INTEGRAL CURRENTS AND PLATEAU'S PROBLEM

RAMAN ALIAKSEYEU

Abstract. Plateau's problem asks to show the existence of a surface with minimal area with a certain boundary. The field of geometric measure theory was born out of a desire to apply methods of modern analysis to this problem. This paper introduces integral currents, a geometric measure theoretic generalization of surfaces, and uses them to formulate and prove the existence of minimizing surfaces with a particular boundary.

Contents

1. Introduction

Geometric measure theory is one of several important theories that were born out of attempts to investigate Plateau's problem, a central question in minimal surface theory. Despite being first posed by Joseph-Louis Lagrange in 1760, and being thoroughly studied by Joseph Plateau in the late 19th century, the first limited solution to the problem was published by Jesse Douglas only in 1931. For this work Douglas was awarded the first ever Fields Medal in 1936.

The first difficulty in solving Plateau's problem comes when one tries to pose it. The most basic statement is: given a boundary of some surface in $\mathbb{R}^n$, find a surface of least possible area with that boundary. However, the existence of such a surface requires the space of surfaces with a given boundary to possess a 'compactness' property, in other words, any sequence of surfaces of decreasing area must have a limit that is also a surface. This is not true if we take the word 'surface' to mean a smooth manifold. Suppose we take our boundary to be a circle, and consider a sequence of surfaces as in Figure 1. Essentially, by extending ever thinner and

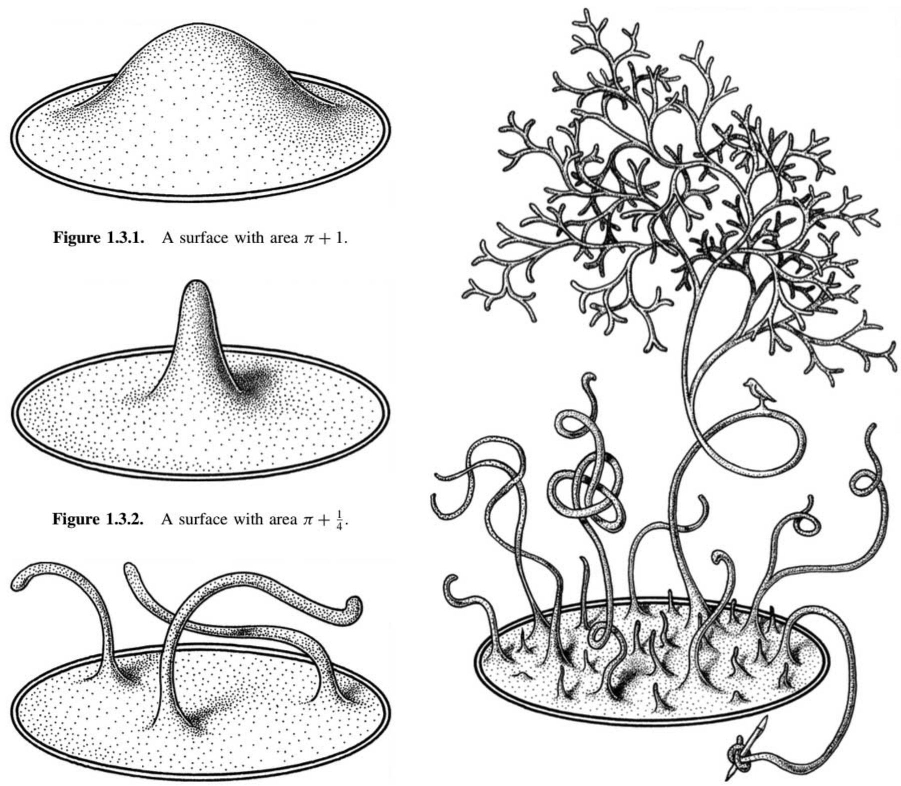

Figure 1.3.1. A surface with area $\pi + 1$.

Figure 1.3.2. A surface with area $\pi + \frac{1}{4}$.

Figure 1.3.3. A surface with area $\pi + \frac{1}{16}$.

Figure 1.3.4. A surface with area $\pi + \frac{1}{64}$.

FIGURE 1. A non-converging minimizing sequence of surfaces with a circular boundary. ([3], pg. 5-7)

longer 'tentacles' out of a surface, we can keep reducing its area, but the limit of our operations will be some surface with discontinuities.

In order to circumvent this issue, Douglas and others from the early 20th century considered only surfaces which are images of sufficiently nice maps from $\mathbb{D}^2$ into $\mathbb{R}^3$, and minimized their energy (L^2 norm of the differential of the map) instead of the area. While that eliminated the problem with tentacles (thin tubes add a lot of energy), images of disks are generally not the correct class of surfaces to consider when minimizing area. For instance, surfaces of non-zero genus pose an issue.

This illustrates the need for a class of surfaces which are as flexible as manifolds in terms of their topology, and also have the desirable convergence and compactness properties. The correct notion to consider here turns out to be that of integral currents, which were developed by Federer and Fleming in the 1960s. The study of these surfaces, and the analytic methods associated with them, grew into the field of geometric measure theory. The goal of this paper is to develop, in broad strokes, the theory of integral currents, convince the reader that currents qualify as generalizations of surfaces, and demonstrate the power of current theory by proving the existence of area-minimizing currents with a particular boundary.

This paper will closely follow the treatment of the subject presented in [4]. In a way, the goal is to abridge, clarify, and motivate the material in that book concerned with the existence of minimizing integral currents. In order to keep this introduction focused, it is assumed the reader is comfortable with measure-theoretic analysis, some functional analysis, and other miscellaneous analytic and differential-geometric prerequisites contained in chapters 1 and 2 of [4], or is willing to trust those results when they are used. They will be named whenever they're used so one can research them independently if needed. Proofs of statements, or sketches of proofs, will be included whenever they are helpful. The amount of detail given will increase as we get closer to examining area-minimizing currents. Complete proofs, or citations of full proofs, may be found in [4]. An effort was made to maintain compatible notation.

2. Rectifiable Sets and Varifolds

2.1. **Rectifiable Sets.** Despite being the correct setting for considering Plateau's problem, which is geometric in nature, currents are not by default geometric objects. There is a particular class of currents however, namely integral currents, that do correspond to varifolds, which are manifold-like sets equipped with a multiplicity function. Before we can talk about currents or varifolds, however, we need to define exactly what 'manifold-like' sets are. They are known in geometric measure theory as rectifiable sets.

Definition 2.1. A set $M \subset \mathbb{R}^{n+l}$ is a *μ-countably n-rectifiable set* if it is a subset of $N_0 \cup (\cup_{j=1}^{\infty} N_j)$, where N_0 has measure zero (with respect to μ) and N_j are C^1 n-dimensional embedded submanifolds of $\mathbb{R}^{n+l}$.

In other words, countably-n-rectifiable sets are collections of countably many subsets of n-manifolds, together with a measure zero set. The following lemma establishes a alternative definition in terms of images of Lipschitz functions, which will be useful soon when we deal with n-rectifiable sets more analytically.

Lemma 2.2. *M is countably-n-rectifiable if and only if it is a subset of $N_0 \cup (\cup_{j=1}^{\infty} F_j(A_j))$, where $F_j : \mathbb{R}^n \to \mathbb{R}^{n+l}$ are Lipschitz functions.*

Proof sketch: The only-if direction of this lemma is clear if one considers coordinate maps ψ of N_j on balls of some sufficiently small radius r around points $p \in M$. Because the range of these functions is restricted, they're Lipschitz. On the other hand, we know we can approximate a Lipschitz function F_j with continuous functions such that they agree outside of an ϵ-measure set. Hence, we can pick a countable sequence of continuous functions whose images (which are C^1 manifolds) contain the image of F_j up to a measure zero set. Repeating this procedure for all j, we can get the desired expression of M in terms of manifolds. □

By this lemma, we can also write an n-rectifiable set as a union of subsets of manifolds, so $M = \cup_{j=1}^{\infty} M_j$ where $M_j \subset N_j$, an n-manifold of $\mathbb{R}^{n+l}$.

Because we want n-rectifiable sets to be a generalization of manifolds, we need a way to do calculus on them. This requires a notion of a tangent space. Of course, n-rectifiable sets don't need to be C^1 anywhere, so we cannot define a tangent space as we would for manifolds. Instead, we generalize based on the fact that tangent spaces are local linear approximations of manifolds. Approximate tangent spaces will thus be defined as linear spaces that 'locally' behave similarly to their n-rectifiable set under integration.

Definition 2.3. Let M be an $\mathcal{H}^n$-measurable subset of $\mathbb{R}^{n+l}$ with $\mathcal{H}^n(M \cap K) < \infty$ for any compact K. Then, an n-dimensional hyperplane $P \subset \mathbb{R}^{n+l}$ is the *approximate tangent space* of M at x if

$$\lim_{\lambda \downarrow 0} \int_{\eta_{x,\lambda}(M)} f(y) d\mathcal{H}^n(y) = \int_P f(y) d\mathcal{H}^n(y) \tag{2.4}$$

Where f is any compactly supported continuous function on $\mathbb{R}^{n+l}$ and $\eta_{x,\lambda}$ is the "zoom-in on x by a factor of λ" function:

$$\eta_{x,\lambda}(y) = \lambda^{-1}(y - x) \tag{2.5}$$

If P exists, we denote it $T_x M$, just like we would denote a tangent space.

Approximate tangent spaces characterize n-rectifiable sets as follows:

Theorem 2.6. *Suppose M is $\mathcal{H}^n$ measurable with $\mathcal{H}^n(M \cap K) < \infty$ for every compact K. Then M is countably n-rectifiable if and only if M has an approximate tangent space at almost every x.*

Proof sketch: For almost every point $x \in M$ there is an associated tangent space $T_x N_j$ for some embedded C^1 manifold N_j. Further, for almost every x in N_j, the density of N_j around x is 1. Hence, by the upper density theorem,

$$\int_{\eta_{x,\lambda}(M)} f(y) d\mathcal{H}^n(y) \xrightarrow{\lambda \to \infty} \int_{\eta_{x,\lambda}(N_j)} f(y) \mathcal{H}^n(y)$$

Further, because $\eta_{x,\lambda}(N_j) \to T_x N_j$, we have that $T_x N_j = T_x M$, if the former exists. This proves the forward direction of the theorem.

The other direction is significantly more technical. It uses the important fact that the existence of an approximate tangent space at x implies that the following are true for μ being $\mathcal{H}^n$ restricted to M:

$$\lim_{\rho \downarrow 0} \frac{\mu(B_\rho(x) \cap M)}{\omega_n \rho^n} = 1 \quad \lim_{\rho \downarrow 0} \frac{\mu(X_{1/2}(\pi_x, x))}{\omega_n \rho^n} = 0$$

Where $X_\alpha(\pi_x, x)$, defined as the set $\{y \in \mathbb{R}^{n+l} \mid \operatorname{dist}(y - x, \pi_x) \leq \alpha |y - x|\}$ is the cone of points that are "over" the cone of slope α around x. If we use Egoroff's theorem, we can find a set E with $\mu(E) \geq \frac{1}{2}\mu(\mathbb{R}^{n+l})$ on which the limits above a uniform. If we find some directions π_j such that the distance between some point y and one of them is at most $1/16$, and partition the set E into E_j corresponding to which π_j the π_x is closest to at x, we can prove that $E_j \cap B_\delta(x) \cap X_{1/4}(\pi_x, x)$ is only x. Hence, $E \cap B_\delta(x)$ is contained in a finite union of Lipschitz graphs. Repeating this procedure for the complement of E gives us a countable set of Lipschitz graphs the union of which contains M up to a measure zero set. $\square$

The theorem and lemma above are sufficient for us to redefine the familiar notions from calculus on n-rectifiable sets. Here, the redefinitions of the gradient, differential of a function, and the Jacobian are given:

Definition 2.7. Let f be a locally Lipschitz function on an open $U \subset \mathbb{R}^{n+l}$ that contains an n-rectifiable set M. Then if N_j is an n-dimensional C^1 manifold, as in (2.1) for almost every $y \in M$ we can define

$$\nabla^M f(y) = \nabla^{N_j} f(y)$$

At all points where both $T_x M$ and $\nabla^M f$ exist, we can define

$$d^M f_x(\tau) = D_\tau f(y) = \langle \tau, \nabla^M f(y) \rangle$$

If $\mathcal{J}(x)$, we can define the Jacobian of M

$$J_M^f(x) = \sqrt{\det \mathcal{J}(x)}$$

The element in the pth row and qth column of $\mathcal{J}(x)$ is given by $D_{\tau_p} f(x) \cdot D_{\tau_q} f(x)$, if $\tau_1, \ldots, \tau_n$ are a basis of $T_x M$.

Finally, we will think about purely unrectifiable sets.

Definition 2.8. A subset $S \subset \mathbb{R}^{n+l}$ is *μ-purely n-unrectifiable* if it contains no μ-rectifiable n-subsets of non-zero measure.

The following lemma clarifies why this concept is useful to us.

Lemma 2.9. *If A is a $\mathcal{H}^n$ σ-finite set, we have the decomposition $A = R \cup P$, where R is countably n-rectifiable, and P is purely n-unrectifiable (If A is a Borel set, then R can be picked to be Borel also).*

Proof. Let $A = \cup_{j=1}^{\infty} A_j$ with A_j of finite measure, and suppose without loss of generality that A is Borel (otherwise pick a union $B = \cup B_j$ that is the same in measure as A and consider $A \cap R$ and $A \cap P$ instead of R and P). Then let α_j be the supremum of $\mathcal{H}^n(S)$ over all $S \subset A_j$ countably n-rectifiable. Choose n-rectifiable Borel sets $R_{i,j}$ such that $\mathcal{H}^n(R_{i,j}) > \alpha_j - 1/i$. Their union R will be countably n-rectifiable, and $A \setminus R$ will be purely unrectifiable, by the fact that α_j are supremums. □

We conclude our discussion of rectifiable sets with a remarkable structure theorem for σ-finite unrectifiable sets, due to Besicovich for $n = l = 1$, and to Federer in general. Its proof is very much beyond the scope of this paper (the theory required to prove it takes up more than a hundred pages of [2]), but it will play an important role in proving the rectifiability theorem for currents in a future section. Additionally, it gives us a geometric intuition for what unrectifiable sets look like, and inspires confidence in our choice of n-rectifiable sets as the geometric setting for varifolds.

Theorem 2.10. *Suppose Q is a σ-finite n-unrectifiable subset of $\mathbb{R}^{n+l}$. Then $\mathcal{H}^n(p(Q)) = 0$ for almost all orthogonal projections p onto n-dimensional subspaces of $\mathbb{R}^n$. Here 'almost-all' is defined in terms of the Haar measure on the group $O(n+l, n)$ of orthogonal projections.*

A simple corollary of this theorem is that every σ-finite rectifiable set R has $\mathcal{H}^n(p(R)) > 0$ for some set of orthogonal projections p with non-zero measure.

Images of projections of a set can be thought of as shadows of a set onto a sheet of paper produced by shining a flashlight on it from different directions. In that case, an unrectifiable set has no shadow from almost every direction.

We can construct a purely 1-unrectifiable set as follows: first take a square $C_0 = [0, 1]^2 \subset \mathbb{R}^2$, remove its central cross and keep the four squares of side length

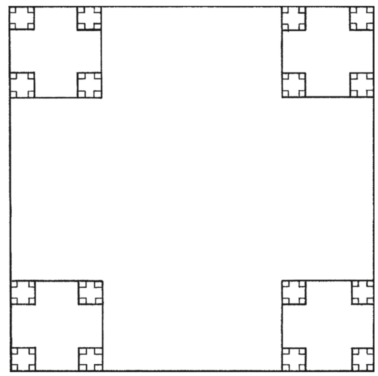

FIGURE 2. Boundaries of C_0, C_1, C_2, C_3, stages in the construction of an unrectifiable set C ([3], pg. 37)

1/4. Call this set C_1. Note that the projections of C_0 and C_1 onto the line $y = x/2$ are both full line segments of length $\frac{3}{\sqrt{5}}$. If we keep removing the middle crosses of squares, letting C_n be the set of 4^n squares of side length 4^{-n} (see Figure 2), the projection onto the line stays invariant. Hence, the limit set $C = \cap_{n=0}^{\infty} C_n$ has positive $\mathcal{H}^1$ measure. However, C's projections onto the x and y axes are Cantor-like sets of measure 0, which cannot happen if C contained a non-zero measure subset of a 2-manifold. This example, and the theorem above, hopefully illustrate how nasty unrectifiable sets are, and why we generally don't want to deal with them.

2.2. **Varifolds.** A slight generalization of an n-rectifiable set will be very useful to us. Sometimes a 'weight' on certain parts of a rectifiable set is necessary to make sure sequences of surfaces properly converge. This is the job varifolds can do for us.

Definition 2.11. Let M be a countably n-rectifiable subset of $\mathbb{R}^{n+l}$, and let θ be a positive locally $\mathcal{H}^n$-integrable function. Then define an *n-(rectifiable)-varifold* $\overline{v}(M, \theta)$ to be the equivalence class of all pairs $(\tilde{M}, \tilde{\theta})$ such that $\mathcal{H}^n((M \setminus \tilde{M}) \cup (\tilde{M} \setminus M)) = 0$ and $\theta = \tilde{\theta}$ $\mathcal{H}^n$-almost everywhere on M. If θ is an integer-valued function, $\overline{v}(M, \theta)$ is called an *integral* varifold.

Note that in literature (like [2] or [3]) it is rare that authors define rectifiable varifolds. They instead deal with *general* n-varifolds over U, defined as Radon measures on the cross product $U \times G(n, m)$, where $G(n, m)$ is the Grassmannian of all m-dimensional linear subspaces of an n-dimensional vector space. It is a theorem that a general n-varifold is rectifiable if it has an approximate tangent space almost everywhere (with respect to the measure it induces), but because we only care about varifolds with this property, and because rectifiable varifolds are more obviously geometric, we will use the terms 'varifold' and 'rectifiable varifold' interchangeably.

Defining varifolds in terms of equivalence classes lets us ignore the measure-zero subset N_0 (as defined in (2.1)) of the corresponding n-rectifiable set M of a particular varifold representative (M, θ), and lets us derive more information by integrating functions over varifolds.

Associated with each varifold V there is a Radon measure μ_T, defined

$$\mu_V(A) = \int_{A\cap M} \theta(x) d\mathcal{H}^n(x) \tag{2.12}$$

For some particular representative (M, θ) of V. The measure doesn't change based on our choice of representative. We define the *mass* of a varifold $\mathbb{M}(V)$ with respect to this measure:

$$\mathbb{M}(V) = \mu_V(\mathbb{R}^{n+l}) = \int_M \theta d\mathcal{H}^n \tag{2.13}$$

The tangent space T_xV for a varifold $V = \overline{v}(M, \theta)$ is simply the tangent space T_xM for x in the corresponding rectifiable set M. We also define the *support of* V $\operatorname{spt} V$ to be equal to the support of its corresponding measure $\operatorname{spt} \mu_V$. The *restriction* $V \llcorner A$ of V to A will be defined by $\overline{v}(M \cap A, \theta \mid_{M\cap A})$.

Let U and W be open sets in $\mathbb{R}^{n+l}$, and suppose $f : U \cap \operatorname{spt} V \to W$ is proper, Lipschitz, and injective. We define the image $f_\# V$ of V under f as the varifold $\overline{v}(f(M), \theta \circ f^{-1})$. Note that as a consequence of the area formula, $\theta \circ f^{-1}$ is locally integrable in W, so $f_\# V$ is indeed a varifold.

If f above is not injective, redefine the multiplicity $\tilde{\theta}$ of $f_\# V$ to account for fibers of $y \in f(M)$:

$$\tilde{\theta}(y) = \sum_{x \in f^{-1}(y)\cap M} \theta(x)$$

In this more general case, if $J_M f$ is the Jacobian of f relative to M by the area formula we have

$$\mathbb{M}(f_\# V) = \int_{f(M)} \tilde{\theta} d\mathcal{H}^n = \int_M (J_M f)\theta d\mathcal{H}^n \tag{2.14}$$

We would now like to define an analogue of minimal surfaces for varifolds, using a first variation formula. This is where the definitions above will be useful. Let $\phi_t : U \times (-\epsilon, \epsilon) \to U$ be a smooth homotopy between two diffeomorphisms that are identity outside a compact subset of U such that $\phi_0(x) = x$, and each ϕ_t is also identity outside of a compact subset of U. Then if $K \subset U$ is compact,

$$\mathbb{M}(\phi_{t\#}(V \llcorner K)) = \int_{M\cap K} J_M \phi_t \theta d\mathcal{H}^n$$

If we set

$$X \mid_x = \frac{\partial}{\partial t}\phi(x, t) \mid_{t=0}$$

The *first variation formula* for a varifold V is

$$\frac{d}{dt}\mathbb{M}(\phi_0(x)) \mid_{t=0} = \int_M \operatorname{div}_M X d\mu_T \tag{2.15}$$

Definition 2.16. A varifold V is stationary if its first variation is 0. So,

$$\frac{d}{dt}\mathbb{M}(\phi_0(x)) \mid_{t=0} = \int_M \operatorname{div}_M X d\mu_T = 0$$

3. Currents

Before we define differential currents, we review the usual results from the theory of differential forms, and establish some notation.

3.1. **Differential Forms.** First we review some definitions relating to differential forms. Consider an orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$. There is a corresponding basis $dx^1, \ldots, dx^n$ of the dual space $\mathbb{R}^{n*}$, such that $dx^i(e_j) = \delta^i_j$. This dual space is otherwise denoted $\Lambda^1(\mathbb{R}^P)$. Then, $\Lambda^k(\mathbb{R}^P)$, the set of *alternating k-tensors*, is defined as the set of alternating linear functions (swapping two adjacent arguments switches the sign) on $\mathbb{R}^P \times \cdots \times \mathbb{R}^P$, k times. If $\omega_1, \ldots, \omega_k \in \Lambda^1(\mathbb{R}^P)$, then we define the wedge product of $\omega_1, \ldots, \omega_k$ as a k-tensor in $\Lambda^k(\mathbb{R}^P)$:

$$\omega_1 \wedge \cdots \wedge \omega_k(v_1, \ldots, v_k) = \det([\omega_i(v_j)])$$

Hence, we know

$$\omega \in \Lambda^k(\mathbb{R}^P) \iff \omega = \sum_{\alpha \in I_{k,n}} \omega_\alpha dx^\alpha$$

where $I_{k,n}$ is the set of multi-indices $(i_1, \ldots, i_k)$ with $1 \leq i_1 \leq \cdots \leq i_k \leq n$, ω_α are coefficients in $\mathbb{R}$, and $dx^\alpha = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. We may define dot products on this space analogously to how we define them for vector spaces:

$$\sum_{\alpha \in I_{k,n}} \omega_\alpha dx^\alpha \cdot \sum_{\alpha \in I_{k,n}} \eta_\alpha dx^\alpha = \sum_{\alpha \in I_{k,n}} \omega_\alpha \eta_\alpha$$

We can define $\Lambda_k(\mathbb{R}^P) = \Lambda^k(\Lambda^1(\mathbb{R}^k))$ as the set of k-tensors over the vector space of linear maps over $\mathbb{R}^k$. By the double dual identification, the objects in this space are spanned by all the $e_1 \wedge \cdots \wedge e^{i_k} = e^\alpha$, where $e_1, \ldots, e_n$ are a basis of $\mathbb{R}^P$ and α is a k multi-index in $I_{k,n}$. This is the space of *k-vectors* over $\mathbb{R}^P$. We define the *differential k-forms* $\Omega^n(U)$ to be the continuous maps from $\Lambda^n(\mathbb{R}^P)$ to U. In other words,

$$\omega \in \Omega^n(U) \iff \omega = \sum_{\alpha \in I_{k,n}} \omega_\alpha dx^\alpha \quad \omega_\alpha \in C^\infty(U)$$

The *exterior derivative* $d : \Omega^n(U) \to \Omega^{n+1}(U)$ of ω is defined

$$d\omega = \sum_{j=1}^{n} \sum_{\alpha \in I_{k,n}} \frac{\partial \omega_\alpha}{dx_j} dx^j \wedge dx^\alpha \tag{3.1}$$

There's also the *pullback* operation on a form $\Omega^n(V)$ ($V \subset \mathbb{R}^m$) with respect to a smooth map $f : U \to V$, defined

$$f^\# \omega := (df_x)^\# (\omega \mid_{f(x)}) \tag{3.2}$$

where the right hand side is defined as usual for pullbacks.

Forms, when combined with using k-vectors to represent oriented k-dimensional tangent spaces, are the most natural things to integrate over manifolds. If $\omega = \omega_\alpha dx^\alpha$ for some k-multi-index α, we define the integral of it over an open subset U of $\mathbb{R}^P$ as

$$\int_U \omega = \int_U \omega_\alpha dx^\alpha \tag{3.3}$$

The integral on the right is defined as a k-iterated integral with respect to the variables prescribed by the multi-index α. If $M \subset \mathbb{R}^P$ is a k dimensional manifold such that $\xi : M \to \Lambda_n(\mathbb{R}^P)$ is a map such that $M(x)$ is the k-vector $v_1 \wedge \cdots \wedge v_k$ that represents the orientation of M at x, then

$$\int_M \omega = \int_M \langle \xi(x), \omega(x) \rangle d\mathcal{H}^n(x) \tag{3.4}$$

The most important property of integration of forms over manifolds is the Stokes' theorem. If ∂M is the $k-1$ dimensional manifold that is a boundary of M, we have

$$\int_{\partial M} \omega = \int_M d\omega \tag{3.5}$$

We will denote the compactly-supported k-forms, i.e. ones with compactly supported coefficients, as $\mathcal{D}^n(U)$. The norm of a compactly-supported k-form is defined similarly to the norm of a vector:

$$|\omega| = \sup_{x\in U} \sqrt{\omega(x)\cdot\omega(x)} \tag{3.6}$$

3.2. **General Currents.** We are now ready to define a general n-current.

Definition 3.7. T is an *n-current* if it is a continuous linear functional on $\mathcal{D}^n(U)$. The set of such n-currents will be denoted $\mathcal{D}_n(U)$ (since it is the dual space of $\mathcal{D}^n(U)$)

Continuity here is defined in terms of the *locally convex topology* on $\mathcal{D}^n(U)$. In it, $\omega_k = \sum \omega_{k\alpha} dx^\alpha$ converges to $\omega = \sum \omega_\alpha dx^\alpha$ if all the coefficients $\omega_{k\alpha}$ are supported on the same compact set K, and $\lim D^\beta \omega_{k\alpha} = D^\beta \omega_\alpha$, where D^β is any m-derivative of ω (for any m) with respect to variables prescribed by the multi-index β.

Despite not looking like it at first glance, currents are natural generalizations of oriented k-manifolds with locally finite $\mathcal{H}^n$ measure, since such manifolds are one class of objects that 'act' on forms in a continuous way. So, given a k-manifold M with an orientation ξ, as in (3.4), we have the corresponding n-current $[[M]]$ defined by

$$[[M]](\omega) := \int_M \langle \xi(x), \omega(x)\rangle d\mathcal{H}^n(x) \tag{3.8}$$

Inspired by the formula above and Stokes' theorem, we define the *boundary* $n-1$-current ∂T of an n-current T like so:

$$\partial T(\omega) := T(d\omega) \quad \omega \in \mathcal{D}^n(U) \tag{3.9}$$

Also motivated by this example, we define *mass of* T over the open set $W \subset U$ as

$$\mathbb{M}_W(T) := \sup_{|\omega|\le 1, \omega\in\mathcal{D}^n(U), \operatorname{spt}\omega\subset W} T(\omega)$$

As such, the mass of $[[M]]$ is its n-measure $\mathcal{H}^n(M)$. If $W = U$, we can denote the mass of T as simply $\mathbb{M}(T)$.

Currents T over U such that $\mathbb{M}_W(T) + \mathbb{M}_W(\partial T)$ is finite for all W compactly supported in U will be the most general setting on which we will prove a lot of our results. So, they deserve a special name, and are commonly referred to as *normal currents* in literature ([3], p. 49)

The *support* of a current T on $\mathcal{D}^n(U)$ is defined as $U \setminus \cup W$, where W are all sets compactly contained in U such that $T(\omega) = 0$ whenever $\omega \in \mathcal{D}^n(U)$ is compactly supported in W. In other words, $\cup W$ is where all the forms in the kernel of T are supported.

Notice we can apply the Riesz Representation Theorem to any current T with a finite mass $\mathbb{M}_W(T)$ for any $W \subset\subset U$ to get a Radon measure μ_T on U and a

μ_T-measurable, $\Lambda_n(\mathbb{R}^n)$-valued function $\vec{T}$, such that T can be expressed as

$$T(\omega) = \int_U \langle \omega(x), \vec{T}(x) \rangle d\mu_T \tag{3.10}$$

The measure μ_T is a nice representation of the current, since $\mu_T(U) = \mathbb{M}(T)$, and $\operatorname{spt} T = \operatorname{spt} \mu_T$ in the usual sense of supports of measures. Similarly to how we defined a restriction of varifolds to sets, we define the restriction $T \llcorner A$ of an n-current T to a μ_T-measurable set $A \subset U$ by

$$(T \llcorner A)(\omega) := \int_A \langle \omega, \vec{T} \rangle d\mu_T \tag{3.11}$$

For all $\omega \in \mathcal{D}^n(U)$, with $\vec{T}$ as above. Similarly, we can apply a locally μ_t-integrable function ϕ to a current:

$$(T \llcorner \phi)(\omega) := \int \langle \omega, \vec{T} \rangle \phi d\mu_T \tag{3.12}$$

We would like to talk about the compactness properties of $\mathcal{D}_n(U)$, hence we will need a topology on this space. We use the *weak* topology* on $\mathcal{D}_n(U)$, in which we say the sequence of currents $\{T_q\}$ converges to T (denoted $T_q \to T$) if $\lim_{q\to\infty} T_q(\omega) = T(\omega)$ for all continuous n-forms on U. Currents, however, are defined in terms of smooth forms, and we say that $\{T_q\}$ *converges weakly* to T (denoted $T_q \rightharpoonup T$) if the limit condition above is true for $\omega \in \mathcal{D}^n(U)$. Note that if all T_q have finite mass on all W compactly supported in U, (3.10) gives us a way to plug in continuous forms by approximating them with smooth ones and taking the limit using dominated convergence, and hence weak pointwise convergence becomes equivalent to weak* convergence. Now, by the Banach-Aloglou theorem combined with the Eberlein-Šmulian theorem, we know $\mathcal{D}^n(U)$ is sequentially compact. Namely,

Lemma 3.13. *If $\{T_q\}$ is a sequence of n-currents, and $\sup_{n\geq 1} \mathbb{M}(T_q) < \infty$ for each W compactly contained in U, then there is a subsequence $\{T_q\}$ and a $T \in \mathcal{D}_n(U)$ such that*

$$\int_U \langle \omega, \vec{T}_{q'} \rangle d\mu_T \to \int_U \langle \omega, \vec{T} \rangle d\mu_T$$

Because mass is defined with the operator norm, it is lower-semicontinuous with respect to weak* topology. In other words, if $T_q \rightharpoonup T$, we have

$$\mathbb{M}_W(T) \leq \lim_{q\to\infty} \mathbb{M}_W(T_q) \tag{3.14}$$

This shows the topology induced by the norm $\mathbb{M}$ is weaker than the topology induced by the pointwise convergence. In other words, $T_q \rightharpoonup T$ implies $\lim_{q\to\infty} \mathbb{M}(T_q - T) = 0$.

In the next section we will want to 'slice' a current by a Lipschitz function, like one would slice some surface by horizontal planes to get level sets. The general definition of a slice is relatively opaque, but slices of integral currents, which we will define soon, have some nice properties.

Definition 3.15. Let f be a Lipschitz function in U, and T be a normal current. Denote

$$L_{(-)}(t) = \{x \in U : f(t) < t\} \quad L_{(+)}(t) = \{x \in U : f(t) > t\}$$

For a normal current $T \in \mathcal{D}_n(U)$, define quantities

$$\langle T, f, t_- \rangle = \partial(T \llcorner L_{(-)}) - (\partial T) \llcorner L_{(-)} \quad \langle T, f, t_+ \rangle = \partial(T \llcorner L_{(+)}) - (\partial T) \llcorner L_{(+)}$$

Whenever $\langle T, f, t_- \rangle = \langle T, f, t_+ \rangle$, their shared value is the $n-1$ current called the *slice of T by f*, denoted $\langle T, f, t \rangle$. Note that this quantity exists for all but countably many t, i.e. those for which $\mathbb{M}(T \llcorner \{f = t\}) + \mathbb{M}((\partial T) \llcorner \{f = t\}) > 0$ (since the current is normal).

We will want the following bound on the integral of slices with respect to f using the derivative of f.

Lemma 3.16. *Let f be Lipschitz on U (hence differentiable almost everywhere), and let $T \in \mathcal{D}_n(U)$ be normal. Then*

$$\int_a^b \mathbb{M}_W(\langle T, f, t \rangle) \leq \operatorname{esssup}_W |Df| \mathbb{M}(T \llcorner \{a < f < b\})$$

Proof Sketch: We will prove this result for continuous maps. Take some smooth increasing function $\gamma : \mathbb{R} \to \mathbb{R}^+$ such that

$$\gamma(t) = \begin{cases} 0 & t < a \\ 1 & t > b \end{cases} \quad 0 \leq \gamma'(t) \leq \frac{1+\epsilon}{b-a} \text{ for } a < t < b$$

for some arbitrary $\epsilon > 0$. Then

$$((\partial T) \llcorner \gamma \circ f)(\omega) - \partial(T \llcorner \gamma \circ f)(\omega) = T(\gamma'(f) df \wedge \omega)$$

The left hand side converges to $\langle T, f, t_+ \rangle$ as $b \to a$, and on the right hand side we have the bound

$$\mathbb{M}(T(\gamma'(f) df \wedge \omega)) \leq \sup_W |Df| \frac{1+\epsilon}{b-a} \mathbb{M}_W(T \llcorner \{a < f < b\}) |\omega|$$

with $\operatorname{spt} \omega$ in W. Then, also letting $\epsilon \to 0$, we have a bound

$$\mathbb{M}(\langle T, f, t_+ \rangle) \leq \operatorname{esssup}_W |Df| \liminf_{h \downarrow 0} h^{-1} \mathbb{M}_W(T \llcorner \{t < f < t+h\}) \tag{3.17}$$

With a similar argument we get

$$\mathbb{M}(\langle T, f, t_- \rangle) \leq \operatorname{esssup}_W |Df| \liminf_{h \downarrow 0} h^{-1} \mathbb{M}_W(T \llcorner \{t-h < f < t\}) \tag{3.18}$$

Notice that

$$\liminf_{h \downarrow 0} h^{-1} \mathbb{M}_W(T \llcorner \{t < f < t+h\}) = \frac{d}{dt} \mathbb{M}_W(T \llcorner \{f < t\})$$

and similarly for $\liminf_{h \downarrow 0} h^{-1} \mathbb{M}_W(T \llcorner \{t-h < f < t\})$. Because $\mathbb{M}_W(T \llcorner \{f < t\})$ is increasing in t, it is differentiable almost everywhere, and so

$$\int_a^b \frac{d}{dt} \mathbb{M}_W(T \llcorner \{f < t\}) dt \leq \mathbb{M}_W(T \llcorner \{a < f < b\})$$

This gives us the result. $\square$

We will also need to know what a pushforward of a current by a map is.

Definition 3.19. If T is an n-current on $\mathcal{D}^n(U)$, $f : U \to V$ for open $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ and f restricted to $\operatorname{spt} T$ is proper, then the m-current $f_\# T$ on $\Omega_c^m(V)$ called the *pushforward of f* is defined as

$$f_\# T(\omega) := T(\xi f^\# \omega) \tag{3.20}$$

for any $\omega \in \mathcal{D}^n(V)$ and smooth ξ on U that is equal to 1 on a neighborhood of $\operatorname{spt} T \cup \operatorname{spt} f^\# \omega$.

By Rademacher's theorem and (3.10) (when the conditions of the latter are satisfied), the pushforward is still valid if f is only Lipschitz. In that case, we have the following representation of the pushforward:

$$f_\# T = \int \langle f^\# \omega, \vec{T} \rangle d\mu_T = \int \langle \omega \mid_{f(x)}, df_{x\#} \vec{T}(x) \rangle d\mu_T \tag{3.21}$$

Notice that the pushforward commutes with the boundary operator: $\partial f_\# T = f_\# \partial T$.

We also want to define products of currents.

Definition 3.22. Let T and S be s and t-currents on $\Omega^s_c(U_1)$ and $\Omega^t_c(U_2)$ respectively ($U_1 \subset \mathbb{R}^{n_1}$ and $U_2 \subset \mathbb{R}^{n_2}$). Express $\omega \in \Omega^{s+t}_c(U_1 \times U_2)$ like

$$\omega = \sum_{\substack{(\alpha,\beta) \in I_{s',n_1} \times I_{t',n_2} \\ s'+t'=s+t}} \omega_{\alpha\beta}(x,y) dx^\alpha dy^\beta$$

Then, define $S \times T$ as a current over $\Omega^{s+t}_c(U_1 \times U_2)$ like

$$S \times T(\omega) := T\left(\sum_\beta S\left(\sum_\alpha \omega_{\alpha\beta} dx^\alpha \right) dy^\beta \right)$$

Notice this definition plays well with manifolds interpreted as currents: if M_1, M_2 are s and t-dimensional manifolds in R^{n_1} and $\mathbb{R}^{n_2}$ respectively, then $[[M_1]] \times [[M_2]] = [[M_1 \times M_2]]$. Also, for any W_1, W_2 compactly contained in U_1, U_2 respectively, we have $\mathbb{M}_{W_1 \times W_2}(S \times T) = \mathbb{M}_{W_1}(S)\mathbb{M}_{W_2}(T)$.

The boundary of a product of currents is computed using the definition of current boundary (3.9):

$$\partial(S \times T) = (\partial S) \times T + (-1)^s S \times \partial T \tag{3.23}$$

Consider the current $[[(0,1)]]$ that corresponds to the 1-current given by the open interval $(0,1) \subset \mathbb{R}^n$ interpreted as a manifold with the usual orientation. Then, denoting the 0-current on smooth maps over U: $\{p\}(\omega) = \omega(p)$ as $\{p\}$, the formula above gives

$$\partial([[(0,1)]] \times T) = \{1\} \times T - \{0\} \times T - [[(0,1)]] \times \partial T \tag{3.24}$$

By computing $\partial h_\#([[(0,1)]] \times T)$ using the identities above we derive the important *homotopy formula*:

Lemma 3.25. *Let $T \in \mathcal{D}_n(U)$, $f, g : U \to V$ be smooth, and $h : [0,1] \times U \to V$ be a homotopy between f and g such that $h\mid_{[0,1]\times \operatorname{spt} T}$ be proper, we have*

$$g_\# T - f_\# T = \partial h_\#([[(0,1)]] \times T) + h_\#([[(0,1)]] \times \partial T)$$

We also have a mass bound on the image of $[[(0,1)]] \times T$ of an affine homotopy.

Lemma 3.26. *If f, g and T are as in (3.25), and h is the affine homotopy $h(x,t) = tg(x) + (1-t)f(x)$ between f and g such that $h\mid_{\operatorname{spt} T}$ is a proper map into V, we have*

$$\mathbb{M}_W(h_\#([[(0,1)]] \times T)) \le \sup_{x \in \operatorname{spt} W \cap W_h} |f(x) - g(x)| \cdot \sup_{x \in \operatorname{spt} t \cap W_h} (|df_x| + |dg_x|)^n \mathbb{M}_{W_h}(T)$$

Where $W_h = p(h^{-1}(W))$ for some $W \subset\subset U$, with $p : (x,t) \mapsto x$.

Proof. Notice that $[[\overrightarrow{(0,1)}]] \times T = e_1 \wedge \vec{T}$, and $\mu_{[[(0,1)]]\times T} = \mathcal{L}^1 \times \mu_T$, so by Fubini's theorem (which we can use since W is compactly contained in U) and (3.21) we have for any n-form ω on V:

$$\begin{aligned}
&\mathbb{M}_W(h_\#([[(0,1)]] \times T)) \\
&= \sup_{|\omega|\le 1, \operatorname{spt}\omega \subset W} \int_0^1 \int_W \langle \omega_{h(t,x)}, dh_{(t,x)\#}(e_1 \wedge \vec{T}(x))\rangle d\mu_T(x)dt \\
&= \sup_{|\omega|\le 1, \operatorname{spt}\omega \subset W} \int_0^1 \int_W \langle \omega_{h(t,x)}, (g(x) - f(x)) \wedge (tdg_x - (1-t)df_x)_\# \vec{T}(x)\rangle d\mu_T(x)dt
\end{aligned}$$

The bound falls out after performing the pushforward. □

This is a good time to mention a characterization of boundaryless currents, called the Constancy Theorem.

Theorem 3.27. *If U is open and connected in $\mathbb{R}^n$, if $T \in \mathcal{D}_n(U)$ and $\partial T = 0$ then there is a constant c such that $T = c[[U]]$.*

Proof Sketch: By using mollifiers, we can express T in the interior of a ball $B_\rho(x_0) \subset U$ as a bounded linear functional, and get a representation of it like

$$T(adx^1 \wedge \cdots \wedge dx^n) = \int \tilde{a}\theta d\mathcal{L}^n$$

for some bounded measurable function θ and $\tilde{a}$ a smooth, compactly supported map. Then we know there's a form $\omega_{j\sigma}$ such that $d\omega_{j\sigma} = (D_{j\tilde{a}})_\sigma dx^1 \wedge \cdots \wedge dx^n$. So, we have

$$\int D_j \tilde{a}\theta = T(d\omega_j) = \partial T(\omega_j) = 0$$

Since T has no boundary. But this implies θ is constant in the interior of $B_\rho(x_0) \subset U$, and thus T is some constant multiple of the current associated with the interior of $B_\rho(x_0)$. Because U is connected and T is continuous, we're done. □

By a similar argument, using the compactness of bounded variation functions $\mathrm{BV}_{\mathrm{loc}}$, we have that if $\mathbb{M}_W(\partial T) < \infty$ then the boundedness of $|\int D_j\tilde{a}\theta|$ implies that there is some $\theta \in \mathrm{BV}_{\mathrm{loc}}(U)$ such that

$$T(\omega) = \int a\theta d\mathcal{L}^n \tag{3.28}$$

3.3. **Integral Currents.** Integral currents are those currents that correspond to integral varifolds with oriented approximate tangent spaces.

Definition 3.29. If $T \in \mathcal{D}_n(U)$ where $U \subset \mathbb{R}^{n+l}$, then T is a *integer multiplicity current* if it can be expressed like

$$T(\omega) =: \underline{\tau}(M, \theta, \xi)(\omega) = \int_M \langle \omega(x), \xi(x)\rangle \theta(x) d\mathcal{H}^n \tag{3.30}$$

Where M is an $\mathcal{H}^n$-measurable, n-rectifiable set, $\theta : M \to \mathbb{Z}$ is locally $\mathcal{H}^n$ integrable, and $\xi : M \to \Lambda^n(\mathbb{R}^{n+l})$ is an $\mathcal{H}^n$-measurable map such that $\xi(x)$ is $\tau_1 \wedge \cdots \wedge \tau_n$ where $\{\tau_i\}$ form an orthonormal basis for the approximate tangent space T_xM. We call θ the *multiplicity*, and ξ the *orientation* of T.

There is an important lemma telling us that the pushforward of an integral current is again an integral current.

Lemma 3.31. *Let $f : U \to W$ be locally Lipschitz, f restricted to* $\operatorname{spt} T$ *be proper, and $T = \tau(M, \xi, \theta) \in \mathcal{D}_n(U)$ be an integer multiplicity current. Then $f_\# T$ is an integer multiplicity current in W.*

Proof. By (3.21), we have

$$f_\# T(\omega) = \int_M \langle \omega_{f(x)}, d^M f_{x\#} \xi(x) \rangle \theta(x) d\mathcal{H}^n(x)$$

We also have $|d^M f_{x\#}\xi(x)| = J^M f(x)$, so by the area formula

$$f_\# T(\omega) = \int_{f(M)} \left\langle \omega \mid_y, \sum_{x \in f^{-1}(y) \cap M_+} \theta(x) \frac{d^M f_{x\#}\xi(x)}{|d^M f_{x\#}\xi(x)|} \right\rangle d\mathcal{H}^n(y)$$

Where $M_+ = \{x \in M : |\nabla^M f(x)| > 0\}$. Since M is an n-rectifiable set, so is $f(M)$. Hence, given an orthonormal basis for $T_y(f(M))$ $\{\tau_1, \dots, \tau_n\}$ for almost every x we have

$$\frac{d^M f_{x\#}\xi(x)}{|d^M f_{x\#}\xi(x)|} = \pm \tau_1 \wedge \cdots \wedge \tau_n$$

Denote the orientation of this approximate tangent space at y by $\eta(y)$. Then

$$f_\# T(\omega) = \int_f (M) \langle \omega(y), \eta(y) \rangle N(y) d\mathcal{H}^n(y)$$

Where

$$N(y)\eta(y) = \sum_{x \in f^{-1}(y) \cap M_+} \theta(x) \frac{d^M f_{x\#}\xi(x)}{|d^M f_{x\#}\xi(x)|}$$

So, $N(y)$, the multiplicity of $f_\# T$, has to be an integer. This proves the lemma. $\square$

Slices over integral currents can be defined more explicitly than with general currents. Here we use $\llcorner$ on $v \in \Lambda_n(T_x M)$ and $w \in T_x M$ as a way of denoting $v \llcorner w \in \Lambda_{n-1}(T_x M)$ such that $\langle v \llcorner w, a \rangle = \langle v, w \wedge a \rangle$ for some $a \in \Lambda_{n-1}(T_x M)$.

Definition 3.32. Let f be Lipschitz on $\mathbb{R}^{n+l}$, and define $M_+ = \{x \in M : |\nabla^M f(x)| > 0\}$. Then, if $T \in \mathcal{D}_n(U)$, for almost every t we define the slice $\langle T, f, t \rangle$ as the $n-1$-current

$$\langle T, f, t \rangle = \underline{\tau}(M_t, \theta_t, \xi_t) \tag{3.33}$$

Where $M_t = f^{-1}(t) \cap M_+$,

$$\theta_t(x) = \begin{cases} 0 & \nabla^M f(x) = 0 \\ \theta(x) & \nabla^M f(x) \neq 0 \end{cases} \text{ restricted to } M_t; \quad \xi_t(x) = \xi(x) \llcorner \frac{\nabla^M f(x)}{|\nabla^M f(x)|}$$

This definition agrees with the more general one in the case that T is normal. This also gives us $\langle \partial T, f, t \rangle = -\partial \langle T, f, t \rangle$ for almost every t. Another important property is that for each open $W \subset U$:

$$\int_{-\infty}^{\infty} \mathbb{M}_W(\langle T, f, t \rangle) dt = \int_{M \cap W} |\nabla^M f| \theta d\mathcal{H}^n \tag{3.34}$$

So, the slices "glue together" to become the whole current over all values of t.

Finally, we prove the completeness of integral currents under the topology induced by the mass norm. This is a crucial step to proving their completeness under the weak* topology.

Lemma 3.35. *The set of integer multiplicity currents in $\mathcal{D}_n(U)$ is complete with respect to the mass norm topology.*

Proof. Let $\{T_Q\} \subset \mathcal{D}_n(U)$ be a Cauchy sequence of integral currents with respect to every semi-norm $\mathbb{M}_W$, $W \subset\subset U$. Denote

$$T_Q = \overline{\tau}(M_Q, \theta_Q, \xi_Q)$$

Then, the Cauchy condition is that for $P \geq Q$,

$$\mathbb{M}_W(T_Q - T_P) = \int_W |\theta_P \xi_P - \theta_Q \xi_Q| d\mathcal{H}^n < \epsilon_W(Q)$$

and $\epsilon_W(Q) \to 0$ as $Q \to \infty$. Since $|\xi_P| = 1$ where it is defined, we get

$$\int_W |\theta_P - \theta_Q| d\mathcal{H}^n < \epsilon_W(Q)$$

Hence θ_P converge in $L^1(\mathcal{H}^n)$ locally in U to some integer values function θ. Notice that M_Q also converges in measure to M_+, defined $\{x \in U \mid \theta(x) > 0\}$. Finally,

$$\int_W \theta_P |\xi_P - \xi_Q| d\mathcal{H}^n < 2\epsilon_W(Q)$$

So ξ_P converges in $L^1(\mathcal{H}^n)$ locally in U to some orientation function ξ on M_+. Finally, $T_x M_+ = T_x M_Q$ except on a set of measure $\leq \epsilon_W(Q)$ in $M_+ \cap W$, hence M_+ is n-rectifiable by (2.6). Hence, $M_W(T - T_P) \to 0$, where $T = \overline{\tau}(M_+, \theta, \xi)$. $\square$

4. Deformation Theorem and its Consequences

The deformation theorem states roughly that we can approximate any normal n-current T by a polyhedral current chain P, consisting of currents corresponding to scaled n-faces of the cubical integer lattice Z^{n+l}. This approximation can be chosen so the supports of P and T are arbitrarily close together, and the masses of P and T are of the same order. The deformation theorem turns out to be extremely useful, and yields surprisingly simple proofs of the important isoperimetric inequality and boundary rectifiability property, which we will need in the next section.

The geometric idea of the proof is illustrated clearly in the case when the current is a 1-manifold in 3-space, approximated by edges of cubes in the lattice $\mathbb{Z}^3$. In order to get the polyhedral chain to approximate the curve, we first project it radially (from the centers of cubes outwards) onto the faces of the cubes via map ψ^2, and then radially from the centers of those faces onto the edges of the cubes via map ψ^1. The union of all edges of the lattice that cover the image of the composition ψ of these two projections, with multiplicity, will be our polyhedral approximation. Of course, because we are approximating a curve with straight line segments of length 1, this approximation will be rough. In order to get a better one, we can first scale the curve by some factor > 1, get a polyhedral chain for that scaled curve, and then scale the approximation back to fit the original curve.

The only possible snag we might hit during this procedure is if the curve wraps very tightly, or passes through, one of the centers of the cubes (as in Figure 4). Then the mass of the resulting approximation could become arbitrarily large. The proof of [4] avoids this issue by instead projecting radially away from some carefully chosen point a close to the center of the cube. Of course, proving that we can pick such a point is difficult.

To begin to generalize and formalize this idea, we first introduce some notation. Let $\mathbf{C} = [0,1]^{n+l} \subset \mathbb{R}^{n+l}$ denote the standard unit cube, let $q = (1/2, \ldots, 1/2)$ be its center, and let $S_1, \ldots S_N$ be all the linear subspaces in $\mathbb{R}^{n+l}$ that contain faces of $\mathbf{C}$. In $\mathbb{R}^3$, those will be the xy, xz, and yz planes. Let p_i denote the orthogonal projection onto S_i. As usual, let $\mathbb{Z}^{n+l}$ denote the integer lattice in $\mathbb{R}^{n+l}$. We can cover $\mathbb{R}^{n+l}$ with the following union of the copies of $\mathbf{C}$:

$$\mathbb{R}^{n+l} = \bigcup_{z \in \mathbb{Z}^{n+l}} (z + \mathbf{C}) \tag{4.1}$$

Denote by L_j the j-skeleton of this decomposition. For example, if $n + l = 3$, $L_0 = \mathbb{Z}^3$ is the union of vertices of the lattice, L_1 is the union of edges (points with at least 2 of the 3 coordinates being integers), and L_2 is the union of faces of the cubes (1 of 3 coordinates is an integer). Denote by $\mathcal{L}_j$ the set of j-faces F in L_j. $\mathcal{L}_j(\rho)$ is the set L_j with all faces scaled by a factor of ρ. Finally, denote by $L_{k-1}(a)$ the shifted $(k-1)$-skeleton $a + L_{k-1}$ for $a \in B_{1/4}(q)$, and by $L_{k-1}(a,\rho)$ the set of all points at most ρ away from the skeleton ($\rho < 1/4$). Note that we can construct $L_{k-1}(a,\rho)$ out of tubes surrounding the inverse images of the projections onto S_j as such:

$$L_{k-1}(a,\rho) = \bigcup_{j=1}^{N} \bigcup_{z \in \mathbb{Z}^{n+l} \cap S_j} p_j^{-1}(B_\rho(p_j(a) + z)) \tag{4.2}$$

This set will be the setting in which to account for 'wrapping' in k dimensions. This way, with say $k = 2$, we could say a 2-current coils around some axis through a point.

Now, we want a map ψ that generalizes the one we described earlier.

Lemma 4.3. *For any $a \in B_{1/4}(q)$ there is a locally Lipschitz map $\psi : \mathbb{R}^{n+l} \setminus L_{k-1}(a) \to \mathbb{R}^{n+l} \setminus L_{k-1}(a)$ such that*

(1) ψ maps $C \setminus L_{k-1}(a)$ to L_n, and is the identity on $C \cap L_n$.

(2) $|D\psi(x)| \leq \frac{c}{\rho}$ for almost every $x \in C \setminus L_{k-1}(a,\rho)$, $0 < \rho < 1/4$, and c dependent on n and k.

Proof. We will first construct an appropriate ψ, (1) and the fact that ψ is locally Lipschitz will then be obvious given the construction. For any $j \geq n+1$, let a_F be the orthogonal projection of a onto a j-face $F \in \mathcal{L}_j$ ($a_F = a$ if $F = C$), and denote by ψ_F the radial retraction of $\overline{F} \setminus \{a_F\}$ onto ∂F. So, $x \in \overline{F} \setminus \{a_F\}$ maps to $y \in \partial F$ such that the line passing through x and y also passes through a_F. Then, define ψ^j, the map from the union of j-faces $\overline{F} \setminus \{a_F\}$ to the union of $j-1$-faces $\overline{G}$ piecewise by $\psi \mid_{\overline{F} \setminus \{a_F\}} = \psi_F$. Finally

$$\psi_0 = \psi^{n+1} \circ \cdots \circ \psi^{n+l} \mid_{C \setminus L_{k-1}(a)} \tag{4.4}$$

ψ_0 is a map only on a subset of C, to get a map ψ on $\mathbb{R}^{n+l} \setminus L_{k-1}(a)$ set $\psi(x+z) = \psi_0(x) + z$ for $x \in C \setminus L_{k-1}(a)$ and $z \in \mathbb{Z}^{n+l}$. By the construction of ψ_F, it should be clear why ψ is the identity on $C \setminus L_n$. It is also Lipschitz because it fixes a given cube $C + z$.

Now we prove the condition (2) on the derivative of ψ by induction on l, the additional dimensions of the ambient space on top of n. Recall that by definition

$$|D\psi(x)| = \limsup_{y \to x} \frac{|\psi(y) - \psi(x)|}{|y - x|} \tag{4.5}$$

If $l = 1$, ψ on C is just ψ^{n+1}, the radial retract away from a. Then we know we have $\sup |D\psi| < c/\rho$.

Now suppose that the statement is true for some $l-1$. Let x be any point in the interior of $C \setminus L_{k-1}(a,\rho)$, and let F be a face that contains $y = \psi^{n+l}(x)$. If p_F is the orthogonal projection onto F, by orthogonality of the lattice frame, $a_F \in L_{k-1}(a)$, so we have

$$|y - a_F| \geq \operatorname{dist}(y, L_{k-1})$$

But also by definition of ψ^{n+l}, and because $\rho < 1/4$, we have

$$|y - a_F| \leq \frac{3}{4}\frac{|p_F(x-a)|}{|x-a|}$$

Let $\tilde{L}_{k-2} = L_{k-1}(a) \cap F$, and let $\tilde{\psi}$ be the radial retract of $F \setminus \tilde{L}_{k-2}(a)$ onto the n-faces of F (so, $\tilde{\psi}$ is what one needs to compose ψ^{n+l} by to get ψ restricted to C). But then we can apply the induction hypothesis together with the two inequalities above to get

$$|D\tilde{\psi}(y)| \leq \frac{c}{\operatorname{dist}(y, \tilde{L}_{k-2}(a))} \leq \frac{4c}{3}\frac{|x-a|}{\operatorname{dist}(x, L_{k-1}(a))}$$

By the base case of our induction we have

$$|D\psi^{n+l}(x)| \leq \frac{c}{|x-a|}$$

Hence by chain rule we conclude

$$|D\psi(x)| \leq |D\tilde{\psi}(y)||D\psi^{n+l}(x)| \leq \frac{c}{\operatorname{dist}(x, L_{k-1}(a))}$$

This proves (2). □

In figure (4) the images of the black 1-current under ψ^3 and $\psi^3 \circ \psi^2 = \psi_0$ are shown in orange and purple respectively. Notice how projection with respect to a eliminates the problem that would be posed by the coiling around q if we projected instead with respect to q.

Condition (2) in the above lemma will be crucial in establishing the mass bounds of the deformation theorem, which is stated formally below.

Theorem 4.6 (Deformation Theorem)**.** *Suppose T is a normal n-current in $\mathbb{R}^{n+l}$. Then we can write*

$$T = P + \partial R + S$$

Where $P = \sum_{F \in \mathcal{L}_n} \beta_F [[F]]$ for some $\beta_F \in \mathbb{R}$ is a polyhedral chain such that

$$\mathbb{M}(P), \mathbb{M}(R) \leq c\mathbb{M}(T) \quad \mathbb{M}(\partial P), \mathbb{M}(S) \leq c\mathbb{M}(\partial T) \tag{4.7}$$

(C dependent on n and k) and

$$\operatorname{spt} P \cup \operatorname{spt} R \subset \{x : \operatorname{dist}(x, \operatorname{spt} T) < 2\sqrt{n+l}\} \tag{4.8}$$

$$\operatorname{spt} \partial P \cup \operatorname{spt} \partial R \subset \{x : \operatorname{dist}(x, \operatorname{spt} \partial T) < 2\sqrt{n+l}\} \tag{4.9}$$

If T and ∂T are integer multiplicity then so are P, R, and S.

For a graphical demonstration of this theorem in the case T is a 1-current contained in $U = \mathbb{R}^3$, see figure (4). In this figure, the lattice has side length 1, $R = S_1 + S_2$.

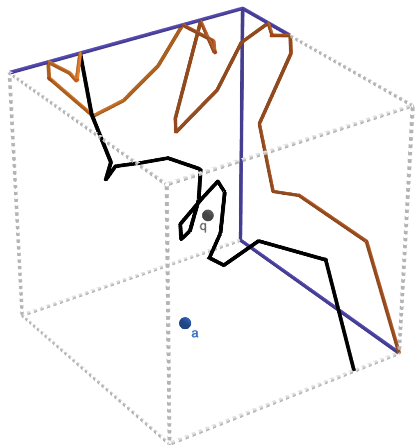

FIGURE 3. Images of the constituent projections of ψ of the deformation map in lemma (4.3).

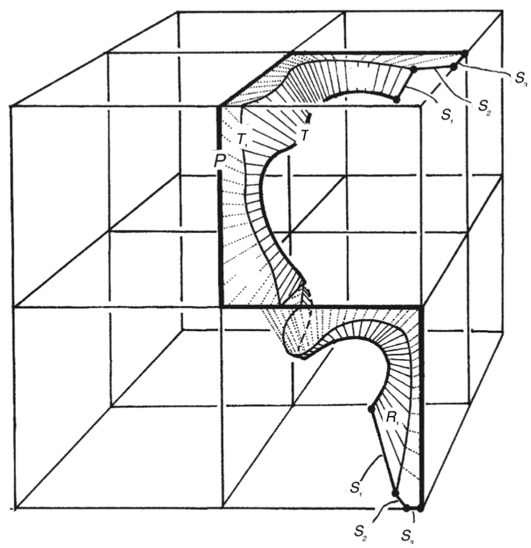

FIGURE 4. Illustration of currents given by the deformation theorem. ([3] pg. 62)

There will be three steps to proving this theorem. First, we will show we can choose a point a such that $L_{k-1}(a, \rho)$ contains portions of T and ∂T bounded by a quantity on the order of ρ^{n+1} (so, we find some a around which T and ∂T don't 'coil', in any copy of C). Then, by taking the limit of $\rho \to 0$, we will find currents Q, R_1 and S_1 so that Q (limit of pushforwards of T without $L_{k-1}(a, \rho)$) is supported on L_n, and the mass/distance conditions of the theorem hold. Finally, we will show that this won't be spoiled if we replace Q with a polyhedral chain P that contains it.

Proof. Let x_j be the central point of the $n+1$-face F_j contained in a subspace S_j of $\mathbb{R}^{n+l}$. Define the 'good' subset $G_j \subset F_j \cap B_{1/4}(x_j)$ as the set of all points g that satisfy

$$M(T \llcorner \cup_{z\in\mathbb{Z}^{n+l}\cap S_j} p_j^{-1}(B_\rho(g+z))) \leq \beta\rho^{n+1}\mathbb{M}(T)$$

So, T doesn't 'coil' around a in the direction normal to S_j in any of the cubes in the lattice. We will now show that the bad 'coiling' set $C_j = F_j \cup B_{1/4}(x_j) \setminus G_j$ has a small $(n+1)$-measure. More speficially, for some β,

$$\mathcal{L}^{n+1}(C_j) \leq 20^{n+1}\beta^{-1}\omega_{n+1}\left(\frac{1}{4}\right)^{1/4} \tag{4.10}$$

We will choose an appropriately large β soon.

Choose a cover $\{B_{\rho_c}(c)\}$ of C_j so that

$$\mathbb{M}(T \llcorner \cup_{z\in\mathbb{Z}^{n+l}\cap S_j} p_j^{-1}(B_{\rho_c}(c+z))) > \beta\rho_c^{n+1}\mathbb{M}(T) \tag{4.11}$$

for each c. By the Vitali 5-covering lemma, there exists a pairwise disjoint subcollection $\{B_{\rho_l}(c_l)\}$ of the cover so that $C_j \subset \cup_l B_{5\rho_l}(c_l)$. Adding up the inequalities (4.11) for every member of the cover, we get the following by disjointness of $B_{\rho_l}(c_l)$:

$$M(T) \geq \beta^{-1}\left(\sum \rho_l^{n+1}\right)\mathbb{M}(T)$$

Which implies $\beta^{-1} \geq \sum \rho_l^{n+1}$. Because $\{B_{5\rho_l}(c_l)\}$ is a cover, we have

$$\mathcal{L}^{n+1}(C_j) \leq \frac{5^{n+1}}{\beta^{-1}}\omega_{n+1}$$

Which, after some algebra, gives us (4.10).

We want a point a around which there's no coiling in any direction, so we will need a way to bound from below the intersection of G_j for all j. For this we use the fact

$$\mathcal{L}^{n+l}(p^{-1}(G_j) \cap B_{1/4}(q)) \geq \left(1 - \frac{\omega_{n+1}}{\omega_{n+l}}20^{n+1}\beta^{-1}\right)\omega_{n+l}\left(\frac{1}{4}\right)^{n+l} \tag{4.12}$$

Then, choosing β small enough so that $20^{n+1}\omega_{n+1}N\beta^{-1} < \frac{\omega_{n+l}}{2(n+l)}$, we know by the above inequality that the set of 'good' a, $G_T = \cap_{j=1}^N (p_j^{-1}(G_j) \cap B_{1/4}(q))$, has non-zero measure. Hence, there is a point a for which

$$\mathbb{M}(T \llcorner L_{k-1}(a,\rho)) \leq N\beta\rho^{n+1}\mathbb{M}(T) \tag{4.13}$$

However, by the way we chose β, the set G_T occupies more than half of $B_{1/4}(q)$. Hence, repeating the argument above with ∂T to get the set $G_{\partial T}$ of good a for ∂T, we find that $G_T \cap G_{\partial T} \neq \emptyset$. So, denoting for brevity

$$T_\rho = T \llcorner L_{k-1}(a,\rho) \quad \partial T_\rho = \partial T \llcorner L_{k-1}(a,\rho)$$

We know there exists an a so that

$$\mathbb{M}(T_\rho) \leq C\rho^{n+1}\mathbb{M}(T) \quad \mathbb{M}((\partial T)_\rho) \leq C\rho^{n+1}\mathbb{M}(\partial T) \tag{4.14}$$

for some $0 < \rho < 1/4$. This concludes the first step of the proof.

Let h be an affine homotopy on $\mathbb{R}^{n+l} \setminus L_{k-1}(a,\sigma)$ $(\sigma > 0)$ between the identity and $\psi(x)$, i.e. $h(x,t) = x + t(\psi(x) - x)$. Then, applying (3.26) and using (4.3), we get

$$
\begin{aligned}
\mathbb{M}(\psi_{\#}(T_\rho - T_{\rho/2})) &\le \mathbb{M}(h_{\#}([[0,1]] \times T)) \\
&\le \sup|\psi(x) - x| \cdot \sup\left(1 + \frac{c}{\rho}\right)^n \mathbb{M}_{W_h}(T) \\
&\le \frac{c}{\rho^n}\rho^{n+1}\mathbb{M}(T) \le c\rho\mathbb{M}(T) \qquad (4.14)
\end{aligned}
$$

By a similar technique

$$
\mathbb{M}(\psi_{\#}(\partial T_\rho - \partial T_{\rho/2})) \le \frac{c}{\rho^{n-1}}\rho^{n+1}\mathbb{M}(\partial T) \le c\rho\mathbb{M}(\partial T) \qquad (4.15)
$$

But then by adding $T_\rho - T_{\rho/2}$, $T_{\rho/2} - T_{\rho/4}$, etc., and applying (4.14) we get

$$
\mathbb{M}(\psi_{\#}(T_\rho - T_{\rho/2^\nu})) \le \mathbb{M}\left(\sum_{\nu=1}^{\infty} \psi_{\#}(T_{\rho/2^{\nu-1}} - T_{\rho/2^\nu})\right) \le 2c\rho\mathbb{M}(T)
$$

A similar condition is true for ∂T. By exploiting the arbitrariness of ρ and η, we establish the following property for any $0 < \sigma < 1$:

$$
\mathbb{M}(\psi_{\#}(T - T_\sigma)) \le c\mathbb{M}(T) \quad \mathbb{M}(\psi_{\#}(\partial T - \partial T_\sigma)) \le c\mathbb{M}(\partial T) \qquad (4.16)
$$

Then, by picking $\rho_k = 1/2^{k+2}$, by (4.14) and (4.15), we get that the following are Cauchy sequences with respect to $\mathbb{M}$:

$$
\begin{gathered}
\psi(T - T_{\rho_k})_k \\
h\#([[(0,1)]] \times (T - T_{\rho_k}))_k \\
\psi(\partial T - \partial T_{\rho_k})_k \\
h\#([[(0,1)]] \times \partial(T - T_{\rho_k}))_k
\end{gathered}
$$

Hence there are currents $Q, S_1 \in \mathcal{D}_n(\mathbb{R}^{n+l})$ and $R_1 \in \mathcal{D}_{n+1}(\mathbb{R}^{n+l})$ that are the limits of the first, second, and last sequences above with respect to the mass norm. By the homotopy formula we get

$$
(T - T_{\rho_k}) - \psi_{\$}(T - T_{\rho_k}) = \partial(h_{\#}([[(0,1)]] \times (T - T_{\rho_k}))) - h_{\#}([[(0,1)]] \times \partial(T - T_{\rho_k}))
$$

Taking the limit $k \to \infty$, (switching the sign of S_1 for convenience) we get

$$
T - Q = \partial R_1 + S_1 \qquad (4.17)
$$

Because $\operatorname{spt}\psi_{\#}(T - T_{\rho_n}) \subset L_n$, $\operatorname{spt} Q \subset L_n$ also. Also, because ψ maps copies of C to themselves, we satisfy the distance requirements

$$
\begin{aligned}
&\operatorname{spt} R_1 \cup \operatorname{spt} Q \subset \{x : \operatorname{dist}(x, \operatorname{spt} T) < \sqrt{n+l}\} \\
&\operatorname{spt} S_1 \subset \{x : \operatorname{dist}(x, \operatorname{spt}\partial T) < \sqrt{n+l}\}
\end{aligned}
$$

and mass requirements (by (4.16) and semicontinuity of mass under weak convergence)

$$
\mathbb{M}(Q) \le c\mathbb{M}(T) \quad \mathbb{M}(R_1) \le c\mathbb{M}(T) \quad \mathbb{M}(S_1) \le c\mathbb{M}(\partial T) \quad \mathbb{M}(\partial Q) \le c\mathbb{M}(\partial T) \qquad (4.18)
$$

This concludes the second step of the proof.

Denote by $\tilde{F}$ the interior of a face $F \in \mathcal{L}_n$. Now realize that by our construction of Q, $p_{\#}(Q \llcorner \tilde{F}) = Q \llcorner \tilde{F}$. Hence, by (3.28), we can express $Q \llcorner \tilde{F}$ as

$$
(Q \llcorner \tilde{F}) = \int_{\tilde{F}} \langle e_1 \wedge \cdots \wedge e_n, \omega(x)\rangle \theta_F(x) d\mathcal{L}^n(x)
$$

Where

$$\mathbb{M}(Q \llcorner \tilde{F}) = \int_{\tilde{F}} |\theta_F| d\mathcal{L}^n \quad \mathbb{M}((\partial Q) \llcorner \tilde{F}) = \int_{\tilde{F}} |D\theta_F| d\mathcal{L}^n(x)$$

Now pick $\beta_F \in \mathbb{Z}$ such that θ_F is either at least or at most β on a set of $\mathcal{L}^n$ measure $1/2$. Then, by the Poincaré inequality, we get

$$\mathbb{M}(Q \llcorner - \beta_F[[F]]) \leq c \int_{\tilde{F}} |D\theta_F| = c\mathbb{M}(\partial Q \llcorner \tilde{F}) \tag{4.19}$$

$$\mathbb{M}(\partial(Q \llcorner - \beta_F[[F]])) \leq c \int_{\tilde{F}} |D\theta_F| = c\mathbb{M}(\partial Q \llcorner \tilde{F}) \tag{4.20}$$

(4.21)

Hence if we define $P = \sum_{F \in \mathcal{L}_n} \beta_F[[F]]$ we have

$$\mathbb{M}(Q - P) \leq c\mathbb{M}(\partial Q) \tag{4.22}$$

$$\mathbb{M}(\partial Q - \partial P) \leq c\mathbb{M}(\partial Q) \tag{4.23}$$

By our choice of β_F we have $|\beta_F| \leq 2 \int |\theta_F|$, so

$$\mathbb{M}(P) \leq c\mathbb{M}(Q) \quad \mathbb{M}(\partial P) \leq c\mathbb{M}(\partial Q)$$

Rewrite (4.17) by setting $R = R_1$ and $S = S_1 + (Q - P)$, and we are done. The additional fact about preservation of integer multiplicity follows by applying the completeness of integral currents in appropriate places throughout the proof. The scaled version follows by the method described earlier. □

The deformation theorem gives us two important facts that we will use frequently in the next section. The first is an isoperimetric inequality for integer multiplicity currents.

Theorem 4.24 (Isoperimetric Theorem)**.** *Suppose $T \in \mathcal{D}_{n-1}(\mathbb{R}^{n+l})$ is an integer multiplicity current, $n \geq 2$, $\operatorname{spt} T$ is compact, and $\partial T = 0$. Then there is an integer multiplicity current $R \in \mathcal{D}_n(\mathbb{R}^{n+l})$ with $\operatorname{spt} R$ compact, $\partial R = T$, and*

$$\mathbb{M}(R)^{\frac{n-1}{n}} \leq C\mathbb{M}(T)$$

where C is a constant dependent on n and k.

Proof. Let P, R, S be integer multiplicity currents given by (4.6) for some $\rho > 0$, and note $S = 0$ because $\partial T = 0$. Because P is a sum of polyhedral currents, for some positive integer $N(\rho)$ we have

$$\mathbb{M}(P) = N(\rho)\rho^{n-1}$$

But also $\mathbb{M}(P) \leq c\mathbb{M}(T)$ for a constant c given by the deformation theorem. Setting $\rho = (2c\mathbb{M}(T))^{1/(n-1)}$, we get $N(\rho) \leq 1/2$, hence $N(\rho) = 0$ and $P = 0$. Thus (4.6) gives $T = \partial R$, and the mass bound gives us $\mathbb{M}(R) \leq c\rho\mathbb{M}(T) = C(\mathbb{M}(T))^{1/(n-1)}$. □

The next fact is the weak polyhedral approximation theorem.

Theorem 4.25 (Weak Polyhedral Approximation)**.** *Given any integer multiplicity normal $T \in \mathcal{D}_n(U)$ there is a sequence $\{P_k\}$ of polyhedral n-currents*

$$P_k = \sum_{F \in \mathcal{L}_n(\rho_k)} \beta_R^k[[F]]$$

With $\rho_k \downarrow 0$, such that $P_k \rightharpoonup T$.

Proof Sketch: In the case that $U = \mathbb{R}^{n+l}$, the sequence of currents P_k given by the scaled version of the deformation theorem using any sequence of $\rho_k \downarrow 0$ is sufficient. If U is some open subset of $\mathbb{R}^{n+l}$, use a non-negative Lipschitz function ϕ that is supported on U, and such that $\{x \in U \mid \phi(x) > \lambda\}$ is compactly contained in U for all $\lambda > 0$.Consider currents $T_\lambda = T \llcorner \{\phi < \lambda\}$ on $\mathbb{R}^{n+l}$. By the above argument, we have a sequence of polyhedral currents converging weakly to every T_λ, hence we have a sequence converging weakly to T as $\lambda \downarrow 0$. □

5. Rectifiability and Compactness

The deformation theorem, weak approximation by polyhedral currents, and the presence of an isoperimetric inequality for normal currents suggest that they have some intrinsic relationship to geometric objects. However, with a minor additional condition on the upper density of their associated measure μ_T, we can prove that all normal currents T are derived from actions of varifolds on forms. This result is called the rectifiability theorem. When proving this theorem, we will be relying heavily on the following lemma and its consequences.

Lemma 5.1. *If $\alpha \in I_{n,P}$, denote by p_α the orthogonal projection of $\mathbb{R}^P$ onto $\mathbb{R}^n$ given by $(x^1, \ldots, x^P) \mapsto (x^{i_1}, \ldots, x^{i_n})$. Suppose E is a closed subset of U, U open in $\mathbb{R}^P$, with $\mathcal{L}^n(p_\alpha(E)) = 0$ for each $\alpha \in I_{n,P}$. Then $T \llcorner E = 0$ whenever $T \in \mathcal{D}_n(U)$ is normal.*

Proof. For $\omega \in \mathcal{D}^n(U)$ as $\omega = \sum_{\alpha \in I_{n,P}} \omega_\alpha dx^\alpha$ for $\omega_\alpha \in C_c^\infty(U)$. Then we write

$$\begin{aligned} T(\omega) = \sum_\alpha T(\omega_\alpha dx^\alpha) = \sum_\alpha (T \llcorner \omega_\alpha)(dx^\alpha) &= \sum_\alpha (T \llcorner \omega_\alpha) p_\alpha^\# dy \\ &= \sum_\alpha p_{\alpha\#}(T \llcorner \omega_\alpha)(dy) \end{aligned}$$

Where $dy = dy^1 \wedge dy^n$ for $y^1, \ldots y^n$ the standard coordinate functions on $\mathbb{R}^n$. Choosing the $\beta \in I_{n,P}$ for which $p_{\beta\#}(T \llcorner \omega_\beta)(dy)$ is maximal among the terms in the sum, and replacing all the other terms with it, we get the inequality

$$\mathbb{M}(T) \leq N \mathbb{M}(p_{\beta\#}(T \llcorner \omega_\beta)) \tag{5.2}$$

For some positive integer N.

We now want to show that $p_{\beta\#}(T \llcorner \omega_\beta)$ has a finite boundary, since that lets us use (3.28). Since for any $\eta \in \mathcal{D}^{n-1}(U)$ product rule yields

$$\partial(T \llcorner \omega_\beta)(\eta) = T(\omega_\beta \eta) - T(d\omega_\beta \wedge \eta)$$

we have

$$\mathbb{M}_W(\partial(T \llcorner \omega_\beta)) \leq \mathbb{M}_W(\partial T)|\omega_\beta| + \mathbb{M}_W(T)|d\omega_\beta|$$

and

$$\mathbb{M}(\partial p_{\beta\#}(T \llcorner \omega_\beta)) = \mathbb{M}(p_{\beta\#}\partial(T \llcorner \omega_\beta)) \leq \mathbb{M}(\partial(T \llcorner \omega_\beta)) < \infty$$

Hence, by (3.28) we can express

$$p_{\beta\#}(T \llcorner \omega_\beta)(\eta) = \int_{p_\beta(U)} \langle \eta, e_1 \wedge \ldots e_n \rangle \theta_\beta dL^n$$

This implies $p_{\beta\#}(T \llcorner \omega_\beta) \llcorner p_\beta(E) = 0$ by the hypothesis.

We then get another inequality involving the mass of $p_{\beta\#}(T\llcorner\omega_\beta)$ for any W such that $\operatorname{spt}\omega \subset W \subset U$:

$$\begin{aligned}\mathbb{M}(p_{\beta\#}(T\llcorner\omega_\beta)) &\leq \mathbb{M}(p_{\beta\#}(T\llcorner\omega_\beta)\llcorner(\mathbb{R}^P \setminus p_\beta(E))) \\ &= \mathbb{M}(p_{\beta\#}((T\llcorner\omega_\beta)\llcorner(\mathbb{R}^P \setminus p_\beta^{-1}p_\beta E)))\end{aligned}$$

By the homotopy formula (3.25), we can continue this chain:

$$\begin{aligned}\mathbb{M}(p_{\beta\#}((T\llcorner\omega_\beta)\llcorner(\mathbb{R}^P \setminus p_\beta^{-1}p_\beta E))) &\leq \mathbb{M}((T\llcorner\omega_\beta)\llcorner(\mathbb{R}^P \setminus p_\beta^{-1}p_\beta E)) \\ &\leq \mathbb{M}_W(T\llcorner(\mathbb{R}^P \setminus E)) \cdot |\omega_\beta| \\ &\leq \mathbb{M}_W(T\llcorner(\mathbb{R}^P \setminus E)) \cdot |\omega_\beta|\end{aligned}$$

Combining this last inequality with (5.2), we get

$$\mathbb{M}_W(T) \leq C\mathbb{M}_W(T\llcorner(\mathbb{R}^P \setminus E))$$

which also implies

$$\mathbb{M}_W(T\llcorner E) \leq C\mathbb{M}_W(T\llcorner(\mathbb{R}^P \setminus E))$$

Which, by definition of μ_T, says

$$\mu_T(W \cap E) \leq C\mu_T(W \setminus E)$$

Let K be some compact subset of E. Then we can choose a decreasing sequence of sets $\{W_q\}$ the intersection of which is K. Using the above inequality on this sequence gives us $\mu_T(K \cap E) \leq 0$, so $\mathbb{M}(T\llcorner K) = 0$ and in fact, $\mu_T(K) = 0$. However, since μ_T is a Radon measure, $\mu_T(E)$ is the supremum of the measures of all possible K, all of which are 0. Thus, $\mu_T(E) = 0$, as desired. □

We know the hypothesis $\mathcal{L}^n(p_\alpha(E))$ is satisfied if $\mathcal{H}^n(E) = 0$. Further, because μ_T is a Radon measure, for any Borel set C, $\mu_T(C) = \sup \mu_T(E)$ where $E \subset W$ is closed. Hence, $H^n(C) = 0$ implies $\mu_T(C) = 0$, so we get

Corollary 5.3. *If T is a normal current, μ_T is absolutely continuous with respect to $\mathcal{H}^n$ in U.*

On the other hand, let Q be any orthogonal transformation of $\mathbb{R}^P$. We have $\mathbb{M}_W(T) = \mathbb{M}_{QW}(Q_\# T)$, so $\mathbb{M}_W(T) < \infty$ implies $\mu_{Q_\# T}(Q(A)) = \mu_T(A)$ for each $A \subset U$, which guarantees

$$\mathcal{L}^n(Q(E)) = 0 \implies \mu_T(E) = 0 \tag{5.4}$$

We can now state and prove the Rectifiability Theorem.

Theorem 5.5 (Rectifiability Theorem)**.** *Suppose T is a normal current, and*

$$\Theta^{*n}(\mu_T, x) > 0$$

for μ_T-almost-every $x \in U$. Then, T is rectifiable, or in other words, $T = \tau(M, \theta, \xi)$, where (M, θ) is a varifold and ξ is an orientation on its tangent space.

Proof. Note that because μ_T is a Radon measure, for $t > 0$ and $A \subset W$ a subset of an open $W \subset\subset U$,

$$\mathcal{H}^n(\{x \in A \mid \Theta^{*n}(\mu_T, x) > t\}) \leq t^{-1}\mu_T(A) \leq t^{-1}\mu_T(W) \tag{5.6}$$

A similar inequality is true for ∂T. Taking $t \to \infty$, and covering U with appropriate W, gives

$$\mathcal{H}^n(\{x \in U \mid \Theta^{*n}(\mu_T, x) = \infty\}) = \mathcal{H}^n(\{x \in U \mid \Theta^{*n}(\mu_{\partial T}, x) = \infty\}) = 0$$

By (5.3), we get

$$\mu_T(\{x \in U \mid \Theta^{*n}(\mu_T, x) = \infty\}) = \mu_T(\{x \in U \mid \Theta^{*n}(\mu_{\partial T}, x) = \infty\}) = 0 \tag{5.7}$$

Denote by M the set of all $x \in U$ such that $\Theta^{*n}(\mu_T, x) > 0$. Writing

$$M = \cup_{j=1}^{\infty} M_j \quad M_j = \{x \in M \mid \Theta^{*n}(\mu_T, x) > 1/j\}$$

and applying (5.6), we realize M must be σ-finite. Suppose $P \subset M$ is H^n purely unrectifiable. Then, by σ-finiteness of M, we can apply the structure theorem (2.10) to get an orthogonal transformation Q of $\mathbb{R}^{n+l}$ corresponding to every $\alpha \in I_{n,n+l}$ such that $\mathcal{H}^n(p_\alpha(QP)) = 0$, where p_α is defined as in (5.1). Then, by (5.4) we conclude $\mu_T(P) = 0$. In other words, every purely unrectifiable subset of M has zero $\mathcal{H}^n$ measure. By (2.9), we have shown that M is rectifiable.

Recalling once more that μ_T is absolutely continuous with respect to $\mathcal{H}^n$, we invoke the Radon-Nikodym theorem to get that for any μ_T measurable A:

$$\mu_T(A) = \int_A \theta d\mathcal{H}^n$$

for some positive locally $\mathcal{H}^n$ integrable θ supported on U. Then by Riesz Representation theorem we get

$$T(\omega) = \int_U \langle \omega, \vec{T} \rangle d\mu_T = \int_U \langle \omega, \xi \rangle \theta d\mathcal{H}^n$$

For some $\mathcal{H}^n$ measurable function with values in $\Lambda_n(\mathbb{R}^{n+l})$ ξ with $|\xi| = 1$. To conclude the proof we only need to show that ξ is an orientation of $T_x M$ for almost every $x \in M$, so $\xi(x) = \pm\tau_1 \wedge \cdots \wedge \tau_n$ almost everywhere for an orthonormal basis $\{\tau_i\}$ of $T_x M$. We do this by using the familiar 'zoom-in' function $\eta_{x,\lambda}$ defined the same way as in (2.3), and proving that $\eta_{x,\lambda\#}T$ for a specific x behaves 'locally' like integration over a linear tangent space for almost every x. This will force ξ to be of the appropriate form.

By (2.1) we may write M as a union of a measure zero set and M_j, pairwise disjoint subsets of some C^1 $\mathbb{R}^{n+l}$-submanifolds N_j. By the upper density theorem, the density of $\mu_T \llcorner ((N_j \setminus M_j) \cup (\cup_{k \neq j} M_k))$ is 0 for almost every $x \in M_j$. Then for every such x, we have

$$\eta_{x,\lambda\#}T(\omega) = T(\eta^{\#}_{x,\lambda}\omega) = \int_{N_j} \langle \xi(y), \eta^{\#}_{x,\lambda}\omega(y) \rangle \theta d\mathcal{H}^n(y) + \epsilon(\lambda)$$

where $\epsilon(\lambda) \downarrow 0$ as $\lambda \downarrow 0$. After performing the substitution $z = \eta_{x,\lambda}(y)$ (equivalently $y = x + \lambda z$), we get

$$\eta_{x,\lambda\#}T(\omega) = \int_{\eta_{x,\lambda}(N_j)} \langle \xi(x + \lambda z), \omega(z) \rangle \theta(x + \lambda z) d\mathcal{H}^n(z) + \epsilon(\lambda)$$

Because N_j is a continuous manifold, the limit of $\eta_{x,\lambda}(N_j)$ as $\lambda \downarrow 0$ is the tangent space $T_x N_j$ of N_j at x, and

$$\lim_{\lambda \downarrow 0} \eta_{x,\lambda\#}T(\omega) = \theta(x) \int_{T_x N_j} \langle \xi(x), \omega(z) \rangle d\mathcal{H}^n(z)$$

for almost every $x \in M_j$. Thus $T_x N_j$ equals $T_x M$, the approximate tangent space of M at x, for all such $x \in M_j$. Examining the boundary $\partial\eta_{x,\lambda\#}T(\omega)$, however, we find that as $\lambda \downarrow 0$,

$$\partial \eta_{x,\lambda\#} T(\omega) = \partial T(\eta_{x,\lambda}^{\#}\omega) = \int_{B_{x,\lambda}(y)} \langle \omega_{\eta_{x,\lambda(x)}}, \eta_{x,\lambda\#}\vec{\partial T}\rangle d\mu_{\partial T}$$
$$\leq C|\omega|\lambda^{1-n}\mu_{\partial T}(B_{\lambda R}(x)) \to 0$$

by (5.7) and the definition of upper density. Thus by the above two statements, we know there is a sequence of factors λ_l such that $\eta_{x,\lambda_l\#}T \rightharpoonup S_x$, where $S_x \in \mathcal{D}_n(\mathbb{R}^{n+l})$ and

$$S_x(\omega) = \theta(x)\int_{T_xM} \langle \xi(x), \omega(z)\rangle d\mathcal{H}^n(z)$$

Hence $\partial S_x = 0$. To see how this ensures ξ orients T_xM, let $\tau_1, \ldots, \tau_n, \ldots, \tau_{n+l}$ be an extension of the orthonormal basis $\tau_1, \ldots, \tau_n$ of T_xM, and select $\omega \in \mathcal{D}^{n-1}(U)$ defined by $\omega(y) = y^j\phi(y)d\tau^\alpha$ where $\alpha \in I_{n-1,n+l}$, $j \geq n+1$, $\{dy^i\}$ are the coordinate functions on $\mathbb{R}^{n+l}$ associated with the $\{\tau_i\}$ basis, and ϕ is an arbitrary smooth compactly supported function on $\mathbb{R}^{n+l}$. Then since $\tau_j = 0$ on T_xM we deduce

$$\partial S_x(\omega) = S_x(d\omega) = \theta(x)\int_{T_xM} \phi(y)\langle \xi(x), dy^j \wedge dy^\alpha\rangle d\mathcal{H}^n =$$
$$\theta(x)\int_{T_xM} \phi(y)\xi(x)\cdot(\tau_j \wedge \tau_\alpha)d\mathcal{H}^n(y)$$

Because $\partial S_x(\omega) = 0$, and ϕ is arbitrary, we have $\xi(x)\cdot(\tau_j\wedge\tau_\alpha) = 0$ for j, α as above. Then, expressing $\xi(x) = \sum_{\beta\in I_{n,n+l}} w^\beta\tau_\beta$, we know every w^β with β containing any j with $j \geq n+1$ must be 0. So, recalling that $|\xi(x)| = 1$, $\xi(x)$ is forced to be $\pm\tau_1 \wedge \cdots \wedge \tau_n$, as required. □

We are now in a position to prove the compactness theorem for integral currents. First, we will prove a weak version of the theorem, and later we will remove the extra assumptions.

Theorem 5.8. *Let $\{T_j\} \subset \mathcal{D}_n(\mathbb{R}^P)$ be a sequence of integral currents with integral boundaries, such that $\sup(\mathbb{M}_W(T_j) + \mathbb{M}_W(\partial T_j)) < \infty$ for all bounded W in $\mathbb{R}^P$, and $\operatorname{spt} T_j \subset K$ for some fixed K. Further, let $\partial T = 0$. Suppose $T_j \rightharpoonup T \in \mathcal{D}_n(U)$. Then T is an integer multiplicity current.*

Proof. Note that the theorem is equivalent to Cauchy completeness of $\mathbb{R}^P$ for $n = 0$. Now we proceed by induction on n, supposing the theorem is true for some $n-1$. Our goal is to show that the upper density condition holds on the limit T, then apply (5.5), and finally show the resulting $T = \overline{\tau}(M, \theta, \xi)$ is integral.

Define for some fixed $\xi \in \mathbb{R}^P$ the function $f(r) = \mathbb{M}(T \llcorner B_r(\xi))$ for $r > 0$. By(3.18), (3.17), the definition (3.15) of a slice, and the hypothesis $\partial T = 0$, we get

$$\mathbb{M}_W(\partial(T \llcorner B_r(\xi))) \leq f'(r) \tag{5.9}$$

Suppose $\Theta^{n*}(\mu_T, \xi) < \eta$ for some $\eta > 0$. Then since by definition $\limsup_{\rho\downarrow 0}\frac{f(\rho)}{\omega_n\rho^n} < \eta$, we have for sufficiently small δ:

$$\frac{1}{\delta}\int_0^\delta \frac{d}{dr}(f^{1/n}(r))dr \leq \delta^{-1}f^{1/n}(\delta) \leq \omega_n^{1/n}\eta$$

In other words,

$$\frac{d}{dr}(f^{1/n}(r)) \leq 2\omega_n^{1/n}\eta \tag{5.10}$$

for a set of $r \in (0,\delta)$ of positive measure. Now, by the inductive hypothesis $\partial(T\llcorner B_r(\xi))$ is an integer multiplicity current for almost every $r > 0$, so using the isoperimetric inequality (4.24), we can find an n-integer multiplicity current S_r such that $\partial S_r = \partial(T\llcorner B_r(\xi))$ and for a set of r of positive $\mathcal{L}^1$ measure in $(0,\delta)$,

$$\mathbb{M}(S_r)^{(n-1)/n} \leq c\mathbb{M}(\partial(T\llcorner B_r(\xi))) \leq c\eta\mathbb{M}(T\llcorner B_r(\xi))^{(n-1)/n}$$

where the last inequality is by (5.9) and (5.10). Now, consider some compact subset C of $\{x \in \mathbb{R}^P \mid \Theta^{*n}(\mu_T, x) < \eta\}$. By the Vitali 5-covering lemma, we may select a disjoint cover $B_j = B_{\rho_j}(\xi_j)$ with $\xi_j \in C$ that covers μ_T-almost all of C, $\cup_j B_j \subset \{x \mid \operatorname{dist}(x, C) < \rho\}$, and (by the above use of the isoperimetric theorem) for some integer multiplicity $S_j^(\rho)$ with (1) $\mathbb{M}(S_j^{(\rho)}) \leq c\eta\mathbb{M}(T\llcorner B_j)$ and (2) $\partial S_j^{(\rho)} = \partial(T\llcorner B_j)$. Because of this we have

$$S_j^\rho = T\llcorner B_j = \partial(h_{\xi_j\#}([[(0,1)]] \times (S_j^\rho - T\llcorner B_j)))$$

where $h_{\xi_j}(x,t) = tx - (1-t)\xi_j$ is the affine homotopy centered at ξ_j. Thus by (3.25) and (3.26), we get for an n-form ω,

$$|(S_j^{(\rho)} - T\llcorner B_j)(\omega)\ | \leq c\rho\mathbb{M}(S_j^{(\rho)} - T\llcorner B_j)|d\omega|$$

and by the isoperimetric theorem bound,

$$|(S_j^{(\rho)} - T\llcorner B_j)(\omega)\ | \leq c\rho\mathbb{M}(T\llcorner B_j)|d\omega|$$

Thus, taking $\rho \downarrow 0$, we get

$$T + \sum_j (S_j^{(\rho)} - T\llcorner B_j) \rightharpoonup T$$

Because the series $\sum_j S_j^{(\rho)}$ and $\sum_j T\llcorner B_j$ are absolutely convergent with respect to $\mathbb{M}$ by (1) and disjointness of B_j, we can split apart the sum to get $T\llcorner(\mathbb{R}^P \setminus \cup_j B_j) + \sum_j S_j^{(\rho)}$ on the left hand side, and hence conclude

$$\mu_T(\{x \mid \operatorname{dist}(x,C) < \rho\}) \leq \mu_T(\{x \mid \operatorname{dist}(x,C) < \rho\}) + c\eta\mu_T(\{x \mid \operatorname{dist}(x,C) < \rho\})$$

By choosing η such that $c\eta < \frac{1}{2}$ and hence

$$\mu_T(\{x \mid \operatorname{dist}(x,C) < \rho\}) \leq 2\mu_T(\{x \mid \operatorname{dist}(x,C) < \rho\})$$

which, as we let $\rho \downarrow 0$, implies that $\mu(C)$ must be 0. Hence, $\Theta^{*n}(\mu_T, x) > 0$ for almost every $x \in \mathbb{R}^P$.

We can now apply (5.5) to get a varifold (M, θ) associated with T. Now the only thing left to prove is that T is integral, i.e. θ is integer valued. To do this first realize that for almost every $x \in M$ by the definition of an approximate tangent space we have $\eta_{x,\lambda\#}T \rightharpoonup \theta(x)[[T_xM]]$ where $[[T_xM]]$ is oriented by the same ξ that orients (M,θ). Set $\pi_\perp$ as the n-dimensional subspace containing x that's normal to T_xM, denote $\Omega = B_1^n(0) \times \pi_\perp$, and let $d(y) = \operatorname{dist}(y, T_xM)$. Then it is possible to find a sequence $\lambda_j \downarrow 0$ and a $\rho > 0$ such that $\mathbb{M}_\Omega(\langle \eta_{x,\lambda_j}T, d, \rho\rangle) \leq c$, and in fact, a subsequence $\{j'\} \subset \{j\}$ and $\rho > 0$ such that $\eta_{x,\lambda_j,\#}T_{j'} \rightharpoonup \theta(x)[[T_xM]]$ and $\mathbb{M}_\Omega(\langle \eta_{x,\lambda_j}T_{j'}, d, \rho\rangle) \leq c$ for each j'. Then, if $S_j = (\eta_{x,\lambda\#})\llcorner\{y \mid d(y) < \rho\}$ then

$$\sup_{j\geq 1}(M_\Omega(S_j) + \mathbb{M}_\Omega(\partial S_j)) < \infty$$

Now let p be the restriction of $\mathbb{R}^P$ to the orthogonal projection onto T_xM, and let $\tilde{S}_j$ be the n current obtained by setting $\tilde{S}_j(\omega) = S_j(\tilde{\omega})$, such that $\tilde{\omega} = \omega$ in Ω but 0 elsewhere. Then by (3.28)

$$p_{\#}\tilde{S}_j(\omega) = \int_{B_1^n(0)} a\theta_j d\mathcal{L}^n$$

where a is such that $\omega = a\xi$, and θ is some integer-valued BV_{loc} function on $B_1^n(0)$ with

$$\mathbb{M}_{\text{int}(B_1^n(0))}(p_{\#}\tilde{S}_j) = \int_{B_1^n(0)} |\theta_j| d\mathcal{L}^n$$
$$\mathbb{M}_{\text{int}(B_1^n(0))}(\partial p_{\#}\tilde{S}_j) = \int_{B_1^n(0)} |D\theta_j|$$

Then, by the above and the supremum of the mass bound on S_j and ∂S_j, we apply the compactness of BV_{loc} functions to deduce that θ_j converge to an integer-valued function θ_*. Finally, since $p_{\#}\tilde{S}_j \rightharpoonup \theta[[T_xM]]$, we know $\theta_* = \theta$, and so we are done. □

Notice that if ∂T is non-zero, it can be replaced with the zero boundary current $h_{\#}([[(0,1)]] \times \partial T) - T$ (h is the affine homotopy centered at 0), and the sequence of T_j by $h_{\#}([[(0,1)]] \times \partial T_j) - T_j$. Since $h_{\#}([[(0,1)]] \times \partial T)$ is an integer multiplicity current, the theorem being true for the above substitutions implies it is true for T, T_j also. Hence, the $\partial T = 0$ condition may be dropped. Also, by replacing T_j with $T_j \llcorner B_r(\xi)$ for some $r > 0$ and $\xi \in U$, and open subset of $\mathbb{R}^P$, we can replace $\mathbb{R}^P$ with an arbitrary open subset instead. Finally, the following lemma removes the condition on ∂T_j being integer multiplicity:

Lemma 5.11 (Boundary Rectifiability). *Suppose T is an integer multiplicity current in $\mathcal{D}_n(U)$ with $\mathbb{M}_W(\partial T) < \infty$ for all W compactly contained in U. Then ∂T is an integer multiplicity $n-1$-current.*

Proof. By the weak polyhedral approximation theorem (4.25) we have a sequence of boundaries of polyhedral (hence integral) currents $\partial P_k \rightharpoonup \partial T$. The weaker version of the compactness theorem is then sufficient for us to conclude that ∂T is also an integral multiplicity current. □

This gives us the Federer-Fleming compactness theorem for currents.

Theorem 5.12 (Federer-Fleming). *If $\{T_j\} \subset \mathcal{D}_n(U)$ is a sequence of normal integer-multiplicity currents then there is an integer multiplicity $T \in \mathcal{D}_n(U)$ and a subsequence $\{T_{j'}\}$ such that $T_{j'} \rightharpoonup T$ in U.*

As promised in the introduction, we will showcase the power of this theorem by proving the existence of area-minimizing currents, proving the existence part of Plateau's problem for currents. Having built up the theory of integral currents, the proof of Plateau's problem is now fairly straightforward.

Theorem 5.13. *Let $S \in \mathcal{D}_{n-1}(\mathbb{R}^{n+l})$ be integer multiplicity with $\operatorname{spt} S$ compact and $\partial S = 0$. Then there is an integer multiplicity current $T \in \mathcal{D}_n(\mathbb{R}^{n+l})$ such that $\operatorname{spt} T$ is compact and $\mathbb{M}(T) \leq M(R)$ for each integer multiplicity $R \in \mathcal{D}_n(\mathbb{R}^{n+l})$ with $\operatorname{spt} R$ compact and $\partial R = S$.*

Proof. Let $\mathcal{I}_S$ be the set of currents R that are integer multiplicity, have compact support, and $\partial R = S$. Note that it is non-empty, the set $h_\#([[(0,1)]] \times S)$ for h the familiar affine homotopy is a member. Then take a sequence $\{R_q\} \subset \mathcal{I}_S$ that converges in mass to the infimum of $\mathbb{M}(R)$ for $R \in \mathcal{I}_S$. We cannot apply the compactness theorem just yet, because that requires our currents to be normal, which R_q don't have to be. So instead, choose $R > 0$ such that $B_R(0)$ contains $\operatorname{spt} S$, and let f be the retraction of $\mathbb{R}^{n+l}$ to the nearest point of $B_R(0)$. Then by (3.26) we have

$$\mathbb{M}(f_\# R_q) \leq \mathbb{M}(R_q)$$

However, $\partial f_\# R_q = f_\# \partial R_q = f_\# S = S$. So $\mathbb{M}(R_q) \to \mathbb{M}(R)$ also implies $\mathbb{M}(f_\# R_q) \to \mathbb{M}(R)$, with $\mathbb{M}(f_\# R_q)$ finite for all q. Hence, by (5.12) we have a subsequence $\{q'\} \subset \{q\}$ and an integer multiplicity current T to which $f_\# R_{q'}$ converge weakly. By the lower-continuity of $\mathbb{M}$ with respect to weak convergence, $\mathbb{M}(T) \leq \inf_{R \in \mathcal{I}_S} \mathbb{M}(R)$, with support R in $B_R(0)$, hence compact, and $\partial T = \lim f_\# \partial R_{q'} = S$. Thus, we are done. $\square$

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