either D_+ or D_- is uniquely determined since at least one of them has at most n crossings and therefore has unknotting number $\leq n$. Thus, the term is uniquely determined as well, and the theorem is proven. $\square$

Note how unknotting number factors into the proof of the Jones polynomial's uniqueness. While it may not be easy to calculate, that does not prevent it from being useful.

To finish this section, we will explicitly calculate the bracket and Jones polynomial of the Hopf link.

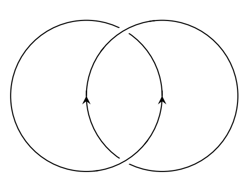


Looking at these diagrams, we notice that the left strand is the overstrand on the first and the right strand is the overstrand on the second (this can be seen by rotating the image). Therefore,

$$= A\left(-A^3 \langle \bigcirc \rangle\right) + A^{-1}\left(-A^{-3} \langle \bigcirc \rangle\right) = -A^4 - A^{-4}$$

Thus, the bracket polynomial of the Hopf link is $-A^4 - A^{-4}$.

Now, to calculate the Jones polynomial. There are two possible orientations on the Hopf link. The first is shown below.



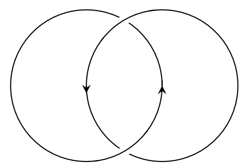
Its top crossing is of the form , so we assign it $+1$. The bottom crossing is the same, so it is also a $+1$. Thus, let this oriented Hopf link be K with diagram D,

$$V(K) = (-A)^{-3w(D)}\langle D \rangle = (-A)^{-3(2)}(-A^4 - A^{-4}) = A^{-6}(-A^4 - A^{-4}) = -A^{-2} - A^{-10}$$

Substituting in $A^{-2} = t^{\frac{1}{2}}$,

$$= -A^{-2} - (A^{-2})^5 = -t^{\frac{1}{2}} - (t^{\frac{1}{2}})^5 = -t^{\frac{1}{2}} - t^{\frac{5}{2}}$$

Now we will calculate it for the other orientation:



Its top crossing is of the form , so we assign it -1. The bottom crossing is the same, so it is also a -1. Thus, let this oriented Hopf link be K' with diagram D',

$$V(K') = (-A)^{-3w(D')}\langle D'\rangle = (-A)^{-3(-2)}(-A^4 - A^{-4}) = A^6(-A^4 - A^{-4}) = -A^{10} - A^2$$

Substituting in $A^{-2} = t^{\frac{1}{2}}$,

$$= -A^{10} - (A^2)^5 = -t^{-\frac{1}{2}} - (t^{-\frac{1}{2}})^5 = -t^{-\frac{1}{2}} - t^{-\frac{5}{2}}$$

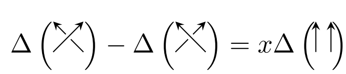
With this, we have a concrete example of orientation affecting the Jones polynomial. Note that the only difference between the two polynomials is the sign of the exponents.

5. Alexander-Conway Polynomial

The Alexander-Conway polynomial will provide many of the same advantages of the Jones polynomial as looked at before. Unlike the Jones polynomial, it is generally formulated in terms of homology theory. This is where the real strength of the Alexander-Conway polynomial is in comparison to the Jones. However, in the interest of accessibility, we will look at John Conway's reformulation, which provides an approach similar to that of the Jones polynomial and which is easier to digest for those with less mathematical experience[2]. Like the Jones polynomial, it may be defined in terms of a skein relation.

Definition 5.1. The **Alexander-Conway polynomial** is an assignment $\Delta : \mathcal{D} \to \mathbb{Z}[x]$, where $\mathcal{D}$ is the set of oriented diagrams, such that the following hold:

(1) $\Delta(D) = \Delta(D')$ if $D \approx D'$;
(2) $\Delta\left(\bigcirc\right) = 1$;
(3) The Conway skein relation holds: $\Delta(L_+) - \Delta(L_-) = x\Delta(L_0)$, or



where the knots are assumed to be identical outside of the crossings considered here.

[2] Information on the homology theory can be found in [Lic97], Chapter 6

By the first axiom, it is a knot invariant. Less obviously, if an extra unknot is added to a given diagram, the Alexander-Conway polynomial is then zero. To see this, consider some diagram D and a unknot. If we connect these by attaching two straight lines between them, we can apply a Type I Reidemeister move. Then, by Conway skein relation, we get the following:

$$x\Delta\left(D \cup \bigcirc\right) = \Delta\left(\;\right) - \Delta\left(\;\right) = \Delta(D) - \Delta(D) = 0$$

by the first axiom (note that the line diagrams represent the way they are connected). Therefore, $\Delta\left(D \cup \bigcirc\right) = 0$. This is a stark difference between the bracket and Jones polynomials.

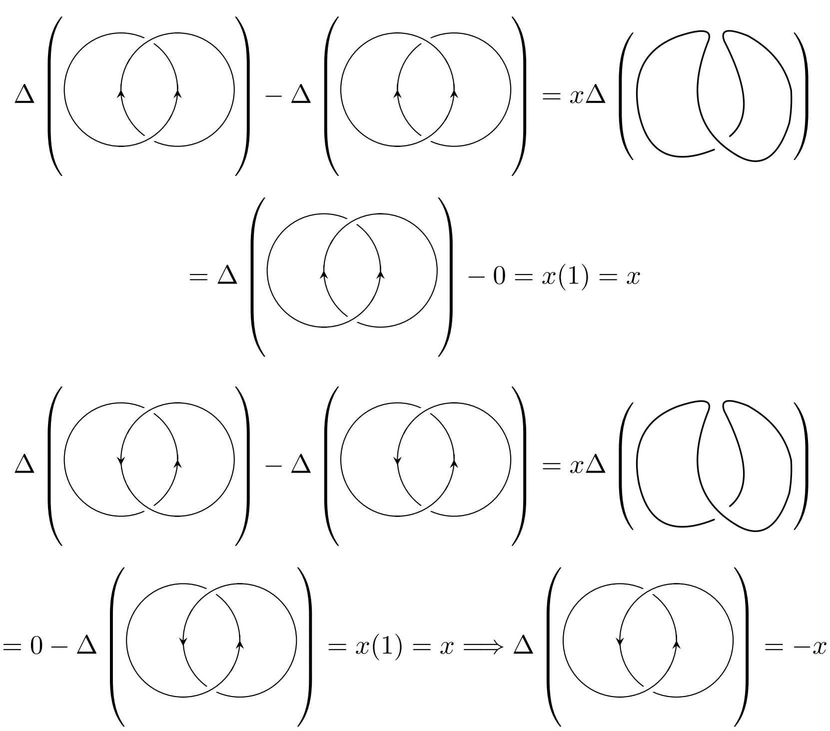
This also has implications beyond just adding trivial components. If a knot or link can be separated into two connected sections, then its Alexander-Conway polynomial is automatically 0 by the same argument given above. These types of knots are called **composite knots**.

Theorem 5.2. *The Alexander-Conway polynomial is unique.*

Proof. The proof follows from the same argument as Theorem 4.7 with the modification of the skein relation for the Alexander-Conway polynomial.

□

As we did before with the Jones polynomial, we will calculate the Alexander-Conway polynomial for the Hopf link with both orientations.



Like with the Jones polynomial, switching the orientation affects the Alexander-Conway polynomial. But, in this case, it affects the sign of the polynomial instead of the sign of the polynomial's exponents.

6. HOMFLY Polynomial

The final polynomial we will look at is the HOMFLY polynomial. Unlike the previous two, it is a polynomial in two variables, and is a generalization of both the Alexander-Conway and Jones polynomials, making it better at distinguishing knots than either of those polynomials.

Unlike the previous two sections, we will not prove the HOMFLY polynomial's uniqueness, as the proof is quite extensive. For those interested, a proof can be found in [Lic97], Theorem 15.2. Rather, this section will focus on the computation of the HOMFLY polynomial.

Definition 6.1. The **HOMFLY polynomial** is a function $P : \{\text{Oriented knots in } S^3\} \to \mathbb{Z}[m, m^{-1}, l, l^{-1}]$ such that P satsifies the following conditions:

(1) $P(D) = P(D')$ if $D \approx D'$;
(2) $P\left(\bigcirc\right) = 1$;
(3) The following skein relation holds: $lP(L_+) + l^{-1}P(L_-) + mP(L_0) = 0$, or

$$lP\left(\nwarrow\!\!\!\!\!\!\nearrow\right) + l^{-1}P\left(\nwarrow\!\!\!\!\!\!\nearrow\right) + mP\left(\uparrow\uparrow\right) = 0$$

where the knots are assumed to be identical outside of the crossings considered here.

Like the Alexander-Conway polynomial, by the first axiom it is a knot invariant. Unlike the Alexander-Conway polynomial, if an extra unknot is added to a given diagram, the polynomial does not become zero. To see this, consider some diagram D and an unknot. If we connect these by attaching two straight lines between them, we can apply a Type I Reidemeister move.

$$lP\left(\nwarrow\!\!\!\!\!\!\nearrow\right) + l^{-1}P\left(\nwarrow\!\!\!\!\!\!\nearrow\right) + mP\left(D \cup \bigcirc\right) = lP\left(D\right) + l^{-1}P\left(D\right) + mP\left(D \cup \bigcirc\right) = 0$$

$$P\left(D \cup \bigcirc\right) = -(l + l^{-1})m^{-1}P(D) \tag{6.2}$$

We will now use this fact to prove the following statement:

Proposition 6.3. *$P(L_1 \cup L_2) = -(l + l^{-1})m^{-1}P(L_1)P(L_2)$ where L_1, L_2 are oriented knots.*

Proof. Apply the skein relation to L_2, and only L_2, to resolve it into various unknots (as we have done before with the Hopf link). Once this is completed, due to the factors introduced from the skein relation, we have $P(L_1 \cup L_2) = P(L_1 \cup \bigcirc)P(L_2)$. Then, by Equation 6.2, $P(L_1 \cup L_2) = -(l + l^{-1})m^{-1}P(L_1)P(L_2)$. □

To show another interesting property of the HOMFLY polynomial, we must first introduce a new operation on knots.

Definition 6.4. A **composition** of two knots is created by removing a small arc from each projection and connecting the endpoints on each knot such that the arcs do not cross. For knots J and K, this is denoted $J\#K$.

Note that this is what we have been doing in the skein relation when adding an extra unknot.

Theorem 6.5. *$P(L_1\#L_2) = P(L_1)P(L_2)$ where L_1, L_2 are oriented knots.*

Proof. Let $L_0 = L_1 \cup L_2$, $L_+ = L_1 \# L_2$ but L_2 flipped such that it introduces a right-hand crossing between them, and $L_- = L_1 \# L_2$ but L_2 flipped such that it introduces a left-hand crossing between them. Note that $L_+ \approx L_-$. Then

$$\begin{aligned} lP(L_+) + l^{-1}P(L_-) + mP(L_0) &= lP(L_1\#L_2) + l^{-1}P(L_1\#L_2) + mP(L_1 \cup L_2) \\ &= (l + l^{-1})P(L_1\#L_2) + m(-(l + l^{-1})m^{-1}P(L_1)P(L_2) \\ = (l + l^{-1})P(L_1\#L_2) - (l + l^{-1})P(L_1)P(L_2) &= 0 \Rightarrow P(L_1\#L_2) = P(L_1)P(L_2) \end{aligned}$$

$\square$

Therefore, the HOMFLY polynomial does not include the limitations that the Alexander-Conway imposes on composite knots, and this rule makes the computation of their HOMFLY polynomial much simpler.

At the beginning of this section, it was mentioned that the HOMFLY polynomial is a generalization of the Alexander-Conway and Jones polynomials. How is this the case? For the Alexander-Conway, consider $l = -i$ and $m = ix$. Apply this just to the skein relation first. Then

$$\begin{aligned} lP(L_+) + l^{-1}P(L_-) + mP(L_0) &= (-i)P(L_+) + (-i)^{-1}P(L_-) + ixP(L_0) = 0 \\ \Rightarrow -i^2P(L_+) + i(-i)^{-1}P(L_-) + i^2xP(L_0) &= P(L_+) - P(L_-) - xP(L_0) = 0 \\ \Rightarrow P(L_+) - P(L_-) = xP(L_0) &\Rightarrow \Delta(L_+) - \Delta(L_-) = x\Delta(L_0) \end{aligned}$$

This gives the same skein relation as the Alexander-Conway polynomial. Likewise, consider $l = it^{-1}$ and $m = i(t^{-1/2} - t^{1/2})$. Apply this just to the skein relation first. Then

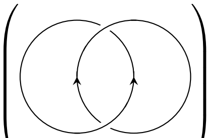
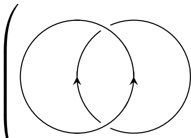
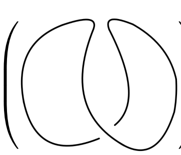
$$lP(L_+)+l^{-1}P(L_-)+mP(L_0) = it^{-1}P(L_+)+(it^{-1})^{-1}P(L_-)+(i(t^{-1/2}-t^{1/2}))P(L_0) = 0$$

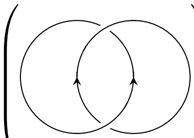
$$it^{-1}P(L_+)-itP(L_-)+i(t^{-1/2}-t^{1/2})P(L_0) = 0 \Rightarrow -i^2t^{-1}P(L_+)+i^2tP(L_-)-i^2(t^{-1/2}-t^{1/2})P(L_0)$$

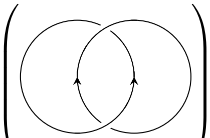
$$= t^{-1}P(L_+)-tP(L_-)+(t^{-1/2}-t^{1/2})P(L_0) = 0 \Rightarrow t^{-1}V(L_+)-tV(L_-)+(t^{-1/2}-t^{1/2})V(L_0) = 0$$

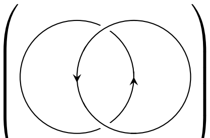
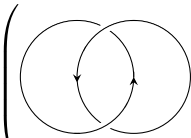
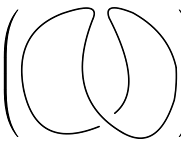
This gives the same skein relation as the Jones polynomial. Thus, both are a special case of the HOMFLY.

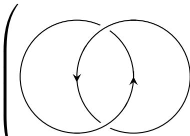
Finally, as we have done with the previous two polynomials, we will calculate the HOMFLY polynomial of the Hopf link.

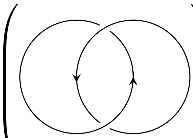
$lP\Bigg($  $\Bigg) + l^{-1}P\Bigg($  $\Bigg) + m\Delta\Bigg($  $\Bigg) = 0$

$= lP\Bigg($  $\Bigg) - l^{-1}(-(l + l^{-1})m^{-1}) + m = 0$

$P\Bigg($  $\Bigg) = l^{-1}(l^{-1}(-(l + l^{-1})m^{-1}) - m) = -(l^{-3} + l^{-1})m^{-1} - l^{-1}m$

$lP\Big($  $\Big) + l^{-1}P\Big($  $\Big) + m\Delta\Big($  $\Big) = 0$

$= l(-(l+l^{-1})m^{-1}) - l^{-1}P\Big($  $\Big) + m = 0$

$P\Big($  $\Big) = l(l(l+l^{-1})m^{-1}) - m) = (l^3+l)m^{-1} - lm$

Note how the exponents change sign on the l variables and the sign changes on the m variables. This is like how changing orientation affected both the Jones and Alexander-Conway polynomials.

7. Conclusion

To close, there are more knot polynomials that were not included in this paper, such as the Kauffman polynomial. Additionally, there is much more to explored with these polynomials than can realistically be fit into a short paper.

For those interested in going further in the topic, I would recommend one of three books: for a lighter introduction, I would recommend [Ada94]; for someone more experienced, [Mur96]; and for those with significant experience, particularly at a graduate level, [Lic97].

Acknowledgments

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