SINGULARITIES, MILNOR FIBRATIONS, AND VANISHING CYCLES

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ABSTRACT. An algebraic variety is the solution set to a system of polynomial equations. The study of algebraic varieties is considerably complicated by the fact that varieties are not, in general, smooth manifolds; indeed, the equation $xy = 0$ in $\mathbb{R}^2$ defines an algebraic variety which is evidently not a manifold. In this paper, we explain tools for understanding the topology of algebraic varieties, focusing on the locally cone-like structure of varieties, and on the Milnor fibration.

CONTENTS

1. INTRODUCTION

Algebraic geometry is the study of *algebraic varieties*, which are solution sets to systems of polynomial equations (in much the same way that linear algebra is the study of solution sets to systems of *linear* equations).

What do algebraic varieties look like?

For instance, consider the complex algebraic variety

$$V = \{(x, y) \in \mathbb{C}^2 \mid y^2 = x^3\}.$$

The classical implicit function theorem can be used to show that V is *mostly* a manifold. Specifically, $V \setminus \{0, 0\}$ is a 1-dimensional complex manifold (that is, it is locally biholomorphic to the unit ball $U = \{z \in \mathbb{C} \mid |z| < 1\}$).

What is the topology of V near $(0, 0)$, though? Brauner (cf. [2], section 1) made the following remarkable observation.

Definition 1.1. Let X be a topological space. The *cone on* X, which we will denote by $\mathrm{Cone}(X)$, is the space

$$\mathrm{Cone}(X) = (X \times [0, 1])/(X \times \{0\})$$

Date: August 15, 2023.

obtained by taking a cylinder on X and then crushing one face of it to a point; this point is called the *cone point* of $\mathrm{Cone}(X)$.

Definition 1.2. A *triple* of topological spaces is a triple (X, Y, Z) where $Z \subseteq Y \subseteq X$. A *homeomorphism of triples*

$$\phi : (X, Y, Z) \to (X', Y', Z')$$

is a homeomorphism $\phi : X \to X'$ so that $\phi|_Y$ is a homeomorphism between Y and Y', and $\phi|_Z$ is a homeomorphism between Z and Z'.

Theorem 1.3 (Brauner)**.** *Let* $\epsilon > 0$, *and let*

$$S_\epsilon = \{z \in \mathbb{C} \mid |z| = \epsilon\},$$

$$D_\epsilon = \{z \in \mathbb{C} \mid |z| \leq \epsilon\}$$

be the radius ϵ *sphere and closed ball, respectively.*

Let $K \subseteq S^3$ *denote a trefoil knot embedded in* S^3. *The map* $K \hookrightarrow S^3$ *induces a natural inclusion* $\mathrm{Cone}(K) \hookrightarrow \mathrm{Cone}(S^3)$. *Let* $*$ *denote the cone point of* $\mathrm{Cone}(S^3)$.

For all sufficiently small ϵ, *there is a homeomorphism of triples*

$$(D_\epsilon, D_\epsilon \cap V, 0) \cong (\mathrm{Cone}(S^3), \mathrm{Cone}(K), *).$$

Proof. See [2], the unnumbered assertion on page 4. But if the reader recalls that a trefoil knot is also a $(2, 3)$-torus knot, it is not so hard to directly prove this. □

In 1968, Milnor published a vast generalization of Theorem 1.3.

Milnor's setting is as follows. Take $V \subseteq \mathbb{C}^{n+1}$ any *complex* algebraic variety; that is, some solution set to a system of polynomial equations over the complex numbers.

Milnor was interested in studying the local topology of V. Specifically, let $p \in V$ be a point, and $U \ni p$ some neighborhood of p in $\mathbb{C}^{n+1}$. Milnor was interested in studying the homeomorphism type of the triple $(U, U \cap V, p)$, just as Brauner did in the case where V was given by $y^2 = x^3$.

The implicit function theorem can be used on the system of polynomial equations defining V to prove that V is mostly a manifold, in the sense that for almost any point p of V, every sufficiently small neighborhood U of p is such that $(U, U \cap V, p) \cong (\mathbb{R}^{2n+2}, \mathbb{R}^d, 0)$, for some integer $d \leq 2n + 2$.

Definition 1.4. A *singularity* of a complex algebraic variety $V \subseteq \mathbb{C}^{n+1}$ is a point p of V so that, for U any small neighborhood of p in $\mathbb{C}^{n+1}$, the intersection $U \cap V$ is *not* a (smooth) submanifold of $\mathbb{C}^{n+1}$.

We write $\Sigma(V)$ for the set of all singularities of V.

Remark 1.5. This definition of singularity is a little different than the one given in a standard course, however over $\mathbb{C}$ our definition is equivalent to the standard one, cf. [2], page 13.

Example 1.6. If V is the variety $y^2 = x^3$ of Theorem 1.3, then

$$\Sigma(V) = \{(0, 0)\}.$$

Theorem 1.3 showed that, despite the origin being a singularity of the variety $y^2 = x^3$, we could still quite concretely describe the topology of that variety near the origin. Quite miraculously, Milnor was able to show that the phenomenon observed in Theorem 1.3 is just the starting point of a very deep story connecting knot theory and algebraic geometry.

In the course of this paper, for *proofs* we will mostly follow Milnor, and study *isolated singularities*. However, there is still an incredibly rich theory for non-isolated singularities, and we will state (mostly without proof) the results in the general case. Our hope is to motivate the general statements by comparison with the special case of isolated singularities.

Acknowledgments

My mentor, Sasha Beilinson, was incredibly helpful throughout the summer: he shared some of his immense mathematical insight with me, gave suggestions on sources and on how to write mathematics well, and gave me moral support. I am very grateful to have been his mentee this summer.

This project could not be possible without the help of Matt Emerton, who first introduced me to so many of the ideas implicit in this paper.

Throughout the summer, I was very lucky to have gotten to work with a number of very kind students – in particular I must thank Matthew Niemiro for many helpful conversations.

Lastly, I must thank Peter May, who organized this summer program, and provided useful mathematical guidance to me and others throughout the summer.

2. The topology of a variety near a singular point

We start with a very general result describing the local topology of algebraic varieties.

Theorem 2.1. *Suppose V is a real or complex affine variety (meaning V is the solution set to a system of polynomial equations in n real or complex variables). Take $x_0 \in V$ an* isolated *point of $\Sigma(V)$, and let $D_\epsilon(x_0)$ denote the* closed *ball of radius ϵ at x_0.*

For every sufficiently small ϵ, the intersection $V \cap D_\epsilon(x_0)$ is homeomorphic to the cone over

$$K = V \cap S_\epsilon(x_0).$$

In fact, there is a homeomorphism of triples

$$(D_\epsilon(x_0), V \cap D_\epsilon(x_0), x_0) \cong (\mathrm{Cone}(D_\epsilon(x_0)), \mathrm{Cone}(K), *), \tag{2.2}$$

for $*$ the cone point of $C(D_\epsilon(x_0))$.

The topological space K will be a manifold. The homeomorphism (2.2) can be chosen such that it becomes a diffeomorphism when you remove the cone point of $\mathrm{Cone}(K)$ and the point x_0 from $V \cap D_\epsilon(x_0)$.

This section will be dedicated to a proof of Theorem 2.1. Before we can state it, we make two remarks.

Firstly, Theorem 2.1 only applies to isolated singularities. However, there is a generalization of Theorem 2.1 to arbitrary varieties; this is called the theory of *Whitney stratifications*, which we explore later.

Secondly, Theorem 2.1 tells us nothing about the topology of the set K. The rest of this paper will contain a more detailed study of K.

As complex varieties are also real varieties of double the dimension, it suffices to prove the result in the real case of Theorem 2.1. We now start our proof of Theorem 2.1, assuming V is a real variety, and x_0 is some isolated singularity.

Notation 2.3. For convenience, we set $M_1 := V \setminus \Sigma(V)$ the set of all nonsingular points in V.

2.1. **Step 1: topological preliminaries.** We start by showing that K is a manifold for small $\epsilon > 0$.

Lemma 2.4. *Every sufficiently small sphere $S_\epsilon(x_0)$ intersects V in a smooth manifold (or the empty set).*

Furthermore, the intersection is transverse*: every vector based at a point on $V \cap S_\epsilon$ can be written (perhaps not uniquely) as a sum of a vector tangent to V and a vector tangent to S_ϵ.*

Proof. Define

$$g(x) = ||x - x_0||^2.$$

The map g is a polynomial in x (which is why we use squared distance instead of distance); it is a classical, but slightly technical, fact that any polynomial on the smooth locus $M_1 = V \setminus \Sigma(V)$ of V has finitely many critical points; see lemma 2.7 of [2].

As there are only finitely many critical points, for sufficiently small ϵ, the map $g|_{M_1 \cap D_\epsilon(x_0)}$ will have no critical points—except possibly for a critical point at x_0—as we may just shrink ϵ until it is smaller than the distance between x_0 and the closest critical point to x_0. The regular value theorem of differential topology then tells us that

$$(g|_{M_1 \cap D_\epsilon(x_0)})^{-1}(\epsilon) = S_\epsilon(x_0) \cap M_1$$

will be a smooth manifold of codimension 1 in V. But recall that x_0 is an isolated singularity, and so for ϵ sufficiently small, we have

$$S_\epsilon(x_0) \cap M_1 = S_\epsilon(x_0) \cap V,$$

as there will be no singularities of V located within ϵ of x_0 except for x_0 itself. In particular,

$$K = S_\epsilon(x_0) \cap V = S_\epsilon(x_0) \cap M_1 = (g|_{M_1 \cap D_\epsilon(x_0)})^{-1}(\epsilon)$$

is a smooth manifold of codimension 1 in V, as desired.

Transversality is equivalent to the claim that, for any point $p \in K = S_\epsilon(x_0) \cap V$, we have

$$T_p S_\epsilon(x_0) + T_p V = T_p \mathbb{R}^n. \tag{2.5}$$

Since $T_p S_\epsilon(x_0)$ has codimension 1 in $\mathbb{R}^n$ (as is immediate from directly computing the tangent space of a sphere!), the only way the sum in (2.5) can fail to span $T_p\mathbb{R}^n$ is if $T_p V \subseteq T_p S_\epsilon(x_0)$.

At the start of the proof of this lemma, we chose ϵ small enough so that p is a regular point of the map

$$r : M_1 \cap D_\epsilon(x_0) \to \mathbb{R}.$$

In particular, the differential of r at p is rank 1. In other words, there is some tangent vector

$$v \in T_p(M_1 \cap D_\epsilon(x_0))$$

so that moving along v causes a change in r; in particular, $v \notin T_p S_\epsilon(x_0)$, since $S_\epsilon(x_0)$ is the level set of r. But, as $M_1 \subseteq V$, we find that

$$T_p(M_1 \cap D_\epsilon(x_0)) \subseteq T_p V,$$

and so $v \in T_p V$ is a tangent vector not contained in $T_p S_\epsilon(x_0)$. Thus $T_p V \not\subseteq T_p S_\epsilon(x_0)$, which as explained above implies

$$T_p S_\epsilon(x_0) + T_p V = \mathbb{R}^n,$$

our desired transversality. □

So, we restrict attention to ϵ sufficiently small so that K is a smooth manifold and $D_\epsilon(x_0)$ contains no singular points of V except x_0.

2.2. **Step 2: constructing a vector field.** The rest of our proof will use a strategy common in Riemannian geometry: to construct the homeomorphism posited by Theorem 2.1, we will construct a certain vector field on $D_\epsilon(x_0)$, and then flow along it.

Definition 2.6. If M is a manifold and $v : M \to TM$ is a vector field on M, then an *integral curve* of v is a curve $c : I \to M$, for some time interval $I \subseteq \mathbb{R}$, so that $c'(t) = v(c(t))$. The *flow* generated by v is the map

$$\Phi : M \times \mathbb{R} \to M$$

generated by sending (p, t) to $c_p(t)$, where $c_p : \mathbb{R} \to M$ is the integral curve obeying $c_p(0) = p$.

We remark that the flow of a vector field does not always exist, since in general solutions to ODEs exist only in a small time interval.

We start by constructing a vector field $\xi : D_\epsilon(x_0) \setminus \{x_0\} \to \mathbb{R}^n$ on the punctured disk $D_\epsilon(x_0) - \{x_0\}$ which has two properties:

(1) for every $x \in D_\epsilon(x_0) \setminus \{x_0\}$, we have $\langle \xi(x), x - x_0 \rangle > 0$,
(2) and $\xi(x)$ always lies tangent to M_1 (recall Notation 2.3) whenever $x \in M_1$.

The first property is used to ensure that flowing along an integral curve of ξ will always get you closer to x_0, and the second property is used to ensure that an integral curve which originates from a point of M_1 will never leave M_1.

To define ξ, we construct vector fields obeying these two conditions locally, and then use a partition of unity to patch the resulting vector fields together. For each $p \in D_\epsilon(x_0) \setminus \{x_0\}$, we will define an open set $U^p \subseteq D_\epsilon(x_0) \setminus \{x_0\}$ and a vector field $\xi^p : U^p \to \mathbb{R}^n$ on U^p as follows.

If $p \notin V$, then just define $U^p = D_\epsilon(x_0) \setminus V$, and let

$$\xi^p(x) = x - x_0$$

This vector field ξ^p will clearly obey our first desired property, since

$$\langle \xi^p(x), x - x_0 \rangle = ||x - x_0||^2 > 0,$$

and it will obey our second desired property vacuously since $U^p \cap M_1 = \emptyset$.

If $p \in V$, then since ϵ is small and $p \in D_\epsilon(x_0) \setminus \{x_0\}$, we have that $p \in M_1 = V \setminus \Sigma(V)$. Choose a chart $u : U^p \to \mathbb{R}^n$ of $\mathbb{R}^n$ so that the submanifold M_1 is cut out by the locus $u_1 = \cdots = u_\rho = 0$, where ρ is the codimension of M_1.

Consider the map

$$r : M_1 \to \mathbb{R},$$
$$r(x) = ||x - x_0||^2,$$

as in the proof of Lemma 2.4 from the previous step of this argument. As explained in the proof of Lemma 2.4, for sufficiently small ϵ, the point $p \neq x_0$ will not be a critical point of r, as the only critical point of r on $M_1 \cap D_\epsilon(x_0)$ is x_0 itself.

We thus find that at least one of the partials $\partial r/\partial u_{\rho+1}, ..., \partial r/\partial u_n$ is nonzero at p.

Say $\partial r/\partial u_h$ is not zero at p. Shrinking U^p if needed, we can assume $\partial r/\partial u_h$ is never 0 on U^p, and so is either always positive or always negative on U^p. Define

$$\xi^p(x) = (-1)^s \left(\frac{\partial x_1}{\partial u_h}, ..., \frac{\partial x_n}{\partial u_h} \right),$$

where $x_i : \mathbb{R}^n \to \mathbb{R}$ are the standard coordinate functions, and we differentiate them with respect to the component u_h of the chart u chosen above. Here, the sign $(-1)^s$ is chosen to be negative if $\partial r/\partial u_h < 0$ on U^p, and positive if $\partial r/\partial u_h > 0$ on U^p.

We then have

$$\begin{aligned}
2\left\langle \xi^p(x), x - x_0 \right\rangle &= (-1)^s \sum_{i=1}^n 2(x^i - x_0^i)\frac{\partial x_i}{\partial u_h}(x) \\
&= (-1)^s \sum_{i=1}^n \frac{\partial r}{\partial x_i}\frac{\partial x_i}{\partial u_h} \\
&= (-1)^s \frac{\partial r}{\partial u_h} > 0,
\end{aligned}$$

since the sign $(-1)^s$ was chosen to make $(-1)^s \partial r / \partial u_h$ positive on U^p. This verifies the first desired property of our vector field.

If $x \in M_1 \cap U^p$, then we wish to check that $\xi^p(x)$ is tangent to M_1. In the chart u, the submanifold M_1 is described by $u_1 = \cdots = u_\rho = 0$. As $\xi^p(x) = \partial/\partial u_h$ at x, and as $h > \rho$, we find that $\xi^p(x)$ is clearly tangent to M_1. Thus ξ^p obeys both desired properties of our vector field.

Choose a partition of unity ρ^p of $D_\epsilon(x_0)$ subordinate to the open cover U^p. We can therefore blend all these ξ^p to get a vector field

$$\xi : D_\epsilon(x_0) \setminus \{x_0\} \to \mathbb{R}^n,$$

$$\xi = \sum_p \rho^p \xi^p,$$

which obeys our two desired properties: it has $\langle \xi(x), x - x_0 \rangle > 0$, and is tangent to M_1. The first property holds since each ρ^p is non-negative, and each ξ^p obeys $\langle \xi^p(x), x - x_0 \rangle > 0$; the second property holds since a linear combination of vectors tangent to M_1 is still tangent to M_1.

To make the explicit computations below slightly easier, we define

$$\nu(x) = \frac{\xi(x)}{2\left\langle \xi(x), x - x_0 \right\rangle}$$

a normalized version of ξ. The quantity in the denominator is never 0, as we verified above that $\langle \xi(x), x - x_0 \rangle > 0$.

2.3. **Step 3: flowing along our vector field.** Armed with a vector field, we will flow along its integral curves. However, integral curves in general only exist locally; we prove that the integral curves of ν exist globally.

Consider the ODE $p'(t) = \nu(p(t))$. Any solution p of this ODE obeys

$$\begin{aligned}
\frac{d}{dt} r(p(t)) &= \frac{d}{dt}\left(\sum_{i=1}^n (p(t)^i - x_0^i)^2\right) \\
&= \sum_{i=1}^n 2p'(t)^i (p(t)^i - x_0^i) \\
&= 2\left\langle p'(t), p(t) - x_0 \right\rangle \\
&= 2\left\langle \nu(p(t)), p(t) - x_0 \right\rangle \\
&= \frac{\langle \xi(p(t)), p(t) - x_0 \rangle}{\langle \xi(p(t)), p(t) - x_0 \rangle} \\
&= 1.
\end{aligned}$$

Thus $r(p(t)) = t+C$. Subtracting a constant from our time parameter (this corresponds to just shifting the interval parametrizing t), we may asssume that $r(p(t)) = t$. In particular, $p(\epsilon^2) \in S_\epsilon(x_0)$.

Lemma 2.7. *Let $a \in S_\epsilon(x_0)$. There is a unique function $p : (0, \epsilon^2] \to D_\epsilon(x_0) \setminus \{x_0\}$ solving the ODE $p'(t) = \nu(p(t))$ with initial condition $p(\epsilon^2) = a$.*

Proof. By the theory of existence of ODEs, there is some unique solution p with our given initial condition defined in $(\alpha, \epsilon^2]$, for some $\alpha > 0$.

We will show that any solution p defined over $(\alpha, \epsilon^2]$ can be extended to $(0, \epsilon^2]$. Indeed, let α be minimal so that our initial value problem $p'(t) = \nu(p(t)), p(\epsilon^2) = a$ has a solution defined over $(\alpha, \epsilon^2]$. Assume for the sake of contradiction that $\alpha > 0$.

As the set $D_\epsilon(x_0)$ is compact, there is some sequence of times $t_n \to \alpha$ so that $p(t_n)$ converges to a point $x' \in D_\epsilon(x_0)$. By continuity of r, this point x' has (recalling that $r(p(t)) = t$ from above)

$$r(x') = \lim_{n\to\infty} r(p(t_n)) = \lim_{n\to\infty} t_n = \alpha > 0.$$

By the existence and uniqueness theorem for ODEs, there is some $\delta > 0$ so that for every x'' within δ of x' and every sufficiently large n, there is a unique solution $q_{x'',n} : (t_n - \delta, t_n + \delta) \to D_\epsilon \setminus \{x_0\}$ to our ODE with initial condition

$$q_{x'',n}(t_n) = x''.$$

Let $q_n := q_{p(t_n),n}$. Then $p = q_n$ on their common domain of definition, by uniqueness of solutions of ODEs, since $q_n(t_n) = p(t_n)$. For sufficiently large n, we will have $t_n - \delta < \alpha$, and therefore we can extend p's domain of definition from $(\alpha, \epsilon]^2$ to $(t_n - \delta, \epsilon^2]$, by setting

$$p(t) = q_n(t)$$

for $t \in (t_n - \delta, t_n + \delta)$. As $p = q_n$ on $(\alpha, t_n + \delta)$, the resulting function will still be smooth, and still solve our ODE.

Thus we can actually extend p past α, contradicting α's definition as the minimal number so that a solution to our initial value problem existed on $(\alpha, \epsilon^2]$.. Thus, there is a solution to our initial value problem defined on $(0, \epsilon^2]$, as desired. □

2.4. **Step 4: the diffeomorphism.** Let $P : S_\epsilon(x_0) \times (0, \epsilon^2] \to D_\epsilon(x_0) \setminus \{x_0\}$ be given by $P(a, t) = p_a(t)$, for p_a the unique solution to the ODE $p_a'(t) = \nu(p_a(t))$ with initial condition $p_a(\epsilon^2) = a$. Such a solution p_a exists over $(0, \epsilon^2]$ by Lemma 2.7.

We can extend P to a continuous map $P : S_\epsilon(x_0) \times [0, \epsilon^2] \to D_\epsilon(x_0)$ by setting

$$P(a, 0) = x_0,$$

since $||p_a(t) - x_0||^2 = r(p_a(t)) = t$, and so as $t \to 0$ we find $p_a(t) \to x_0$. As P maps all of $S_\epsilon(x_0) \times \{0\}$ to x_0, it induces a map

$$\widetilde{P} : \mathrm{Cone}(S_\epsilon(x_0)) \to D_\epsilon(x_0).$$

Lemma 2.8. *This function P maps $S_\epsilon(x_0) \times (0, \epsilon^2]$ diffeomorphically onto $D_\epsilon(x_0) \setminus \{x_0\}$.*

Proof. The map P is smooth. It also has a smooth inverse: for any $x \in D_\epsilon(x_0) \setminus \{x_0\}$, there's a unique solution p_x to our ODE with $p_x(|x - x_0|^2) = x$. Via the method of Lemma 2.7, we can extend this solution to one defined on $(0, \epsilon^2]$, which will force $x = P(p_x(\epsilon^2), |x - x_0|^2)$. This map

$$x \mapsto (p_x(\epsilon^2), |x - x_0|^2)$$

is smooth by smooth dependence of the solution of an ODE on the initial condition, and so we conclude that P has a smooth inverse, and hence is a diffeomorphism. $\square$

Lemma 2.9. *Any solution $p : (0, \epsilon^2] \to D_\epsilon(x_0) \setminus \{x_0\}$ of our ODE which lies in V at some time is in fact contained in V.*

Proof. Assume that $p(t_0) \in V$ for some time t_0. Then

$$p'(t_0) = \nu(p(t_0))$$

is tangent to V, by our assumption that ν sends points in V to vectors tangent to V. Thus by existence and uniqueness applied to V, we find that there's a local solution $\widetilde{p}$ to our ODE defined in the hypersurface V.

The argument of Lemma 2.7 can be used to extend $\widetilde{p}$ to a solution to the ODE on $(0, \epsilon^2]$ which lies in V. But $\widetilde{p} = p$ by uniqueness of solutions to ODEs (since both have the same value at time t_0), and so we deduce that the integral curve p is contained in V, as desired. $\square$

It follows from Lemma 2.8 and Lemma 2.9 that P maps $K \times (0, \epsilon^2]$ diffeomorphically onto $(D_\epsilon(x_0) \setminus \{x_0\}) \cap V$. The induced map

$$\widetilde{P} : \operatorname{Cone}(K) \to D_\epsilon(x_0) \cap V$$

obtained by sending the cone point of $\operatorname{Cone}(K)$ to x_0 will then be a homeomorphism: it is continuous; it is a bijection, since P maps $K \times (0, \epsilon^2]$ bijectively onto $D_\epsilon(x_0) \setminus \{x_0\}) \cap V$ and we just added an extra cone point to $K \times (0, \epsilon^2]$ to hit x_0, and $\operatorname{Cone}(K)$, $D_\epsilon(x_0) \cap V$ are compact Hausdorff spaces, so $\widetilde{P}$ being a continuous bijection implies it is a homeomorphism.

The map $\widetilde{P}$ is then the homeomorphism which Theorem 2.1 claims exists; thus we conclude the proof.

We introduce a piece of notation.

Definition 2.10. If V is an algebraic variety with an isolated singularity at a point x_0, for small ϵ, we say that the *Milnor link* of V at x_0 is the intersection

$$K = V \cap S_\epsilon(x_0),$$

which recall by Theorem 2.1 is a manifold which, up to diffeomorphism, does not depend on ϵ.

3. Whitney stratifications

Theorem 2.1 extends beyond the case of isolated singularities, albeit becoming more complicated in the process.

The idea starts with a simple observation of Whitney: while a variety X is not a manifold, $X \setminus \Sigma(X)$ is; likewise, while $\Sigma(X)$ is not a manifold, $\Sigma(X) \setminus \Sigma(\Sigma(X))$ is; continuing in this way, we can filter X as

$$X \supseteq \Sigma(X) \supseteq \Sigma(\Sigma(X)) \supseteq \cdots$$

in such a way that the difference between any two adjacent sets in the filtration is a manifold. This inspires the following notion.

Definition 3.1. A *filtered space* is a Hausdorff space X with a filtration

$$\emptyset = X^{-1} \subseteq X^0 \subseteq \cdots \subseteq X^n = X$$

where each X^i is closed.

We write $X_i = X^i \setminus X^{i-1}$ if $i \geq 0$. The connected components of the X_i are called the *strata* of X. The strata of X_n are called the *regular strata*, and the rest are called the *singular strata*, and the union of the singular strata is called the *singular locus* of X, and we typically write Σ_X or $\Sigma(X)$ to mean the singular locus. Note that $\Sigma_X = X^{n-1}$.

Unfortunately, the story does not end here. Let us recall Lemma 2.4, which showed that an algebraic variety with an isolated singularity at 0 intersects every sufficiently small sphere transversally. The transversality of this intersection is a significant fact, and it is desirable that we should have some transversality in general. This requires us to make sense of transversality for filtered spaces, but that is not so hard.

Definition 3.2. Let M be a manifold, and take $X, Y \subseteq M$ two closed filtered subsets of M, with the property that each stratum of X, Y is a submanifold of M. We say that *X and Y intersect transversally* if each strata of X intersects each strata of Y transversally.

One might hope that every algebraic variety, with its filtration by $\Sigma(X), \Sigma(\Sigma(X))$, etc. intersects sufficiently small spheres transversally, as in Lemma 2.4. Unfortunately, Whitney observed that this is *false*. To save it, he needed to introduce more complicated stratifications of algebraic varieties. We attempt to briefly explain what he needed to do.

First, we introduce the *frontier hypothesis*, a condition on filtered spaces capturing the idea that strata should either be separated from each other, or one should be on the boundary of the other.

Definition 3.3. A *stratified space* is a filtered space X obeying the *frontier hypothesis*: for every two strata S, T, if $S \cap \overline{T} \neq \emptyset$, then $S \subseteq \overline{T}$.

We write $S \prec T$ to denote that $S \cap \overline{T} \neq \emptyset$.

Simplicial complexes are a very good mental model of stratified spaces.

Example 3.4. If X is an n-dimensional simplicial complex, then we can give X the structure of a filtered space by letting X^i denote the union of the i-dimensional simplices. The strata are then just the interiors of the simplices. If two strata S, T obey $S \cap \overline{T} \neq \emptyset$, then either S is the interior of some face of T, or $S = T$, and so $S \subseteq \overline{T}$.

We are now going to state two additionally hypotheses, bearing Whitney's name.

Definition 3.5 (Whitney regularity conditions)**.** Let M be a manifold, and let X, Y be two disjoint C^1 submanifolds of M.

We say that (X, Y) obeys the *Whitney condition (a)* at a point $x \in X \cap \overline{Y}$ if, for any sequence $y_n \in Y$ so that $y_n \to x$ and so that $T_{y_n}Y$ converges to a hyperplane T_∞, we have $T_xX \subseteq T_\infty$ (take this limit by taking a chart of M at x and then using a Grassmannian).

We say that (X, Y) obeys the *Whitney condition (b)* at $x \in X \cap \overline{Y}$ if, for any sequence $(x_n, y_n) \in X \times Y$ such that $x_n \to x, y_n \to y$, the lines ℓ_n from x_n to y_n converge to some line ℓ_∞, and the subtangent spaces $T_{y_n}Y$ converge to T_∞, then we have $\ell_\infty \subseteq T_\infty$.

If $S \subseteq M$ is a stratified space whose strata are C^1 submanifolds of M, then we say that S obeys Whitney's condition (a) (resp. (b)) if each pair of its strata do. We say that S is a *Whitney stratified space* if S obeys both of Whitney's conditions.

At first, these conditions seem complicated and unruly. We are going to state without proof a few results about these two conditions, to hopefully motivate them.

Recall that the whole problem with the naive stratification $X \supseteq \Sigma(X) \supseteq \Sigma(\Sigma(X)) \supseteq \cdots$ was that we did not have good transversality properties. Whitney's condition (a) is equivalent to stating that transversality along S is an open condition.

Theorem 3.6. *Let M be a manifold, and $S \subseteq M$ a stratified space whose strata are C^1 submanifolds of M. Then S obeys Whitney's condition (a) if and only if for every C^1 submanifold N, the set*

$$\{C^1 \text{ function which are transverse to the strata of } S\}$$

is open in the set $C^1(N, M)$.

Proof. See [4]'s theorem 2.12. □

Theorem 3.6 helps explain the meaning of Whitney's condition (a). The meaning of Whitney's condition (b) is a little more complicated, but we try to explain. Crucial to our proof of Theorem 2.1 was the construction of a certain vector field which we could flow along. Thom invented a beautiful notion of a *controlled vector field*, allowing him to generalize the properties of the vector field we constructed in the proof of Theorem 2.1. With this notion, it is possible to prove the following, which evidently generalizes Theorem 2.1 to Whitney stratified spaces.

Theorem 3.7 (Thom-Mather)**.** *Let M be a C^2 manifold, and $X \subseteq M$ a Whitney stratified space whose strata are C^2 submanifolds of M. Then for each stratum Σ and each point $x_0 \in \Sigma$, there exists some open neighborhood U of x_0 inside of M, and a stratified space $L \subseteq S^{k-1}$ (the sphere), and a homeomorphism*

$$h : (U, U \cap X, U \cap \Sigma) \to (U \cap \Sigma) \times (B^k, \mathrm{Cone}(L), *),$$

for $*$ the cone point of $\mathrm{Cone}(L)$ and B^k the ball.

Proof. See [4]'s theorem 2.17. □

It is true that every algebraic variety admits a Whitney stratification, but the proof is quite hard.

4. The Milnor fibration

The local topology of an algebraic variety V near an isolated singularity x_0 is, by Theorem 2.1, completely determined by the Milnor link K at x_0. In the coming sections, we will explain an assortment of useful and foundational results on the topology of K, all first proven by Milnor.

Milnor's study of the topology of K was inspired by the fact that, in the simplest cases (as in Theorem 1.3), the manifold K is a knot (in the case where V is a surface, K will be a compact 1-manifold embedded in S^3; but the only compact 1-manifolds are disjoint unions of circles, and a circle embedded in S^3 is exactly a knot). In knot theory, one often studies a knot by looking at its complement.

To imitiate this knot theoretic idea, Milnor devised a certain technical notion: the *Milnor fibration*. First, we remind the reader what a fiber bundle is.

Definition 4.1. Let $\pi : E \to B$ be a continuous surjection, with B connected, and F a topological space. We say that π *is a fiber bundle with fiber F, total space E, and base space B* if for every $x \in B$, there is some open $U \ni x$ and a homeomorphism $\phi : \pi^{-1}(U) \to U \times F$ so that the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\phi"] \arrow[d, "\pi"] & U \times F \arrow[dl] \\ U & \end{tikzcd}$$

commutes. We say that E *is fibered over* B.

Recall that the Milnor link K at x_0 is defined as $K = S_\epsilon(x_0) \cap V$, for sufficiently small ϵ. Milnor proved that the complement $S_\epsilon(x_0) \setminus K$ of the Milnor link in the sphere is fibered over S^1. Using this fiber bundle, called the *Milnor fibration*, he was able to prove great results on the homotopy and homology groups of K.

4.1. **The Milnor fibration of a hypersurface.** Fix V an *affine hypersurface*. That is, V is an algebraic variety of the form

$$V = \{(x_1, ..., x_{n+1}) \in \mathbb{C}^{n+1} \mid f(x_1, ..., x_{n+1}) = 0\}, \tag{4.2}$$

for f a non-constant polynomial.

Warning 4.3. In this section, we will always assume V is the zero locus of a single nonconstant polynomial f, as in (4.2). We call such a V an *affine hypersurface*.

We introduce some notation from complex calculus.

Definition 4.4. If $f : \mathbb{C}^m \to \mathbb{C}$ is an analytic function, then we define

$$\operatorname{grad} f = \left(\overline{\partial f/\partial z_1}, ..., \overline{\partial f/\partial z_m}\right).$$

Remark 4.5. The conjugates are chosen so that if $p : [0,1] \to \mathbb{C}^m$ is a curve, then

$$\frac{df(p(t))}{dt} = \left\langle \frac{dp}{dt}, \operatorname{grad} f \right\rangle.$$

Finally, our main object of study.

Definition 4.6. Let $f : \mathbb{C}^m \to \mathbb{C}$ be a polynomial, and let

$$K := f^{-1}(0) \cap S_\epsilon$$

denote the Milnor link of f at 0.

The *Milnor map* of f at 0 is the the function $\phi : S_\epsilon \setminus K \to S^1$ given by

$$\phi(z) = \frac{f(z)}{|f(z)|},$$

where the division is always defined since K is excluded from the domain of ϕ.

We will prove that the Milnor map is a smooth fiber bundle. We first show that, for all small ϵ, the Milnor map has no critical points. Fix $f : \mathbb{C}^m \to \mathbb{C}$ a polynomial, $K = f^{-1}(0) \cap S_\epsilon$, and $\phi : S_\epsilon \setminus K \to S^1$ the Milnor map.

While $\log f(z)$ is multivalued, since any two branches differ by a constant, the derivative $\operatorname{grad} \log f(z)$ is single valued. As you might guess from the definition of ϕ, it is closely related to $\operatorname{grad} \log f(z)$.

The next two lemmas are all exercises in multivariable calculus, and so we omit the proofs. For complete proofs, see section 4 of [2], but the reader is encouraged to work out the details for themselves.

Lemma 4.7. *The critical points of the Milnor map are precisely those points $z_0 \in S_\epsilon \setminus K$ so that $i \operatorname{grad} \log f(z_0)$ is a real multiple of z_0.*

Lemma 4.8. *Assume that f is a polynomial, and that $f(0) = 0$. Then there is some $\epsilon_0 > 0$ so that, for all $z \in \mathbb{C}^m \setminus V$ with $||z|| \leq \epsilon_0$, the two numbers $z, \operatorname{grad} \log f(z)$ are either linearly independent over $\mathbb{C}$, or*

$$\operatorname{grad} \log f(z) = \lambda z$$

where $\lambda \neq 0$ is a complex number whose argument has absolute value less than $\pi/4$.

With these two lemmas, we can prove that ϕ has no singularities.

Corollary 4.9. *Assume that $f(0) = 0$. Then for every $\epsilon \ll 1$, the Milnor map ϕ has no critical points.*

Proof. By Lemma 4.7, it suffices to prove that for every $z \in \mathbb{C}^m \setminus V$ which is sufficiently close to 0, then z and $i \operatorname{grad} \log f(z)$ are $\mathbb{R}$-linearly independent. But Lemma 4.8 immediately implies this. □

Thus the Milnor map ϕ associated to a polynomial has no critical points. In particular, for every point of S^1, the fiber

$$F_\theta = \phi^{-1}(e^{i\theta}) \subseteq S_\epsilon \setminus K$$

of the Milnor map above θ is a smooth $(2m-2)$ dimensional manifold, by the regular value theorem.

We are going to prove that ϕ is actually the projection map of a locally trivial fibration. We need some more analysis. The proof will be similar to the proof of Theorem 2.1, which used a very clever argument involving ODEs to construct certain homeomorphisms. The idea of that proof was to construct a vector field on $V \cap D_\epsilon$ which always pointed away from x_0 and was always tangent to V.

We start with a similar step: we produce a tangent vector field which obeys a certain inner product condition, analogous to the pointing away condition.

Lemma 4.10. *If $\epsilon \ll 1$, then there exists a smooth vector field $V(z)$ on $S_\epsilon \setminus K$, tangent everywhere to S_ϵ, so that, for every $z \in S_\epsilon \setminus K$, the complex inner product*

$$\langle V(z), i \operatorname{grad} \log f(z) \rangle$$

is nonzero, and has argument less than $\pi/4$ in absolute value.

Proof. As in Theorem 2.1, it suffices to construct the vector field locally, since then we can use a partition of unity to blend them together. This argument is extremely similar to Theorem 2.1, so we omit the details; see [2], lemma 4.6 for a complete proof. □

Next we normalize, as we did in the proof of Theorem 2.1. Set

$$W(z) = \frac{V(z)}{\Re \langle V(z), i \operatorname{grad} \log f(z) \rangle}.$$

Then W is a tangential vector field on $S_\epsilon \setminus K$ so that

$$\Re\left(\langle W(z), i \operatorname{grad} \log f(z) \rangle\right) \equiv 1$$

identically, but

$$|\Im(\langle W(z), i \operatorname{grad} \log f(z) \rangle)| < 1.$$

As in Theorem 2.1's proof, we prove an existence and uniqueness theorem.

Lemma 4.11. *Let $z_0 \in S_\epsilon \setminus K$. Then there exists a unique smooth curve*

$$p : \mathbb{R} \to S_\epsilon \setminus K$$

so that $p(0) = z_0$ and $p'(t) = W(p(t))$.

Proof. As in Theorem 2.1, we have by standard theory that such a solution exists locally, and can be extended to a maximal interval $I \subseteq \mathbb{R}$. We prove that $I = \mathbb{R}$ by assuming for sake of contradiction that it wasn't, and then showing how to extend; the argument is analogous to the one employed in Theorem 2.1, so we omit the proof. Refer [2], lemma 4.7 to see a complete argument. □

Let $P : (S_\epsilon \setminus K) \times \mathbb{R} \to S_\epsilon \setminus K$ be the function given by

$$P(a,t) = p_a(t),$$

for p_a the unique solution to $p_a'(t) = W(p(t))$ with $p_a(0) = a$.

Define $h_t : S_\epsilon \setminus K \to S_\epsilon \setminus K$ as $h_t(a) = P(a,t)$.

We claim that each h_t is a diffeomorphism. Indeed, it has inverse map h_{-t}. More interestingly, the fiber $F_\theta = \phi^{-1}(e^{i\theta})$ of the Milnor map deforms in a simple way under h_t.

Suppose that $a \in F_\theta$. Then by definition, $f(a) = e^{i\theta}|f(a)|$. We have $h_t(a) = p_a(t)$. The argument of $f(p_a(t))$ is just the imaginary part of $\log f(p_a(t))$, and an easy computation proves that the derivative of this imaginary part is identically 1, and so we find that $h_t(a) \in F_{\theta+t}$. Thus $h_t(F_\theta) \subseteq F_{\theta+t}$, and combining this with the same observation on h_{-t}, we get that $h_t : F_\theta \to F_{\theta+t}$ is a diffeomorphism.

This easily implies that the Milnor map is a smooth fiber bundle.

Theorem 4.12. *Let V be a complex hypersurface defined by some polynomial equation f. Assume also that $f(0) = 0$, and that 0 is an isolated singularity of V. Then, for every sufficiently small ϵ, the Milnor map $\phi : S_\epsilon \setminus K \to S^1$, defined by $\phi(z) = f(z)/|f(z)|$ on the complement of*

$$K = \{z \in S_\epsilon \mid f(z) = 0\}.$$

Then ϕ is a smooth fiber bundle.

Proof. Set $F_\theta = \phi^{-1}(\theta)$.

Suppose that $U = S^1 \setminus \{\theta\}$. We can then define a continuous map $\alpha : U \to \mathbb{R}$ so that $e^{i\alpha(p)} = p$ for $p \in U$. We define a smooth map

$$\psi : \phi^{-1}(U) \to U \times F_0$$

by

$$\psi(a) = (\phi(a), h_{-\alpha(\phi(a))}(a)).$$

This map is a diffeomorphism, with inverse

$$(a, f) \mapsto h_{\alpha(a)}(f).$$

These maps ψ for different choices of θ witness the fact that ϕ is a fiber bundle, and so we conclude. ☐

Thus, we have proven the existence of the Milnor fibration. The Milnor fibration will be an incredibly useful technical tool to us; to get the most out of it, we start by studying its fibers F_θ in more detail.

Definition 4.13. The *Milnor fiber* of f is the diffeomorphism type of any fiber F_θ of the corresponding Milnor fibration ϕ_f, recalling that any two fibers of a smooth fiber bundle are diffeomorphic.

We end this section by presenting an equivalent version of the Milnor fibration.

Theorem 4.14. *Let V be a complex affine hypersurface cut out by the equation $f = 0$, and suppose that V has an isolated singularity at 0. Fix a small $\epsilon > 0$, and let*

$$K = V \cap S_\epsilon$$

be the associated Milnor link. For $\delta > 0$, set

$$A_\delta = \{z \in \mathbb{C} \mid |z| = \delta\}.$$

Then, for every sufficiently small $\delta \ll \epsilon$, the map

$$f : f^{-1}(A_\delta) \cap B_\epsilon \to A_\delta$$

is a smooth fiber bundle; this fiber bundle is isomorphic to the Milnor fibration

$$\phi_f : S_\epsilon \setminus K \to S^1,$$

in the sense that there is a commutative diagram

$$\begin{array}{ccc} f^{-1}(A_\delta) \cap B_\epsilon & \xrightarrow{\alpha} & S_\epsilon \setminus K \\ \downarrow & & \downarrow \phi_f \\ A_\delta & \xrightarrow{\beta} & S^1, \end{array}$$

where the top and bottom horizontal arrows α, β are diffeomorphisms.

In particular, these two fiber bundles have diffeomorphic fibers, and so the Milnor fiber F_θ of ϕ_f is canonically diffeomorphic to $f^{-1}(c) \cap B_\epsilon$, for c some small complex number of argument θ. In particular, since every complex variety is automatically a complex manifold and hence orientable, each fiber F_θ of the Milnor fibration is orientable.

Proof. We omit the proof of this result, but remark that it can be found as the main result of section 5 of [2]. Above, we saw two arguments of the form 'construct a vector field, and then flow along it to get an ODE'; the proof of this theorem goes in exactly the same way, by constructing a third vector field, and then flowing along it—all the ideas are present in the previous two cases. □

Theorem 4.14 is an incredibly useful reinterpretation of the Milnor fibration, as it relates the topology of the singular fiber $f^{-1}(0)$ of f with the topology of the nearby smooth fibers $f^{-1}(c)$ for small values of c.

5. The topology of the Milnor link and the Milnor fibers

As in the previous section, fix $f : \mathbb{C}^{n+1} \to \mathbb{C}$ a polynomial with $f(0) = 0$, and let V denote the affine hypersurface described by $f = 0$. Suppose that 0 is an isolated singularity of V. Thus our previous results apply, and so we have a Milnor link K, a Milnor fibration $\phi : S_\epsilon \setminus K \to S^1$ with Milnor fibers $F_\theta := \phi^{-1}(\theta)$.

In this section, we give a collection of results on the topology of K and F_θ.

Our results in this section will typically suppose $n \geq 1$, that is, $\mathbb{C}^{n+1}$ is not just $\mathbb{C}$. This is because the zero set of a polynomial equation in one variable is just a finite discrete set, and so there is not much of interest to say about the topology.

5.1. **Computing the homology.**

Lemma 5.1. *Each Milnor fiber is a $2n$-dimensional manifold. The closure $\overline{F_\theta}$ is a manifold with boundary, having interior F_θ and boundary K. (In particular, $\dim K = \dim F_\theta - 1 = 2n - 1$.)*

Proof. See lemma 6.1 of [2]. The proof is not complicated, but requires a technical tool due to Milnor called the curve selection lemma which we wish to avoid for purposes of space. □

Lemma 5.1 has some powerful corollaries.

Corollary 5.2. *The spaces $\overline{F_\theta}$ and $S_\epsilon \setminus \overline{F_\theta}$ are homotopy equivalent.*

Proof. We have the Milnor fibration

$$\phi : S_\epsilon \setminus K \to S^1.$$

Note that $S^1 \setminus \{\theta\} \cong \mathbb{R}$ is contractible, and therefore the pullback of ϕ to the bundle

$$\phi' : S_\epsilon \setminus (K \cup F_\theta) \to S^1 \setminus \{\theta\}$$

is trivial, as every smooth fiber bundle over $\mathbb{R}$ is trivial; thus

$$S_\epsilon \setminus (K \cup F_\theta) \cong \mathbb{R} \times F_{\theta'}$$

for any fiber $F_{\theta'}$ of ϕ'. But $\overline{F_\theta} = K \cup F_\theta$ by Lemma 5.1 and $F_{\theta'} \cong F_\theta$ as all Milnor fibers are diffeomorphic. Thus $S_\epsilon \setminus \overline{F_\theta}$ is homeomorphic to the product of F_θ and $\mathbb{R}$, so that $S_\epsilon \setminus \overline{F_\theta}$ is homotopy equivalent to F_θ, which is in turn homotopy equivalent to $\overline{F_\theta}$, as any compact manifold with boundary is homotopy equivalent to its interior. ☐

Corollary 5.3. *Suppose $n \geq 1$. The Milnor fiber F_θ has the homology of a point in degrees $0, 1, ..., n-1$, and in degrees $n+1, n+2, ..., 2n$. (Recall $\dim F_\theta = 2n$.) In particular, the only homology group of F_θ which has a chance of being nontrivial is $H_n(F_\theta)$.*

Proof. By Alexander duality,

$$\widetilde{H}_{2n-i}(S_\epsilon \setminus \overline{F_\theta}) \cong \widetilde{H}^i(\overline{F_\theta}).$$

By Theorem 4.14, each F_θ is homeomorphic to the intersection of an open ball with a smooth hypersurface. There is a classical, and beautiful, theorem of Andreotti-Frankel [1] that such a manifold is homotopy equivalent to a CW complex of dimension at most n. Thus for $i \geq n+1$, we find that $\widetilde{H}^i(\overline{F_\theta}) = 0$, and so

$$\widetilde{H}_q(S_\epsilon \setminus \overline{F_\theta}) = 0$$

for $q = 0, 1, ..., n-1$. But Corollary 5.2 tells us that

$$F_\theta \simeq \overline{F_\theta} \simeq S_\epsilon \setminus \overline{F_\theta},$$

and so we conclude.

The result about degrees $n+1$ and higher follows immediately from Andreotti-Frankel. ☐

5.2. **Vanishing cycles.** Corollary 5.3 leaves open the question of computing the middle homology $H_n(F_\theta)$ of the Milnor fiber. To determine this, we first state without proof the following result of Milnor.

Theorem 5.4. *Suppose $n \geq 1$. The Milnor fiber F_θ is $(n-1)$-connected, and the Milnor link K is $(n-2)$-connected and orientable.*

Proof. The argument relies on a certain lengthy analysis using Morse theory; see theorem 5.2 and lemma 6.4 of [2]. ☐

Corollary 5.5. *Assume $n \geq 1$. The Milnor fiber has the homotopy type of a wedge sum of some finite number of n-spheres.*

Proof. First assume $n \geq 2$.

By the Hurewicz theorem, since F_θ is $(n-1)$-connected, the Hurewicz homomorphism

$$h : \pi_n(F_\theta) \to H_n(F_\theta)$$

is surjective (in fact it is also an isomorphism, but we only need the surjectivity).

The homology $H_n(F_\theta)$ must be free abelian, as any torsion classes would (by the universal coefficients theorem) give rise to a class in $H^{n+1}(F_\theta; \mathbb{Z})$. But F_θ is, by the

Andreotti-Frankel theorem mentioned in the proof of Corollary 5.3, homotopy equivalent to a *finite* CW complex of dimension at most n, and hence must have trivial cohomology in degree $n+1$. Thus $H_n(F_\theta)$ is (by finiteness of the CW complex in Andreotti-Frankel) some finitely generated free abelian group, so that $H_n(F_\theta) = \mathbb{Z}^\mu$ for some $\mu \geq 0$.

Take maps $a_1, ..., a_\mu : S^n \to F_\theta$ so that $h(a_1), ..., h(a_\mu)$ generate $H_n(F_\theta) = \mathbb{Z}^\mu$. This is possible since h is surjective. Then the induced map

$$a : S^n \vee \cdots \vee S^n \to F_\theta$$

is a continuous function between two simply connected spaces (as $n \geq 2$) which induces an equivalence on homology (this is obvious in degree n, and both domain and codomain have trivial homology in all other degrees). Whitehead's theorem then implies that a is a homotopy equivalence, and so we deduce that F_θ has the homotopy type of a wedge sum of spheres, as desired.

In the case of $n = 1$, the argument is even simpler: we already know by Andreotti-Frankel that F_θ has the homotopy type of a 1-dimensional CW complex, so F_θ is homotopy equivalent to a connected graph. But every connected graph is homotopy equvialent to a wedge sum of circles (this is a classical result; to prove it, just collapse a maximal spanning tree). ☐

By Corollary 5.5, $H_n(F_\theta) \cong \mathbb{Z}^\mu$ for some integer $\mu \geq 0$. This integer μ is often called the *Milnor number* of f. Elements of $H_n(F_\theta)$ are often called *vanishing cycles*. To motivate this name, recall that $F_\theta \cong f^{-1}(\delta e^{i\theta}) \cap B_\epsilon$ (for B_ϵ the open ball) whenever $\delta \ll 1$. As we let $\delta \to 0$, this intersection approaches $f^{-1}(0) \cap B_\epsilon$, which recall is homeomorphic to $\mathrm{Cone}(K)$, and in particular contractible. Thus the cycles in $H_n(F_\theta) \cong H_n(f^{-1}(\delta e^{i\theta}) \cap B_\epsilon)$ 'vanish,' in the sense that they become homologous to 0 as you let $\delta \to 0$.

Example 5.6. Consider the case where $n = 1$ and $f(x, y) = y^2 - x^3 - x^2$. In this case, the associated variety is an elliptic curve; for $c \neq 0$, the level set $f^{-1}(c)$ is just the once punctured torus; the intersection of the torus with a small ball is homeomorphic then to a cylinder. The cylinder has first homology $\mathbb{Z}$. The zero set $f^{-1}(0)$ is a pinched torus: the homology class of the cylinder is pinched down to a point, and hence becomes trivial in homology, as $c \to 0$. Thus, our nonzero homology cycle vanishes in the $c \to 0$ limit.

We can also use Theorem 5.4 to say something about the homology of K.

Theorem 5.7. *The Milnor link K has the homology of a point in all degrees except possibly $n-1$ and n.*

Proof. For each integer $0 < k \leq n-2$, Poincare duality tells us that there is an isomorphism

$$H^{2n-1-k}(M; \mathbb{Z}) \cong H_k(M) = 0.$$

In particular, the integral cohomology groups in degrees $n+1$ to $2n-2$ are trivial. But by the universal coefficients theorem for cohomology, there are split exact sequences

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{k-1}(K), \mathbb{Z}) \to H^k(K; \mathbb{Z}) \to \mathrm{Hom}(H_k(K), \mathbb{Z}) \to 0 \tag{5.8}$$

for each k.

For $n+1 \leq k \leq 2n-2$, we have $H^k(K; \mathbb{Z}) = 0$, and in particular the only way that (5.8) can be a split exact sequence is if $\mathrm{Hom}(H_k(K), \mathbb{Z}) = 0$ and $\mathrm{Ext}^1_{\mathbb{Z}}(H_{k-1}(K), \mathbb{Z}) = 0$.

As $\mathrm{Hom}(H_k(K), \mathbb{Z}) = 0$, the abelian group $H_k(K)$ must be torsion. And as

$$\mathrm{Ext}^1_{\mathbb{Z}}(H_{k-1}(K), \mathbb{Z}) = 0,$$

we deduce that $H_{k-1}(K)$ is torsion free (as any torsion elements of $H_{k-1}(K)$ would create a nonzero class in $\mathrm{Ext}^1_{\mathbb{Z}}(H_{k-1}(K), \mathbb{Z})$).

This creates a tension: for $n+1 \leq k \leq 2n-2$, the group $H_k(K)$ is torsion, but for $n \leq k \leq 2n-3$, the group $H_k(K)$ is torsion free.

In particular, for $n+1 \leq k \leq 2n-3$, the group $H_k(K)$ must be both torsion and torsionfree, implying $H_k(K) = 0$ for $n+1 \leq k \leq 2n-3$.

For $k = 2n-2$, we can at least deduce that $H_{2n-2}(K)$ is torsion. But K is orientable, and every orientable compact manifold has torsionfree integral homology in degree one less than its dimension; thus $H_{2n-2}(K)$ is torsion and torsionfree, so it is 0.

As K is connected and orientable, we also have that $H_{2n-1}(K) = \mathbb{Z}$. We conclude. □

Thus, just as with the Milnor fiber, the only two homology groups of the Milnor link K which we need to compute are the middle two, $H_{n-1}(K)$ and $H_n(K)$. In the general case, computing these two homology groups is still an open problem.

6. Connection to exotic spheres

We started this paper by giving a fascinating result, Theorem 1.3, stating that the singularity of $y^2 = x^3$ looked like the cone on a trefoil knot. In the previous sections, we generalized this to get a very interesting description of the local topology of a variety near any isolated singularity. There are many more things one can say about this story, but to end we will return to the beginning in some sense, by explaining another fascinating geometric phenomenon that can happen: sometimes, singularities can be cones on exotic spheres.

Recall that an *exotic sphere* is a smooth n-manifold which is homeomorphic to S^n, but not diffeomorphic to S^n.

Let's take the same setup as before: $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a polynomial so that $V = f^{-1}(0)$ has an isolated singularity at the origin; K is the Milnor link of this singularity, which we have proven is some smooth $(2n-1)$-manifold.

We will be hunting for polynomials f making K an exotic sphere. There are two steps to this: first, we need to be able to tell if K is a topological sphere, and then we need to prove that it has a non-standard smooth structure. For the first step, we use the following easy theorem.

Theorem 6.1. *Suppose $n \geq 3$. The Milnor link K is homeomorphic to the $(2n-1)$ sphere if and only $H_{n-1}(K) = H_n(K) = 0$.*

Proof. If K is a sphere, then it has trivial homology in every degree except 0 and $(2n-1)$, and so one implication is trivial.

If $H_{n-1}(K) = H_n(K) = 0$, then (by our earlier computation of the other homology groups), we find that K is a simply connected manifold (simply connected since K is $(n-2)$-connected and $n \geq 3$) with the homology of S^n. Using Hurewciz's theorem and Whitehead's theorem similarly to the proof of Corollary 5.5, this implies that K is homotopy equivalent to S^n. But the Poincare conjecture, proven in large dimensions by Smale [8], then tells us that K is homeomorphic to the sphere, as desired. □

Hirzebruch [9], building on work of Breiskhorn, Pham, and Milnor, produced a family of examples whose Milnor links were topological spheres. Let $n \geq 3$ be a fixed integer, and take $a = (a_0, a_1, ..., a_n)$ a tuple of integers $a_i \geq 2$. Set

$$f_a(z_0, ..., z_n) = z_0^{a_0} + \cdots + z_n^{a_n},$$
$$V_a := f_a^{-1}(0).$$

Hirzebruch was able to show that, for certain values of the tuple a, the Milnor link K of V_a at 0 was a topological sphere.

It is a theorem (which we did not prove) that every Milnor link bounds a parallelizable manifold (in fact, the Milnor fiber is parallelizable fiber and bounded by K).

In their study of exotic spheres, Milnor-Kervaire showed that the operation of connected sum makes the set of diffeomorphism classes of manifolds homeomorphic to S^k into a finite abelian group whenever $k \geq 5$. This group, denoted Θ_k, has a subgroup BP_{k+1} consisting of those k-spheres which bound a parallelizable manifold.

We focus on the case $k = 4m - 1$. The group BP_{4m} is then known to be cyclic; furthermore, if $\Sigma \in BP_{4m}$ is an exotic $(4m-1)$-sphere, with intersection form I_Σ, then it is known that there is a class $g_m \in BP_{4m}$ generating BP_{4m} so that $\tau(I_\Sigma)/8 \cdot g_m$ represents Σ in BP_{4m}, where τ denotes the signature of a quadratic form (to compute τ, diagonalize I_Σ over $\mathbb{R}$; then τ is the number of positive entries of the diagonalized matrix minus the number of negative entries).

Thus, to determine 'which' exotic sphere our Milnor link K is, we need only study its intersection form.

In this way, Hirzebruch was able to construct varieties whose Milnor links are exotic spheres. For instance, consider the case of $n = 3$. There are 28 distinct exotic 7-spheres; these exotic 7-spheres end up being the Milnor links of the varieties

$$z_0^3 + z_1^{6k-1} + z_2^2 + z_3^2 = 0,$$

where $k \in \{1, ..., 27, 28\}$.

7. Further directions

The local topology of an algebraic variety near its singularities is an incredibly deep and fascinating topic; as we just saw, algebraic varities can resemble phenomenon from knots to exotic spheres. In this paper, we focused on the case of isolated singularities; however, the general case is also incredibly deep and fascinating.

In the more topological direction, there are the theories of intersection homology and stratified Morse theory, developed by Goresky-MacPherson in [3] and [5]. These play a major role in modern algebraic geometry.

In the more algebraic direction, Deligne and Grothendieck developed a theory of Milnor fibrations over an arbitrary field (and using their theory, they were able to prove results that Milnor had conjectured but could not prove). Beilinson-Bernstein-Deligne-Gabber developed this further into the theory of *perverse sheaves*, which have proven to be an incredibly useful tool in algebraic geometry and representation theory. The idea of the topological story is that, while fiber bundles are hard to make sense of in complete generality, there are two topological invariants of a fibration which you can make sense of: the cohomology of the fiber, and the monodromy of the fibration. Deligne and Grothendieck found an algebraic tool, working over any field, that could encode both the cohomology groups and the monodromy. [6] and [7] are the original references on these theories.

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