COMPARISON AND RIGIDITY STATEMENTS IN GEOMETRY

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Abstract. In this expository paper, we present the basic ideas behind recent rigidity and comparison statements in geometry in relation to minimal surfaces.

Contents

1. Introduction

The question we're interested in for this paper (and one which is very natural to ask) is as follows: given the existence of minimal surfaces (or other non-trivial submanifolds) in a manifold, and prescribed values for their volumes or areas, can we extract any information about the metric on the manifold, or the manifold itself? In a recent publication, L. Mazet and H. Rosenberg [3] demonstrated that a minimal two sphere Σ satisfies a lower area bound when immersed in a manifold with sectional curvatures bounded between 0 and 1. Furthermore, when equality occurs, they uniquely determine the manifold, which can be either the standard three sphere S^3 with Σ being a totally geodesic submanifold, or as a quotient of $S^2 \times \mathbb{R}$. These lines of inquiry were prompted by Calabi's characterization of the 2-sphere, which will be the first of the results we present. These rigidity statements are an active area of research (see [7]), which is the reason for which we make an effort to present this condensed version of recent important results to serve as a guide into the subject.

2. Preliminaries

Let us start by giving a brief review of minimal surfaces by following [1].

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2.1. **First Variation Formula.** Let (M^n, g) be a Riemannian manifold and $\Sigma^k \subset M$ a submanifold. Consider $(x_1, \dots, x_k)$ local coordinates on Σ and let

$$g_{ij}(x) = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$$

for $1 \leq i, j \leq k$, be the components of $g|_\Sigma$. The (Riemannian) volume element of Σ is denoted by $d\Sigma$. The volume of Σ is given by

$$\mathrm{Vol}(\Sigma) = \int_\Sigma d\Sigma.$$

Consider the variation of Σ given by a smooth map $F : \Sigma \times (-\varepsilon, \varepsilon) \to M$ (which we assume is a diffeomorphism onto its image). We use $F_t(x) = F(x,t)$ and $\Sigma_t = F_t(\Sigma)$.

Definition 2.1. Let X be an arbitrary vector field on $\Sigma^k \subset M$. We define its **divergence** as

$$\mathrm{div}_\Sigma X(p) = \sum_{i=1}^k \langle \nabla_{e_i} X, e_i \rangle,$$

where $\{e_1, \dots, e_k\} \subset T_p\Sigma$ is an orthonormal basis and ∇ is the Levi-Civita connection with respect to g.

Lemma 2.2. *We have that*

$$\frac{\partial}{\partial t} d\Sigma_t = div_{\Sigma_t}\left(\frac{\partial F}{\partial t}\right) d\Sigma_t.$$

Proof. Note that

$$\frac{\partial}{\partial t} \det g = \mathrm{tr}(g^{-1}\partial_t g) \det g,$$

where $g^{-1} = (g^{ij}) = (g_{ij})^{-1}$. Then

$$\frac{\partial}{\partial t} \det g = \sum_{i,j} \left(g^{ij} \partial_t g_{ij}\right) \det g.$$

We can calculate the first derivative of the metric using the compatibility of ∇ with respect to g

$$\partial_t g_{ij} = g\left(\nabla_{\partial F/\partial t} \partial_i F, \partial_j F\right) + g\left(\partial_i F, \nabla_{\partial F/\partial t} \partial_j F\right),$$

where $\partial_i F = \partial F/\partial x_i$. Use the symmetry of ∇ to commute $\nabla_{\partial F/\partial t} \partial_i F = \nabla_{\partial_i F} \partial F/\partial t$. Put everything together to obtain

$$\frac{\partial}{\partial t} \det g = 2 \sum_{i,j} g^{ij} g(\nabla_{\partial_i F} \partial F/\partial t, \partial_j F) \det g = 2\mathrm{div}_{\Sigma_t}\left(\frac{\partial F}{\partial t}\right) \det g.$$

$\square$

We denote the area of Σ_t by $|\Sigma_t|$.

Theorem 2.3 (**First Variation Formula I**). *We have that*

$$\boxed{\frac{d}{dt}|\Sigma_t| = \int_{\Sigma_t} div_{\Sigma_t}\left(\frac{\partial F}{\partial t}\right) d\Sigma_t.}$$

Lemma 2.4. *We have*

$$div_\Sigma X = div_\Sigma X^T + \sum_{i=1}^{k} \langle \nabla_{e_i} X^N, e_i \rangle.$$

Theorem 2.5 (**First variation formula II**).

$$\frac{d}{dt}|\Sigma_t| = -\int_{\Sigma_t} \langle \partial F/\partial t, H \rangle d\Sigma_t = +\int_{\partial \Sigma_t} \langle \partial F/\partial t, \nu \rangle d\sigma_t.$$

Moreover, if $X = \frac{\partial F}{\partial t}$ *vanishes on* $\partial\Sigma$ *at* $t = 0$*, then*

$$\left.\frac{d}{dt}\right|_{t=0} |\Sigma_t| = -\int_\Sigma \langle X, H \rangle d\Sigma,$$

where H *is the mean curvature and* ν *the outer-pointing normal to* Σ*.*

Corollary 2.6.

$$\left.\frac{d}{dt}\right|_{t=0} |\Sigma_t| = 0, \text{ for any } X, \text{ with } X = 0 \text{ on } \partial\Sigma, \text{ iff } H = 0.$$

Definition 2.7. $\Sigma^k \subset M$ is a **minimal submanifold** of M if $H = 0$.

As one might intuit, geodesics are 1-dimensional minimal submanifolds, which leads us to the first instance of our discussion.

3. Comparison of manifolds given embedded minimal surfaces

In a letter sent to the authors of [2], Calabi presented a proof for the following first statement (although it is worth mentioning that the inequality part of the proof is due to Pogorelov).

Theorem 3.1. *Let* (S^2, g) *be the two dimensional sphere with a metric of class* $C^{1,1}$ *whose Gaussian curvature satisfies* $0 \le K \le 1$*. Then any simple closed geodesic* γ *on* (S^2, g) *has length at least* 2π*. If the length of* γ *is* 2π*, then either* (S^2, g) *is isometric to the standard round sphere* (S^2, g_0) *and* γ *is a great circle on* (S^2, g_0) *or* (S^2, g) *is isometric to a circular cylinder of circumference* 2π *capped by two unit hemispheres and* γ *is a belt around the cylinder. Thus, if* K *is continuous or if* $K > 0$*, then* (S^2, g) *is isometric to the standard round sphere.*

Lemma 3.2. *Let* $k(t)$ *be an* L^∞ *function on* $[0, \infty)$ *so that* $k(t)$ *and* $y(t)$ *with* $0 \le k(t) \le y(t)$*, are defined by the initial value problem*

$$y''(t) + k(t)y(t) = 0, \;\; y(0) = 1, \;\; y'(0) = 0.$$

Let us denote the smallest positive zero of $y(t)$ *by* β *(it may be the case that* $\beta = \infty$*). Then* $0 \le -y'(t) \le 1$ *for* $0 \le t \le \beta$*. Moreover, if* $y'(t_0) = -1$ *for some* $t_0 \in [0, \beta]$*, then* $t_0 = \beta < \infty$*,* $\beta \ge \pi/2$*, and*

$$y(t) = \begin{cases} 1, \;\; 0 \le t < \beta - \pi/2, \\ \cos(t - (\beta - \pi/2)), \;\; \beta - \pi/2 < t \le \beta, \end{cases}$$

$$k(t) = \begin{cases} 0, \;\; 0 \le t < \beta - \pi/2, \\ 1, \;\; \beta - \pi/2 < t \le \beta. \end{cases}$$

Proof. First let us note that on $[0, \beta)$, we have that

$$y''(t) = -k(t)y(t) \leq 0$$

by assumption. This means that $y'(t)$ is monotone-decreasing, and since $y'(0) = 0$, we conclude that $y' \leq 0$ on $[0, \beta)$. Moreover, we have that

$$(y^2 + (y')^2)' = 2yy' + 2y'y'' = 2yy' - 2y'yk = 2yy'(1-k) \leq 0$$

on $[0, \beta)$. Using the initial conditions, this means that $y^2 + (y')^2 \leq 1$, from which we conclude that

$$0 \leq -y'(t) \leq 1,$$

as desired. Now, if $t_0 \in [0, \beta]$ is such that $y'(t_0) = -1$, Then the previous inequality implies that $y^2(t_0) = 0$, so by construction this means that $t_0 = \beta$. Moreover, the second inequality implies that $y'(1-k) = 0$ on $[0, \beta)$. Following the sequence of inequalities and solving the differential equation gives the desired result. $\square$

Proof of Theorem. Let $c : [0, L] \to S^2$ be a unit speed parametrization of the closed geodesic γ, and let n be a unit normal along c. For each $s \in [0, L]$ let $\beta(s)$ be the cut distance from the curve γ along the geodesic $t \mapsto \exp_{c(s)}(tn(s))$. Define a map $F(s,t)$ on the set of ordered pairs (s,t) with $s \in [0, L]$ and $0 \leq t \leq \beta(s)$ by

$$F(s,t) = \exp_{c(s)}(tn(s)), \quad 0 \leq s \leq l, \quad 0 \leq t \leq \beta(s).$$

Then s, t are Fermi coordinates on the disk M bounded by γ and with inner normal n. In these coordinates the metric g, Gaussian curvature K and the area form dA are given by

$$g = F^2 ds^2 + dt^2, \quad K = \frac{-F_{tt}}{F}, \quad dA = F\, ds\, dt.$$

And because c is a geodesic, $F(s, 0) \equiv 1$ and $F_t(s, 0) \equiv 0$. Thus for fixed s the function $y(t) := F(s,t)$ satisfies $y'' + Ky = 0$, $y(0) = 1$, and $y'(0) = 0$. Notice that these are exactly the conditions of the previous lemma, from which we conclude that

$$\begin{aligned} 2\pi &= \int_M K\, dA = \int_0^L \int_0^{\beta(s)} -F_{tt}\, dt ds \\ &= \int_0^L (-F_t(s, \beta(s)))\, ds \\ &\leq \int_0^L 1\, ds \\ &= L. \end{aligned}$$

This gives the desired bound on the length of γ. If $L = 2\pi$, then $E_t(s, \beta(s)) = -1$ for all $s \in [0, L]$. Again, by the lemma we have that

$$K(s,t) = \begin{cases} 0, & 0 \leq t < \beta(s) - \pi/2, \\ 1, & \beta(s) - \pi/2 < t \leq \beta(s). \end{cases}$$

Let M_{+1} denote the interior of the set $\{K(x) = +1\}$ so that

$$M_{+1} = \left\{ \exp_{c(s)}(tn(s)) \,:\, s \in [0, 2\pi], \;\; \beta(s) - \pi/2 < t \leq \beta(s) \right\}.$$

Let $s_0 \in [0, 2\pi]$ be a point where $\beta(s)$ is maximal. Then the open disk $B(x_0, \pi/2)$ of radius $\pi/2$ about $x_0 := \exp_{c(s_0)}(\beta(s_0)n(s_0))$ is contained in M_{+1}, for if not it would meet ∂M_{+1} at some point $\exp_{c(s)}((\beta(s) - \pi/2)n(s))$ and this point is a distance of $\beta(s) - \pi/2$ from γ. Thus the distance of $x_0 = \exp_{c(s_0)}(\beta(s_0)n(s_0))$ to γ is less than $\pi/2 + (\beta(s) - \pi/2) = \beta(s)$, which contradicts the maximality of $\beta(s_0)$. Thus $B(x_0, \pi/2) \subseteq M_{+1}$. But using the Gauss-Bonnet theorem and $K \equiv +1$ on M_{+1}

$$2\pi \geq \int_{M_{+1}} K\, dA = \text{Area}(M_{+1}) \geq \text{Area}(B(x_0, \pi/2)) = 2\pi.$$

This means that $M_{+1} = B(x_0, \pi/2)$, and therefore $s \mapsto \beta(s)$ is constant, which implies that the disk M is bounded by γ and with inner normal n is a cylinder of circumference 2π capped at one end with a hemisphere. The same argument applied to the disk bounded by γ and having $-n$ as inward normal shows (S^2, g) is two of these capped cylinders glued together along γ, which is equivalent to the statement of the problem. □

3.1. **3-dimensional version.** In [3], Mazet and Rosenberg proved an analogous theorem to Calabi's for 2-spheres in 3-manifolds, which we present here (all the proofs which we do not present here can be found in the authors' paper). Following a similar setting as above, let us consider what happens in a complete 3-manifold M with sectional curvatures between 0 and 1.

Let Σ be an embedded minimal 2-sphere in M. Then the Gauss-Bonnet theorem and the Gauss equation tells us that the area of S is at least 4π:

$$4\pi = \int_\Sigma \overline{K}_\Sigma = \int \det(A) + K_{T\Sigma} \leq \int_\Sigma 1 = A(\Sigma),$$

with $\det(A)$ the determinant of the shape operator which is non positive because Σ is minimal (and also using the Gauss equation).

We denote by S^n_1 the sphere of dimension n with constant sectional curvature 1. We then have the following result.

Theorem 3.3. *Let M be a complete Riemannian 3-manifold whose sectional curvatures satisfy $0 \leq K \leq 1$. Assume that there exists an embedded minimal sphere Σ in M with area 4π. Then the manifold M is isometric either to the sphere S^3_1 or to a quotient of $S^2_1 \times \mathbb{R}$.*

Proof. Let Φ be the map $\Sigma \times \mathbb{R} \to M$ given by $(p, t) \mapsto \exp_p(tN(q))$ where N is a unit normal vector field along Σ. In the following, we focus on $\Sigma \times \mathbb{R}_+$; by symmetry, the analysis is similar for $\Sigma \times \mathbb{R}_-$. Since Σ is compact, there is $\varepsilon > 0$ such that Φ is an immersion on $\Sigma \times [0, \varepsilon)$. Let ε_0 be the supremum of all such ε's (so it is possible that it equals $+\infty$). This metric can be written as $ds^2 = d\sigma_t^2 + dt^2$, where $d\sigma_t^2$ is a smooth family of metrics on Σ. With this metric, Φ becomes a local isometry from $\Sigma \times [0, \varepsilon_0)$ to M, and $\Sigma \times [0, \varepsilon_0)$ has sectional curvatures bounded between 0 and 1. Let us denote by $\Sigma_t = \Sigma \times \{t\}$ the equidistant surfaces. We denote by $H(p, t)$ the mean curvature of Σ_t at the point (p, t) with respect to the unit normal vector ∂_t. First note that Σ_0 is minimal and has area 4π. Now we prove that $d\sigma_0^2$ has constant sectional curvature 1 so $(\Sigma, d\sigma_0^2)$ is isometric to S^2_1. Moreover, we have two cases

(1) $\varepsilon_0 = \pi/2$ and $d\sigma_t^2 = \sin^2 t d\sigma_0^2$ or
(2) $\varepsilon_0 = +\infty$ and $d\sigma_t^2 = d\sigma_0^2$.

Indeed, we define $\lambda(p,t) \geq 0$ such that $H+\lambda$ and $H-\lambda$ are the principal curvature of Σ_t at (p,t). We notice that $\lambda = 0$ if Σ_t is umbilical at (p,t). The surfaces Σ_t are spheres, so using the Gauss equation, the Gauss-Bonnet formula implies

$$4\pi = \int_{\Sigma_t} \overline{K}_{\Sigma_t} = \int_{\Sigma_t} (H+\lambda)(H-\lambda) + K_t = \int_{\Sigma_t} H^2 - \lambda^2 + K_t$$

where $\overline{K}_{\Sigma_t}$ is the intrinsic curvature of Σ_t and K_t is the sectional curvature of the ambient manifold of the tangent space to Σ_t. Since $K_t \leq 1$, we obtain

$$\int_{\Sigma_t} H^2 + K_t - 4\pi \leq \int_{\Sigma_t} H^2 + A(\Sigma_t) - 4\pi$$

where $A(\Sigma_t)$ is the area of Σ_t. In the following, we denote by $F(t)$ the right hand side of this inequality, and we show that it vanishes on $[0, \varepsilon_0)$. Since Σ_0 is minimal and has area 4π, we have that $F(0) = 0$. This means that $\lambda(p,0) = 0$, so Σ_0 is umbilical and $K_{T\Sigma} = 1$, which implies that $(\Sigma_0, d\sigma_0)$ is isometric to S^2_1. Now, the first and second variation formula give

$$\frac{\partial}{\partial t} A(\Sigma_t) = -\int_{\Sigma_t} 2H \text{ and } \frac{\partial H}{\partial t} = \frac{1}{2}(\mathrm{Ric}(\partial_t) + |A_t|^2),$$

where A_t is the shape operator of Σ_t and Ric is the Ricci curvature of $\Sigma \times [0, \varepsilon_0)$. Using the fact that the sectional curvatures of $M \times [0, \varepsilon_0)$ are non-negative, we conclude that Ric is non-negative too. Thus the second formula implies that H is increasing, and therefore $H \geq 0$ everywhere. Moreover, we have that

$$\begin{aligned}
F'(t) &= \int_{\Sigma_t} \left(2H\frac{\partial H}{\partial t} - 2H^3\right) - \int_{\Sigma_t} 2H \\
&= \int_{\Sigma_t} H(Ric(\partial_t) + |A_t|^2 - 2H^2 - 2) \\
&= \int_{\Sigma_t} H\left((Ric(\partial_t) - 2) + (H+\lambda)^2 + (H-\lambda)^2 - 2H^2\right) \\
&= \int_{\Sigma_t} H\left((Ric(\partial_t) - 2) + 2\lambda^2\right) \\
&\leq 2\int_{\Sigma_t} H\lambda^2
\end{aligned}$$

where the last inequality follows from $Ric(\partial_t) - 2 \leq 0$ by hypothesis on the sectional curvatures. Now, choosing $\varepsilon < \varepsilon_0$, there is a constant $C \geq 0$ such that $H \leq C$ on $\Sigma \times [0, \varepsilon]$. So for $t \in [0, \varepsilon]$, we obtain that $F'(t) \leq 2CF(t)$. Then $F(t) \leq F(0)e^{2Ct} = 0$ on $[0, \varepsilon]$. Therefore $F \leq 0$ on $[0, \varepsilon_0)$, and therefore $F = 0$ on $[0, \varepsilon_0)$. As a consequence, we have that all the equidistant surfaces Σ_t are umbilical, so $\lambda \equiv 0$. Taking the derivative of F, this implies that

$$\int_{\Sigma_t} H(Ric(\partial_t) - 2) = 0,$$

and by the inequality derived from the hypothesis above, we obtain

$$H(Ric(\partial_t) - 2) = 0 \text{ everywhere.}$$

Moreover, umbilicity and the variation formulas imply that $\frac{\partial H}{\partial t} = \frac{1}{2} Ric(\partial_t) + H^2$. We now prove that given $(p,t) \in \Sigma \times [0, \varepsilon_0)$ such that $H(p,t) > 0$, then $H(q,t) > 0$ for any $q \in \Sigma$. Indeed, consider $\Omega = \{q \in \Sigma \,|\, H(q,t) > 0\}$ which is a nonempty open subset of Σ. Let $q \in \Omega$. Since $H(q,t) > 0$, then $Ric(\partial_t)(q,t) = 2$. Thus $Ric(\partial_t)(r,t) = 2$ for any $r \in \overline{\Omega}$. So if $r \in \overline{\Omega}$, $Ric(\partial_t)(r,s) > 0$ for $s < t$, and therefore $H(r,t) > 0$ and $r \in \Omega$. Therefore Ω is closed and therefore $\Omega = \Sigma$.

Now assume that there is some $\varepsilon_1 > 0$ such that $H(p,t) = 0$ for $(p,t) \in \Sigma \times [0,\varepsilon_1]$ and $H(p,t) > 0$ for any $(p,t) \in \Sigma \times (\varepsilon_1, \varepsilon_0)$. Because of the evolution equation of H, this implies that $Ric(\partial_t) = 0$ on $\Sigma \times [0,\varepsilon_1]$, but on $\Sigma \times (\varepsilon_1,\varepsilon_0)$ we have $Ric(\partial_t) = 2$ by the application of Gauss' equation above, which is a contradiction by the continuity of $Ric(\partial_t)$. Thus we have two possibilities:

(1) $H = 0$ on $\Sigma \times [0,\varepsilon_0)$ and $Ric(\partial_t) = 0$ on $\Sigma \times [0,\varepsilon_0)$,
(2) $H > 0$ on $\Sigma \times (0,\varepsilon_0)$ and $Ric(\partial_t) = 2$ on $\Sigma \times [0,\varepsilon_0)$.

In the first case, this means that the sectional curvature of any 2-plane orthogonal to Σ_t is zero, and therefore $d\sigma_t^2 = d\sigma_0^2$. Since Φ only fails to be an immersion if $d\sigma_t^2$ becomes singular, it follows that in this case $\varepsilon_0 = +\infty$. Therefore $\Sigma \times \mathbb{R}_+$ is isometric to $S_1^2 \times \mathbb{R}_+$ with the induced metric, and Φ is a local isometry $S_1^2 \times \mathbb{R}_+$.

In the second case, the sectional curvature of any 2-plane orthogonal to Σ_t is equal to 1. Therefore $d\sigma_t^2 = \sin^2 t d\sigma_0$ and $\varepsilon_0 = \pi/2$. This also implies that $\Phi(p,\pi/2)$ is a point, and $\Sigma \times [0,\pi/2]$ with the metric ds^2 is isometric to a hemisphere of S_1^3, and Φ is a local isometry from that hemisphere to M.

We can perform the same analysis for $\Sigma \times \mathbb{R}_-$, for which we get that in the first case Φ is a local isometry $S_1^2 \times \mathbb{R} \to M$, and in the second case a local isometry $\Phi : S_1^3 \to M$. Since $S_1^2 \times \mathbb{R}$ and S_1^3 are simply connected, Φ is then the universal cover of M and M is then isometric to a quotient of $S_1^2 \times \mathbb{R}$ or S_1^3. Since Φ is injective on Σ, in the second case we see that actually Φ is injective and therefore a global isometry. □

3.2. **Higher codimension case.** Perhaps not so surprisingly, this result can be extended to higher codimension. This result is presented in [4] by Mazet.

Theorem 3.4. *Let M be a Riemannian $n \geq 3$-manifold whose sectional curvature is bounded above by 1. Let us assume that M contains an immersed minimal 2-sphere of area 4π which has index at least $n-2$. Then the universal cover of M is isometric to the unit sphere S_1^n.*

While the proof of this theorem is quite involved for the purposes of this paper, the idea of the proof is as follows: if S is an immersed 2-sphere, we can define a function F by

$$F(S) = A(S) + \int_S |H|^2 - 4\pi,$$

where $A(S)$ is the area of S and H is the mean curvature of S. If $F(S)$ vanishes, S is totally umbilical and we can extract information on the sectional curvature of M along S. Therefore, if S_0 is the minimal 2-sphere given by the statement of the theorem, then $F(S_0) = 0$. The idea is then to explore the geometry of M by computing $F(S_t)$ along a deformation $\{S_t\}_t$ of S_0, and the proof in the paper produces the family $\{S_t\}$ as a mean curvature flow out of S_0. More precisely, the author constructs non trivial ancient solutions $\{S_t\}_{t\in(-\infty,b)}$ of the mean curvature flow such that $S_t \to S_0$ as $t \to -\infty$.

4. Results on capillary surfaces

Similar to the statements we've considered so far, it is possible to consider analogous statements for non-minimal surfaces. These results are presented by Espinar and Rosenberg in [5]. Let us first set up the notation and basic definitions.

Let $(M, \partial M)$ be a complete Riemannian 3-manifold with boundary. Throughout this section, η will always stand for the inward normal along ∂M. Now, let Σ be an oriented compact surface with boundary $\partial\Sigma$ and unit normal N, N chosen so that $||H'||N = H$ when $H \neq 0$; $H' = 2HN$, where H' and H are the mean curvature vector and mean curvature function respectively. If H is constant along Σ, we say that Σ is a H-surface. Also, denote by II_Σ the second fundamental form of Σ in M with respect to N and by K_e and K_Σ its extrinsic and Gaussian curvature, the extrinsic curvature K_e is nothing but the product of the principal curvatures. Associated to the mean and extrinsic curvatures one can define the **skew curvature** as $\Phi = \sqrt{H^2 - K_e}$ that measures how far the surface is from being umbilic. Throughout this section we will denote by $|\Sigma|$ and $|\partial\Sigma|$ the area of Σ and the length of $\partial\Sigma$ respectively.

We assume that $\Sigma \subset M$ and $\partial\Sigma \subset \partial M$. We say that Σ is a **capillary surface of angle** β in M if the outer conormal ν along $\partial\Sigma$ and the unit normal along ∂M make a constant angle β along $\partial\Sigma$, *i.e.* there exists a constant $\beta \in [0, \pi/2)$ so that $\langle \nu, \eta\rangle = -\cos\beta$. In particular, when $\beta = 0$ or, equivalently, Σ meets orthogonally ∂M, we say that Σ is a free boundary surface.

4.1. Manifolds with umbilic boundary and $0 \leq K_{sect} \leq 1$.

Lemma 4.1. *Let $(M, \partial M)$ be a complete orientable Riemannian 3-manifold with boundary. Let $\Sigma \subset M$ be a compact oriented H-disk ($H \geq 0$) with boundary $\partial\Sigma \subset \partial M$. Assume that*

- *The sectional curvatures of M satisfy $K_{sect} \leq 1$,*
- *∂M is umbilic, with umbilicity factor $\alpha \in \mathbb{R}$,*
- *Σ is a capillary disk of angle $\beta \in [0, \pi/2)$,*

then

$$2\pi \leq (1 + H^2)|\Sigma| + \frac{\alpha + (H + \max_{\partial\Sigma}\Phi)\sin\beta}{\cos\beta}|\partial\Sigma|.$$

Moreover, equality holds if, and only if, Σ is umbilic, $K_\Sigma = 1 + H^2$ and $K_{sect} \equiv 1$ along Σ.

Proof. First, by the Gauss equation and the AM-GM inequality, we obtain

$$K_\Sigma = K_e + K_{sect} \leq H^2 + 1,$$

hence, integrating over Σ, the Gauss-Bonnet formula yields

$$2\pi = \int_\Sigma K_\Sigma + \int_{\partial\Sigma} k_g \leq (1 + H^2)|\Sigma| + \int_{\partial\Sigma} k_g.$$

Now let t denote a unit tangent vector field along $\partial\Sigma$, clearly $t \in (\partial M)$, and let $n = Jt$ be the rotation by an angle of $\pi/2$ on ∂M. On the one hand, since $\{t, n\}$ is an orthonormal frame along ∂M, we have

$$2H_{\partial M} = -\langle t, \overline{\nabla}_t \eta\rangle - \langle \eta, \overline{\nabla}_n \eta\rangle,$$

and since ∂M is umbilic, we obtain

$$-\langle t, \overline{\nabla}_t \eta \rangle = \alpha.$$

On the other hand, by the capillary condition, $-\eta = \cos\beta\nu + \sin\beta N$ along $\partial\Sigma$, hence

$$-\langle t, \overline{\nabla}_t \eta \rangle = \cos\beta \langle t, \overline{\nabla}_t \nu \rangle + \sin\beta \langle t, \overline{\nabla}_t N \rangle = \cos\beta k_g - \sin\beta\, II_\Sigma(t,t).$$

Therefore, combining both equations we get

$$\cos\beta k_g = \alpha + \sin\beta\, II_\Sigma(t,t)$$

along $\partial\Sigma$. Finally, since Σ has constant mean curvature H, we have that $II_\Sigma(t,t) \leq H + \Phi$ and hence we obtain

$$2\pi\cos\beta \leq (1+H^2)\cos\beta|\Sigma| + (\alpha + (H + \max_{\partial\Sigma}\Phi)\sin\beta)|\partial\Sigma|,$$

as claimed. Moreover, equality holds if and only if it holds in the second equation, that is $K_\Sigma = H^2+1$, which implies that Σ is umbilic and $K_{sect} \equiv 1$ along Σ. From the Gauss equation we then deduce that $K_\Sigma = 1 + H^2$. □

When Σ is a free boundary disk and ∂M is totally geodesic, the above inequality reads as $2\pi \leq (1+H^2)|\Sigma|$, and equality holds with the same conditions as above. Now let us describe the model cases of Riemannian manifolds $(M, \partial M)$ where free boundary disks achieve the equality. In this case we have two distinct models:

- **Model 1**: Let $S^3 \subset \mathbb{R}^4$ be the standard unit three-sphere embedded in the four dimensional Euclidean space with the standard Euclidean metric $\langle \cdot, \cdot \rangle_0$. Then, the upper hemisphere, given by

  $$S^3_+ = \{(x_1, x_2, x_3, x_4) \in S^3 \,:\, x_4 > 0\},$$

  is a complete manifold with constant sectional curvatures equal to 1 and totally geodesic boundary $\partial S^3_+ = \{x \in S^3_+ \,:\, x_4 = 0\}$, which is isometric to a two-sphere S^2. From now on, we denote by $S^n(r)$ the standard n-dimensional sphere of constant sectional curvatures $1/r^2$.

  For any $x \in \partial S^3_+$, let $B_x(R)$ be the geodesic ball of the standard three-sphere S^3 centered at x of radius R. Fix $H \geq 0$ a constant and set $R_H = \frac{\pi}{2} - \arctan H$. Then, we have

  $$D_H := \partial B_x(R_H) \cap S^3_+$$

  is a umbilic H-disk orthogonal to ∂S^3_+ such that

  $$|D_H| = \frac{2\pi}{1+H^2}.$$

- **Model 2**: Let S^2_+ be the upper hemisphere of the standard 2-sphere. Clearly, $\mathbb{R} \times S^2_+$ with the standard product hemisphere is a complete manifold with sectional curvatures between 0 and 1 and totally geodesic boundary

  $$\partial(\mathbb{R} \times S^2_+) = \mathbb{R} \times \partial S^2_+.$$

  For any $t \in \mathbb{R}$, $D_t = \{t\} \times S^2_+$ is a totally geodesic minimal disk orthogonal to $\mathbb{R} \times \partial S^2_+$ such that

  $$|D_t| = 2\pi.$$

Now we can state the following theorem.

Theorem 4.2. *Let $(M, \partial M)$ be a complete orientable Riemannian 3-manifold with boundary. Assume that M has sectional curvatures $0 \leq K_{sect} \leq 1$ and ∂M is connected and totally geodesic. If there exists a compact oriented embedded H-disk $\Sigma \subset M$ orthogonal to ∂M such that*

$$|\Sigma| = \frac{2\pi}{1+H^2}.$$

Then:

- *If $H > 0$, the mean convex side of $M \backslash \Sigma$, call it U, is isometric to $B_x(R_H) \cap S^3_+ \subset S^3$ with the standard metric, $R_H = \pi/2 - \arctan H$ and $x \in \partial S^3_+$. Moreover, Σ is a disk D_H described in the model 1.*
- *If $H = 0$, M is isometric to either S^3_+ with its standard metric of constant sectional curvature one, or a quotient of $\mathbb{R} \times S^2_+$ with the standard product metric. Moreover, Σ is a disk D_H or D_t as described in the models 1 and 2.*

4.2. **Manifolds with umbilic boundary and $K_{sect} \leq -1$.** In this section we'll study compact capillary H-surfaces of non-positive Euler characteristic immersed in a three-manifold with umbilic boundary and section curvatures less or equal to -1. We obtain an upper bound for the area of such capillary surfaces with a corresponding rigidity statement.

Lemma 4.3. *Let $(M, \partial M)$ be a complete orientable Riemannian 3-manifold with boundary. Let $\Sigma \subset M$ be a compact oriented H-surface $(H \geq 0)$ with boundary $\partial\Sigma \subset \partial M$. Assume that*

- *The sectional curvatures of M satisfy $K_{sect} \leq -1$,*
- *∂M is umbilic, with umbilicity factor $\alpha \in \mathbb{R}$,*
- *Σ is a capillary surface of angle $\beta_i \in [0, \pi/2)$ at each connected component $\partial\Sigma_i$ of the boundary $\partial\Sigma$.*

Then

$$-2\pi\chi(\Sigma) + \sum_{i=1}^{k} \frac{\alpha + \sin\beta_i(H + \max_{\partial\Sigma}\Phi)}{\cos\beta_i}|\partial\Sigma_i| \geq (1-H^2)|\Sigma|,$$

where k is the number of connected components of $\partial\Sigma$. Moreover, equality holds if and only if Σ is umbilic, $K_\Sigma = H^2 - 1$ and $K_{sect} \equiv -1$ along Σ.

Proof. As we did in the first lemma, by the Gauss-Bonnet formula and the hypothesis we get

$$\begin{aligned} 2\pi\chi(\Sigma) &= \int_\Sigma K_\Sigma + \int_{\partial\Sigma} k_g \\ &= \int_\Sigma H^2 - \int_\Sigma \Phi^2 + \int_\Sigma K_{sect} + \sum_{i=1}^{k} \int_{\partial\Sigma_i} k_g \\ &\leq (H^2-1)|\Sigma| + \sum_{i=1}^{k} \frac{\alpha + \sin\beta_i(H + \max_{\partial\Sigma}\Phi)}{\cos\beta_i}|\partial\Sigma_i|, \end{aligned}$$

as claimed. □

As in the previous lemma, we will describe the model cases when equality is achieved by a free boundary H-surface, $H \in [0, 1]$, in a complete manifold with

$K_{sect} \leq -1$ and totally geodesic boundary.

In both cases, $H = 1$ and $H \in [0, 1)$, M will be a hyperbolic manifold with totally geodesic boundary and Σ will be a constant intrinsic curvature umbilical H-surface orthogonal to ∂M.

- **Model 3**: We begin with the case $H \in [0, 1)$. Let S be a closed oriented surface of curvature -1. S is a quotient of $\mathbb{H}^2$ by a cocompact Fuchsian group Γ of isometries of $\mathbb{H}^2$. Consider $P_0 \equiv \mathbb{H}^2$ as isometrically embedded in $\mathbb{H}^3$ as a totally geodesic plane, with n a unit normal vector field to P_0 in $\mathbb{H}^3$. We parametrize $\mathbb{H}^3$ in Fermi coordinates by $F : \mathbb{R} \times P_0 \to \mathbb{H}^3$, where
$$F(t, x) := \exp_x(tn(x)).$$
The metric of $\mathbb{H}^3$ in these coordinates is $dt^2 + \cosh^2(t)g_{-1}$, where g_{-1} is the standard hyperbolic metric of curvature -1. Observe that
$$P_t := \{F(t, x) \,:\, x \in H_0\}$$
is an equidistant surface of P_0 of constant mean curvature $\tanh(t)$. The group Γ extends to isometries of $\mathbb{H}^3$: for $\gamma \in \Gamma$, $t \in \mathbb{R}$ and $x \in H_0$, define
$$\gamma(F(t, x)) = F(t, \gamma(x)).$$
The extended action leaves each P_t invariant and $P_t/\Gamma = S_t$ is a constant sectional curvature umbilical surface in $\mathbb{H}^3/\Gamma$. $\mathbb{H}^3/\Gamma$ is homeomorphic to $\mathbb{R} \times S$.
- **Model 4**: we describe the case $H = 1$. Consider a horosphere H_0 of $\mathbb{H}^3$ and a $\mathbb{Z}^2$ group Γ of parabolic isometries leaving H_0 invariant. Let n be a unit normal vector field to H_0 in $\mathbb{H}^3$, pointing to the mean convex side of H_0. $\mathbb{H}^3$ has the Fermi coordinates $F : \mathbb{R} \times H_0 \to \mathbb{H}^3$, where
$$F(t, x) := \exp_x(tn(x)),$$
here exp denotes the exponential map in $\mathbb{H}^3$. the metric of $\mathbb{H}^3$ in these coordinates is $dt^2 + e^{2t}g_e$, where g_e is the standard Euclidean metric of curvature 0. As before, Γ acts on $\mathbb{H}^3$, leaving each horosphere H_t invariant. Each H_t has mean curvature 1. $\mathbb{H}^3/\Gamma$ is a hyperbolic three manifold of constant section curvature -1 isometric to $\mathbb{R} \times \mathbb{T}^2$ with the metric $dt^2 + e^{-2t}g_e$, where g_e is the standard flat metric of T^2.

Now we can characterize Riemannian manifolds $(M, \partial M)$ with totally geodesic boundary and sectional curvatures $K_{sect} \leq -1$ assuming the existence of an oriented H-surface meeting ∂M orthogonally and of greatest area. By the lemma we just proved, if $\Sigma \subset M$ is an oriented compact H-surface, $H^2 \leq 1$, such that $\partial\Sigma \subset \partial M$ and meets ∂M orthogonally, then
$$(1 - H^2)|\Sigma| \leq 2\pi|\chi(\Sigma)|,$$
where we are assuming that ∂M is totally geodesic in M. Note that, in the case $\chi(\Sigma) = 0$, this says that if there exists a H-surface Σ, $H^2 \leq 1$, orthogonal to the boundary with $\chi(\Sigma) = 0$ then Σ has constant mean curvature $H = 1$, it is umbilic, $K_\Sigma = 0$ and $K_{sect} \equiv -1$ along Σ, without any area information. It is remarkable that we cannot characterize the manifold when Σ is minimal.

Theorem 4.4. *Let $(M, \partial M)$ be a complete orientable Riemannian 3-manifold with boundary. Assume that M has sectional curvatures $K_{sect} \leq -1$ and ∂M is totally geodesic. Assume that there exists a compact oriented embedded H-surface $\Sigma \subset M$, $H \in (0,1]$, orthogonal to ∂M with non-positive Euler characteristic. Then:*

- *If $H \in (0,1)$, Σ separates and $|\Sigma| = \frac{2\pi|\chi(\sigma)|}{1-H^2}$, $\chi(\Sigma) < 0$, then there exists a totally geodesic minimal surface Σ_m orthogonal to ∂M and an isometry*
$$F : ([0, arctanh H) \times \Sigma, dt^2 + \cosh^2(t) g_{-1}) \to M,$$
where g_{-1} denotes the standard metric of constant curvature -1, such that
  - *$F(0, \Sigma) = \Sigma$ and $F(arctanh H, \Sigma) = \Sigma_m$, and*
  - *$F(t, \Sigma) = \Sigma_t$ is an embedded totally umbilic H-surface, $H = \tanh(arctanh\ H - t)$, orthogonal to ∂M for all $t \in [0, arctanh\ H)$.*

  Moreover, if Σ_m is non-orientable, a tubular neighborhood of Σ_m is foliated by its equidistants Σ_t.
- *If $H = 1$ and $\chi(\Sigma) = 0$, the mean convex side of $M \setminus \Sigma$, call this component U, is isometric to $[0, \infty) \times \Sigma$ endowed with the product metric $g = dt^2 + e^{-2t} g_e$, where g_e is the standard Euclidean metric of curvature 0. That is, U is isometric to a cusp hyperbolic end P corresponding to Model 4 and Σ is a slice.*

The reader is referred to the paper to see a proof of the theorems.

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