# Finite Spaces and Larger Contexts

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# Contents

## Introduction

A finite space is a topological space that has only finitely many points. At first glance, it seems ludicrous to think that such spaces can be of any interest. In fact, from the point of view of homotopy theory, they are equivalent to finite simplicial complexes. Therefore they support the entire range of invariants to be found in classical algebraic topology. For a striking example that sounds like nonsense, there is a space with six points and infinitely many non-zero homotopy groups. That is like magic: it sounds impossible until you know the trick, when it becomes obvious. We usually restrict attention to finite $T_0$-spaces[1], and those are precisely equivalent to finite posets (partially ordered sets). Therefore finite spaces are also of interest in combinatorics. In fact, there is a large and growing literature about finite spaces and their role in other areas of mathematics and science.

My own interest in the subject was aroused by 1966 papers by McCord [**50**] and Stong [**65**] that are the starting point of this book. However, I should admit that I came upon these papers while casting about for material to teach in Chicago's large scale REU, which I organize and run. I wanted something genuinely fascinating, genuinely deep, and genuinely accessible, with lots of open problems. Finite spaces provide a perfect REU topic for an algebraic topologist. Most experts in my field know nothing at all about finite spaces, so the material is new even to the experts, and yet it really is accessible to smart undergraduates. This book will feature several contributions made by undergraduates, some from Chicago's REU and some not.

When I first started talking about finite spaces, in the summer of 2003, my interest had nothing at all to do with my own areas of research, which seemed entirely disjoint. However, it has gradually become apparent that finite spaces can be integrated seamlessly into a global picture of how posets, simplicial complexes, simplicial sets, topological spaces, small categories, and groups are interrelated by a web of adjoint pairs of functors with homotopical meaning. The undergraduate may shudder at the stream of undefined terms!

The intention of this book is to introduce the algebraic topology of finite topological spaces and to integrate that topic into an exposition of a global view of a large swathe of modern algebraic topology that is accessible to undergraduates and yet has something new for the experts. A slogan of our REU is that "all concepts will be carefully defined", and we will follow that here. However, proofs will be selective. We aim to convey ideas, not all of the details. When the results are part of the mainstream of other subjects (group theory, combinatorics, point-set topology, and algebraic topology) we generally quote them. When they are particular to our main topics and not to be found on the textbook level, we give complete details.

These notes started out entirely concretely, without even a mention of things like categories or simplicial sets. Chicago students won't stand for oversimplification, and their questions always led me into deeper waters than I intended. They were also impatient with the restriction to finite spaces and finite simplicial complexes, one reason being that as soon as their questions forced me to raise the level of discourse, the restriction to finite things seemed entirely unnatural to them.

The infinite version of finite topological spaces is readily defined and goes back to a 1937 paper of Alexandroff [**2**]. We call these spaces Alexandroff spaces, and we use the abbreviation $A$-space for Alexandroff $T_0$-space. To go along with this, we

[1]The $T_0$ property means that the topology distinguishes points.

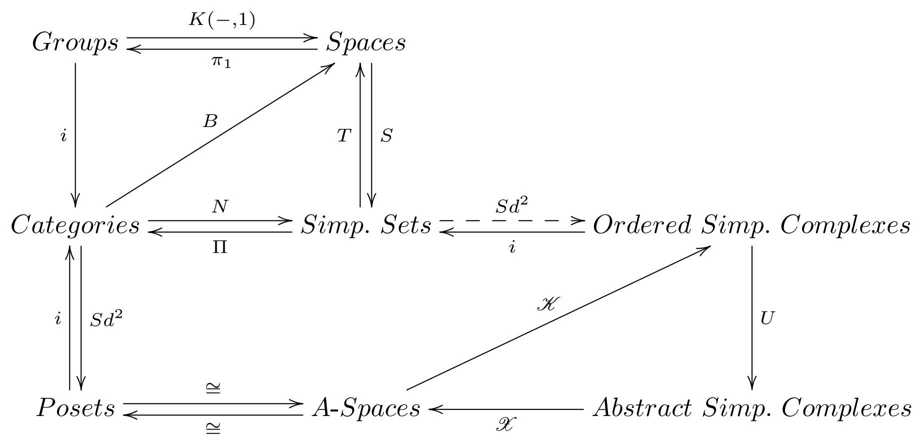
also use the abbreviation $F$-space for finite $T_0$-space. Just as $F$-spaces are equivalent to finite posets, so $A$-spaces are equivalent to general posets. Similarly, from the point of view of homotopy theory, $F$-spaces are equivalent to finite simplicial complexes and $A$-spaces are equivalent to general simplicial complexes.

Roughly speaking, the first part of the book focuses on the homotopy theory of $F$-spaces and $A$-spaces. A central theme is the difference between weak homotopy equivalences and homotopy equivalences. A continuous map $f\colon X \longrightarrow Y$ is a homotopy equivalence if there is a map $g\colon Y \longrightarrow X$ such that the composite $g \circ f$ is homotopic to the identity map of $X$ and the composite $f \circ g$ is homotopic to the identity map of $Y$. The map $f$ is a weak homotopy equivalence (usually abbreviated to weak equivalence) if for every choice of basepoint $x \in X$ and every $n \geq 0$, the induced map $f_*\colon \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$ is an isomorphism (of sets if $n = 0$, of groups if $n = 1$, and of abelian groups if $n \geq 2$).

Every homotopy equivalence is a weak homotopy equivalence. A map between nice spaces, namely CW complexes, is a homotopy equivalence if it is a weak homotopy equivalence. All of the spaces that one encounters in standard introductions to algebraic topology are nice, so that the distinction seems parenthetical and of minor interest. It is by now very well understood by algebraic topologists that the definitively "right" notion of equivalence is weak equivalence, not homotopy equivalence. However, to get a feel for the strength of the distinction, one needs to see serious examples where the two notions are genuinely different.

The first half of the book offers just such a perspective. The work of Stong makes it very easy to understand homotopy equivalences of finite spaces. The work of McCord relates weak equivalences of Alexandroff spaces to weak equivalences, and therefore homotopy equivalences, of simplicial complexes. As we shall explain, a reinterpretation in terms of finite spaces of a conjecture of Quillen about the poset of non-trivial elementary subgroups of a finite group illuminates precisely this distinction between weak homotopy equivalences and actual homotopy equivalences. Another open problem also illuminates the distinction. The problem of enumerating homotopy equivalences of finite spaces combinatorially has been solved by a pair of Chicago undergraduates, Alex Fix and Stephan Patrias. The problem of enumerating weak homotopy equivalences combinatorially is still open.

The second half of the book guides the reader through the following oversimplified diagram of categories and functors between them.




The connections among these categories are remarkably close. It has been understood since the 1950's that topological spaces and simplicial sets can in principle be used interchangeably in the study of homotopy theory. In fact, except that groups only model very special spaces, called $K(\pi, 1)$'s, all of these categories can in principle be used interchangeably in the study of homotopy theory. We'd like people outside algebraic topology to become more aware of these interconnections.

One thing that is largely new is a careful combinatorial analysis of exactly how subdivision ties together the categories of simplicial sets, (small) categories, and posets, alias $A$-spaces. This is due in large part to Rina Foygel, a recent Chicago PhD and now faculty member in Statistics, and her work is included with her permission. In particular, we give a careful explanation of the classical result that the second subdivision of a suitably well-behaved simplicial set is a simplicial complex and the folklore result that the second subdivision of any (small) category is a poset. One striking result is that, when regarded as a simplicial set, any classical (ordered) simplicial complex is the nerve of a category. As far as I know, that has never before been noticed. We ask the novice not to be intimidated. We will go slow! We ask the expert to be patient. There will be new things along the way.

There are all sorts of possible choices of material and presentation for a book on this general topic, and I'll explain, but not justify, my choices rather flippantly. The main justification is that the REU is supposed to be fun, and so is this book.

It is a standard saying that one picture is worth a thousand words. It is a defect of the author that he is not good at drawing pictures and is too lazy to learn. That is one among many reasons that this book, although started by the senior author, the one who is writing this introduction, has been joined by his friend and student Elle Pishawar as a coauthor. She has drawn all of the pictures, edited all of the contributions by REU participants, and helped in countless other ways. Mistakes that remain are due to the senior author.

In mathematics, it is perhaps fair to also say that one good definition is worth a thousand calculations. The author likes to make up definitions and to see relations between seemingly unrelated concepts, so we will do lots of that. However, to quote a slogan from a T-shirt worn by one of the author's students, "calculation is the way to the truth". There is a need for more calculational understanding of the subject here, and the author, being too lazy to compute himself, hopes that readers will be inspired.

In fact, the author's notes on this subject have been online since 2003, and a number of people have been inspired by them. In particular, Gabriel Minian, in Buenos Aires, and his students have followed up problems in my notes. His student Jonathan Barmak wrote a 2009 thesis, now a book [**7**], that has a good deal of overlap with the first half of this book.[2] I'll content myself with the basic theory and refer to Barmak's book for more recent advances made in Argentina.

Pedagogically, I've been using this material as a device to offer beginning undergraduates capsule introductions to point-set topology, algebraic topology, and category theory. I've also used the evolution of concepts as a means to help students gain an intuition for abstraction and conceptualization in modern mathematics.[3] These twin purposes pervade and guide the exposition.

Elle and I have drawn inspiration from a number of REU papers over the years, and we have included several with the permission of their authors. The topics were often suggested during REUs, and several are original related research. We will highlight contributions as they appear, but here is a list of contributors.

Adam Black.
*The Euler Characteristic of Finite Topological Spaces.*
*http://math.uchicago.edu/ may/REU2015/REUPapers/Black.pdf* (2015), [18].

Cathy (Xi) Chen.
*Cores of Alexandroff Spaces.*
*http://math.uchicago.edu/ may/REU2015/REUPapers/Chen,Xi(Cathy).pdf* (2015), [14].

Alex Fix and Stephen Patrias.
*Enumeration of Homotopy Classes of Finite $T_0$ Topological Spaces.*
*https://math.uchicago.edu/ may/VIGRE/VIGRE2008/REUPapers/Fix.pdf* (2008), [2.5, 15].

Isaac Friend.
*Finite Connected H-Spaces are Contractible.*
*http://math.uchicago.edu/ may/REU2015/REUPapers/Friend.pdf* (2015), [8].

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[2] I'll quote from his introduction. "In 2003, Peter May writes a series of unpublished notes in which he synthesizes the most important ideas on finite spaces until that time. In these articles, May also formulates some natural and interesting questions and conjectures which arise from his own research. May was one of the first to note that Stong's combinatorial point of view and the bridge constructed by McCord could be used together to attack algebraic topology problems using finite spaces. Those notes came to the hands of my PhD advisor Gabriel Minian, who proposed me to work on this subject. May's notes and problems, jointly with Stong's and McCord's papers, were the starting point of our research on the Algebraic Topology of Finite Topological Spaces and Applications."

[3] Entirely independent of this book, an advertisement for just such a use of the subject of finite spaces as a pedagogical tool has been published by two students of a student of mine [**32**].

Joseph (Nico) Marin Gamboa.
*Morse Theory for Partially Ordered Sets in Context. http://math.uchicago.edu/ may/REU2022/REUPapers/Mar*
(2022), [16.4].

Doron L. Grossman-Naples.
*Finite Manifolds and Minimal Finite Models of Closed Surfaces.*
*https://math.uchicago.edu/ may/REU2018/REUPapers/Grossman-Naples.pdf* (2018),
[19].

Ryan Hopkins.
*Finite Metric Spaces and their Embedding into Lebesgue Spaces.*
*http://math.uchicago.edu/ may/REU2015/REUPapers/Hopkins.pdf* (2015), [15.0.4].

Eleanor McSpirit.
*Homotopy Theory from Subdivision and A-Space Models.*
*http://math.uchicago.edu/ may/REU2018/REUPapers/McSpirit.pdf* (2018), [22].

Adele Padgett.
*An Investigation into Covers of Some Finite Spaces.*
*https://math.uchicago.edu/ may/REU2015/REUPapers/Padgett.pdf* (2015), [21].

David Ramsey.
*A Result on Representations of Homology Manifolds by Finite Spaces.*
*http://www.math.uchicago.edu/ may/VIGRE/VIGRE2010/REUPapers/RamseyD.pdf*
(2010), [19].

Jasha Sommer-Simpson.
*Barycentric Subdivision and Isomorphisms of Groupoids.*
*http://math.uchicago.edu/ may/REU2015/REUPapers/Sommer-Simpson.pdf* (2015),
[**??**].

Daping Weng.
On Minimal Finite Models.
*http://www.math.uchicago.edu/ may/VIGRE/VIGRE2010/REUPapers/Weng.pdf*
(2010), [19].

Zhenghui (Sunny) Zhang.
*Characterization of Nerves of Posets.*
*http://math.uchicago.edu/ may/REU2022/REUPapers/Zhang,Zhenghui.pdf* (2022),
[12.7].

# Part 1

# Alexandroff spaces, posets, and simplicial complexes

CHAPTER 1

# Alexandroff spaces and posets

## 1.1. The basic definitions of point set topology

The intuitive notion of a set in which there is a prescribed description of nearness of points is obvious. So is the intuitive notion of a function that takes nearby points to nearby points. However, formulating the "right" general abstract notion of what a "topology" on a set should be and what a "continuous map" between topological spaces should be is not so obvious. Since, intuitively, nearness is thought of in terms of distance, the most immediate way to make the intuition precise is to use distance functions. That leads to metric spaces and the $\varepsilon$-$\delta$ description of continuity, which is how we usually think of spaces and maps. Hausdorff came up with a much more abstract and general notion that is now universally accepted.

**Definition 1.1.1.** A *topology* on a set $X$ consists of a set $\mathscr{U}$ of subsets of $X$, called the "open sets of $X$ in the topology $\mathscr{U}$", with the following properties.

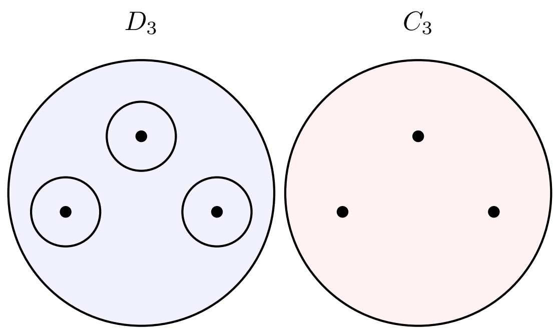
(i) The empty set $\emptyset$ and the set $X$ are in $\mathscr{U}$.
(ii) A finite intersection of sets in $\mathscr{U}$ is in $\mathscr{U}$.
(iii) An arbitrary union of sets in $\mathscr{U}$ is in $\mathscr{U}$.

A *neighborhood* of a point $x \in X$ is an open set $U$ such that $x \in U$.

We write $(X, \mathscr{U})$ for the set $X$ with the topology $\mathscr{U}$. More usually, when the topology $\mathscr{U}$ is understood, we just say that $X$ is a topological space. We say that a topology $\mathscr{U}$ is *finer* than a topology $\mathscr{V}$ if every set in $\mathscr{V}$ is also in $\mathscr{U}$ ($\mathscr{U}$ has more open sets). We then say that $\mathscr{V}$ is *coarser* than $\mathscr{U}$. We have two obvious and uninteresting topologies on any set $X$.

**Definition 1.1.2.** The *discrete topology* on $X$ is the topology in which all sets are open. It is the finest topology on $X$. The *trivial* or *coarse* or *indiscrete* topology on $X$ is the topology in which $\emptyset$ and $X$ are the only open sets. It is the coarsest topology on $X$. We write $D_n$ and $C_n$ for the discrete and coarse topologies on a set with $n$ elements. These are the largest and the smallest possible topologies (in terms of the number of open subsets).

**Example 1.1.3.** In pictures, we shall display non-empty open sets as the set of points interior to circles drawn on a space. We only draw circles around the smallest open sets, remembering that the union of open sets is open. The following figure depicts $D_3$ and $C_3$, each contained within a large circle.




**Definition 1.1.4.** Let $X$ be a topological space. A subset of $X$ is *closed* if its complement is open. The closed sets satisfy the following conditions.

(i) The empty set $\emptyset$ and the set $X$ are closed.
(ii) An arbitrary intersection of closed sets is closed.
(iii) A finite union of closed sets is closed.

We shall make little or no use of the following definition, but it may help make clear how the abstract definitions correspond to common notions in calculus.

**Definition 1.1.5.** Let $A$ be a subset of a topological space $X$. The *interior* $\mathring{A}$ of $A$ is the union of the open subsets of $X$ contained in $A$. The *closure* $\bar{A}$ of $A$ is the intersection of the closed sets containing $A$. A point $x \in X$ is a *limit point* of $A$ if every neighborhood of $x$ contains a point $a \neq x$ of $A$. $A$ is *dense* in $X$ if $\bar{A} = X$.

We shall omit proofs of many standard results that are part of basic point-set topology, such as the next one. While this result is not too hard and can safely be left as an exercise, other omitted proofs will be more substantial.

**Proposition 1.1.6.** *A point $x \in X$ is in $\bar{A}$ if and only if every neighborhood of $x$ contains a point of $A$, and $\bar{A}$ is the union of $A$ and the set of limit points of $A$. The set $A$ is closed if and only if it contains all of its limit points.*

## 1.2. Alexandroff and finite spaces

It is very often interesting to see what happens when one takes a standard definition and tweaks it a bit. The following tweaking of the notion of a topology is due to Alexandroff [2], except that he used a different name for the notion[1].

**Definition 1.2.1.** A topological space $X$ is an *Alexandroff space* if the set $\mathscr{U}$ is closed under arbitrary intersections, not just finite ones.

**Remark 1.2.2.** The notion of an Alexandroff space has a pleasing complementarity. If $X$ is an Alexandroff space, then the closed subsets of $X$ give it a new topology in which it is again an Alexandroff space. We write $X^{op}$ for $X$ with this opposite topology. Then $(X^{op})^{op}$ is the space $X$ back again.

A space is *finite* if the set $X$ is finite. Since any intersection in a finite space is finite, the following observation is immediate.

**Lemma 1.2.3.** *A finite space is an Alexandroff space.*

[1] His name was Diskrete Räume, which translates as discrete spaces.

It turns out that a great deal of what can be proven for finite spaces applies equally well more generally to Alexandroff spaces, with exactly the same proofs. When that is the case, we will prove the more general version. However, finite spaces have recently captured people's attention. Since digital processing and image processing start from finite sets of observations and seek to understand pictures that emerge from a notion of nearness of points, finite topological spaces seem a natural tool in many such scientific applications. There are quite a few papers on the subject, although few of much mathematical depth, starting from the 1980's.

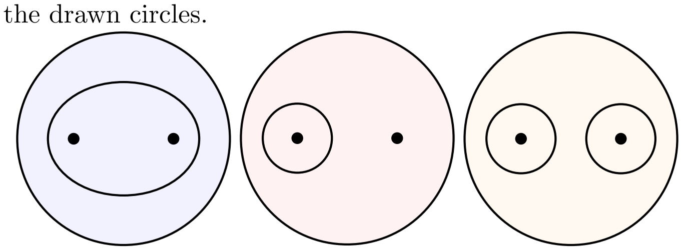
There was a brief early flurry of beautiful mathematical work on this subject. Two independent papers, by McCord and Stong [**50, 65**], both published in 1966, are especially interesting. We will work through them. We are especially interested in questions that are raised by the union of these papers but are answered in neither. These questions have only recently been pursued. We are also interested in calculational questions about the enumeration of finite topologies.

There is a hierarchy of "separation properties" on spaces, and intuition about finite spaces is impeded by too much habituation to the stronger of them.

**Definition 1.2.4.** Let $(X, \mathscr{U})$ be a topological space.

(i) $X$ is a $T_0$*-space* if for any two points of $X$, there is an open neighborhood of one that does not contain the other. That is, the topology distinguishes points.
(ii) $X$ is a $T_1$*-space* if each point of $X$ is a closed subset.
(iii) $X$ is a $T_2$*-space*, or *Hausdorff space*, if any two points of $X$ have disjoint open neighborhoods.[2]

**Example 1.2.5.** The following are examples of spaces with the aforementioned separation properties; keep in mind that the smallest open sets are pictured in the interiors of the drawn circles.



not $T_0$ $\qquad$ $T_0$, *not* $T_1$ $\qquad$ $T_2$

**Lemma 1.2.6.** *If $X$ is a $T_2$-space, then it is a $T_1$-space. If $X$ is a $T_1$-space, then it is a $T_0$-space.*

There are still stronger separation properties. In most of topology, the spaces considered are at least Hausdorff. For example, metric spaces are Hausdorff. We discuss them briefly in the final section. It is commonplace to use the following property.

**Proposition 1.2.7.** *Let $A$ be a subset of a Hausdorff space $X$ and let $x \in X$. Then $x$ is a limit point of $A$ if and only if every neighborhood of $x$ contains infinitely many points in $A$.*

[2]The terminology is due to a 1935 paper of Alexandroff and Hopf [**3**]. The German word for separation is "Trennung", hence the letter $T$ for the hierarchy of separation properties.

Obviously, intuition gained from thinking about Hausdorff spaces is likely to be misleading when thinking about finite spaces! In fact, there are no interesting spaces that are both Alexandroff and $T_1$, let alone $T_2$.

**Lemma 1.2.8.** *If an Alexandroff space is $T_1$, then it is discrete.*

PROOF. Every subset of any set is the union of its subsets with a single element. In an Alexandroff space, all unions of closed subsets are closed. In a $T_1$-space, all singleton subsets are closed. If both of these conditions hold, every subset is closed. Therefore every subset is open. $\square$

In contrast, Alexandroff $T_0$-spaces are very interesting. The following warm-up problem might seem a bit difficult right now, but its solution will shortly become apparent.

**Exercise 1.2.9.** Show that a finite $T_0$-space has at least one point which is a closed subset.

**Notation 1.2.10.** As in the introduction, we define an $F$-space to be a finite $T_0$-space and an $A$-space to be an Alexandroff $T_0$-space.

## 1.3. Bases and subbases for topologies

Alexandroff spaces have canonical minimal bases, which we describe in this section. We first recall the notions of a basis and a subbasis for a topology. The idea is that one often has a preferred collection of "small" or canonical open sets, a"basis" from which all other open sets are generated.

**Definition 1.3.1.** A *basis* for a topology on a set $X$ is a set $\mathscr{B}$ of subsets of $X$ such that

(i) For each $x \in X$, there is at least one $B \in \mathscr{B}$ such that $x \in B$.
(ii) If $x \in B' \cap B''$ where $B', B'' \in \mathscr{B}$, then there is at least one $B \in \mathscr{B}$ such that $x \in B \subset B' \cap B''$.

The topology $\mathscr{U}$ *generated* by the basis $\mathscr{B}$ is the set of subsets $U$ such that, for every point $x \in U$, there is a $B \in \mathscr{B}$ such that $x \in B \subset U$. Equivalently, a set $U$ is in $\mathscr{U}$ if and only if it is a union of sets in $\mathscr{B}$.

In the definition, we did not assume that we started with a topology on $X$. If we do start with a given topology $\mathscr{U}$, then it usually admits many different bases. We can easily characterize which subsets of $\mathscr{U}$ give bases.

**Lemma 1.3.2.** *Let $(X, \mathscr{U})$ be a topological space. A subset $\mathscr{B}$ of $\mathscr{U}$ is a basis that generates $\mathscr{U}$ if and only if for every $U \in \mathscr{U}$ and every $x \in U$, there is a $B \in \mathscr{B}$ such that $x \in B \subset U$.*

We can generate bases for topologies from subbases.

**Definition 1.3.3.** A *subbasis* for a topology on a set $X$ is a set $\mathscr{S}$ of open subsets of $X$ whose union is $X$; that is, $\mathscr{S}$ is a *open cover* of $X$. The set of finite intersections of sets in $\mathscr{S}$ is the basis generated by $\mathscr{S}$. If $(X, \mathscr{U})$ is a topological space, a subbasis $\mathscr{S}$ for the topology $\mathscr{U}$ is a subset of $\mathscr{U}$ such that every set in $\mathscr{U}$ is a union of finite intersections of sets in $\mathscr{S}$.

**Example 1.3.4.** The set of singleton sets $\{x\}$ is a basis for the discrete topology on $X$. The set of open balls $B(x, r) = \{y | d(x, y) < r\}$ is a basis for the topology on a metric space $X$.

Returning to Alexandroff spaces, we find that such a space has a canonical basis which is minimal in the strong sense that the open sets in the canonical basis are open sets in *any* basis for the topology on $X$.

**Definition 1.3.5.** Let $X$ be an Alexandroff space. For $x \in X$, define $U_x$ to be the intersection of the open sets that contain $x$. Define a relation $\leq$ on the set $X$ by $x \leq y$ if $x \in U_y$ or, equivalently, $U_x \subset U_y$. Write $x < y$ if the inclusion is proper.

**Lemma 1.3.6.** *The set of open sets $U_x$ is a basis $\mathscr{B}$ for $X$. If $\mathscr{C}$ is any other basis, then $\mathscr{B} \subset \mathscr{C}$. Therefore $\mathscr{B}$ is the unique minimal basis for $X$.*

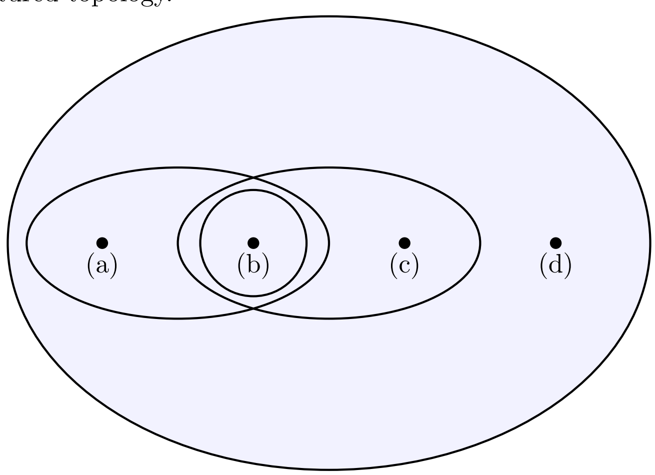
Proof. The first statement is clear from the definitions. If $\mathscr{C}$ is another basis and $x \in X$, then there is a $C \in \mathscr{C}$ such that $x \in C \subset U_x$. This implies that $C = U_x$, so that $U_x \in \mathscr{C}$. □

As you may have guessed, we can detect whether or not an Alexandroff space is $T_0$ in terms of its minimal basis. This is formalized as follows.

**Lemma 1.3.7.** *Two points $x$ and $y$ in $X$ have the same neighborhoods if and only if $U_x = U_y$. Therefore $X$ is $T_0$ if and only if $U_x = U_y$ implies $x = y$.*

Proof. If $x$ and $y$ have the same neighborhoods, then obviously $U_x = U_y$. Conversely, suppose that $U_x = U_y$. If $x \in U$ where $U$ is open, then $U_y = U_x \subset U$ and therefore $y \in U$. Similarly if $y \in U$, then $x \in U$. Thus $x$ and $y$ have the same neighborhoods. □

**Exercise 1.3.8.** Identify the inclusion relations among $U_a$, $U_b$, $U_c$, and $U_d$ in the following pictured topology.




## 1.4. Operations on spaces

There are many standard operations on spaces that we shall have occasion to use. We record four of them now and will come back to others later.

**Definition 1.4.1.** The *subspace topology* on $A \subset X$ is the set of all intersections $A \cap U$ for open sets $U$ of $X$.

Subspace topologies are defined for injective functions. There is a perhaps less intuitive analogue for surjective functions.

**Definition 1.4.2.** Let $X$ be a topological space and $q : X \longrightarrow Y$ be a surjective function. The *quotient topology* on $Y$ is the set of subsets $U$ such that $q^{-1}(U)$ is open in $X$.

**Definition 1.4.3.** The *topology of the union* on the disjoint union $X \amalg Y$ has as open sets the unions of an open set of $X$ and an open set of $Y$. More generally, for a set $\{X_i | i \in I\}$ of topological spaces, the topology of the union on the disjoint union $\coprod_{i\in I} X_i$ has as open sets the unions of open sets $U_i \subset X_i$. Note that a subset is closed if and only if it intersects each $X_i$ in a closed subset.

**Definition 1.4.4.** The *product topology* on the cartesian product $X \times Y$ is the topology with basis the products $U \times V$ of an open set $U$ in $X$ and an open set $V$ in $Y$. More generally, for a set $\{X_i | i \in I\}$ of topological spaces, the product topology on the product set $\prod_{i\in I} X_i$ is the topology generated by the basis consisting of all products $\prod_{i\in I} U_i$ where $U_i$ is open in $X_i$ and $U_i = X_i$ for all but finitely many $i$.

There is a consistency observation relating the subspace and product topologies.

**Proposition 1.4.5.** *If $A \subset X$ and $B \subset Y$, then the subspace and product topologies on $A \times B \subset X \times Y$ coincide.*

For Hausdorff spaces, we have the following observations, the proofs of which make good exercises.

**Proposition 1.4.6.** *A space $X$ is Hausdorff if and only if the diagonal subspace $\{(x, x)\} \subset X \times X$ is closed.*

**Proposition 1.4.7.** *A subspace of a Hausdorff space is Hausdorff. A quotient of a Hausdorff space need* not *be Hausdorff. A disjoint union of Hausdorff spaces is Hausdorff. Any product of Hausdorff spaces is Hausdorff.*

**Exercise 1.4.8.** Verify the following analogue for Alexandroff spaces.

**Proposition 1.4.9.** *A subspace of an Alexandroff space is an Alexandroff space. A quotient of an Alexandroff space is an Alexandroff space. A disjoint union of Alexandroff spaces is an Alexandroff space. A product of finitely many Alexandroff spaces is an Alexandroff space.*

Here is a thought exercise for you.

**Problem 1.4.10.** *Is the product of infinitely many Alexandroff spaces an Alexandroff space?*

### 1.5. Continuous functions and homeomorphisms

**Definition 1.5.1.** Let $X$ and $Y$ be spaces. A function $f : X \longrightarrow Y$ is *continuous* if $f^{-1}(U)$ is open in $X$ for all open subsets $U$ of $Y$. A continuous function is often called a *map*.

It suffices that $f^{-1}(U)$ be open for each $U$ in a basis for the topology on $Y$, or even for each $U$ in a subbasis. The reader is encouraged to use that to verify that the abstract definition of continuity just given coincides with the usual $\varepsilon$-$\delta$ definition of continuity on metric spaces; see §16.1. By passage to complements, a function $f$ is continuous if and only if $f^{-1}(C)$ is closed in $X$ for all closed subsets $C$ of $Y$. This can be reinterpreted in terms of closures (and thus in terms of limit points).

**Lemma 1.5.2.** *A function $f : X \longrightarrow Y$ is continuous if and only if, for all $A \subset X$, $f(\bar{A}) \subset \overline{f(A)}$.*

**Lemma 1.5.3.** *Let $A$ be a subspace of a space $X$. A continuous function from $A$ to a Hausdorff space $Y$ admits at most one extension to a continuous map $\bar{A} \longrightarrow Y$.*

Identity functions and composites of continuous functions are continuous.

**Lemma 1.5.4.** *Let $X$ be a space, let $A \subset X$, and give $A$ the subspace topology. Then the inclusion $i : A \longrightarrow X$ is a continuous function. If $B$ is a space and $j : B \longrightarrow A$ is a function such that $i \circ j$ is continuous, then $j$ is continuous.*

**Lemma 1.5.5.** *Let $X$ be a space, let $q : X \longrightarrow Y$ be a surjective function, and give $Y$ the quotient topology. Then $q$ is a continuous function. If $Z$ is a space and $r : Y \longrightarrow Z$ is a function such that $r \circ q$ is continuous, then $r$ is continuous.*

**Lemma 1.5.6.** *Let $X_i$ be spaces and let $\iota_i \colon X_i \longrightarrow \coprod X_i$ be the inclusion. Then $\iota_i$ is a continuous function. If $Z$ is a space and $\eta_i \colon X_i \longrightarrow Z$ are continuous functions, then the unique function $\coprod X_i \longrightarrow Z$ that restricts to $\eta_i$ on $X_i$ is continuous.*

**Lemma 1.5.7.** *Let $X_i$ be spaces and let $\pi_i : \prod_i X_i \longrightarrow X_i$ be the projection. Then $\pi_i$ is a continuous function. If $Y$ is a space and $\rho_i : Y \longrightarrow X_i$ are continuous functions, then the unique function $Y \longrightarrow \prod X_i$ with $i^{th}$ coordinate $\rho_i$ is continuous.*

The four previous propositions state that the subspace, quotient, union, and product topologies satisfy certain "universal properties". In each of these results, the specified topology is the only topology for which the last statement is true.

Continuity is a local condition on a function.

**Lemma 1.5.8.** *A function $f : X \longrightarrow Y$ is continuous if and only if for each $x \in X$ and each neighborhood $V$ of $f(x)$, there is a neighborhood $U$ of $x$ such that $f(U) \subset V$.*

**Lemma 1.5.9.** *A function $f : X \longrightarrow Y$ is continuous if and only if its restriction to each set in an open cover of $X$ is continuous.*

There is an analogue for finite closed covers.

**Lemma 1.5.10.** *A function $f : X \longrightarrow Y$ is continuous if and only if its restriction to each set in a finite closed cover of $X$ is continuous.*

In particular, if $X = A \cup B$ where $A$ and $B$ are closed subsets of $X$, then continuous functions $A \longrightarrow Y$ and $B \longrightarrow Y$ that agree on $A \cap B$ induce a continuous function $X \longrightarrow Y$.

**Definition 1.5.11.** A continuous bijection $f : X \longrightarrow Y$ is a *homeomorphism* if its inverse $f^{-1}$ is also continuous. That is, a homeomorphism is a continuous bijection with a continuous inverse. Equivalently, a map $f \colon X \longrightarrow Y$ is a homeomorphism if there is a map $g \colon Y \longrightarrow X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. An *inclusion* or *embedding* is a continuous injection that is a homeomorphism onto its image. We write $X \cong Y$ to indicate that $X$ is homeomorphic to $Y$.

Intuitively, homeomorphism is the topological counterpart of the algebraic notion of isomorphism. Topologists are interested in properties of spaces that are invariant under homeomorphism. We shall later (Lemma 1.7.1, Theorem 16.2.7) give conditions on $X$ and $Y$ that ensure that a continuous bijection is a homeomorphism.

## 1.6. Alexandroff spaces, preorders, and partial orders

Here we relate Alexandroff spaces to the combinatorial notions of preorder and partial order.

**Definition 1.6.1.** A *preorder* on a set $X$ is a reflexive and transitive relation, denoted $\leq$. This means that $x \leq x$ and that $x \leq y$ and $y \leq z$ imply $x \leq z$. A preorder is a *partial order* if it is antisymmetric, which means that $x \leq y$ and $y \leq x$ imply $x = y$. Then $(X, \leq)$ is called a *poset*. A poset is totally ordered if for all $x, y \in X$, either $x \leq y$ or $y \leq x$.

Recall from Definition 1.3.5 that, in an Alexandroff space $X$, $x \leq y$ means that $U_x \subset U_y$.

**Lemma 1.6.2.** *The relation $\leq$ on an Alexandroff space $X$ is reflexive and transitive, so that the relation $\leq$ is a preorder. The relation is also antisymmetric, so that $(X, \leq)$ is a poset, if and only if the space $X$ is $T_0$.*

Proof. The first statement is clear and the second holds by Lemma 1.3.7. □

**Lemma 1.6.3.** *A preorder $(X, \leq)$ determines a topology $\mathscr{U}$ on $X$ with basis the set of all sets $U_x = \{y | y \leq x\}$. It is called the order topology on $X$. The space $(X, \mathscr{U})$ is an Alexandroff space. It is a $T_0$-space if and only if $(X, \leq)$ is a poset.*

Proof. If $x \in U_y$ and $x \in U_z$, then $x \leq y$ and $x \leq z$, hence $x \in U_x \subset U_y \cap U_z$. Therefore $\{U_x\}$ is a basis for a topology. The intersection $U$ of a set $\{U_i\}$ of open subsets is open since if $x \in U$, then $U_x \subset U_i$ for each $i$ and therefore $U$ is the union of these $U_x$. Therefore $(X, \mathscr{U})$ is an Alexandroff space with minimal basis $\{U_x\}$. Since $U_x = U_y$ if and only if $x \leq y$ and $y \leq x$, Lemma 1.3.7 implies that $(X, \mathscr{U})$ is $T_0$ if and only if $(X, \leq)$ is a poset. □

We put things together to obtain the following conclusion.

**Proposition 1.6.4.** *For a set $X$, the Alexandroff space topologies on $X$ are in bijective correspondence with the preorders on $X$. The topology $\mathscr{U}$ corresponding to $\leq$ is $T_0$ if and only if the relation $\leq$ is a partial order.*

**Remark 1.6.5.** If $\leq$ is a preorder on $X$, the opposite preorder is given by $x \leq^{op} y$ if and only if $y \leq x$. The corresponding Alexandroff space is $X^{op}$.

The real force of the comparison between Alexandroff spaces and preorders comes from the fact that continuous maps correspond precisely to order-preserving functions.

**Definition 1.6.6.** Let $X$ and $Y$ be preorders. A function $f: X \longrightarrow Y$ is order-preserving if $w \leq x$ in $X$ implies $f(w) \leq f(x)$ in $Y$.

**Lemma 1.6.7.** *A function $f: X \longrightarrow Y$ between Alexandroff spaces is continuous if and only if it is order preserving.*

Proof. Let $f$ be continuous and suppose $w \leq x$. Then $w \in U_x \subset f^{-1}U_{f(x)}$ and thus $f(w) \in U_{f(x)}$. This means that $f(w) \leq f(x)$. For the converse, let $f$ be order preserving and let $V$ be open in $Y$. If $f(x) \in V$, then $U_{f(x)} \subset V$. If $w \in U_x$, then $w \leq x$ and thus $f(w) \leq f(x)$ and $f(w) \in U_{f(x)} \subset V$, so that $w \in f^{-1}(V)$. Thus $f^{-1}(V)$ is the union of these $U_x$ and is therefore open. □

## 1.7. Finite spaces and homeomorphisms

In this section we specialize the theory above to finite spaces. Thus let $X$ be a finite space and write $|X|$ for the number of points in $X$. One might think that finite spaces are uninteresting since they are just finite preorders in disguise, but that turns out to be far from the case.

Topologists are only interested in spaces up to homeomorphism, and we proceed to classify finite spaces up to homeomorphism.

**Lemma 1.7.1.** *A map $f\colon X \longrightarrow X$ is a homeomorphism if and only if $f$ is either one–to–one or onto.*

PROOF. By finiteness, one–to–one and onto are equivalent. Assume they hold. Then $f$ induces a bijection $2^f$ from the set $2^X$ of subsets of $X$ to itself. Since $f$ is continuous, if $f(U)$ is open, then so is $U$. Therefore the bijection $2^f$ must restrict to a bijection from the topology $\mathscr{U}$ to itself. Alternatively, observe that the function $f$ is a permutation of the set $X$ and the set of permutations of $X$ is a finite group. Therefore $f^n$ is the identity for some $n$, and the continuous function $f^{n-1}$ is $f^{-1}$. □

The previous lemma fails if we allow different topologies on $X$: there are continuous bijections between different topologies. We proceed to describe how to enumerate the distinct topologies up to homeomorphism. We say that two topologies $\mathscr{U}$ and $\mathscr{V}$ on $X$ are equivalent if there is a homeomorphism $(X, \mathscr{U}) \longrightarrow (X, \mathscr{V})$. There are quite a few papers on this enumeration problem in the literature, although some of them focus on enumeration of all topologies, rather than homeomorphism classes of topologies [**14**, **20**, **24**, **24**, **37**, **38**, **39**, **41**, **59**, **60**]. The difference already appears for two point spaces, where there are four distinct topologies but three inequivalent topologies, that is three non-homeomorphic two point spaces. Here is a table lifted straight from Wikipedia that gives an idea of the enumeration.

| $n$ | Distinct topologies | Distinct $T_0$-topologies | Inequivalent topologies | Inequivalent $T_0$-topologies |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 4 | 3 | 3 | 2 |
| 3 | 29 | 19 | 9 | 5 |
| 4 | 355 | 219 | 33 | 16 |
| 5 | 6942 | 4231 | 139 | 63 |
| 6 | 209,527 | 130,023 | 718 | 318 |
| 7 | 9,535,241 | 6,129,859 | 4,535 | 2,045 |
| 8 | 642,779,354 | 431,723,379 | 35,979 | 16,999 |
| 9 | 63,260,289,423 | 44,511,042,511 | 363,083 | 183,231 |
| 10 | 8,977,053,873,043 | 6,611,065,248,783 | 4,717,687 | 2,567,284 |

Through $n = 9$, a published source for the fourth column is [**39**]. However, this is not the kind of enumeration problem for which one expects to obtain a precise answer for all $n$. Rather, one expects bounds and asymptotics. There is a precise formula relating the second column to the first column, but we are really only interested in the last column. In fact, we are far more interested in refinements of the last column that shrink its still inordinately large numbers to smaller numbers of far greater interest to an algebraic topologist.

We shall explain how to reduce the determination of the third and fourth columns to a matrix computation, using minimal bases. For this purpose, it is convenient to describe minimal bases for a topology on $X$ without reference to their enumeration by the elements $x \in X$, since the latter can give redundant information when the space is not $T_0$. The following sequence of lemmas applies to the study of general Alexandroff spaces, not necessarily finite.

**Lemma 1.7.2.** *A set $\mathscr{B}$ of nonempty subsets of $X$ is the unique minimal basis for an Alexandroff topology $\mathscr{U}$ if and only if the following conditions hold.*

*(i) Every point of $X$ is in some set $B$ in $\mathscr{B}$.*
*(ii) The intersection of two sets in $\mathscr{B}$ is a union of sets in $\mathscr{B}$.*
*(iii) If a union of sets $B_i$ in $\mathscr{B}$ is again in $\mathscr{B}$ and if $x \in B \subset \cup B_i$ with $B \in \mathscr{B}$, then $B = B_i$ for some $i$.*

Proof. Conditions (i) and (ii) are equivalent to saying that $\mathscr{B}$ is a basis for a topology, which we call $\mathscr{U}$. We suppose this topology is Alexandroff. Then each $B$ in $\mathscr{B}$ must be a union of sets of the form $U_x$ and each $U_x$ must be in $\mathscr{B}$ by Lemma 1.3.6. If $\mathscr{B}$ is the minimal basis $\{U_y\}$, then each given set $B_i$ in (iii) must be $U_y$ for some $y \in X$. If the union of these $U_y$ is also in $\mathscr{B}$, then the union must be $U_x$ for some $x \in X$. But then $x$ is in $U_y$ for some $y$ and thus $U_x = U_y$, so that (iii) holds. If $\mathscr{B}$ is a possibly larger basis, we still have that any open set $B$ is a union of sets $U_y$. If that union is in $\mathscr{B}$ and not of the form $U_x$ for any $x$, then $\mathscr{B} \setminus \{B\}$ is still a basis, so that $\mathscr{B}$ is not minimal. □

This result implies the following relationships between minimal bases and subspaces, quotients, disjoint unions, and products of Alexandroff spaces.

**Lemma 1.7.3.** *If $A$ is a subspace of $X$, the minimal basis of $A$ consists of the intersections $A \cap U$, where $U$ is in the minimal basis of $X$.*

**Lemma 1.7.4.** *If $Y$ is a quotient space of $X$ with quotient map $q\colon X \longrightarrow Y$, the minimal basis of $Y$ consists of the subsets $U$ of $Y$ such that $q^{-1}(U)$ is in the minimal basis of $X$.*

**Lemma 1.7.5.** *The minimal basis of $X \amalg Y$ is the union of the minimal basis of $X$ and the minimal basis of $Y$.*

**Lemma 1.7.6.** *The minimal basis of $X \times Y$ is the set of products $U \times V$, where $U$ and $V$ are in the minimal bases of $X$ and $Y$.*

Returning to finite spaces $X$, we shall show how to enumerate the homeomorphism classes of spaces with finitely many elements. This is meant only to illustrate how such an enumeration problem can be reduced to computationally accessible form. To allow spaces that are not $T_0$, the finite number to focus on is not the number of elements in $X$ but rather the number of elements in the minimal basis for the topology on $X$. These numbers are equal if and only if $X$ is a $T_0$-space.

**Definition 1.7.7.** Consider square matrixes $M = (a_{i,j})$ with integer entries that satisfy the following properties.

(i) $a_{i,i} \geq 1$.
(ii) $a_{i,j}$ is $-1$, 0, or 1 if $i \neq j$.

(iii) $a_{i,j} = -a_{j,i}$ if $i \neq j$.
(iv) $a_{i_1,i_s} = 0$ if there is a sequence of distinct indices $\{i_1, \cdots, i_s\}$ such that $s > 2$ and $a_{i_k,i_{k+1}} = 1$ for $1 \leq k \leq s-1$.

Say that two such matrices $M$ and $N$ are equivalent if there is a permutation matrix $T$ such that $T^{-1}MT = N$ and let $\mathscr{M}$ denote the set of equivalence classes of such matrices.

**Theorem 1.7.8.** *The homeomorphism classes of finite spaces are in bijective correspondence with $\mathscr{M}$. If the homeomorphism class of $X$ corresponds to the equivalence class of an $r \times r$ matrix $M$, then $r$ is the number of sets in a minimal basis for $X$, and the trace of $M$ is the number of elements of $X$. Moreover, $X$ is a $T_0$-space if and only if the diagonal entries of $M$ are all one.*

PROOF. We work with minimal bases for the topologies rather than with elements of the set. For a minimal basis $U_1, \cdots, U_r$ of a topology $\mathscr{U}$ on a finite set $X$, define an $r \times r$ matrix $M = (a_{i,j})$ as follows. If $i = j$, let $a_{i,i}$ be the number of elements $x \in X$ such that $U_x = U_i$. Define $a_{i,j} = 1$ and $a_{j,i} = -1$ if $U_i \subset U_j$ and there is no $k$ (other than $i$ or $j$) such that $U_i \subset U_k \subset U_j$. Define $a_{i,j} = 0$ otherwise. Clearly (i)–(iv) hold, and a reordering of the basis results in a permutation matrix that conjugates $M$ into the matrix determined by the reordered basis. Thus $X$ determines an element of $\mathscr{M}$.

If $f\colon X \longrightarrow Y$ is a homeomorphism, then $f$ determines a bijection from the basis for $X$ to the basis for $Y$. This bijection preserves inclusions and the number of elements that determine corresponding basic sets, hence $X$ and $Y$ determine the same element of $\mathscr{M}$. Conversely, suppose that $X$ and $Y$ have minimal bases $\{U_1, \cdots, U_r\}$ and $\{V_1, \cdots, V_r\}$ that give rise to the same element of $\mathscr{M}$. Reordering bases if necessary, we can assume that they give rise to the same matrix. For each $i$, choose a bijection $f_i$ from the set of elements $x \in X$ such that $U_x = U_i$ and the set of elements $y \in Y$ such that $V_y = V_i$. We read off from the matrix that the $f_i$ together specify a homeomorphism $f\colon X \longrightarrow Y$. Therefore our mapping from homeomorphism classes to $\mathscr{M}$ is one–to–one.

To see that our mapping is onto, consider an $r \times r$-matrix $M$ of the sort under consideration and let $X$ be the set of pairs of integers $(u, v)$ with $1 \leq u \leq r$ and $1 \leq v \leq a_{u,u}$. Define subsets $U_i$ of $X$ by letting $U_i$ have elements those $(u, v) \in X$ such that either $u = i$ or $u \neq i$ but $u = i_1$ for some sequence of distinct indices $\{i_1, \cdots, i_s\}$ such that $s \geq 2$, $a_{i_k,i_{k+1}} = 1$ for $1 \leq k \leq s-1$, and $i_s = i$. We see that the $U_i$ give a minimal basis for a topology on $X$ by verifying the conditions specified in Lemma 1.3.6.

Condition (i) is clear since $(u, v) \in U_u$. To verify (ii) and (iii), we observe that if $(u, v) \in U_i$ and $u \neq i$, then $U_u \subset U_i$. Indeed, we certainly have $(u, v) \in U_i$ for all $v$, and if $(k, v) \in U_u$ with $k \neq u$, then we must have a sequence connecting $k$ to $u$ and a sequence connecting $u$ to $i$. These can be concatenated to give a sequence connecting $k$ to $i$, which shows that $(k, v)$ is in $U_i$. To see (ii), if $(u, v) \in U_i \cap U_j$, then $U_u \subset U_i \cap U_j$, which implies that $U_i \cap U_j$ is a union of sets $U_u$. To see (iii), if a union of sets $U_i$ is a set $U_j$, there is an element of $U_j$ in some $U_i$ and then $U_j \subset U_i$, so that $U_j = U_i$. A counting argument for the diagonal entries and consideration of chains of inclusions show that the matrix associated to the topology whose minimal basis is $\{U_i\}$ is the matrix $M$ that we started with. □

## 1.8. Spaces with at most four points

We describe the homeomorphism classes of spaces with at most four points, with just a start on taxonomy. Recall from Definition 1.1.2 that $D_n$ and $C_n$ denote the discrete and coarse topologies on an $n$-element set.

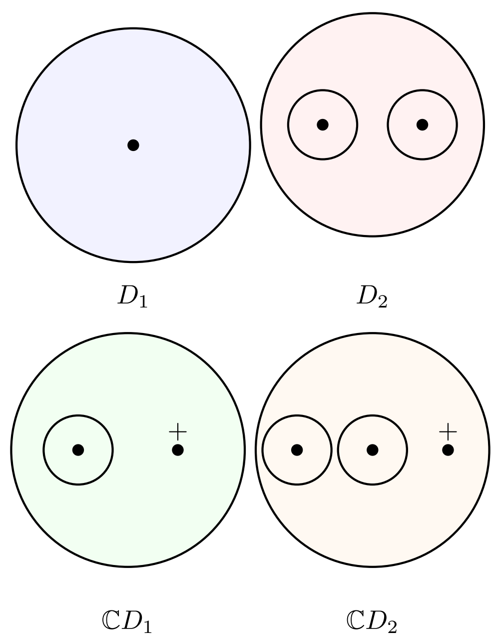
- There is a unique space with one point, namely $C_1 = D_1$.
- There are three spaces with two points, namely $C_2$, $P_2 = \mathbb{C}D_1$, and $D_2$.

Proper subsets of $X$ are those not of the form $\emptyset$ or $X$. Since $\emptyset$ and $X$ are in any topology, we often restrict to proper subsets when specifying topologies. The following definitions prescribe the two names for the second space in the short list just given.

**Definition 1.8.1.** We define certain topologies on a set $S_n$ with $n$ elements. Let $P_n = P_{1,n}$ be the space (unique up to homeomorphism) which has only one proper open set, containing only one point $s \in S_n$; for $1 < m < n$, let $P_{m,n}$ be the space whose proper open subsets are all of the non-empty subsets of a given subset $S_m$ of $S_n$ with $m$ elements.

**Definition 1.8.2.** For a space $X$ define the *non-Hausdorff cone* by $\mathbb{C}X := X \coprod \{*\}$, where $\{*\}$ is a disjoint added basepoint. We let the open subsets of $\mathbb{C}X$ be the open subsets of $X$ along with the set $X \cup \{*\}$.

**Example 1.8.3.** We observed earlier that $P_{1,2} = \mathbb{C}D_1$. That is the start of a pattern. We claim that $\mathbb{C}D_{n-1}$ is homeomorphic to $P_{n-1,n}$ for any $n$. We see that by identifying $D_{n-1}$ with $S_{n-1} \subset S_n$ and identifying the cone point $+$ with the point of $S_n$ not in $S_{n-1}$.




We shall see that $\mathbb{C}X$ is contractible in Lemma 2.3.2 below. This means that it is a point to the eyes of homotopy theory or algebraic topology.

Here is a table of the nine homeomorphism classes of topologies on a three point set $X = \{a, b, c\}$. All of these spaces are disjoint unions of contractible spaces. A space that is not the disjoint union of proper open and closed subspaces is *connected*.

| Proper open sets | Name | $T_0$? | connected? |
|---|---|---|---|
| all | $D_3$ | yes | no |
| a, b, (a,b), (b,c) | $D_1 \amalg P_2$ | yes | no |
| a, b, (a,b) | $P_{2,3} \cong \mathbb{C}D_2$ | yes | yes |
| a | $P_3$ | no | yes |
| a, (a,b) | $\mathbb{C}P_2 \cong (\mathbb{C}P_2)^{op}$ | yes | yes |
| a, (b,c) | $D_1 \amalg C_2$ | no | no |
| a, (a,b), (a,c) | $(\mathbb{C}D_2)^{op}$ | yes | yes |
| (a,b) | $\mathbb{C}C_2 \cong P_3^{op}$ | no | yes |
| none | $C_3 = D_3^{op}$ | no | yes |

**Exercise 1.8.4.** Check that the spaces said to be homeomorphic in the above list are in fact homeomorphic.

We tabulate the proper open subsets of the thirty-three homeomorphism classes of topologies on a four point space $X = \{a, b, c, d\}$. That is, these topologies are obtained by adding in the empty set and the whole set. The list is ordered by decreasing number of singleton sets in the topology, and, when that is fixed, by decreasing number of two-point subsets and then by decreasing number of three-point subsets.[3]

[3] I thank Mark Bowron for sending me a correction and suggesting a reordering.

| | |
|---|---|
| 1 | all |
| 2 | a, b, c, (a,b), (a,c), (b,c), (a,d), (a,b,c), (a,b,d), (a,c,d) |
| 3 | a, b, c, (a,b), (a,c), (b,c), (a,b,c), (a,b,d) |
| 4 | a, b, c, (a,b), (a,c), (b,c), (a,b,c) |
| 5 | a, b, (a,b), (a,c), (a,d), (a,b,c), (a,b,d), (a,c,d) |
| 6 | a, b, (a,b), (a,c), (b,d), (a,b,c), (a,b,d) |
| 7 | a, b, (a,b), (a,c), (a,b,c), (a,b,d) |
| 8 | a, b, (a,b), (a,c), (a,b,c), (a,c,d) |
| 9 | a, b, (a,b), (c,d), (a,c,d), (b,c,d) |
| 10 | a, b, (a,b), (a,c), (a,b,c) |
| 11 | a, b, (a,b), (a,b,c), (a,b,d) |
| 12 | a, b, (a,b), (a,b,c) |
| 13 | a, b, (a,b), (a,c,d) |
| 14 | a, b, (a,b) |
| 15 | a, (a,b), (a,c), (a,d), (a,b,c), (a,b,d), (a,c,d) |
| 16 | a, (a,b), (a,c), (a,b,c), (a,b,d) |
| 17 | a, (a,b), (a,c), (a,b,c) |
| 18 | a, (a,b), (c,d), (a,c,d) |
| 19 | a, (a,b), (a,b,c), (a,b,d) |
| 20 | a, (b,c), (a,b,c), (b,c,d) |
| 21 | a, (a,b), (a,b,c) |
| 22 | a, (a,b), (a,c,d) |
| 23 | a, (b,c), (a,b,c) |
| 24 | a, (a,b) |
| 25 | a, (a,b,c) |
| 26 | a, (b,c,d) |
| 27 | a |
| 28 | (a,b), (c,d) |
| 29 | (a,b), (a,b,c), (a,b,d) |
| 30 | (a,b), (a,b,c) |
| 31 | (a,b) |
| 32 | (a,b,c) |
| 33 | none |

**Problem 1.8.5.**

(1) *Determine which of these spaces are* $T_0$ *and which are connected.*
(2) *Give a taxonomy in terms of explicit general constructions that accounts for all of these topologies. That is, determine appropriate "names" for all of these spaces.*
(3) *How many are not contractible spaces or disjoint unions of contractible spaces? (Hint: there is one connected* 4*-point space that is not contractible; which one of the 33 is it?)*

CHAPTER 2

# Homotopy equivalences of Alexandroff and finite spaces

## 2.1. Connectivity and path connectivity

We begin the exploration of homotopy properties of Alexandroff spaces by discussing connectivity and path connectivity. We recall the general definitions. We let $I = [0,1]$ denote the unit interval with its usual metric topology as a subspace of $\mathbb{R}$. A *path* in a space $X$ is a map $f : I \longrightarrow X$; it is said to connect the points $f(0)$ and $f(1)$.

**Definition 2.1.1.** Let $X$ be a space.

(i) $X$ is *connected* if the only subspaces of $X$ that are both open and closed are $\emptyset$ and $X$.
(ii) $X$ is *path connected* if any two points of $X$ can be connected by a path.

A path connected space is connected, but not conversely. The following results can be found in any text in point-set topology, such as [**54**]. They also make good exercises.

**Lemma 2.1.2.** *Let $Y$ be a subspace of a space $X$ and let $Y = A \cup B$. Then $A$ and $B$ are both open and closed in $Y$ if and only if $\bar{A} \cap B$ and $A \cap \bar{B}$ are both empty or, equivalently, $A$ contains no limit point of $B$ and $B$ contains no limit point of $A$. We then say that $Y = A \cup B$ is a* separation *of $Y$. Thus $Y$ is connected if and only if it has no separation.*

The following consequence is used very frequently.

**Proposition 2.1.3.** *Let $X = A \cup B$ be a separation. If $Y \subset X$ is connected, then $Y$ is contained in either $A$ or $B$.*

**Proposition 2.1.4.** *A union of connected or path connected spaces that have a point in common is connected or path connected.*

**Proposition 2.1.5.** *If $f : X \longrightarrow Y$ is a continuous map and $X$ is connected or path connected, then the image of $f$ is connected or path connected.*

For example, $I$ is a connected space, hence the image of a path in $X$ is a connected subspace of $X$.

**Proposition 2.1.6.** *Any product of connected or path connected spaces is connected or path connected.*

**Definition 2.1.7.** Define two equivalence relations $\sim$ and $\approx$ on $X$.

(i) $x \sim y$ if $x$ and $y$ are both in some connected subspace of $X$. A *component* of $X$ is an equivalence class of points under $\sim$. Let $\pi_0'(X)$ denote the set of components of $X$.

(ii) $x \approx y$ if there is a path connecting $x$ and $y$. A *path component* of $X$ is an equivalence class of points under $\approx$. Let $\pi_0(X)$ denote the set of path components of $X$.

If $x \approx y$, then $x \sim y$ since the image of a path connecting $x$ and $y$ is a connected subspace. Therefore each component of $X$ is the union of some of its path components. For nice spaces, components and path components are the same.

**Definition 2.1.8.** Let $X$ be a space.

(i) $X$ is *locally connected* if for each $x \in X$ and each neighborhood $U$ of $x$, there is a connected neighborhood $V$ of $x$ contained in $U$.
(ii) $X$ is *locally path connected* if for each $x \in X$ and each neighborhood $U$ of $x$, there is a path connected neighborhood $V$ of $x$ contained in $U$.

**Proposition 2.1.9.** *Let $X$ be a space.*

*(i) $X$ is locally connected if and only if every component of an open subset $U$ is open in $X$.*
*(ii) $X$ is locally path connected if and only if every path component of an open subset $U$ is open in $X$.*
*(iii) If $X$ is locally path connected, then the components and path components of $X$ coincide.*

Now return to a finite or, more generally, Alexandroff space $X$. At first sight, one might imagine that there are no continuous maps from $I$ to a finite space, but that is far from the case. The most important feature of finite spaces is that they are surprisingly richly related to the "real" spaces that algebraic topologists care about.

**Lemma 2.1.10.** *Let $X$ be an Alexandroff space. Then each $U_x$ is connected. If $X$ is connected and $x, y \in X$, there is a finite sequence of points $z_i$, $1 \leq i \leq q$, such that $z_1 = x$, $z_q = y$ and either $z_i \leq z_{i+1}$ or $z_{i+1} \leq z_i$ for $i < q$.*

PROOF. Suppose that $U_x = A \amalg B$, where $A$ and $B$ are open and disjoint. We may as well assume that $x$ is in $A$. Then $U_x \subset A$ and therefore $B = \emptyset$ and $U_x = A$. Therefore $U_x$ is connected. Now assume that $X$ is connected. Fix $x$ and consider the set $A$ of points $y$ that are connected to $x$ by some sequence of points $z_i$, as in the statement. We see that $A$ is open since if $z$ is in $A$ then the open set $U_z$ of points $w \leq z$ is contained in $A$. We see that $A$ is closed since if $y$ is not connected to $x$ by a finite sequence of points, then neither is any point of $U_y$, so that the complement of $A$ is open. Since $X$ is connected, it follows that $A = X$. $\square$

**Lemma 2.1.11.** *If $x \leq y$ in an Alexandroff space $X$, then there is a path $p\colon I \longrightarrow X$ connecting $x$ and $y$.*

PROOF. Define $p(t) = x$ if $t < 1$ and $p(1) = y$. We claim that $p$ is continuous. Let $V$ be an open set of $X$. If neither $x$ nor $y$ is in $V$, then $p^{-1}(V) = \emptyset$. If $x$ is in $V$ and $y$ is not in $V$, then $p^{-1}(V) = [0, 1)$. If $y$ is in $V$, then $x$ is in $U_y \subset V$ since $x \leq y$, hence $p^{-1}(V) = I$. In all cases, $p^{-1}(V)$ is open. $\square$

**Proposition 2.1.12.** *An Alexandroff space is connected if and only if it is path connected.*

PROOF. The previous two lemmas, the second generalized by concatenation of paths to finite sequences as in the first, imply that $x \sim y$ if and only if $x \approx y$. $\square$

## 2.2. Function spaces and homotopies

An open cover of a space $X$ is any set of open subsets whose union is all of $X$. The following notion is fundamental to point-set topology. It is discussed in more detail in §15.0.4.

**Definition 2.2.1.** A space is *compact* if every open cover has a finite subcover.

For example, a classical result called the Heine-Borel theorem says that a subspace of $\mathbb{R}^n$ is compact if and only if it closed and bounded.

**Definition 2.2.2.** Let $X$ and $Y$ be spaces and consider the set $Y^X$ of maps $X \longrightarrow Y$. The *compact–open topology* on $Y^X$ is the topology in which a subset is open if and only if it is a union of finite intersections of sets

$$W(C,U) = \{f | f(C) \subset U\},$$

where $C$ is compact in $X$ and $U$ is open in $Y$. This means that the set of all $W(C,U)$ is a subbasis for the topology.

Ignoring topology, for sets $X$, $Y$, and $Z$, functions $f\colon X \times Y \longrightarrow Z$ are in bijective correspondence with functions $\hat{f}\colon X \longrightarrow Z^Y$ via the relation

$$f(x,y) = \hat{f}(x)(y).$$

Returning to topology, and so restricting $Z^Y$ to consist only of the continuous functions $Y \longrightarrow Z$, one would like to have that $f$ is continuous if and only if $\hat{f}$ is continuous. The compact-open topology, which at first sight seems to be unmotivated, is designed to minimize conditions on $X$, $Y$, and $Z$ which force this conclusion. In fact, there are several different criteria which guarantee the conclusion. We recall one due to Fox [**25**] which applies to both Alexandroff spaces and metric spaces.

**Definition 2.2.3.** A space is *first countable* if every point $x$ has a countable neighborhood basis $\mathscr{B}_x$. This means that if $U$ is a neighborhood of $x$, then there is a $B \in \mathscr{B}_x$ such that $x \in B \subset U$.

**Example 2.2.4.** An Alexandroff space $X$ is first countable since the singleton set $\{U_x\}$ is a neighborhood basis for $x$. A metric space is first countable since the $\varepsilon$-neighborhoods $B(x,\varepsilon) = \{y | d(x,y) < \varepsilon\}$ for positive rational numbers $\varepsilon$ form a countable neighborhood basis.

**Proposition 2.2.5.** *Let $X$ and $Y$ be first countable spaces. Then a function $f\colon X \times Y \longrightarrow Z$ is continuous if and only if $\hat{f}\colon X \longrightarrow Z^Y$ is continuous.*

We shall use function spaces to study the notion of homotopy.

**Definition 2.2.6.** A *homotopy* $h\colon f \simeq g$ is a map $h\colon X \times I \longrightarrow Y$ such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$. Two maps are homotopic, written $f \simeq g$, if there is a homotopy between them.

It is impossible to overstate the importance of this notion. We will be studying the homotopy theory of finite topological spaces. For finite spaces, the use of function spaces allows us to recognize homotopic maps in a very simple way. The first statement of the following result is clear, and the reader should check the second statement from the definitions. The conclusion reduces the determination of whether or not two maps are homotopic to the determination of whether or not they are in the same path component of $Y^X$.

**Corollary 2.2.7.** *If $X$ is first countable, then homotopies $h\colon X \times I \longrightarrow Y$ correspond bijectively to paths $j\colon I \longrightarrow Y^X$ via $h \leftrightarrow j$ if $h(x,t) = j(t)(x)$. Therefore the homotopy classes of maps $X \longrightarrow Y$ are in canonical bijective correspondence with the path components of $Y^X$.*

When $Y$ is Alexandroff, we can use its preorder to compare maps $X \longrightarrow Y$ for any space $X$.

**Definition 2.2.8.** If $Y$ is Alexandroff, define the pointwise ordering of maps $X \longrightarrow Y$ by $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$.

**Proposition 2.2.9.** *If $Y$ is Alexandroff, then the intersection $V_g$ of the open sets in $Y^X$ that contain a map $g$ is $\{f | f \leq g\}$.*

Proof. Let $f \in V_g$ and $x \in X$. Since $g \in W(\{x\}, U_{g(x)})$ and $\{x\}$ is compact, $f \in W(\{x\}, U_{g(x)})$, so $f(x) \in U_{g(x)}$ and $f(x) \leq g(x)$. Since $x$ was arbitrary, $f \leq g$. Conversely, let $f \leq g$. Consider any $W(C,U)$ that contains $g$ and let $x \in C$. Then $g(x) \in U$ and, since $f(x) \leq g(x)$, $f(x) \in U_{g(x)} \subset U$. Therefore $f \in W(C,U)$ and $f$ is in all open subsets of $Y^X$ that contain $g$. □

Unfortunately, however, $V_g$ need not be open in $Y^X$ in general. This problem is addressed in work of Kukieła [42]. Since our primary interest is in finite spaces, we shall not go into detail, but the following remarks indicate the subtleties here.

**Remark 2.2.10.** Michał Kukieła [42] studied the behavior of the compact open topology on $Y^X$ when $X$ and $Y$ are possibly infinite Alexandroff spaces.[1] He showed that $Y^X$ is rarely an Alexandroff space. In particular $X^X$ is never an Alexandroff space if $X$ is infinite, which contradicts an assumption made by Arenas [4]. However, Kukieła proved that $Y^X$ is Alexandroff if $X$ is finite. For any $X$ we have an ordering on the set $Y^X$, hence we have the Alexandroff topology on $Y^X$ that it determines. However the Alexandroff topology is generally finer (has more open sets) than the compact open topology.

When $X$ and $Y$ are both finite, so is $Y^X$, and then Proposition 2.2.9 has the following interpretation.

**Corollary 2.2.11.** *If $X$ and $Y$ are finite, then the pointwise ordering on $Y^X$ coincides with the preordering associated to its compact open topology.*

Here, finally, is our easy way to recognize homotopic maps between finite spaces. Part of the result holds for all Alexandroff spaces.

**Proposition 2.2.12.** *If $X$ and $Y$ are Alexandroff spaces and $f \leq g$, then $f \simeq g$ by a homotopy $h$ such that $h(x,t) = f(x)$ for all $t$ and all points $x \in X$ such that $f(x) = g(x)$. Conversely, if $X$ and $Y$ are finite and $f \simeq g$, then there is a sequence of maps $\{f = f_1, f_2, \cdots, f_q = g\}$ such that either $f_i \leq f_{i+1}$ or $f_{i+1} \leq f_i$ for $i < q$.*

Proof. For the first statement, we have the path $p$ connecting $f$ to $g$ in $Y^X$ that is specified by $p(t) = f$ if $t < 1$ and $p(1) = g$. By Lemma 2.1.11, it is continuous if we give $Y^X$ the Alexandroff topology associated to $\leq$. Since that topology has more open sets than the compact open topology, by Kukieła's result

[1] Kukieła made his contribution as an undergraduate at Nicolaus Copernicus University, in Toruń, Poland. Quoting from an email from him, "my study of Alexandroff spaces was in a great degree inspired by your notes on finite spaces".

just mentioned, it is also continuous if we give $Y^X$ the compact open topology. By Proposition 2.2.9, the corresponding function $X \times I \longrightarrow Y$ is also continuous, giving us the claimed homotopy. For the second statement, Corollary 2.2.7 shows that homotopies between maps $X \longrightarrow Y$ are paths in $Y^X$, hence two maps are homotopic if and only if they are in the same path component. Now Lemma 2.1.10 and Corollary 2.2.11 give the conclusion. □

## 2.3. Homotopy equivalences

We have seen that enumeration of finite sets with reflexive and transitive relations $\leq$ amounts to enumeration of the topologies on finite sets. We have refined this to consideration of homeomorphism classes of finite spaces. We are much more interested in the enumeration of the homotopy types of finite spaces. We will come to a still weaker and even more interesting enumeration problem later, one which is still unsolved.

**Definition 2.3.1.** Two spaces $X$ and $Y$ are *homotopy equivalent* if there are maps $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. A space is *contractible* if it is homotopy equivalent to a point.

This relationship can change the number of points. We have a first example.

**Lemma 2.3.2.** *If $X$ is a space containing a point $y$ such that the only open (or only closed) subset of $X$ containing $y$ is $X$ itself, then $X$ is contractible. In particular, the non-Hausdorff cone $\mathbb{C}X$ is contractible for any $X$.*

Proof. This is a variation on a theme we have already seen twice. Let $*$ denote a space with a single point, also denoted $*$. Define $r\colon X \longrightarrow *$ by $r(x) = *$ for all $x$ and define $i\colon * \longrightarrow X$ by $i(*) = y$. Clearly $r \circ i = \mathrm{id}$. Define $h\colon X \times I \longrightarrow X$ by $h(x,t) = x$ if $t < 1$ and $h(x,1) = y$. Then $h$ is continuous. Indeed, let $U$ be open in $X$. If $y \in U$, then $U = X$ and $h^{-1}(U) = X \times I$, while if $y \notin U$, then $h^{-1}(U) = U \times [0,1)$. The argument when $X$ is the only closed subset containing $y$ is the same. Clearly $h$ is a homotopy $\mathrm{id} \simeq i \circ r$. □

**Definition 2.3.3.** A point $x$ of an Alexandroff space $X$ is *maximal* if there is no $y > x$ in $X$; minimal points are defined similarly.

**Corollary 2.3.4.** *If $X$ is an Alexandroff space and $x \in X$, then $U_x$ is contractible. If $X$ is finite and has a unique maximal point or a unique minimal point, then $X$ is contractible.*

Proof. The only open subset of $U_x$ that contains $x$ is $U_x$ itself. If $X$ is finite and $x$ is the unique maximal point in $X$, then $X = U_x$. If $x$ is the unique minimal point in $X$, then the only closed set containing $x$ is $X$. □

A result of McCord [**50**, Thm. 4] says that, when studying finite or, more generally, Alexandroff spaces up to homotopy type, there is no loss of generality if we restrict attention to $T_0$-spaces, that is, to posets. The proof is based on use of the Kolmogorov quotient of a space.

**Definition 2.3.5.** Let $X$ be any space. Define an equivalence relation $\sim$ on $X$ by $x \sim y$ if $x$ and $y$ have the same open neighborhoods. The Kolmogorov quotient $X_0$ of $X$ is the quotient space $X/(\sim)$ obtained by identifying equivalent points. It is a $T_0$ space. Let $q_X\colon X \longrightarrow X_0$ be the quotient map.

The Kolmogorov quotient satisfies a universal property.

**Lemma 2.3.6.** *Let $Z$ be a $T_0$-space and $f\colon X \longrightarrow Z$ be a map. Then there is a unique map $f_0\colon X_0 \longrightarrow Z$ such that $f_0 \circ q_X = f$. Therefore, if $f\colon X \longrightarrow Y$ is any map, there is a unique map $f_0\colon X_0 \longrightarrow Y_0$ such that $q_Y \circ f = f_0 \circ q_X$.*

Proof. Since the topology on $Z$ separates points, $f$ must take equivalent points to the same point. Therefore $f$ factors through a function $f_0\colon X_0 \longrightarrow Y_0$, and $f_0$ is continuous by the universal property of the quotient topology. □

**Theorem 2.3.7.** *For an Alexandroff space $X$, the quotient map $q_X\colon X \longrightarrow X_0$ is a homotopy equivalence.*

Proof. The equivalence relation $\sim$ on $X$ is given by $x \sim y$ if $U_x = U_y$, or, equivalently, if $x \leq y$ and $y \leq x$. The relation $\leq$ on $X$ induces a relation $\leq$ on $X_0$. We claim that $q(U_x) = U_{q(x)}$ for all $x \in X$. To see this, observe first that $q^{-1}q(U_x) = U_x$ since if $q(y) = q(z)$ where $z \in U_x$, then $y \in U_y = U_z \subset U_x$. Therefore $q(U_x)$ is open, hence it contains $U_{q(x)}$. Conversely, $U_x \subset q^{-1}(U_{q(x)})$ by continuity and thus $q(U_x) \subset U_{q(x)}$.

We conclude that the quotient topology on $X_0$ agrees with the topology determined by $\leq$. It follows that $q(x) \leq q(y)$ if and only if $x \leq y$. Indeed, $q(x) \leq q(y)$ implies $q(x) \in U_{q(y)} = q(U_y)$. Thus $q(x) = q(z)$ for some $z \in U_y$ and $U_x = U_z \subset U_y$, so that $x \leq y$. Conversely, if $x \leq y$, then $U_x \subset U_y$ and therefore $U_{q(x)} \subset U_{q(y)}$, so that $q(x) \leq q(y)$.

To prove that $q$ is a homotopy equivalence, let $f\colon X_0 \longrightarrow X$ be any function such that $q \circ f = \mathrm{id}$. That is, we choose a point from each equivalence class. By what we have just proven, $f$ preserves $\leq$ and is therefore continuous.[2] Let $g = f \circ q$. We must show that $g$ is homotopic to the identity. We see that $g$ is obtained by first choosing one $x_u$ with $U_{x_u} = U$ for each $U$ in the minimal basis for $X$ and then letting $g(x) = x_u$ if $U_x = U$. Thus $U_{g(x)} = U_x$ and $g(x) \in U_x$, which means that $g \leq \mathrm{id}$. Now Proposition 2.2.12 gives the required homotopy $h\colon \mathrm{id} \simeq g$. Note that $h(g(x), t) = g(x)$ for all $t$. □

We conclude that to classify Alexandroff spaces up to homotopy equivalence, it suffices to classify $A$-spaces up to homotopy equivalence.

## 2.4. Cores of finite spaces

Stong [**65**, §4] has given an interesting way of studying homotopy types of finite spaces. An attempt to extend his results to Alexandroff spaces was made by Arenas [**4**], but his work had a mistake that was noticed and corrected by Kukieła [**42**]; see Remark 2.2.10. Since the generalization is not an immediate one, we give proofs for the finite space case in this section, turning to Alexandroff spaces in Chapter 14. However, we give the basic definitions in full generality. We change Stong's language a bit in the following exposition. We first single out an especially nice class of homotopy equivalences.

**Definition 2.4.1.** Let $Y$ be a subspace of a space $X$, with inclusion denoted by $i\colon Y \longrightarrow X$. We say that $Y$ is a *deformation retract* of $X$ if there is map $r\colon X \longrightarrow Y$

[2] I have seen it claimed in an undergraduate thesis (not at the University of Chicago, which does not have undergraduate theses) that Theorem 2.3.7 holds for any space $X$, not necessarily Alexandroff. However, there need not be a continuous function $f\colon X_0 \longrightarrow X$ such that $q \circ f = \mathrm{id}$.

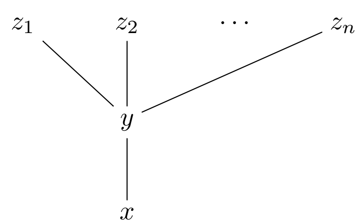
such that $r \circ i$ is the identity map of $Y$ and there is a homotopy $h\colon X \times I \longrightarrow X$ from the identity map of $X$ to $i \circ r$ such that $h(y,t) = y$ for all $y \in Y$ and $t \in I$.

**Definition 2.4.2.** Let $X$ be a finite space.

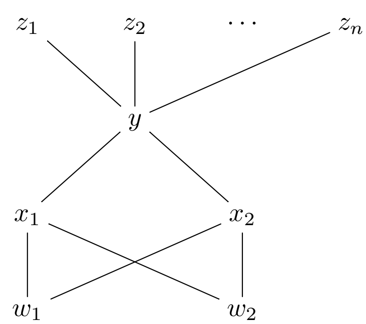
(a) A point $x \in X$ is *upbeat* if there is a $y > x$ such that $z > x$ implies $z \geq y$. Note that $y$ is unique if $X$ is $T_0$.
(b) A point $x \in X$ is *downbeat* if there is a $y < x$ such that $z < x$ implies $z \leq y$.
(c) A point $x \in X$ is a *beat point* if it is either an upbeat point or a downbeat point.

$X$ is a *minimal finite space* if it is a $T_0$-space and has no beat points. A *core* of a finite space $X$ is a subspace $Y$ that is a minimal finite space and a deformation retract of $X$.

**Remark 2.4.3.** If we draw a graph of a poset by drawing a line downwards from $y$ to $x$ if $x < y$, we see that, above an upbeat point $x$, the graph of those edges with $y$ as a vertex looks like




For a more complicated example, both $x_1$ and $x_2$ are upbeat points in the poset




Turning the pictures upside down, we see what the graphs below downbeat points look like. The essential point is that a beat point has either exactly one edge connecting to it from above or exactly one edge connecting to it from below.

Intuitively, identifying $x$ and $y$ and erasing the line between them should not change the homotopy type. We say this another way in the proof of the following result, looking at inclusions rather than quotients in accordance with our definition of a core.

**Theorem 2.4.4.** *Any finite space $X$ has a core.*

PROOF. With the notations of the proof of Theorem 2.3.7, identify $X_0$ with its image $f(X_0) \subset X$. The proof of Theorem 2.3.7 shows that $X_0$, so interpreted, is a deformation retract of $X$. Thus we may as well assume that $X$ is $T_0$. Suppose that

$X$ has an upbeat point $x$. We claim that the subspace $X - \{x\}$ is a deformation retract of $X$. To see this define $f\colon X \longrightarrow X - \{x\} \subset X$ by $f(z) = z$ if $z \neq x$ and $f(x) = y$, where $y > x$ is such that $z > x$ implies $z \geq y$. Clearly $f \geq \mathrm{id}$. We claim that $f$ preserves order and is therefore continuous. Thus suppose that $u \leq v$. We must show that $f(u) \leq f(v)$. If $u = v = x$ or if neither $u$ nor $v$ is $x$, there is nothing to prove. When $u = x < v$, $f(u) = y$ and $f(v) = v \geq y$. When $u < x = v$, $f(u) = u < x < y = f(v)$. Now Proposition 2.2.12 gives the required deformation. A similar argument applies to show that $X - \{x\}$ is a deformation retract of $X$ if $x$ is a downbeat point. Starting with $X_0$, define $X_i$ from $X_{i-1}$ by deleting one upbeat or downbeat point. After finitely many stages, there are no more upbeat or downbeat points left, and we arrive at the required core. □

**Theorem 2.4.5.** *If $X$ is a minimal finite space and $f\colon X \longrightarrow X$ is homotopic to the identity, then $f$ is the identity.*

Proof. First suppose that $f \geq \mathrm{id}$. For all $x$, $f(x) \geq x$. If $x$ is a maximal point, then $f(x) = x$. Let $x$ be any point of $X$ and suppose inductively that $f(z) = z$ for all $z > x$. Then, by continuity, $z > x$ implies $z = f(z) \geq f(x) \geq x$. If $f(x) \neq x$, this implies that $x$ is an upbeat point, contradicting the minimality of $X$. Therefore $f(x) = x$. By induction, $f(x) = x$ for all $x$. A similar argument shows that $f \leq \mathrm{id}$ implies $f = \mathrm{id}$. By Proposition 2.2.12, it now follows that the component of the identity map in the finite space $X^X$ consists only of the identity map. That is, any map homotopic to the identity is the identity. □

**Corollary 2.4.6.** *If $f\colon X \longrightarrow Y$ is a homotopy equivalence of minimal finite spaces, then $f$ is a homeomorphism.*

Proof. If $g\colon Y \longrightarrow X$ is a homotopy inverse, then $g \circ f \simeq \mathrm{id}$ and $f \circ g \simeq \mathrm{id}$. By the theorem, $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$. □

**Corollary 2.4.7.** *Finite spaces $X$ and $Y$ are homotopy equivalent if and only if they have homeomorphic cores. In particular, the core of $X$ is unique up to homeomorphism.*

Proof. This is immediate since the cores of $X$ and $Y$ are minimal finite spaces that are homotopy equivalent to $X$ and $Y$. □

**Remark 2.4.8.** In any homotopy class of finite spaces, there is a representative with the least possible number of points. This representative must be a minimal finite space, since its core is a homotopy equivalent subspace. The minimal representative is homeomorphic to a core of any finite space in the given homotopy class.

In an appendix (Section 14) is an exposition on cores of Alexandroff Spaces, included from an REU paper written by Xi (Cathy) Chen in 2015.

### 2.5. Hasse diagrams and homotopy equivalence

This section is taken from an REU paper written by Alex Fix and Stephen Patrias in 2008. While this portion of their paper is expository, they went on to do original research on the enumeration of homotopy types of finite spaces. That portion of their work will appear in the appendix 15. Their remarkable conclusion is that, as $n$ grows large, the number of homotopy classes of $F$-spaces is asymptotically equivalent to the number of homeomorphism classes of $F$-spaces. We urge the skeptical reader to read the final paragraph of Chapter 15.

Conceptually, the reason for this is that homotopy equivalence of finite spaces, in contrast to homotopy equivalence between the usual spaces of algebraic topology, is far too strict. The notion of weak homotopy equivalence, studied in the following chapter is the right one.

**2.5.1. Hasse diagrams.** The correspondence between $F$-Spaces and partial orders leads to a graphical visualization of $F$-spaces.

**Definition 2.5.1.** For a partial order $P$, we define its associated *Hasse diagram* $H$, a directed graph which captures all of the relevant order information of $P$. Let the vertices of $H$ be the points of $P$ and let there be a directed edge from $y$ to $x$ whenever $x < y$ but there is no other vertex $z$ such that $x < z < y$. We then say that $y$ is a predecessor of $x$ and $x$ is a successor of $y$.

If there is a path $y \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_k \longrightarrow x$ then $y > x_1 > x_2 > \ldots > x_k > x$ so $y > x$. Conversely, if $y > x$ and $x$ is not a sucessor $y$, then we can find $z$ so that $y > z > x$ and by doing this recursively (since the graph is finite), we can find $y > x_1 > \ldots > x_k > x$ so that each step is to a successor, and thus there is a path $y \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_k \longrightarrow x$ in $H$. From this, we also see that the Hasse diagram is necessarily acyclic, that is, there are no directed cycles $x \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_k \longrightarrow x$ or else we would have $x > x$.

We can also go the other way, from a directed acyclic graph $G$ back to a partial order $P$, by saying $y \geq x$ in $P$ whenever there is a path (including trivial paths) from $y$ to $x$ in $G$. However, to do this uniquely, we need the following definitions.

**Definition 2.5.2.** We say that an edge $y \longrightarrow x$ is a *shortcut* in a directed graph $G$ if there is also a path $y \longrightarrow x_1 \longrightarrow \cdots \longrightarrow x_k \longrightarrow x$ with at least two edges between $x$ and $y$. We say that a directed acyclic graph is a *partial order diagram* if it has no shortcuts.

**Theorem 2.5.3.** *The above construction of the Hasse diagram gives a bijection from partial orders to partial order diagrams. Furthermore, there is an isomorphism of partial orders between two posets $P$ and $Q$ if and only if there is a graph isomorphism between the associated diagrams $H_P$ and $H_Q$.*

Proof. It is easy to check that these two constructions are in fact inverses of each other, so that we have a bijection on objects. Then, a bijection $\sigma : P \longrightarrow Q$ is order-preserving if and only if it preserves successors and predecessors, that is it preserves edges in the graph. Therefore $\sigma$ is an isomorphism of posets if and only if it is also a graph isomorphism of the associated Hasse diagrams. □

**Corollary 2.5.4.** *We have a bijection between $F$-space topologies and Hasse diagrams, so that homeomorphism of $F$-spaces is equivalent to graph isomorphism of their diagrams.*

It is also useful to have a convention for drawing these diagrams, as having an orderly presentation allows both a consistent visual understanding of their structure and an additional handle for computation with these graphs.

**Definition 2.5.5.** The height $h(X)$ of a poset $X$ is the maximal length $k$ of a chain $x_1 < \cdots < x_k$ in $X$. If we fix a vertex $v$, we define the *level* $\ell$ of $v$ as the maximal length of the chain that ends at $v$.

We have the following important facts about levels:

(1) The level of a vertex $v$ is the length of the longest downward path beginning at $v$.
(2) There is always an edge from a point $v$ with level $\ell$ to some $v'$ with level $\ell - 1$.
(3) There is never an edge from a point $v$ with level $\ell$ to any $v'$ in level $\ell' \geq \ell$.
(4) Level 1 consists of precisely the minimal points of the graph.

**Remark 2.5.6.** When drawing the Hasse diagram of a poset, we always draw level 1 at the bottom, and each subsequent level $\ell$ immediately above its predecessor, level $\ell - 1$. Thus, all edges in the graph point downwards in the graph, allowing us to omit specifying the directions of edges.

Our theorems about cores and minimal finite spaces have the following immediate corollary:

**Corollary 2.5.7.** *In order to enumerate all the finite spaces with $n$ points up to homotopy equivalence, it suffices to enumerate the minimal spaces with at most $n$ points up to homeomorphism.*

Proof. Since any finite space $X$ on $n$ points has a core, and this core is a deformation retract of the original space, $X$ is homotopy equivalent to a minimal space on no more than $n$ points. Thus, there is at least one minimal space in every homotopy equivalence class. Additionally, if there are two minimal spaces $X$ and $Y$ in the same class, then there is a homotopy equivalence $f : X \longrightarrow Y$. But then $f$ is a homeomorphism. So if we enumerate the minimal spaces up to homeomorphism, we pick exactly one representative from each homotopy class. □

**2.5.2. Minimal Spaces as Graphs.** We now begin the process of converting these topological notions into graph theory, from which actual computations can be made. Primarily, we wish to categorize minimal spaces via a property of the associated Hasse diagram. We start first with a description of upbeat and downbeat points as they appear in the graph.

**Theorem 2.5.8.** *A point $x$ in a finite space $X$ is an upbeat point if and only if it has in-degree one in the associated Hasse diagram (that is, it has only one incoming edge). Similarly, $x$ is downbeat if and only if it has out-degree one (it has only one outgoing edge).*

Proof. Assume that $x$ is upbeat. Then there exists $y > x$ such that for all $z > x$, $z \geq y$. First, we have that $y$ is a successor of $x$, since there cannot be any $z$ with $y > z > x$. Thus, there is an edge $y \longrightarrow x$ in the Hasse diagram. We claim that there is no other edge $y' \longrightarrow x$ with $y' \neq y$. If there were, then $y' > x$ so since $x$ is upbeat, $y' > y$. But since $>$ is equivalent to the existence of a path, we have that there exists a path $y' \longrightarrow \cdots \longrightarrow y$. Hence there is both a path $y' \longrightarrow \cdots \longrightarrow y \longrightarrow x$ and an edge $y' \longrightarrow x$ which violates the requirement that the Hasse diagram have no shortcuts. Thus, $x$ has exactly one incoming edge.

Conversely, assume there is exactly one $y$ such that $y \longrightarrow x$. Then for any $z > x$ we have that there is a path $z \longrightarrow \cdots \longrightarrow x$. But since there is only one vertex $y$ such that $y \longrightarrow x$, this path must actually be $z \longrightarrow \cdots \longrightarrow y \longrightarrow x$ so there is also a path from $z$ to $y$ so $z \geq y$. Thus $x$ is upbeat.

The proof for the second claim is exactly symmetric. □

**Corollary 2.5.9.** *A space is minimal if and only if for every vertex in its associated Hasse diagram, the in-degree and out-degree are both not equal to one.*

**Definition 2.5.10.** Henceforth, we will refer to such a graph as a *minimal graph* for brevity.

We can derive several useful consequences from this classification. For starters, be can begin enumerating the minimal spaces by explicitly constructing graphs which satisfy the above condition (which we will do in Chapter 15). However, we can also use this theorem to derive additional facts about the structure of minimal graphs which might otherwise be difficult to derive using only topological arguments.

**Proposition 2.5.11.** *Let $G$ be a minimal graph with at least two vertices. Then each level of $G$ contains at least two vertices.*

PROOF. Assume first that level 1 has exactly one vertex $v$. Then, since $G$ has at least two vertices, there is some vertex $v'$ in level 2. But every vertex in level 2 has an edge to a vertex in level 1, so $v' \longrightarrow v$ is an edge in the graph. But then $v'$ has exactly one downwards edge, contradicting the minimality of $G$.

Now, assume that some level $\ell > 1$ has exactly one vertex $v$. This vertex has a neighbor $v'$ on level $\ell - 1$, so $v \longrightarrow v'$ is an edge in the graph. Now, assume there is some other $w \neq v$ such that $w \longrightarrow v'$ is also an edge in the graph. Since all edges proceed downwards, we have that $w$ is on some level $k > \ell - 1$. Level $\ell$ has exactly one vertex and $w$ is not it, so $k > \ell$. We claim that this implies that there is in fact a path $w \longrightarrow \cdots \longrightarrow v$ in the graph, so that the edge $w \longrightarrow v'$ is a shortcut of the path $w \longrightarrow \cdots \longrightarrow v \longrightarrow v'$, which is not allowed.

To prove this claim, we induct on $k$: for a vertex $w$ on level $k = \ell + 1$, $w$ must have a neighbor on level $\ell$, so $w \longrightarrow v$ is an edge in the graph, and hence also a path. Then, for $w$ on level $k > \ell + 1$, we again have that $w$ has a neighbor on the next lowest level, so there is some $w'$ on level $k - 1$ such that $w \longrightarrow w'$ is an edge. By induction, there is a path $w' \longrightarrow \cdots \longrightarrow v$ in $G$, so $w \longrightarrow w' \longrightarrow \cdots \longrightarrow v$ is also a path in $G$. ☐

CHAPTER 3

# Homotopy groups and weak homotopy equivalences

## 3.1. Homotopy groups

We recall the definition of the homotopy groups $\pi_n(X, x)$ of a space $X$ at $x \in X$. We shall not give adequate motivation here. This is the first of several places where the first author will advertise his book [48] as a source for a more complete treatment, but in fact all standard textbooks in algebraic topology treat these definitions. For $n = 0$, we define $\pi_0(X)$ to be the set of path components of $X$, with the component of $x$ taken as a basepoint (and there is no group structure). When $n = 1$, we define $\pi_1(X, x)$, or $\pi_1(X)$ when the basepoint is assumed, to be the fundamental group of $X$ at the point $x$.

For all $n \geq 0$, $\pi_n(X)$ can be described most simply by considering the standard sphere $S^n$ with a chosen basepoint $*$. One considers all maps $\alpha\colon S^n \longrightarrow X$ such that $f(*) = x$. One says that two such maps $\alpha$ and $\beta$ are based homotopic if there is a based homotopy $h\colon \alpha \simeq \beta$. Here a homotopy $h$ is based if $h(*, t) = x$ for all $t \in I$. If $n = 1$, the map $\alpha$ is a loop at $x$, and we can compose loops to obtain a product which makes $\pi_1(X, x)$ a group. The homotopy class of the constant loop at $x$ gives the identity element, and the loop $\alpha^{-1}(t) = \alpha(1 - t)$ represents the inverse of the homotopy class of $\alpha$. There is a similar product on the higher homotopy groups, but, in contrast to the fundamental group, the higher homotopy groups are abelian.

A path $p$ from $x$ to $x'$ induces an isomorphism $\pi_n(X, x) \longrightarrow \pi_n(X, x')$. On the fundamental group, it maps a loop $\alpha$ to the composite $p \circ \alpha \circ p^{-1}$, where $p^{-1}$ is the reverse path $p^{-1}(t) = p(1 - t)$ from $x'$ to $x$.

A map $f\colon X \longrightarrow Y$ induces a function $f_*\colon \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$. One just composes maps $\alpha$ and homotopies $h$ as above with the map $f$. If $n \geq 1$, $f_*$ is a homomorphism.

## 3.2. Weak homotopy equivalences

**Definition 3.2.1.** A map $f\colon X \longrightarrow Y$ is a *weak homotopy equivalence* if

$$f_*\colon \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$$

is an isomorphism for all $x \in X$ and all $n \geq 0$. If $n = 0$, this means that components are mapped bijectively. Two spaces $X$ and $Y$ are weakly homotopy equivalent if there is a finite chain of weak homotopy equivalences $Z_i \longrightarrow Z_{i+1}$ or $Z_{i+1} \longrightarrow Z_i$ starting at $X = Z_1$ and ending at $Z_q = Y$.

The definition may seem strange at first sight, but it has gradually become apparent that the notion of a weak homotopy equivalence is even more important in algebraic topology than the notion of a homotopy equivalence. The notions are related. We state some theorems that the reader can take as reference points. Proofs can be found in [**48**]. We mention CW complexes in the following result because they give the appropriate level of generality. They will be defined later, in Definition 10.10.1. However, all the reader needs to know here is that the geometric realizations of simplicial complexes, which will be defined in Definition 4.2.5, are special cases of CW complexes.

**Theorem 3.2.2.** *A homotopy equivalence is a weak homotopy equivalence. Conversely, a weak homotopy equivalence between CW complexes (for example, between simplicial complexes) is a homotopy equivalence.*

**Theorem 3.2.3.** *Spaces $X$ and $Y$ are weakly homotopy equivalent if and only if there is a space $Z$ and weak homotopy equivalences $Z \longrightarrow X$ and $Z \longrightarrow Y$. Moreover, there is such a $Z$ which is a CW complex.*

That is, the chains that appear in the definition need only have length two. For those who know about homology and cohomology, we record the following result.

**Theorem 3.2.4.** *A weak homotopy equivalence induces isomorphisms of all singular homology and cohomology groups.*

### 3.3. A local characterization of weak equivalences

An essential point in our work, which we will take for granted, is that weak homotopy equivalence is a local notion in the sense of the following theorem. McCord [**50**] relies on point-by-point comparison with arguments in the early paper [**22**] which proves the result using quasifibrations. More modern references are [**47, 69**].

**Theorem 3.3.1.** *Let $f\colon A \longrightarrow B$ be a continuous map. Suppose that $B$ has a basis $\mathscr{O}$ such that for each $U \in \mathscr{O}$, the restriction $f\colon f^{-1}(U) \longrightarrow U$ is a weak homotopy equivalence. Then $f$ is a weak homotopy equivalence.*

proof?

### 3.4. The non-Hausdorff suspension

The suspension is one of the most basic constructions in all of topology. Following McCord [**50**], we show that it comes in two weakly equivalent versions, the classical one and a non-Hausdorff analogue that preserves finite spaces. For the purposes of this book, we shall use the following unbased variant of the classical suspension.

**Definition 3.4.1.** Define the *cone* $CX$ of a topological space $X$ to be the quotient space $X \times I/X \times \{1\}$ obtained by identifying $X \times \{1\}$ to a single point, denoted $+$. Define the *suspension* $SX$ of $X$ to be the quotient space obtained from $X \times [-1, 1]$ by identifying $X \times \{1\}$ to a single point $+$ and identifying $X \times \{-1\}$ to another single point, denoted $-$. Thus $SX$ can be thought of as obtained by gluing together the bases of two cones on $X$. For a map $f\colon X \longrightarrow Y$, define $Sf\colon SX \longrightarrow SY$ by $(Sf)(x,t) = (f(x), t)$.

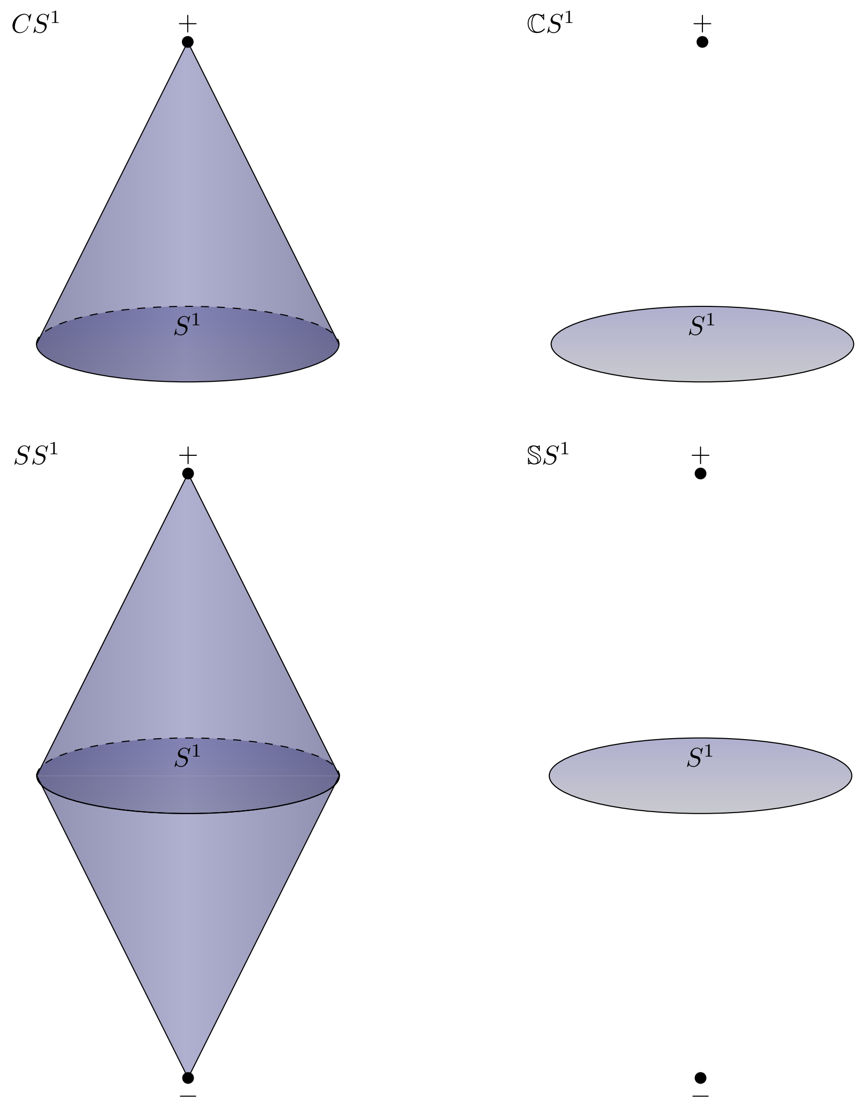
It should be clear that $CX$ is contractible to its cone point $+$. We defined the non-Hausdorff cone $\mathbb{C}X$ by adjoining a new cone point $*$ and letting the proper open subsets of $\mathbb{C}X$ be all of the open subsets of $X$, and we saw that $\mathbb{C}X$ is contractible. We now change the notation for the cone point $*$ and call it $+$.

**Definition 3.4.2.** The *non-Hausdorff suspension of* $X$ is defined to be

$$\mathbb{S}X := X \amalg \{+\} \amalg \{-\}.$$

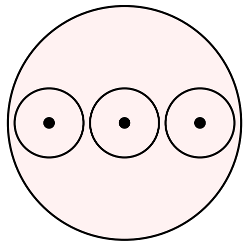
Here $\{+\}$ and $\{-\}$ are new points disjoint from $X$. We let all of the proper open sets of $\mathbb{S}X$ be the open subsets of $X$ along with the sets $X \cup \{+\}$ and $X \cup \{-\}$.

**Remark 3.4.3.** A visualization of $CS^1$ gives geometric meaning to the description of the cone construction. In fact, if we let $X := S^1$, the unit circle, we can visualize and compare all of the constructions just defined.



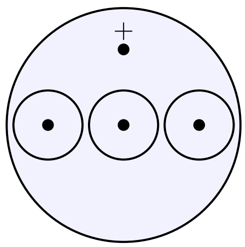

**Example 3.4.4.** Observe that if $X$ is a $T_0$-space, then so are $\mathbb{C}X$ and $\mathbb{S}X$. If, for example, $X = D_3$, the finite space of 3 points endowed with the discrete topology, then the above constructions produce the following $T_0$ spaces (pictured left), while the associated Hasse diagram is shown on the right.

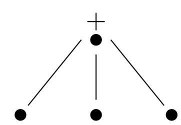
$D_3$





$\mathbb{C}D_3$




**Remark 3.4.5.** When $X$ is an $A$-space, $x < +$and $x < -$ for all $x \in X$. Notice that the only open set containing both of the points $\{+\}$ and $\{-\}$ in $\mathbb{S}X$ is the entire space. In the language of posets, the non-Hausdorff suspension just adds two elements on top of the Hasse diagram of $X$.

Given a map $f : X \longrightarrow Y$, we construct a map $\mathbb{S}f : \mathbb{S}X \longrightarrow \mathbb{S}Y$ by defining

$$\mathbb{S}f(x) = \begin{cases} f(x) & \text{if } x \in X \\ + & \text{if } x = + \\ - & \text{if } x = - \end{cases}$$

**Exercise 3.4.6.** Check that $\mathbf{S}f$ is continuous since $f$ is continuous.

With these definitions in hand, we can relate the classical suspension of a space with the non-Hausdorff suspension by defining the following comparison map.

**Definition 3.4.7.** Define $\gamma_X : SX \longrightarrow \mathbb{S}X$ by

$$\gamma(x,t) = \begin{cases} x & \text{if } -1 < t < 1 \\ + & \text{if } t = 1 \\ - & \text{if } t = -1 \end{cases}$$

**Exercise 3.4.8.** We've defined $\gamma$ so that it is continuous and $\gamma_Y \circ Sf = \mathbb{S}f \circ \gamma_X$. Check that these statements are true.

**Lemma 3.4.9.** *The map $\gamma_X : SX \longrightarrow \mathbb{S}X$ is a weak homotopy equivalence. For any weak homotopy equivalence $f : X \longrightarrow Y$, the maps $Sf : SX \longrightarrow SY$ and $\mathbb{S}f : \mathbb{S}X \longrightarrow \mathbb{S}Y$ are weak homotopy equivalences.*

Proof. This is an application of Theorem 3.3.1. Take the three subspaces $X$, $X \cup \{+\}$, and $X \cup \{-\}$ as our open cover of $\mathbb{S}X$. This is a basis since the intersection of any two of these open sets is $X$. The inverse images under $\gamma$ of these open subsets are $X \times (-1, 1)$, $X \times [-1, 1)$, and $X \times (-1, 1]$. We claim that if we restrict $\gamma$ to each of these subspaces, $\gamma$ becomes a homotopy equivalence and, therefore, $\gamma$ is a weak homotopy equivalence. The proof of this claim is clear: $\gamma(X \times (-1, 1)) = X$ and the domain and target are both contractible in the other two cases.

Similarly, taking the three subspaces $Y$ , $Y \cup \{+\}$, and $Y \cup \{-\}$ as our basis of $\mathbb{S}Y$, their inverse images under $\mathbb{S}f$ are $X$, $X \cup \{+\}$, and $X \cup \{-\}$, and the restrictions of $\mathbb{S}f$ on these three subspaces are weak homotopy equivalences. Finally, take the images in $SY$ of $Y \times (-1/2, 1/2)$, $Y \times [-1, 1/2)$, and $Y \times (-1/2, 1]$ as our basis of $SY$. Their inverse images under $Sf$ are the corresponding subspaces of $SX$, and the restrictions of $Sf$ to these subspaces are weak homotopy equivalences. □

**Definition 3.4.10.** The *nth non-Hausdorff suspension of* $X$ is $\mathbb{S}^n X := \mathbb{S}(\mathbb{S}^{n-1} X)$. The *nth classical suspension of* $X$ is $S^n X := S(S^{n-1} X)$. Inductively, we have a map $\gamma^n : S^n X \longrightarrow \mathbb{S}^n X$.

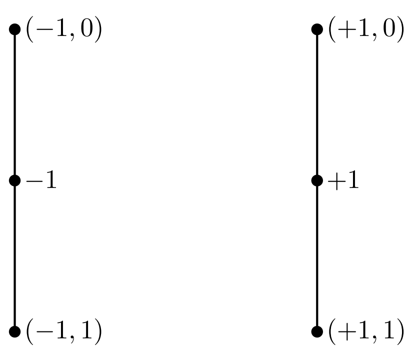
**Theorem 3.4.11.** *For a space* $X$*, the map* $\gamma^n : S^n X \longrightarrow \mathbb{S}^n X$ *is a weak homotopy equivalence.*

Proof. We appeal to the diagram

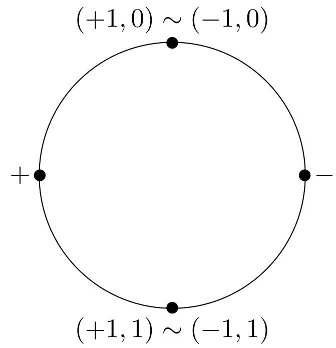
$$\begin{array}{ccc} S^n X & \xrightarrow{S\gamma^{n-1}} & S\mathbb{S}^{n-1} X \\ {\scriptstyle\gamma}\downarrow & \searrow^{\gamma^n} & \downarrow{\scriptstyle\gamma} \\ \mathbb{S}S^{n-1} X & \xrightarrow[\mathbb{S}\gamma^{n-1}]{} & \mathbb{S}^n X \end{array}.$$

The commutativity of the diagram follows from Exercise 3.4.8. We may assume inductively that $\gamma^{n-1}$ is a homotopy equivalence. It follows that $S\gamma^{n-1}$ and $\mathbb{S}\gamma^{n-1}$ are also weak homotopy equivalences by the preceding lemma. By the commutativity of the diagram, we have that $\gamma^n$ is also a weak homotopy equivalence. □

We apply the previous theorem to a simple example. Consider $S^0$, which is just a two-point space. Building $SS^0$ is a process that's pictured below. Namely, we first cross the two points with the unit interval, obtaining:

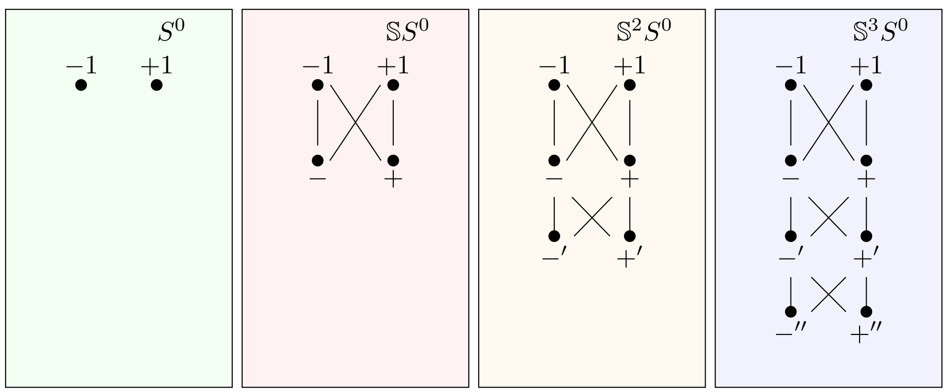



We then identify $(+1, 1) \sim (-1, 1)$ and $(+1, 0) \sim (-1, 0)$, which produces what's pictured in the following diagram.




It's then pictorially clear that $SS^0 \simeq S^1$. A little visualization, gluing two hollow cones onto a circle (one upwards, one downwards), will convince the reader that $SS^1 \cong S^2$. This then gives that $SS^1 = S^2S^0 \simeq S^2$. This result holds in fact for all $n$: the $n$-fold classical suspension of $S^0$ is homeomorphic to $S^n$.

The non-Hausdorff suspension of a finite space is easy to visualize, and we can draw the iteration of this process as follows in the case $X = S^0$.




...

Thus, $n$ iterations of the non-Hausdorff suspension of $S^0$ yields a finite space with $2n$ new points, in addition to the 2 that we started with. The clear implication is stated as follows.

**Theorem 3.4.12.** *Each $\mathbb{S}^nS^0$ is a finite minimal space with $2n + 2$ points.*

The minimality holds since $\mathbb{S}^nS^0$ has no upbeat or downbeat points. Instead, each point has incomparable points above and below it in the partial ordering.

We then have the following result.

**Theorem 3.4.13.** *The $n$-sphere $S^n$ is weak homotopy equivalent to the finite minimal space $\mathbb{S}^nS^0$ with $2n + 2$ points.*

Proof. By Lemma 3.4.11, we have that $\gamma_n : S^nS^0 \longrightarrow \mathbb{S}^nS^0$ is a weak homotopy equivalence. As mentioned before, $S^nS^0 \cong S^n$. □

It is classical that infinitely many of the homotopy groups of $S^2$ are non-zero. Thus we have a six point space with infinitely many non-zero homotopy groups!

The following example gives two weakly equivalent five point spaces that are not homotopy equivalent.

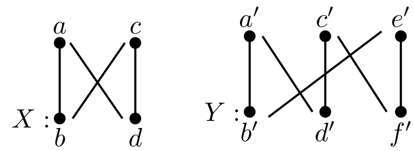
**Example 3.4.14.** Consider the Hasse diagram of $\mathbb{S}D_3$, as shown in Example 3.4.4, and the opposite space $\mathbb{S}D_3$ (pictured below), which has two minimal points.

$\mathbb{S}D_3$ $(\mathbb{S}D_3)^{op}$

We can check that $SD_3$ is homotopy equivalent to the wedge, or one-point union, of two circles. We know that $\gamma_{D_3} : SD_3 \longrightarrow \mathbb{S}D_3$ is a weak homotopy equivalence, and we have a very similar weak equivalence $SD_3 \longrightarrow (\mathbb{S}D_3)^{op}$. It will later become clear that $X$ and $X^{op}$ have the same weak homotopy type for any finite space $X$. However, a comparison of the minimal Hasse diagrams of $(\mathbb{S}D_3)^{op}$ and $\mathbb{S}D_3$ shows that the two spaces are not homotopy equivalent.

Thus, our five point example gives two weakly homotopy equivalent minimal finite spaces with the same number of points that are not homotopy equivalent. Moreover, there is no direct weak homotopy equivalence from one to the other: one needs a chain, like $\mathbb{S}D_3 \longleftarrow SD_3 \longrightarrow (\mathbb{S}D_3)^{op}$.

**Example 3.4.15.** There are minimal finite spaces with more than $2n+2$ points that are also weakly homotopy equivalent to $S^n$. For example, consider the four-point circle and the six point space pictured below. Both $X$ and $Y$ can be seen to be minimal, and they are clearly not homeomorphic and therefore not homotopy equivalent. However, these spaces are weak homotopy equivalent.




Take the unit circle in the complex plane. Let $f : S^1 \longrightarrow X$ and $g : S^1 \longrightarrow Y$ be given by

$$f(x) = \begin{cases} a & \text{if } x = 1 \\ b & \text{if } x = e^{i\theta}, 0 < \theta < \pi \\ c & \text{if } x = -1 \\ d & \text{if } x = e^{i\theta}, \pi < \theta < 2\pi \end{cases}$$

and

$$g(x) = \begin{cases} a' & \text{if } x = 1 \\ b' & \text{if } x = e^{i\theta}, 0 < \theta < 2\pi/3 \\ c' & \text{if } x = e^{2\pi i\theta/3} \\ d' & \text{if } x = e^{i\theta}, 2\pi/3 < \theta < 4\pi/3 \\ e' & \text{if } x = e^{4\pi i\theta/3} \\ f' & \text{if } x = e^{i\theta}, 4\pi/3 < \theta < 2\pi \end{cases}$$

One can verify both $f$ and $g$ are weak homotopy equivalences, and thus $X$ and $Y$ are both finite models of the circle. It should be clear that $X$ is the unique minimal finite model of the circle.

### 3.5. 6-point spaces and height

Up to homeomorphism, the only minimal connected spaces with at most five points are the one point space, the 4-point circle, and the two 5-point minimal spaces described in Example 3.4.14.

**Proposition 3.5.1.** *Up to homeomorphism, there are seven connected minimal 6-point spaces $X$, and none of them are weakly contractible. One is the six point two sphere $\mathbb{S}^2 S^0$, two are $\mathbb{S}D_4$ and its opposite. The remaining four have three maximal and three minimal points.*

Proof. We must have at least two minimal and at least two maximal points. Indeed, if we have just one intermediate point $y$, any point greater or less than it is upbeat or downbeat. If we have two intermediate points, they cannot be comparable without again contradicting minimality, and if they are incomparable we arrive by minimality at $\mathbb{S}^2 S^0$, which is homeomorphic to its opposite. The only remaining cases have all points either minimal or maximal. By the minimality of $X$, each minimal point must be less than at least two maximal points and each maximal point must be greater than at least two minimal points. There is only one example with two minimal points, and its opposite is the only example with four minimal points. We are left with the case when there are three minimal and three maximal points. Here each minimal point must be less than at least two maximal points and zero, one, two, or all three of them can be less than all three maximal points. In all four cases, the resulting space is homeomorphic to its opposite. □

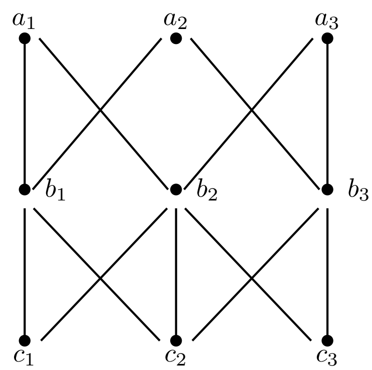
Recall that, by definition, minimal finite spaces can contain neither upbeat nor downbeat points. As said before, any non-maximal point in the Hasse diagrams of such models must have at least two points below, and similarly any non-minimal point must contain two points above. When the senior author first taught finite spaces in the REU, in 2003, he asked if $2n+2$ was the least number of points in a finite space of the weak homotopy type of $S^n$. Barmak and Minian [8] proved that using homology, but we shall give a direct elementary proof. Recall the definition of the height $h(X)$ of a poset $X$ (definition Definition 2.5.5).

**Proposition 3.5.2.** *Let $X \neq *$ be a minimal finite space. Then $X$ has at least $2h(X)$ points. It has exactly $2h(X)$ points if and only if it is homeomorphic to $\mathbb{S}^{h(X)-1} S^0$ and therefore weakly homotopy equivalent to $S^{h(X)-1}$.*

Proof. Let $x_1 < \cdots < x_h$ be a maximal chain in $X$. Since $X$ cannot have a minimimum point, there is a $y_1$ which is not greater than $x_1$. Since no $x_i$ is an upbeat point, $1 \leq i < h$, there must be some $y_{i+1} > x_i$ such that $y_{i+1}$ is not greater than $x_{i+1}$. The points $y_i$ are easily checked to be distinct from each other and from the $x_j$. Now suppose that $X$ has exactly these $2h$ points. By the maximality of our chain, the $x_i$ and $y_j$ are incomparable. For $i < j$, we started with $x_i < x_j$, and we check by cases from the absence of upbeat and downbeat points that $y_i < x_j$, $y_i < y_j$, and $x_i < y_j$. Comparing with the iterated suspension, we see that this implies that $X$ is homeomorphic to $\mathbb{S}^{h-1} S^0$. □

Drawing posets, and thinking about them, leads to lots of eliminations from the list of $F$-spaces that might not be contractible or weakly contractible (weakly homotopy equivalent to a point).

It presents a rather challenging exercise to determine the least number of points $n$ giving an $n$-point weakly contractible space that is not contractible. The reference ([**17**]) gives the answer as $n = 9$ with the following example.



CHAPTER 4

# Simplicial complexes

### 4.1. Abstract and ordered simplicial complexes

Simplicial complexes provide a general class of spaces that is sufficient for most purposes of basic algebraic topology. There are more general classes of spaces, in particular the CW complexes, that are more central to the modern development of the subject, but they give exactly the same collection of homotopy types, as we shall recall. We shall give a quick introduction to simplicial complexes here, largely restricting ourselves to what we shall use later. More detail can be found in many textbooks in algebraic topology (although not in my own book [**48**]). However, it is hard to find as precise a demarkation between simplicial complexes and ordered simplicial complexes as is needed for conceptual understanding, and this will become increasingly important as we go on. We implicitly focus on finite simplicial complexes, waiting for simplicial sets for full rigor in the infinite case.

**Definition 4.1.1.** An *abstract simplicial complex* $K$ is a set $V = V(K)$, whose elements are called *vertices*, together with a set $K$ of (non-empty) finite subsets of $V$, whose elements are called *simplices*, such that every vertex is an element of some simplex and every subset of a simplex is a simplex; such a subset is called a *face* of the given simplex. We say that $K$ is finite if $V$ is a finite set. The *dimension* of a simplex is one less than the number of vertices in it.

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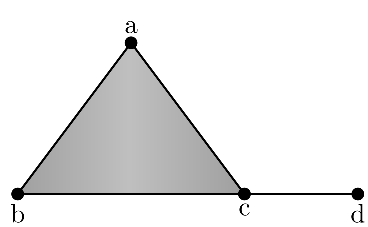
**Example 4.1.2.** Abstract complexes can be understood in a diagrammatic way. Consider for example, the abstract simplicial complex whose vertex set and simplices are given by

$$V(K) = \{a, b, c, d\}, K = \{a, b, c; ab, bc, ac, cd; abc\}.$$

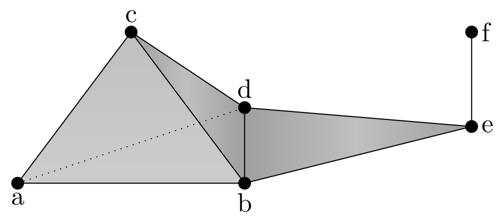
The vertices contain one simplex, and thus have dimension zero. They can be drawn simply as points. The simplices composed of two vertices can be drawn as lines as they are one-dimensional objects. Finally, faces (simplices containing three vertices) are shaded to indicate solidity. This produces the diagram pictured:



**Example 4.1.3.** Notice that $K$ must contain all points in the vertex set, as well as all vertexes included in higher dimensional simplices. Thus, we note the following examples of vertex sets that are NOT abstract simplical complexes.

(1) $V = \{a, b, c\}$, $K = \{a, b; ab\}$
(2) $V = \{a, b, c\}$, $K = \{a, b, c; ab, bc; abc\}$

**Exercise 4.1.4.** Write down the set of simplices of the simplicial complex $X$ given below:




**Definition 4.1.5.** A map $g\colon K \longrightarrow L$ of abstract simplicial complexes is a function $g\colon V(K) \longrightarrow V(L)$ that takes simplices to simplices. We say that $K$ is a *subcomplex* of $L$ if the vertices and simplices of $K$ are some of the vertices and simplices of $L$. We say that $K$ is a *full subcomplex* of $L$ if, further, every simplex of $L$ whose vertices are in $K$ is a simplex of $K$.

As already said, there is a very important distinction to be made between simplicial complexes as we have just defined them and *ordered* simplicial complexes.

**Definition 4.1.6.** An *ordering* of an abstract simplicial complex $K$ is a partial order on the vertices of $K$ that restricts to a total order on the vertices of each simplex of $K$. A map of ordered simplicial complexes is a map of simplicial complexes that is given by an order preserving map on its poset of vertices.

While imposition of an ordering may seem artificial, since we have no canonical choice, it is essential to a serious calculational theory. We shall later introduce simplicial sets, which generalize simplicial complexes and elegantly systematize orderings. Many of the definitions below have evident ordered variants. We shall not belabor the point. However, orderings will be essential to understanding the relationship between simplicial complexes and finite spaces. Of course, this is not surprising since finite spaces are essentially the same as finite posets.

Unless otherwise stated, simplicial complexes without an adjective (such as ordered or geometric) means abstract simplicial complexes henceforward.

## 4.2. Geometric simplicial complexes

Following geometric intuition, we must first define geometric simplices.

**Definition 4.2.1.** Let $\{v_0, \ldots, v_n\}$ be a set of points in some $\mathbb{R}^N$ such that the vectors

$$\{(v_1 - v_0), (v_2 - v_0) \ldots (v_n - v_0)\}$$

are linearly independent. The (geometric) *$n$-simplex* $\sigma$ spanned by $\{v_0, \ldots, v_n\}$ is the set of all points $\sum_{t=0}^{n} t_i v_i$, where $0 \le t_i \le 1$ and $\sum t_i = 1$. The $t_i$ are called the *barycentric coordinates* of the point $x$. When each $t_i = 1/(n+1)$, the point $x$ is called the *barycenter* of $\sigma$. The points $v_i$ are the *vertices* of $\sigma$. A simplex spanned by a subset of the vertices is a *face* of $\sigma$; it is a *proper face* if the subset is proper.

**Definition 4.2.2.** The *standard* $n$-simplex $\Delta[n]$ is the $n$-simplex spanned by the standard basis of $\mathbb{R}^{n+1}$. Thus the standard 0-simplex is the point $1 \in \mathbb{R}$, the standard 1-simplex is the line$\{t, 1-t\} \subset \mathbb{R}^2$, and so forth. Later, when necessary for clarity, we will sometimes denote these topological $n$-simplices by $\Delta[n]^t$ to distinguish them from other kinds of $n$-simplices that will appear.

As we noticed before, $n$-simplices are easy to visualize for small $n$.

**Exercise 4.2.3.** The vertices of a 3-dimensional simplex $\sigma$ can be labelled $\{a, b, c, d\}$. How many 0,1 and 2-dimensional simplices does $\sigma$ contain?

**Definition 4.2.4.** A *geometric simplicial complex* $K$ is a set of simplices in some $\mathbb{R}^N$ such that every face of a simplex in $K$ is a simplex in $K$ and the intersection of two simplices in $K$ is a simplex in $K$. The set of vertices of $K$ is the union of the sets of vertices of its simplexes. Note that although we require all vertices to lie in some $\mathbb{R}^N$ and we require each set of vertices that spans a simplex of $K$ to be geometrically independent, we do not require the entire set of vertices to be geometrically independent. For example, we can have three vertices on a single line in $\mathbb{R}^N$, as long as the two vertices furthest apart do not span a 1-simplex of $K$. A subcomplex $L$ of a simplicial complex $K$ is a simplicial complex whose simplices are some of the simplices of $K$. It is a full subcomplex if every simplex of $K$ with vertices in $L$ is in $L$.

The simplices of a geometric simplicial complex are the building blocks of a subspace of $\mathbb{R}^N$.

**Definition 4.2.5.** The *geometric realization* $|K|$ of a geometric simplicial complex $K$ is the union of the simplices of $K$, each regarded as a subspace of $\mathbb{R}^N$, with the topology whose closed sets are the sets that intersect each simplex in a closed subset. If $K$ is finite, but not in general otherwise, this is the same as the topology of $|K|$ as a subspace of $\mathbb{R}^N$. The open simplices of $|K|$ are the interiors of its simplices (where a vertex is an interior point of its 0-simplex), and every point of $|K|$ is an interior point of a unique simplex. The *boundary* $\partial\sigma$ of a simplex $\sigma$ is the subcomplex given by the union of its proper faces. The *closure* of a simplex is the union of its interior and its boundary. A space homeomorphic to $|K|$ for some $K$ is called a *polytope*.

The *dimension* of a simplicial complex is the maximal dimension of its simplices, and that of course corresponds to our geometric intuition.

**Definition 4.2.6.** A map $g \colon K \longrightarrow L$ of simplicial complexes is a function from the vertex set $V(K)$ to the vertex set $V(L)$ such that, for each subset $S$ of $V(K)$ that spans a simplex of $K$, the set $g(S)$ is the set of vertices of a simplex of $L$. The same definition applies to geometric simplicial complexes. Then $g$ determines the continuous map $|g| \colon |K| \longrightarrow |L|$ that sends $\sum t_i v_i$ to $\sum t_i g(v_i)$. Note that although we do not require $g$ to be one–to–one on vertices, $|g|$ is nevertheless well-defined and continuous. If $g$ is a bijection on vertices and simplices, we say that it is an isomorphism, and then $|g|$ is a homeomorphism.

It is usual to abbreviate $|g|$ to $g$ and to refer to it as a simplicial map, but for now we prefer to keep the distinction between $g$ and $|g|$ clear.

**Remark 4.2.7.** The reader can and should object to our insistence that all of the vertices of $K$ are in some $\mathbb{R}^N$. Why not allow an infinite set of vertices with no bound on the allowed size of the simplices? The idea is to take the topological space given by the disjoint union of the simplices of a geometric simplicial complex, ignoring their embeddings in Euclidean space, and to then form a quotient space by glueing them together along their common faces. We might instead think of sets of standard $n$-simplices $\Delta[n]$, and we might think of taking their disjoint union and then gluing together along prescribed faces to construct the geometric realization more abstractly. We shall allow ourselves to think of such infinite dimensional simplicial complexes, but it is best not to take them too seriously for now. We shall come back to them under the guise of simplicial sets, which are best treated later. In that context, we will make the intuition precise and show how best to define geometric realization in general.

## 4.3. Comparison of abstract and geometric simplicial complexes

**Definition 4.3.1.** The abstract simplicial complex $aK$ determined by a geometric simplicial complex $K$ has vertex set the union of the vertex sets of the simplices of $K$. Its simplices are the subsets that span a simplex of $K$. An abstract finite simplicial complex $K$ determines a geometric finite simplicial complex $gK$ by choosing any bijection between the vertices of $K$ and a geometrically independent subset of some $\mathbb{R}^N$. For specificity, we can take the standard basis elements of $\mathbb{R}^N$ where $N$ is the number of points in the vertex set $V(K)$. The geometric simplices are spanned by the images of the simplices of $K$ under this bijection. For an abstract simplicial complex $K$, $agK$ is isomorphic to $K$, the isomorphism being given by the chosen bijection. Similarly, for a finite geometric simplicial complex $K$, $gaK$ is isomorphic to $K$.

We could remove the word finite from the previous definition by defining geometric simplicial complexes more generally, without reference to a finite dimensional ambient space $\mathbb{R}^N$, as in Remark 4.2.7. We also note that we do not have to realize in such a high dimensional Euclidean space as a count of vertexes would dictate. The following result holds no matter how many vertices there are. It is rarely used, but it is conceptually attractive. A proof can be found in [**34**, 1.9.6].

**Theorem 4.3.2.** *Any finite simplicial complex $K$ of dimension $n$ can be geometrically realized in $\mathbb{R}^{2n+1}$.*

In view of the discussion above, abstract and geometric finite simplicial complexes can be used interchangeably. In particular, the geometric realization of an abstract simplicial complex $K$ is understood to mean the geometric realization of any $gK$.

We need a criterion for when the geometric realizations of two simplicial maps are homotopic.

**Definition 4.3.3.** Continuous maps $f$ and $g$ from a topological space $X$ to the geometric realization $|K|$ of a simplicial complex are *simplicially close* if, for each $x \in X$, both $f(x)$ and $g(x)$ are in the closure of some simplex $\sigma(x)$ of $K$.

**Proposition 4.3.4.** *If $f$ and $g$ are simplicially close continuous maps from a topological space $X$ to some $|K| \subset \mathbb{R}^N$, then $f$ and $g$ are homotopic.*

PROOF. Define $h\colon X \times I \longrightarrow \mathbb{R}^N$ by

$$h(x,t) = (1-t)f(x) + tg(x).$$

Since $h(x,t)$ is in the closure of $\sigma(x)$ and therefore in $|K|$, we see that it is continuous and specifies a homotopy as required. □

## 4.4. Cones and subdivisions of simplicial complexes

We've so far seen cones on topological spaces, as well as cones on $F-$spaces. The notion of a cone also exists for abstract and finite geometric simplicial complexes.

**Definition 4.4.1.** The *cone* $K * x$ on an abstract simplicial complex $K$ is constructed by adding a new vertex $x$ and taking the simplices to be all subsets of all unions of $x$ with a simplex in $K$.

If $K$ is instead a finite geometric simplicial complex in $\mathbb{R}^n$, consider $x$ as a point of $\mathbb{R}^N - K$ such that each ray starting at $x$ intersects $|K|$ in at most one point. Observe that the union of $\{x\}$ and the set of vertices of a simplex of $K$ is a geometrically independent set. Define the cone $K * x$ on $K$ with vertex $x$ to be the geometric simplicial complex whose simplices are all of the faces of the simplices spanned by such unions.

**Remark 4.4.2.** Notice that $K$ is a subcomplex of $K * x$, $x$ is the only vertex not in $K$, and $|K * x|$ is homeomorphic to $C|K|$.

**Example 4.4.3.** A simplex is the cone of any one of its vertices with the subcomplex spanned by the remaining vertices (the opposite face).

Subdivisions of simplicial complexes will play a central role in our work.

**Definition 4.4.4.** A simplicial complex $L$ is a *subdivision* of a simplicial complex $K$ if $V(K) \subset V(L)$, each simplex of $K$ is contained in a simplex of $L$, and each simplex of $K$ is the union of finitely many simplices of $L$. If $L$ is ordered, then the partial order on $V(L)$ restricts to a partial order on $V(K)$ that gives $K$ an ordering.

**Definition 4.4.5.** The *canonical subdivision* $K'$ of an abstract simplicial complex $K$ is the ordered simplicial complex whose vertices are the simplices of $K$, partially ordered by inclusion, and whose simplices are the totally ordered finite subsets $\{\sigma_0, \ldots, \sigma_n\}$ of simplices of $K$. With $\sigma_0 < \cdots < \sigma_n$ we call $\sigma_n$ the barycenter of the simplex $\{\sigma_0, \ldots, \sigma_n\}$.

**Definition 4.4.6.** A subdivision $L$ of a finite geometric simplicial complex $K$ is a geometric simplicial complex such that each simplex of $L$ is contained in a simplex of $K$ and each simplex of $K$ is the union of finitely many simplices of $L$.

The following observation should be clear.

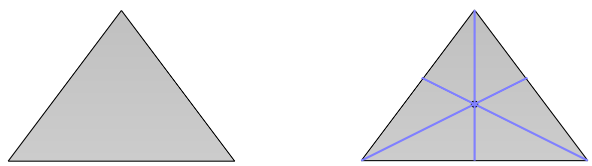
**Lemma 4.4.7.** *If $L$ is a subdivision of $K$, then $|L| = |K|$ (as spaces).*

The $n$-skeleton $K^n$ of $K$ is the union of the simplices of $K$ of dimension at most $n$. It is a subcomplex. There are many ways to subdivide both abstract and geometric simplicial complexes, and in applications there can be advantages to one or another of them. However, we will focus on the standard canonical choices. We have defined that already for abstract simplicial complexes. We give a somewhat pedantic inductive geometric construction for geometric simplicial complexes that should make the idea clear and then reexpress the answer combinatorially, proving that the canonical choices agree under the passages $a$ and $g$ back and forth.

**Construction 4.4.8.** We construct the *barycentric subdivision* $K'$ of a geometric simplicial complex $K$. We subdivide the skeleta of $K$ inductively. Let $L_0 = K^0$. Suppose that a subdivision $L_{n-1}$ of $K^{n-1}$ has been constructed. Let $b_\sigma$ be the barycenter of an $n$-simplex $\sigma$ of $K$. The space $|\partial\sigma|$ coincides with $|L_\sigma|$ for a subcomplex $L_\sigma$ of $L_{n-1}$, and we can define the cone $L_\sigma * b_\sigma$. Clearly $|L_\sigma * b_\sigma| = |\sigma|$ and $|L_\sigma * b_\sigma| \cap |L_{n-1}| = |L_\sigma| = |\partial\sigma|$.

If $\tau$ is another $n$-simplex, then $|L_\sigma * b_\sigma| \cap |L_\tau * b_\tau| = |\sigma \cap \tau|$, which is the realization of a subcomplex of $L_{n-1}$ and therefore of both $L_\sigma$ and $L_\tau$. Define $L_n$ to be the union of $L_{n-1}$ and the complexes $L_\sigma * b_\sigma$, where $\sigma$ runs over all $n$-simplices of $K$. Our observations about intersections show that $L_n$ is a simplicial complex which contains $L_{n-1}$ as a subcomplex. The union of the $L_n$ is denoted $K'$ and called the barycentric subdivision of $K$.

**Example 4.4.9.** The barycentric subdivision of a 2-simplex is easily visualized pictorially.



The second barycentric subdivision of $K$ is the barycentric subdivision of the first barycentric subdivision, and so on inductively.

We can enumerate the simplices of $K'$ explicitly rather than inductively.

**Proposition 4.4.10.** *Define $\sigma < \tau$ if $\sigma$ is a proper face of $\tau$. Then $K'$ is the geometric simplicial complex whose vertices are the barycenters of simplices of $K$ and whose $n$-simplices $\sigma'$ are the spans of the geometrically independent sets $\{b_{\sigma_0}, \cdots, b_{\sigma_n}\}$, where $\sigma_0 > \cdots > \sigma_n$. The vertex $b_{\sigma_0}$ is called the* leading vertex *of the simplex $\sigma'$.*

Proof. We show this inductively for the subcomplexes $L_n$. Since $L_0 = K^0$, this is clear for $L_0$. Assume that it holds for $L_{n-1}$. If $\tau$ is a simplex of $L_n$ such that $|\tau|$ is contained in $|K^n|$ but not contained in $K^{n-1}$, then $\tau$ is a simplex in the cone $L_\sigma * b_\sigma$ for some $n$-simplex $\sigma$. By the induction hypothesis and the definition of $L_\sigma$, each simplex of $L_\sigma$ is the span of a set $\{b_{\sigma_0}, \cdots, b_{\sigma_m}\}$, where $\sigma > \sigma_0 > \cdots > \sigma_m$. Therefore $\tau$ is the span of a set $\{b_\sigma, b_{\sigma_0}, \cdots, b_{\sigma_m}\}$. □

**Proposition 4.4.11.** *There is a simplicial map $\xi = \xi_K : K' \longrightarrow K$ whose realization is simplicially close to the identity map and therefore homotopic to the identity map.*

Proof. Let $\xi$ map each vertex $b_\sigma$ of $K'$ to any chosen vertex of $\sigma$. If $\sigma'$ is a simplex of $K'$ with leading vertex $b_{\sigma_0}$, then all other vertices of $\sigma'$ are barycenters of faces of $\sigma_0$, hence are mapped under $\xi$ to vertices of $\sigma_0$. Therefore the images under $\xi$ of the vertices of $\sigma'$ span a face of $\sigma_0$, so that $\xi$ is a simplicial map. With these notations, if $x \in |K'|$ is an interior point of the simplex $\sigma'$, then it is mapped under $|\xi|$ to a point of $\sigma_0 \supset \sigma'$, and we let $\sigma(x) = \sigma_0$. Since $\xi$ maps every vertex of $\sigma'$ to a vertex of $\sigma_0$, $x$ and $\xi(x)$ are both in the closure of $\sigma_0$. □

**Definition 4.4.12.** Just as for abstract simplicial complexes, we say that a geometric simplicial complex is ordered if its vertices are partially ordered and the partial order restricts to a total order of the vertices of each simplex. For an ordered geometric simplicial complex $K$, define the *standard simplicial map* $\xi\colon K' \longrightarrow K$ by letting $\xi(b_\sigma)$ be the maximal vertex $x_n$ of the simplex $\sigma = \{x_0, \cdots, x_n\}$.

**Remark 4.4.13.** Observe that $K'$ has a canonical ordering even when $K$ does not. Explicitly, the partial ordering of the set of vertices $\{b_\sigma\}$ of $K'$ is given by $b_\sigma \le b_\tau$ if $\sigma$ is a face of $\tau$. Notice for this that a vertex of $K$, regarded as a simplex, is its own barycenter. This partial order clearly restricts to a total order on the vertices of each simplex.

**Proposition 4.4.14.** *If $K$ is a geometric simplicial complex with canonical subdivision $K'$, then $aK'$ is isomorphic to the canonical subdivision of $aK$.*

Proof. Letting the vertex $b_\sigma$ in $K'$ correspond to the vertex $\sigma$ in $aK$, this is an immediate comparison of definitions. □

**Remark 4.4.15.** The barycenters of the simplices of $K$ that are not vertices correspond to the vertices of $aK'$ that are not vertices of $aK$. All simplices of $aK'$ with more than one vertex have at least one vertex that is not in $aK$. Thus the only simplices in $aK'$ that are also simplices in $aK$ are the vertices of $aK$. However, if we think geometrically, then every simplex $\tau$ of $K'$ is contained in a unique simplex $\sigma$ of $K$, as must be so since $K'$ is a subdivision and is also clear from a picture of the barycentric subdivision. The simplex $\sigma$ is called the *carrier* of $\tau$.

**Proposition 4.4.16.** *A simplicial map $g\colon K \longrightarrow L$ induces a subdivided simplicial map $g'\colon K' \longrightarrow L'$ whose realization is simplicially close to $|g|$ and hence homotopic to $|g|$. Moreover, $g'$ is order-preserving.*

Proof. The images under $g$ of the vertices of a simplex $\sigma$ of $K$ span a simplex $g(\sigma)$, of possibly lower dimension than $\sigma$, and we define $g'(b_\sigma) = b_{g(\sigma)}$ on vertices. If $b_{\sigma_0}$ is the leading vertex of a simplex $\sigma'$ of $K'$, then all other vertices of $\sigma'$ are barycenters of faces of $\sigma_0$. Their images under $g'$ are barycenters of faces of $g(\sigma_0)$. If $x$ is an interior point of $\sigma'$, then both $g(x)$ and $g'(x)$ are in the closure of $g(\sigma_0)$. □

**Remark 4.4.17.** When $K$ and $L$ are ordered and $g$ is an order-preserving simplicial map, the following "naturality" diagram commutes if we use the standard simplicial maps $\xi$ for $K$ and $L$.

$$\begin{array}{ccc} K' & \xrightarrow{g'} & L' \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \xi} \\ K & \xrightarrow[g]{} & L \end{array}$$

## 4.5. The simplicial approximation theorem

The classical point of barycentric subdivision is its use in the simplicial approximation theorem, which in its simplest form reads as follows. Starting with $K^{(0)} = K$, let $K^{(n)} = K^{(n-1)'}$ be the $n$th barycentric subdivision of a simplicial complex $K$. By iteration of $\xi\colon K' \longrightarrow K$, we obtain a simplicial map $\xi^{(n)}\colon K^{(n)} \longrightarrow K$ whose geometric realization is homotopic to the identity map.

**Theorem 4.5.1.** *Let $K$ be a finite simplicial complex and $L$ be any simplicial complex. Let $f\colon |K| \longrightarrow |L|$ be any continuous map. Then, for some sufficiently large $n$, there is a simplicial map $g\colon K^{(n)} \longrightarrow L$ such that $f$ is homotopic to $|g|$.*

This means that, for the purposes of homotopy theory, general continuous maps may be replaced by simplicial maps. Since this is proved in so many places, we shall content ourselves with a slightly sketchy proof. It relies on the classical Lebesgue lemma, whose proof is not hard but just a little far afield.

**Lemma 4.5.2** (Lebesgue lemma). *Let $(X, d)$ be a compact metric space with a given open cover $\mathscr{U}$. Then there exists a number $\lambda > 0$ such that every subset of $X$ with diameter less than $\lambda$ is contained in some set $U \in \mathscr{U}$. The smallest such $\lambda$ is called the Lebesgue number of the cover.*

**Definition 4.5.3.** For a vertex $v$ of a simplicial complex $K$, define $star(v)$ to be the union of the interiors of all simplices of $|K|$ that contain $v$ as a vertex. For a subcomplex $L$ of $K$, define $star(L) \subset |K|$ to be the union over $v \in L$ of the open spaces $star(v)$.

Proof of the simplicial approximation theorem. We are given a map $f\colon |K| \longrightarrow |L|$. Give $|K|$ the open cover by the sets $f^{-1}(star(w))$, where $w$ runs over the vertices of $L$. Since $|K|$ is a compact subspace of a metric space, the Lebesgue lemma ensures that there is a number $\lambda$ such that any subset of $|K|$ of diameter less than $\lambda$ is contained in one of the open sets $f^{-1}(star(w))$. The diameter of a (closed) simplex is easily seen to be the maximal length of a one-dimensional face. Each barycentric subdivision therefore has the effect of decreasing the maximal diameter of a simplex. Precisely, the maximal diameter of the subdivision of a $q$-simplex turns out to be $q/q+1$ times the maximal diameter of the given simplex (e.g. [**63**, p.124], [**34**, p.24], [**31**, p. 120]), but the precise estimate is not important.

What is important is that, since $K$ is finite, for any $\delta > 0$ there is a large enough $n$ such that every simplex of $K^{(n)}$ has diameter less than $\delta/2$. Then each $star(v)$ for a vertex $v$ of $K^{(n)}$ has diameter less than $\delta$, and we conclude that $f(star(v)) \subset star(w)$ for some vertex $w$ of $L$. Define $g\colon V(K^{(n)}) \longrightarrow V(L)$ by letting $g(v) = w$ for some $w$ such that $f(star(v)) \subset star(w)$. One checks that $g$ maps simplices to simplices and so specifies a map of simplicial complexes. If $u$ is an interior point of a simplex $\sigma$ of $K$, then $f(x)$ is an interior point of some simplex $\tau$ of $L$. One can check that $g$ maps each vertex of $\sigma$ to a vertex of $\tau$. This implies that $|g|$ is simplicially close to $f$ and therefore homotopic to $f$. □

## 4.6. Contiguity classes and homotopy classes

We are interested not just in representing maps up to homotopy as simplicial maps, but in enumerating the resulting homotopy classes of maps. For two spaces $X$ and $Y$, we define the set $[X, Y]$ of homotopy classes of maps $X \longrightarrow Y$ to be the set of equivalence classes of maps $f\colon X \longrightarrow Y$, where two maps are equivalent if they are homotopic. We write $[f]$ for the homotopy class of $f$. This notion has a number of variants. For example, we can consider based spaces, base-point preserving maps, and homotopies that preserve the basepoints. We write $[X, Y]_*$ for the resulting set of based homotopy classes of based maps. Thus, with this notation, $\pi_n(X) = [S^n, X]_*$.

We want to understand the relationship between simplicial maps $K \longrightarrow L$ and the set $[|K|, |L|]$, where $K$ is finite. Thus we fix $K$ and $L$ in the rest of this section, taking $K$ to be finite.

We know that any homotopy class is represented by a simplicial map $f\colon K \longrightarrow L$, provided that we first subdivide $K$ sufficiently, and we ask for a simplicial description of when two simplicial maps $f, g\colon K \longrightarrow L$ have homotopic geometric realizations. The notion of "contiguity" can be used to give an answer. If $q > n$, we agree to write $\xi\colon K^{(q)} \longrightarrow K^{(n)}$ for the map obtained by iteration of maps $\xi$.

**Definition 4.6.1.** Let $f, g\colon K \longrightarrow L$ be simplicial maps between (geometric) simplicial complexes. We say that $f$ is *contiguous* to $g$ if for every simplex $\sigma$ of $K$, the union $f(\sigma) \cup g(\sigma)$ is contained in a simplex of $L$. More generally, let $f\colon K \longrightarrow L$ and $g\colon K^{(n)} \longrightarrow L$ be simplicial maps. We say that $f$ is contiguous to $g$ if for each simplex $\tau$ of $K^{(n)}$ with carrier $\sigma$ in $K$, $f(\sigma) \cup g(\tau)$ is contained in a simplex of $L$.

If $q > n$, a check of definitions shows that if $f$ and $g$ are contiguous, then so are $f$ and $g \circ \xi$. Similarly, if $q > 0$ and $f$ and $g$ are contiguous, then so are $f \circ \xi$ and $g$, where now $\xi\colon K^{(q)} \longrightarrow K$. The relation of contiguity is reflexive and symmetric, but it is not transitive. We let $\sim$ denote the equivalence relation generated by contiguity. Thus $f \sim g$ if there is a sequence of simplicial maps $\{f = f_1, f_2, \cdots, f_q = g\}$ such that $f_i$ is contiguous to $f_{i+1}$ for $i < q$.

**Proposition 4.6.2.** *If $f, g\colon K \longrightarrow L$ are contiguous simplicial maps, then $|f| \simeq |g|\colon |K| \longrightarrow |L|$.*

Proof. In fact, $|f|$ and $|g|$ are simplicially close by a comparison of definitions. Therefore this result is a special case of Proposition 4.3.4: the same simplex by simplex linear homotopy does the trick. □

Remember that two simplicially close maps $f, g\colon X \longrightarrow |L|$ have homotopic realizations, where $X$ is any space, not necessarily a simplicial complex. We used that fact to show that if $K$ is finite, then any map $f\colon |K| \longrightarrow |L|$ is homotopic to the realization of a simplicial map $g\colon K^{(n)} \longrightarrow L$ for some sufficiently large $n$. It is natural to ask how unique that simplicial approximation is, and the notion of contiguity gives a useful answer.

**Proposition 4.6.3.** *If $g$ and $g'$ are simplicial approximations of the same continuous map $f\colon |K| \longrightarrow |L|$, $K$ finite, then $g$ and $g'$ are contiguous.*

Proof. To see this, just look back at the proof of the simplicial approximation theorem. □

**Theorem 4.6.4.** *If $f$ and $f'$ are homotopic maps $|K| \longrightarrow |L|$, $K$ finite, and $g$ and $g'$ are simplicial approximations to $f$ and $f'$, then $g$ is contiguous to $g'$. Therefore, for every pair of homotopic maps $f, f'\colon |K| \longrightarrow |L|$, there is a sufficiently large $n$ such that $f$ and $f'$ are represented by contiguous simplicial maps $K^{(n)} \longrightarrow L$.*

Maybe revisit this proof to make clear L need not be finite; compare Thibault's passage to limits in his Thm 2.5.30

Sketch proof. Two slightly different detailed proofs may be found in [**34**, p. 40], [**63**, p. 132]. We follow [**34**]. Remember that $|L|$ is a subspace of some $\mathbb{R}^N$, so that we can talk about the distance between two points of $|L|$. We define the distance between two maps $f, g\colon |K| \longrightarrow |L|$ to be the maximum of the distances between $f(x)$ and $g(x)$ for $x \in |K|$. Let $\lambda$ be the Lebesgue number of the covering of $|L|$ by the open stars of its vertices and let $\varepsilon = (1/3)\lambda$. Then $\varepsilon$ is small enough that if the distance between $f$ and $g$ is less than $\varepsilon$, then there is a simplicial map $g$ that is a simplicial approximation of both $f$ and $f'$. The precise estimate $\varepsilon$ is unimportant. It is clear from the proof of the simplicial approximation theorem that some small enough $\epsilon$ will have the stated property.

Returning to the hypotheses of the theorem, let $h\colon |K| \times I \longrightarrow |L|$ be a homotopy from $f = h_0$ to $f' = h_1$, where $h_t(x) = h(x,t)$. The claim is that there is a simplicial approximation $g$ to $f$, a simplicial approximation $g'$ to $f'$, and a sequence of simplicial maps $\{g = g_1, g_2, \cdots, g_q = g'\}$ such that $g_i$ is contiguous to $g_{i+1}$ for $i < q$. We use an $\varepsilon$, $\delta$ proof. There is a $\delta > 0$ such that $|h_s(x), h_t(x)| < \varepsilon$ for all $x \in |K|$ and all $s, t \in I$ such that $|t - s| < \delta$. Choose $q > 1/\delta$. Then, for $i < q$, the distance between $h_{(i-1)/q}$ and $h_{i/q}$ is less than $\varepsilon$. Therefore these two maps have a common simplicial approxmation $g_i$. Since $g_i$ and $g_{i+1}$ are both simplicial approximations of $h_{i/q}$, they are contiguous and we have chosen our maps so that $g = g_1$ is a simplicial approximation of $f$ and $g' = g_q$ is a simplicial approximation to $f'$. By the previous result, they are contiguous to any other such simplicial approximations. □

**Remark 4.6.5.** In the next chapter we will define simplicial complexes $|\mathscr{K}(X)|$ associated to finite spaces. The simplicial complex assigned to $\mathbb{S}^2 S^0$ is homeomorphic to $S^2$. The simplicial complex assigned to the remaining connected minimal 6-point spaces are graphs that are homotopy equivalent to the wedge (or 1-point union) of one, two, three, or four circles.

CHAPTER 5

# The relation between A-spaces and simplicial complexes

Following McCord [**50**], we are going to relate $A$-spaces, and in particular $F$-spaces, with simplicial complexes, explaining how to go back and forth between them. Since any Alexandroff space is homotopy equivalent to a $T_0$-space, there is no loss of generality if we restrict attention to $A$-spaces. As usual, the reader may prefer to think only in terms of $F$-spaces.

## 5.1. The construction of simplicial complexes from A-spaces

**Definition 5.1.1.** Let $X$ be an $A$-space. Define $\mathscr{K}(X)$ to be the abstract simplicial complex whose vertex set is $X$ and whose simplices are the finite totally ordered subsets of the poset $X$; $\mathscr{K}(X)$ is often called the *order complex* of $A$. Observe that the partial order of $X$ gives an ordering of $\mathscr{K}(X)$, since it restricts to a total order on each simplex. Observe too that if $V$ is a subspace of $X$, then $\mathscr{K}(V)$ is a full subcomplex of $\mathscr{K}(X)$ since any totally ordered subset of $X$ whose points are in $V$ is a totally ordered subset of $V$. Since a map $f\colon X \longrightarrow Y$ is an order–preserving function, it may be regarded as a simplicial map $\mathscr{K}(f)\colon \mathscr{K}(X) \longrightarrow \mathscr{K}(Y)$.

**Theorem 5.1.2.** *For an $A$-space $X$, there is a weak homotopy equivalence*

$$\psi = \psi_X \colon |\mathscr{K}(X)| \longrightarrow X$$

*such that the following diagram commutes for each map $f\colon X \longrightarrow Y$.*

$$\begin{array}{ccc} |\mathscr{K}(X)| & \xrightarrow{|\mathscr{K}(f)|} & |\mathscr{K}(Y)| \\ {\scriptstyle \psi_X}\downarrow & & \downarrow{\scriptstyle \psi_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Proof. Each point $u \in |\mathscr{K}(X)|$ is an interior point of a simplex $\sigma$ spanned by some strictly increasing sequence $x_0 < x_1 < \cdots < x_n$ of points of $X$. We define $\psi(u) = x_0$. For $f\colon X \longrightarrow Y$, $\mathscr{K}(f)(u)$ is in the simplex spanned by the $f(x_i)$ and $f(x_0) \leq f(x_1) \leq \cdots \leq f(x_n)$. Omitting repetitions, we see that $f(x_0)$ is the minimal vertex of this simplex, so that $\psi(f(u)) = f(x_0) = f(\psi(u))$, which proves that the diagram commutes. We must still prove that $\psi$ is continuous and that it is a weak homotopy equivalence.

For $x \in X$, let $star(x)$ denote the union of the interiors of the simplices of $\mathscr{K}(X)$ that have $x$ as a vertex; it is an open neighborhood of $x$ in $|\mathscr{K}(X)|$. For an open subset $V$ of $X$, define the open star, $star(V)$, to be the union over the vertices $v \in V$ of the open subspaces $star(v)$. It is the complement of $|\mathscr{K}(X - V)|$ in $|\mathscr{K}(X)|$. To see that $\psi$ is continuous, we show that $\psi^{-1}(V) = star(V)$. If

$\psi(u) = v \in V$, then $v$ is the initial vertex $x_0$ of a simplex $\sigma$. Since a vertex $v$ is the unique interior point of the simplex $\{v\}$, $u \in star(V)$. Conversely, suppose that $u \in star(v)$, where $v \in V$. Then $u$ is an interior point of a simplex $\sigma$ determined by an increasing sequence $x_0 < x_1 < \cdots < x_n$ such that some $x_i = v \in V$. Since $x_0 \leq v$, $x_0 \in U_v$. Since $V$ is open, $U_v \subset V$. Thus $\psi(u) = x_0$ is in $V$.

It remains to prove that $\psi$ is a weak homotopy equivalence. We shall do so by applying Theorem 3.3.1 to the minimal open cover $\{U_x\}$ of $X$. If $x$ is in $U_y \cap U_z$, then $x$ is in both $U_y$ and $U_z$, so that $U_x$ is contained in both $U_y$ and $U_z$. This verifies the first hypothesis of the cited theorem. For the second hypothesis, we know that each $U_x$ is a contractible subspace of $V$. We also know that each $|\mathscr{K}(U_x)|$ is a contractible space. In fact, $\mathscr{K}(U_x)$ is a simplicial cone, in the sense that for every simplex $\sigma$ of $\mathscr{K}(U_x)$ which does not contain $x$, $\sigma \cup \{x\}$ is a simplex of $\mathscr{K}(U_x)$. The realization of such a simplicial cone is contractible to the cone vertex $x$ since $h(y,t) = (1-t)y + tx$ gives a well-defined contracting homotopy. Specializing the following general result to $L = \mathscr{K}(U_x)$, we see that $star(U_x)$ is also contractible. Therefore each restriction $\psi\colon \psi^{-1}(U_x) \longrightarrow U_x$ is a weak homotopy equivalence and Theorem 3.3.1 applies to show that $\psi$ is a weak equivalence. □

**Proposition 5.1.3.** *Let $L$ be a full subcomplex of a simplicial complex $K$. Then $|L|$ is a deformation retract of its open star, $starL$, in $|K|$.*

Proof. Again, $starL$, is defined to be the union of the open stars of the vertices of $L$. This result is a standard fact in the theory of simplicial complexes, and a more detailed proof than we shall given can be found in [**62**, 70.1 and p. 427]. Consider a simplex $\sigma$ that is in the closure of $star(L)$. Then $\sigma$ has vertex set the disjoint union of a set of vertices in $L$ and a set of vertices in $K - L$. Each point $u$ of $\sigma$ that is neither in the span $s$ of the vertices in $L$ nor in the span $t$ of the vertices not in $L$ is on a unique line segment joining a point in $t$ to a point in $s$. Define the required retraction $r$ by sending $u$ to the end point in $s \subset L$ of this line segment, letting $r$ be the identity map on $L$ and thus on $s$. Deformation along such line segments gives the required homotopy showing that $i \circ r$ is homotopic to the identity, where $i$ is the inclusion of $|L|$ in its open star. □

**Example 5.1.4.** Suppose that $|\mathscr{K}(X)|$ is homotopy equivalent to a sphere $S^n$. Then the dimension of $|\mathscr{K}(X)|$, which is $h(X) - 1$, must be at least $n$. Thus $h(X) \geq n+1$. Therefore, by Proposition 3.5.2, $X$ has at least $2n+2$ points and, if $X$ has exactly $2n+2$ points, then it is homeomorphic to $\mathbb{S}^n S^0$.

## 5.2. The construction of A-spaces from simplicial complexes

Now let $K$ be a finite geometric simplicial complex with first barycentric subdivision $K'$. Remember that $|K| = |K'|$.

**Definition 5.2.1.** Define an $A$-space $\mathscr{X}(K)$ as follows. The points of $\mathscr{X}(K)$ are the barycenters $b_\sigma$ of the simplices of $K$, that is, the vertices of $K'$. The required partial order $\leq$ is defined by $b_\sigma \leq b_\tau$ if $\sigma \subset \tau$. The open subspace $U_{b_\sigma}$ coincides with $\mathscr{X}(\sigma)$, where $\sigma$ (together with its faces) is regarded as a subcomplex of $K$. For a simplicial map $g\colon K \longrightarrow L$, define $\mathscr{X}(g)\colon \mathscr{X}(K) \longrightarrow \mathscr{X}(L)$ by $\mathscr{X}(g)(b_\sigma) = b_{g(\sigma)}$, and note that this function is order–preserving and therefore continuous. Using the barycenters themselves to realize the vertices geometrically, we see from the description of $K'$ in Proposition 4.4.10 that $\mathscr{K}\mathscr{X}(K) = K'$ and $\mathscr{K}\mathscr{X}(g) = g'$.

We use Theorem 5.1.2 to obtain the following complementary result.

**Theorem 5.2.2.** *For a simplicial complex $K$, there is a weak homotopy equivalence*

$$\phi = \phi_K \colon |K| \longrightarrow \mathscr{X}(K)$$

*such that the following diagram is commutative*

$$\begin{array}{ccc} |K'| & \xrightarrow{|g'|} & |L'| \\ {\scriptstyle \phi_K}\downarrow & & \downarrow{\scriptstyle \phi_L} \\ \mathscr{X}(K) & \xrightarrow[\mathscr{X}(g)]{} & \mathscr{X}(L) \end{array}$$

Proof. Define

$$\phi_K = \psi_{\mathscr{X}(K)} \colon |K'| = |\mathscr{K}\mathscr{X}(K)| \longrightarrow \mathscr{X}(K).$$

Then $\phi_K$ is a weak homotopy equivalence and the diagram commutes by Theorem 5.1.2. Since $|K| = |K'|$ and $|L| = |L'|$, we can replace $|g'|$ by $|g|$ in the diagram. By Proposition 4.4.16, $|g'|$ is simplicially close to $|g|$ and hence homotopic to $|g|$. Therefore, after the replacement, the diagram would only be homotopy commutative, in the sense that the two composite maps in the diagram would be homotopic. □

## 5.3. Mapping spaces

For completeness, we record results of Stong [**65**, §6] that were obtained about the same time as the results of McCord recorded above and that give a quite different approach to the relationship between finite simplicial complexes and finite spaces. Since the proofs are fairly long and combinatorial in flavor, and since the statements do not have quite the same immediate impact as those in McCord's work, we shall not work through the details here.

Recheck: add? Expository paper topic?

Rather than constructing finite models for finite simplicial complexes, Stong studies all maps from the geometric realizations of simplicial complexes $K$ into finite spaces $X$ by studying the properties of the function space $X^K \equiv X^{|K|}$. More generally, he fixes a subcomplex $L$ of $K$ and a basepoint $* \in X$ and studies the subspace $(X,*)^{(K,L)}$ of maps $f\colon |K| \longrightarrow X$ such that $f(|L|) = *$. Homotopies relative to $|L|$ between such maps are homotopies $h$ such that $h(p,t) = *$ for $p \in |L|$.

**Theorem 5.3.1.** *Let $L$ be a subcomplex of a finite simplicial complex $K$, let $X$ be a finite space with basepoint $*$, and let $F = (X,*)^{(K,L)}$ denote the subspace of $X^K$ consisting of those maps $f\colon |K| \longrightarrow X$ such that $f(|L|) = *$.*

- *(i) For any $f \in F$, there is a map $g \in F$ such that the set $V = \{h | h \leq g\} \subset F$ is a neighborhood of $f$ in $F$; that is, there is an open subset $U$ such that $f \in U \subset V$.*
- *(ii) If $f \simeq f'$ relative to $L$, then there is a sequence of elements $\{g_1, \cdots, g_s\}$ in $F$ such that $g_1 = f$, $g_s = f'$, and either $g_i \leq g_{i+1}$ or $g_{i+1} \leq g_i$ for $1 \leq i < s$.*

The essential point of this analysis is the following consequence.

**Corollary 5.3.2.** *The path components and components of $F$ coincide. That is, the homotopy classes of maps $f\colon (K,L) \longrightarrow (X,*)$ are in bijective correspondence with the components of $F$.*

## 5.4. Simplicial approximation and A-spaces

There are two papers, [29, 30], that start with the simplicial approximation theorem and take up where McCord and Stong leave off. In view of the explicit constructions of $\mathscr{K}(X)$ and $\mathscr{X}(K)$, the following definition is reasonable.

earlier, in section defining $\mathscr{X}$, and clarify what it looks like without barycenter terminology

**Definition 5.4.1.** Define the *barycentric subdivision* of an $A$-space $X$ to be $X' = \mathscr{X}\mathscr{K}(X)$. For a map $f\colon X \longrightarrow Y$, define $f'\colon X' \longrightarrow Y'$ to be $\mathscr{X}\mathscr{K}(f)$. Iterating the construction, define $X^{(n)} = (X^{(n-1)})'$, where $X^{(0)} = X$. Observe inductively that $\mathscr{K}(X^{(n)}) = \mathscr{K}(X)^{(n)}$ since $\mathscr{K}\mathscr{X}(K) = K'$.

**Proposition 5.4.2.** *There is a map $\zeta = \zeta_X\colon X' \longrightarrow X$ that makes the following diagram commute, and $\zeta$ is a weak homotopy equivalence.*

$$\begin{array}{ccccc} |\mathscr{K}\mathscr{X}\mathscr{K}(X)| & = & |\mathscr{K}(X)'| & \xrightarrow{|\xi_{\mathscr{K}(X)}|} & |\mathscr{K}(X)| \\ \downarrow{\scriptstyle \psi_{\mathscr{X}\mathscr{K}(X)}} & & & & \downarrow{\scriptstyle \psi_X} \\ X' = \mathscr{X}\mathscr{K}(X) & & \xrightarrow[\zeta_X]{} & & X. \end{array}$$

*The simplicial map $\xi_{\mathscr{K}(X)}$ coincides with $\mathscr{K}(\zeta_X)\colon \mathscr{K}(X') \longrightarrow \mathscr{K}(X)$. The following diagram commutes for a map $f\colon X \longrightarrow Y$.*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle \zeta_X}\downarrow & & \downarrow{\scriptstyle \zeta_Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Proof. The points of $\mathscr{X}\mathscr{K}(X)$ are the barycenters of the simplices of $\mathscr{K}(X)$. These simplices $\sigma$ are spanned by increasing sequences $x_0 < \cdots < x_n$ of elements of $X$. Let $\zeta(b_\sigma) = x_n$. Since $b_\sigma \le b_\tau$ implies $\sigma \subset \tau$ and thus $\zeta(b_\sigma) \le \zeta(b_\tau)$, $\zeta$ is continuous. We understand $\xi_{\mathscr{K}(X)}$ to be the standard choice specified in Definition 4.4.12. Inspection of definitions shows that $\xi_{\mathscr{K}(X)} = \mathscr{K}(\zeta_X)$. The commutativity of the first diagram follows from the "naturality" of $\psi$ with respect to the map $\zeta_X$. That is, this diagram is a specialization of the commutative diagram of Theorem 5.1.2, with $f$ there taken to be $\zeta_X$ here. That $\zeta_X$ is a weak homotopy equivalence follows from the diagram, since all other maps in it are weak homotopy equivalences. The last statement is clear by inspection of definitions. □

**Theorem 5.4.3.** *Let $X$ be an $F$-space and $Y$ be an $A$-space, and let $f\colon |\mathscr{K}(X)| \longrightarrow |\mathscr{K}(Y)|$ be any map. Then for some sufficiently large $n$ there is a map $g\colon X^{(n)} \longrightarrow Y$ such that $f$ is homotopic to $|\mathscr{K}(g)|$. We call $g$ a* finite approximation *to $f$.*

Proof. By the classical simplicial approximation theorem for simplicial complexes, for a sufficiently large $n$ there is a simplicial approximation

$$j\colon \mathscr{K}(X^{(n-1)}) = \mathscr{K}(X)^{(n-1)} \longrightarrow \mathscr{K}(Y)$$

to $f$. Let $g$ be the composite

$$X^{(n)} = \mathscr{X}\mathscr{K}(X^{(n-1)}) \xrightarrow{\mathscr{X}(j)} \mathscr{X}\mathscr{K}(Y) = Y' \xrightarrow{\zeta_Y} Y.$$

Then

$$\mathscr{K}(g) = \mathscr{K}(\zeta_Y) \circ \mathscr{K}\mathscr{X}(j) = \mathscr{K}(\zeta_Y) \circ j'.$$

We have $|j'| \simeq |j|$ by Proposition 4.4.16 and $|j| \simeq f$ by assumption. Since we also have $|\mathscr{K}(\zeta_Y)| = |\xi_{\mathscr{K}(Y)}| \simeq \mathrm{id}$, we have $|\mathscr{K}(g)| \simeq f$, as required. □

The point to emphasize here is that finite models for spaces have far too few maps between them. For example, $\pi_n(S^n, *) = \mathbb{Z}$, but there are only finitely many distinct maps from any finite model for $S^n$ to itself. The theorem says that, after subdividing the domain sufficiently, we can realize any of these homotopy classes in terms of maps between (different) finite models for $S^n$.

## 5.5. Contiguity of maps between A-spaces

Remembering the definition of $\mathscr{K}(X)$, we may as well refer to points of an $A$-space $X$ as vertices and to finite ordered subsets of $X$ as simplices. Thus "simplex" is just a convenient abbreviation of "finite totally ordered subset". We use that language in translating the notion of contiguity from simplicial complexes to finite spaces. If $q > n$, we agree to write $\zeta$ for the composite $X^{(q)} \longrightarrow X^{(n)}$ determined by iteration of maps $\zeta$.

**Definition 5.5.1.** Let $f, g\colon X \longrightarrow Y$ be continuous maps between $A$-spaces. We say that $f$ is *contiguous* to $g$ if for every simplex $\sigma$ of $X$, there is a simplex of $Y$ that contains both $f(\sigma)$ and $g(\sigma)$. More generally, let $f\colon X \longrightarrow Y$ and $g\colon X^{(n)} \longrightarrow Y$ be continuous maps. We say that $f$ is contiguous to $g$ if for each simplex $\sigma$ of $X^{(n)}$, there is a simplex of $Y$ that contains both $(f \circ \zeta)(\sigma)$ and $g(\sigma)$. If $q > n$, a check of definitions shows that if $f$ and $g$ are continguous, then so are $f$ and $g \circ \zeta$. Similarly, if $q > 0$ and $f$ and $g$ are contiguous, then so are $f \circ \zeta$ and $g$, where now $\zeta\colon K^{(q)} \longrightarrow K$. The relation of contiguity is reflexive and symmetric, but it is not transitive. We let $\sim$ denote the equivalence relation generated by contiguity. Thus $f \sim g$ if there is a sequence of continuous maps $\{f = f_1, \cdots, f_q = g\}$ such that $f_i$ is contiguous to $f_{i+1}$ for $i < q$.

**Proposition 5.5.2.** *If $f\colon X \longrightarrow Y$ and $g\colon X^{(n)} \longrightarrow Y$ are contiguous maps between $A$-spaces, then $f \circ \zeta \simeq g\colon X^{(n)} \longrightarrow Y$.*

The analogue for simplicial maps used the notion of simplicially close maps from an arbitrary space to a simplicial complex. We have an analogous notion for maps to $A$-spaces. The term "approximate map" is sometimes used for either of these notions.

**Definition 5.5.3.** Let $X$ be any space and let $Y$ be an $A$-space. Two maps $f, g\colon X \longrightarrow Y$ are *simplicially close* if for each $x \in X$ there is a simplex $\tau = \tau_x$ of $Y$ such that $f(x)$ and $g(x)$ are both in $\tau$.

Clearly contiguous maps between $A$-spaces are simplicially close in this sense. Therefore the following result implies Proposition 5.5.2.

**Proposition 5.5.4.** *At least if both $X$ and $Y$ are $A$-spaces, simplicially close maps $f, g\colon X \longrightarrow Y$ are homotopic.*

Proof. Define $h\colon X \times I \longrightarrow Y$ by

$$h(x,t) = f(x) \quad \text{if} \quad 0 \le t < 1/2$$

$$h(x,1/2) = \begin{cases} g(x) & \text{if } f(x) \le g(x) \\ f(x) & \text{if } g(x) \le f(x). \end{cases}$$

$$h(x,t) = g(x) \quad \text{if} \quad 1/2 < t \le 1$$

Since $f(x)$ and $g(x)$ are both in a simplex $\tau_x$, either $f(x) \leq g(x)$ or $g(x) \leq f(x)$. Therefore $h$ is well-defined, and it suffices to prove that $h$ is continuous. One way to study the problem is to introduce the three point space $J = \{0, 1/2, 1\}$ whose proper open subsets are $\{0\}$, $\{1\}$, and their union $\{0, 1\}$. Define $\pi\colon I \longrightarrow J$ by

$$\pi([0, 1/2)) = 0, \quad \pi(1/2) = 1/2, \quad \pi((1/2, 1]) = 1.$$

Certainly $\pi$ is continuous, hence so is $\mathrm{id} \times \pi\colon X \times I \longrightarrow X \times J$. There is an obvious function $j\colon X \times J \longrightarrow Y$ such that $h = j \circ (\mathrm{id} \times \pi)$, namely

$$j(x, 0) = f(x), \quad j(x, 1/2) = h(x, 1/2), \quad j(x, 1) = g(x).$$

It suffices to prove that $j$ is continuous. When $X$ is an $A$-space, this can be done by giving $X \times J$ the product order, namely $(x, i) \leq (x', i')$ if and only if both $x \leq x'$ and $i \leq i'$, and checking that $j$ is order-preserving since $f$ and $g$ are order preserving. Since both $0 < 1/2$ and $1 < 1/2$ and since $x \leq x'$ implies both $f(x) \leq f(x')$ and $g(x) \leq g(x')$, the check is easy and can be left to the reader. □

Comparing our two definitions of simplicially close maps, for simplicial complexes and for Alexandroff spaces, we see the following properties of the constructions $\mathscr{K}$ and $\mathscr{X}$.

**Proposition 5.5.5.** *If $f\colon \mathscr{K}(X^{(m)}) \longrightarrow \mathscr{K}(Y)$ and $g\colon \mathscr{K}(X^{(n)}) \longrightarrow \mathscr{K}(Y)$ are contiguous maps of simplicial complexes, then $\zeta_Y \circ \mathscr{X}(f)\colon X^{(m+1)} \longrightarrow Y$ and $\zeta_Y \circ \mathscr{X}(g)\colon X^{(n+1)} \longrightarrow Y$ are contiguous maps of $A$-spaces. If $f\colon X^{(m)} \longrightarrow Y$ and $g\colon X^{(n)} \longrightarrow Y$ are contiguous maps of $A$-spaces, then $\mathscr{K}(f)$ and $\mathscr{K}(g)$ are contiguous maps of simplicial complexes.*

Now the simplicial results Theorems 4.6.3 and 4.6.4 have the following immediate consequences.

**Proposition 5.5.6.** *If $g\colon X^{(m)} \longrightarrow Y$ and $g'\colon X^{(n)} \longrightarrow Y$ are finite approximations of the same map $f\colon |\mathscr{K}(X)| \longrightarrow |\mathscr{K}(Y)|$, then $g$ and $g'$ are contiguous.*

**Theorem 5.5.7.** *If $f$ and $f'$ are homotopic maps $|\mathscr{K}X| \longrightarrow |\mathscr{K}Y|$ and $g$ and $g'$ are finite approximations to $f$ and $f'$, then $g$ is contiguous to $g'$. Therefore, for every pair of homotopic maps $f, f'\colon |\mathscr{K}X| \longrightarrow |\mathscr{K}Y|$, there is a sufficiently large $n$ such that $f$ and $f'$ have contiguous finite approximations $X^{(n)} \longrightarrow Y$.*

We have focused on understanding homotopy classes of maps between finite simplicial complexes in terms of contiguity classes of simplicial maps and contiguity classes of continuous maps between finite spaces, but one can also ask the relationship between homotopy classes and contiguity classes of maps between finite spaces. We have seen that contiguous maps are homotopic, but the converse is also true. To see that, we refine Proposition 2.2.12, following [**7**, 2.1.1].

**Definition 5.5.8.** Maps $f, g\colon X \longrightarrow Y$ between Alexandroff spaces are very close if $f = g$ on all but one point $x \in X$, and either $f(x) < g(x)$ or $g(x) < f(x)$. The maps $f, g$ are closely equivalent if there is a sequence of maps $\{f = f_1, f_2, \cdots, f_q = g\}$ such that $f_i$ is very close to $f_{i+1}$ for $i < q$.

**Lemma 5.5.9.** *If $f, g\colon X \longrightarrow Y$ are very close, then they are contiguous.*

Proof. Without loss of generality, we may assume that $f(x) < g(x)$ for the unique point $x$ on which $f$ and $g$ differ. For a simplex $\sigma$ of $X$ that does not contain $x$, we have $f(\sigma) = g(\sigma)$, which is clearly contained in a simplex of $Y$. If $x$ is in a simplex $\sigma = \{x_0 < x_1 < \cdots < x_n\}$, then $x = x_i$ for some $i$ and $f(\sigma) \cup g(\sigma)$ is the simplex obtained by deleting repetitions from the ordered set

$$\{f(x_0) \leq f(x_1) \leq \cdots \leq f(x_i) \leq g(x_i) \leq g(x_{i+1}) \leq \cdots \leq g(x_n)\}$$ □

**Theorem 5.5.10.** *If $f, g\colon X \longrightarrow Y$ are homotopic maps between finite spaces, then $f$ and $g$ are very closely equivalent and are therefore contiguous.*

Proof. By Proposition 2.2.12, we may assume without loss of generality that $f \leq g$. Let $A \subset X$ be the set of points $x$ such that $f(x) \neq g(x)$. Of course, we may assume that $A$ is non-empty, and we let $x$ be a maximal point in $A$, so that $x' > x$ implies $f(x') = g(x')$. Define $f_2$ by $f_2(x') = f(x')$ for $x' \neq x$ and $f_2(x) = g(x)$. Certainly $f_2$ is order–preserving and thus continuous. It differs from $g$ at one less point than $f = f_1$ differs from $g$. Repeating the construction, we arrive at $f_q = g$ after finitely many steps since $X$ and $Y$ are finite. □

## 5.6. Products of simplicial complexes

We here discuss several important constructions that we shall use later. The discussion focuses on how these concepts compare in the worlds of posets, simplicial complexes, and general spaces.

Inclusions of posets and simplicial complexes have an obvious meaning, and they are characterized as in Lemma 1.5.4. Quotients are more subtle and we shall return to them when we discuss simplicial sets.

We defined disjoint unions $X \amalg Y$ of topological spaces in Definition 1.4.3 and characterized the disjoint union by a universal property in Lemma 1.5.6. Similarly, we defined the product $X \times Y$ of topological spaces in Definition 1.4.4 and characterized the product by a universal property in Lemma 1.5.7. We can ask similarly for disjoint unions, often called "'coproducts", and products of other kinds of objects. Since posets are "the same" as $A$-spaces, we can translate the definitions of their coproducts and products to obtain the following definitions.

**Definition 5.6.1.** The disjoint union of posets $X$ and $Y$ is the set $X \amalg Y$ with the partial order specified by requiring $X$ and $Y$ to be subposets, with no relations $x \leq y$ or $y \leq x$ for $x \in X$ and $y \in Y$. If $f\colon X \longrightarrow Z$ and $Y \longrightarrow Z$ are order-preserving functions to a poset $Z$, then there is a unique order-preserving function $X \amalg Y \longrightarrow Z$ that restricts to $f$ and $g$ on $X$ and $Y$.

**Definition 5.6.2.** The product of posets $X$ and $Y$ is the set $X \times Y$ with the partial order specified by $(x, y) \leq (x', y')$ if $x \leq x'$ and $y \leq y'$. The projections to $X$ and $Y$ are order-preserving and if $f\colon W \longrightarrow X$ and $g\colon W \longrightarrow Y$ are order-preserving maps defined on a poset $W$, then the unique function $W \longrightarrow X \times Y$ with coordinates $f$ and $g$ is order-preserving.

The specified partial orders on $X \amalg Y$ and $X \times Y$ are the only ones that satisfy the specified universal property. We shall discuss definitions like this formally when we discuss categories, but this categorical point of view can be inconsistent with properties we might like, as we illustrate by considering products of simplicial complexes. Disjoint unions behave as one would expect and require no discussion.

**Definition 5.6.3.** The product $K \times L$ of two abstract simplicial complexes $K$ and $L$ has $V(K \times L) = V(K) \times V(L)$ and has simplices all subsets of products $\sigma \times \tau$ of sets $\sigma$ and $\tau$ that prescribe simplices of $K$ and $L$. We must take subsets here since a general subset of $\sigma \times \tau$ is not a product of subsets of $\sigma$ and $\tau$. The projections from $V(K \times L)$ to $V(K)$ and $V(L)$ prescribe simplicial maps and if $f: J \longrightarrow K$ and $g: J \longrightarrow L$ are maps of simplicial complexes then the unique function $V(J) \longrightarrow V(K) \times V(L)$ with coordinates $V(f)$ and $V(g)$ prescribes a map of simplicial complexes. The product of geometric simplicial complexes in $\mathbb{R}^M$ and $\mathbb{R}^N$ is defined similarly as a geometric simplicial complex in $\mathbb{R}^{M+N} = \mathbb{R}^M \times \mathbb{R}^N$.

It is important to distinguish beween ordered and unordered simplicial complexes here. If we construct realizations directly, without introducing orderings, it is *not* true that the realization of a product of abstract simplicial complexes is homeomorphic to the product of their realizations. The former just has too many simplices. The difference already appears when $K$ and $L$ each have just two vertices and their subsets. However, the difference disappears in the presence of orderings.

**Proposition 5.6.4.** *Let $X$ and $Y$ be posets. Then $\mathscr{K}(X \times Y)$ is a subdivision of $\mathscr{K}(X) \times \mathscr{K}(Y)$, hence both have the same geometric realization, and their common realization is homeomorphic to $|\mathscr{K}(X)| \times |\mathscr{K}(Y)|$.*

Proof. Clearly $\mathscr{K}(X) \times \mathscr{K}(Y)$ and $\mathscr{K}(X \times Y)$ have the same finite set of vertices. Inspection shows that every simplex of $\mathscr{K}(X \times Y)$ is contained in a product of simplices of $\mathscr{K}(X)$ and $\mathscr{K}(Y)$ and that every simplex of $\mathscr{K}(X) \times \mathscr{K}(Y)$ is a union of finitely many simplices of $\mathscr{K}(X \times Y)$. In more detail, the $n$-simplices of $\mathscr{K}(X \times Y)$ are all sets of pairs $\tau = \{(x_i, y_i) | 0 \le i \le n\}$ such that $(x_i, y_i) < (x_{i+1}, y_{i+1})$. This means that $x_i \le x_{i+1}$ and $y_i \le y_{i+1}$, with not both equal. If there are $p+1$ distinct $x_i$ and $q+1$ distinct $y_j$, then $\rho = \{x_i\}$ is a $p$-simplex of $\mathscr{K}(X)$, $\sigma = \{y_j\}$ is a $q$-simplex of $\mathscr{K}(Y)$, and $\tau$ is contained in $\rho \times \sigma$. There are many choices of $\tau$ that determine the same $\rho$ and $\sigma$. Thus every simplex of $\mathscr{K}(X \times Y)$ is contained in a simplex of $\mathscr{K}(X) \times \mathscr{K}(Y)$. The projections $X \times Y \longrightarrow X$ and $X \times Y \longrightarrow Y$ induce the coordinates of a map

$$|\mathscr{K}(X \times Y)| \longrightarrow |\mathscr{K}(X)| \times |\mathscr{K}(Y)|.$$

A point on the right is a pair $(u, v)$ where $u$ is an interior point of some simplex $\sigma$ of the geometric simplicial complexe $g\mathscr{K}(X)$ and $v$ is an interior point of some simplex $\tau$ of $g\mathscr{K}(Y)$. Since all simplices on the left are subsimplices of some $\sigma \times \tau$, this map is a homeomorphism. □

**Definition 5.6.5.** Let $K$ and $L$ be ordered simplicial complexes (abstract or geometric). Order the elements of $V(K) \times V(L)$ by $(x, y) \le (x', y')$ if $x \le x'$ and $y \le y'$. The simplices of the *ordered* simplicial complex $K \times L$ are the sets of pairs $\tau = \{(x_i, y_i) | 0 \le i \le n\}$ such that $(x_i, y_i) < (x_{i+1}, y_{i+1})$, $\{x_0, \ldots, x_n\}$ is a simplex of $K$ and $\{y_0, \ldots, y_n\}$ is a simplex of $L$.

With this definition in place, the last statement of Proposition 5.6.4 generalizes, with the same proof.

**Proposition 5.6.6.** *Let $K$ and $L$ be ordered (geometric) simplicial complexes. Then the projections induce a homeomorphism*

$$|K \times L| \longrightarrow |K| \times |L|.$$

Intuitively, the point is that the product of two geometric simplices is not a geometric simplex (a square is not a triangle) but can be subdivided into geometric simplices. In effect, the displayed homeomorphism carries out this subdivision consistently over all of the simplices of a product of simplicial complexes.

## 5.7. The join operation

The join operation played a very substantial role in the early decades of algebraic topology and is a very natural operation in the context of simplicial complexes. We shall only use it peripherally, when we relate simplicial complexes to finite groups, but it is best introduced here, where comparisons with disjoint unions and with products can be seen clearly.

**Definition 5.7.1.** The *join* $X * Y$ of posets $X$ and $Y$ is the poset given by the disjoint union of the posets $X$ and $Y$, together with the additional relations $x < y$ if $x \in X$ and $y \in Y$.

As something of a joke, consider the opposite choice available in Definition 5.7.1.

**Definition 5.7.2.** Define the *antijoin* $(X*Y)^-$ of posets $X$ and $Y$ to be the poset given by the disjoint union of the posets $X$ and $Y$, together with the additional relations $y < x$ if $x \in X$ and $y \in Y$.

There is no order-preserving function relating $X*Y$ and $(X*Y)^-$, but we have the following illuminating observation.

**Proposition 5.7.3.** *The subdivisions of $X * Y$ and $(X * Y)^-$ are isomorphic.*

PROOF. Remember that $X' = \mathscr{X}\mathscr{K}X$. We define an isomorphism $f\colon (X * Y)' \longrightarrow (\mathrm{Sd}(X*Y)^-)'$ that restricts to the identity map between the subcomplexes $X'$ and $Y'$ of each. A typical point of $(X*Y)'$ that is in neither $X'$ nor $Y'$ has the form

$$(x_0 < \cdots < x_m < y_0 < \cdots < y_n)$$

where $m \geq 0$, $n \geq 0$, $x_i \in X$, and $y_j \in Y$. Define

$$f(x_0 < \cdots < x_m < y_0 < \cdots < y_n) = (y_0 < \cdots < y_n < x_0 < \cdots < x_m).$$

It is visibly clear that $f$ is a well-defined isomorphism of posets with inverse given by

$$f^{-1}(y_0 < \cdots < y_m < x_0 < \cdots < x_n) = (x_0 < \cdots < x_n < y_0 < \cdots < y_m).$$

□

If $Y$ is a single point, then $X*Y$ is the cone $CX$ as we defined it earlier. Quillen defines $CX = (X*Y)^-$. The choice is arbitrary and we have just seen that the two choices have isomorphic subdivisions and therefore homeomorphic realizations.

**Remark 5.7.4.** It is perhaps illuminating to use both choices, and we write $C^+X$ for the first choice and $C^-X$ for the second. There is a canonical map $i$ from $X * Y$ to the poset $C^+X \times C^-Y - \{(c_X, c_Y)\}$, where $c_X$ and $c_Y$ denote the cone points. Indeed, we set $i(x) = (x, c_Y)$ and $i(y) = (c_X, y)$. Since $x < c_X$ and $c_Y < y$, $i(x) < i(y)$ for all $x$ and $y$, while $i(x) \leq i(x')$ if and only $x \leq x'$ and $i(y) \leq i(y')$ if and only if $y \leq y'$.

Just as for products, the precise definition of which is different when we consider products of posets, of simplicial complexes, and of topological spaces, we have different meanings of the notion of join, all of which are denoted by $*$. However, unlike products, which are characterized by a universal property, the different definitions of the join are primarily motivated by the comparisons among them.

**Definition 5.7.5.** The *join* $K * L$ of abstract simplicial complexes $K$ and $L$ has vertex set $V(K * L)$ the disjoint union of $V(K)$ and $V(L)$ and has simplices the simplices of $K$, the simplices of $L$, and all disjoint unions of simplices of $K$ and $L$.

The join of geometric simplicial complexes is defined similarly, requiring the disjoint union of $V(K)$ and $V(L)$ to be a linearly independent set.

Conceptually, it is helpful to note that, just like the product, where $X \times Y$ is not literally the same as $Y \times X$ but only isomorphic to it, we should think of disjoint union as an operation only commutative up to isomorphism. Then the evident choice of order on the join of ordered geometric simplicial complexes corresponds to the analogous choice we had when defining the join of posets in Definition 5.6.2.

**Definition 5.7.6.** The join of topological spaces $X$ and $Y$ is the quotient space of $X \times I \times Y$ obtained by identifying $(x, 0, y)$ with $(x', 0, y)$ and $(x, 1, y)$ with $(x, 1, y')$ for all $x, x' \in X$ and $y, y' \in Y$. It is the space of lines connecting $X$ to $Y$. If $X$ and $Y$ are geometrically independent subspaces of some large Euclidean space, $X * Y$ is defined geometrically as the subspace of points $tx + (1-t)y$ for $x \in X$, $y \in Y$, and $0 \leq t \leq 1$, noting that the point is independent of $x$ if $t = 0$ and of $y$ if $t = 1$.

**Lemma 5.7.7.** *For spaces $X$ and $Y$, $X * Y$ is homeomorphic to the union $(CX \times Y) \cup_{X \times Y} (X \times CY)$ where the notation indicates that we identify the copies of $X \times Y$ in $CX \times Y$ and $X \times CY$.*

PROOF. We identify $X * Y$ and $(CX \times Y) \cup_{X \times Y} (X \times CY)$ as homeomorphic quotients of subspaces of $X \times Y \times I \times I$. Let $J$ be the diagonal $\{(s,t) | s + t = 1\}$ in the square. Then $X * Y$ is homeomorphic to the quotient of $X \times Y \times J$ obtained from the equivalence relation given by

$$(x, y, (1, 0)) \sim (x', y, (1, 0)) \text{ and } (x, y, (0, 1)) \sim (x, y', (0, 1)).$$

Think of the cone coordinates of $CX$ and $CY$ as the edges $I_1 = [(0,0),(1,0)]$ and $I_2 = [(0,0),(0,1)]$ of $I \times I$. Let $K = I_1 \cup I_2 \subset I \times I$. Then the space

$$(CX \times Y) \cup_{X \times Y} (X \times CY)$$

is homeomorphic to the quotient of $X \times Y \times K$ obtained from precisely the same equivalence relation. Radial projection from the point $(1, 1)$ gives a deformation

$$I \times I - \{1, 1\} \longrightarrow K$$

that restricts to a homeomorphism $J \longrightarrow K$ and thus induces the claimed homeomorphism. □

**Proposition 5.7.8.** *For posets $X$ and $Y$,*

$$\mathscr{K}(X * Y) \cong \mathscr{K}(X) * \mathscr{K}(Y).$$

*For abstract simplicial complexes $K$ and $L$,*

$$g(K * L) \cong gK * gL.$$

*For ordered geometric simplicial complexes $K$ and $L$,*

$$|K * L| \cong |K| * |L|.$$

We give another way to think about the join $|K| * |L|$ in $\mathbb{R}^N$, where $K$ and $L$ are geometric simplicial complexes. The notion of $X - \{x\}$, $x \in X$, is clear for a poset. For a simplicial complex $K$, $K - \{v\}$ for $v \in V(K)$ means the simplicial complex that is obtained from $K$ by deleting all simplices which have $v$ as a vertex, and

$\mathscr{K}(X - \{x\}) = \mathscr{K}(X) - \{x\}$. However, $|K - \{v\}|$ is quite different from $|K| - v$. The cone $CK$ of a geometric simplicial complex $K$ is obtained by by adding a vertex $c_K$ that is geometrically independent of all vertices in $K$ and adding a new simplex spanned by the union of $c_K$ and the vertices of $\sigma$ for each simplex $\sigma$ of $K$. If $K$ is ordered, then $CK$ is ordered by requiring $c_K$ to be greater than all other vertices.

**Proposition 5.7.9.** *Let $K$ and $L$ be ordered (geometric) simplicial complexes. Then*

$$CK \times CL - \{(c_K, c_L)\} = (CK \times L) \cup_{K \times L} (K \times CL)$$

*as subcomplexes of $CK \times CL$. Therefore*

$$|K| * |L| \cong |CK \times CL - \{(c_K, c_L)\}|$$

.

Proof. The simplices of $CK \times CL$ that do not have $(c_K, c_L)$ as a vertex are the simplices in either $CK \times L$ or $K \times CL$. The gives the first conclusion. Geometric realization commutes up to homeomorphism with cones, products and unions, so that

$$|(CK \times L) \cup_{K \times L} (K \times CL)| \cong (C|K| \times |L|) \cup_{|K| \times |L|} (|K| \times C|L|).$$

Now Lemma 5.7.7 gives the second conclusion. □

## 5.8. Reduction methods of finite spaces

The manipulation of a finite space through removal of points presents a space weakly homotopy equivalent to the original. The exposition presented in the next section follows the work of Sharon Zhou, in her 2020 REU paper on homotopy types of finite spaces and simplicial complexes.

As observed, if two finite spaces $X$ and $Y$ are homotopy equivalent, then so are their corresponding order complexes $\mathscr{K}(X)$ and $\mathscr{K}(Y)$. In fact, T. Osaki [**55**] showed that $\mathscr{K}(X)$ and $\mathscr{K}(Y)$ are actually simple homotopy equivalent, which is a more refined notion of homotopy equivalence in the world of simplicial complexes.

In order to understand this result, the following definition of simple homotopy is presented:

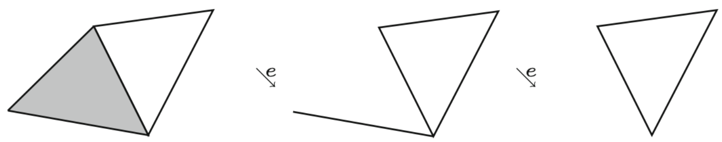
**Definition 5.8.1.** Let $K$ be a finite simplicial complex and $L \subset K$ be a subcomplex. We say that $K$ collapses to $L$ via an *elementary simplicial collapse* and write $K \searrow^e L$ if there exists a simplex $S \in K$ and a vertex $a \in K$ that is not contained in $S$ such that

$$K = L \cup aS \text{ and } L \cap aS = a\partial S.$$

In other words, $K$ collapses to $L$ via an elementary simplicial collapse if there are only two simplices $S, S' \in K$ disjoint from $L$ such that $S$ is a free face of $S'$, i.e., $S'$ is the only simplex disjoint from $L$ that contains $S$ as a face.

**Definition 5.8.2.** We say that $K$ *(simplicially) collapses* to $L$ or $L$ *(simplicially) expands* to $K$ if $L$ can be obtained from $K$ via a sequence of elementary collapses. We denote this by $K \searrow L$ or $L \nearrow K$. Two complexes $K$ and $L$ have the same *simple homotopy type* if there exists a sequence of simplicial complexes $K = K_1, K_2, \ldots, K_n = L$ such that $K_i \searrow K_{i+1}$ or $K_i \nearrow K_{i+1}$ for all $1 \leq i \leq n$.

For a concrete example, consider the sequence of elementary collapses below, which can be found in [7].




We say that a simplicial complex $K$ is *collapsible* if it collapses to one of its vertices. For example, any simplicial cone $aK$ is collapsible. The key observation here is that simple homotopy equivalence is a special case of homotopy equivalence, as we show below.

**Proposition 5.8.3.** *If two simplicial complexes are simple homotopy equivalent, then they are homotopy equivalent.*

PROOF. Let $K, L$ be two simplicial complexes. Without loss of generality, let $L \subset K$ be a subcomplex and suppose that $K$ collapses to $L$ via an elementary simplicial collapse. Then there exists some simplex $S \in K$ and a vertex $a \in K, a \notin S$ such that $K = L \cup aS$ and $L \cap aS = a\partial S$. Note that the inclusion $i : L \cap aS \hookrightarrow aS$ is a homotopy equivalence. Applying the gluing theorem (Theorem A.2.5 in [7]) to the diagram below,

$$\begin{array}{ccc} L \cap aS & \xrightarrow{i} & aS \\ \cap\downarrow & & \downarrow \\ L & \xrightarrow[I]{} & K \end{array}$$

we see that the inclusion $I : L \hookrightarrow K$ is also a homotopy equivalence. □

This yields the following proper containment of types of homotopies between simplicial complexes, where $\mathcal{S}$ denotes the set of simple homotopy equivalences. A theorem by Whitehead shows that a homotopy equivalence between simplicial complexes is a simple homotopy equivalence precisely when the Whitehead torsion $\tau$ vanishes (see [52] for details).

$$\mathcal{S} \subset \{\text{Homotopy equivalence}\} = \{\text{Weak Homotopy Equivalence}\}$$

In finite spaces, as we will soon show, a different relation holds:

$$\{\text{Homotopy equivalence}\} \subset \mathcal{S} \subset \{\text{Weak equivalence}\}$$

In both cases, the containment is proper. A natural question to ask is whether there exists some kind of homotopy equivalence between simple homotopy and homotopy equivalence of CW complexes. In other words, one might hope to define a new class of homotopy equivalences that will “fill in” the first chain of set containment. The close correspondence between finite spaces and simplicial complexes suggests that we may find an answer by examining the hierarchy of homotopy equivalences of finite spaces.

The following question may then be presented: *Can we further refine the notion of homotopy equivalence in the world of simplicial complexes to obtain some formal class of homotopy equivalence between simple homotopy equivalence and general homotopy equivalence by using the homotopy theory of finite spaces?*

Although this remains an open problem, several methods of examination are presented in Section 4 of the source paper.

What follows presents the effect of one-point reductions on the order complex of the space. In particular, results will show that removing beat points from a finite space $X$ does not affect the homotopy type of either $X$ or $\mathscr{K}(X)$.

## 5.9. One-point reduction of finite spaces

In this section, we study three types of one-point reductions of finite spaces, namely the removal of beat points, weak points, and $\gamma$-points, and consider what kind of homotopy equivalences they induce on the corresponding order complexes.

**5.9.1. Beat points.** Recall that two finite spaces are homotopy equivalent if and only if one can be obtained from another by successively removing or adding beat points. We have the following useful corollary, directly implied from Theorem 2.4.4.

**Corollary 5.9.1.** *A finite space $X$ is contractible if and only if one can remove beat points from $X$ one at a time to obtain a space consisting of only one point.*

We now consider what kind of homotopy equivalence a beat point removal will induce on the order complex associated to the original finite space. As one would reasonably expect, removing a beat point from a finite space $X$ does not change the simple homotopy type of $\mathscr{K}(X)$. This result was first proved by Osaki [**55**].

**Theorem 5.9.2** (Osaki)**.** *If $x$ is a beat point, then $\mathscr{K}(X)$ collapses to $\mathscr{K}(X \backslash \{x\})$.*

Since two finite spaces are homotopy equivalent if and only if one can be obtained from another by successively removing and adding beat points, this theorem generalizes readily to the following corollary.

**Corollary 5.9.3.** *If $X$ and $Y$ are homotopy equivalent, then $\mathscr{K}(X)$ and $\mathscr{K}(Y)$ have the same simple homotopy type.*

There is one thing unsatisfactory, however, about this corollary: its converse is false. To see that, consider the following example given by Barmak and Minian [**7**].

**Example 5.9.4.** The finite space $W$ (inspired by its resemblance to a wallet), which we draw below, has no beat points and is therefore non-contractible. Nevertheless, if one follows the definition of a order complex and draws out $\mathscr{K}(W)$, one sees that $\mathscr{K}(W)$ is contractible. In fact, it will soon be shown that $\mathscr{K}(W)$ is simple homotopy equivalent to a point.

This example suggests that homotopy equivalence of finite spaces is a "stronger" relation than simple homotopy equivalence of simplicial complexes. To put it more precisely, the set of homotopy equivalences in simplicial complexes that are induced by removal of beat points from finite spaces, which we sometimes call *strong homotopy equivalence*, is a proper subset of the set of simple homotopy equivalences of simplicial complexes. Accordingly, the removal of a beat point from a finite space is a "stronger" move than an elementary collapse in simplicial complexes.

This observation naturally gives rise to the following question: does there exist an "elementary move" in finite spaces that would precisely correspond to an

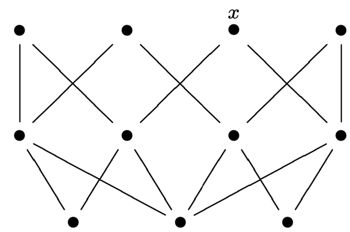



FIGURE 1. $W$

elementary collapse in simplicial complexes? It is for precisely this reason that Barmak and Minian [7] introduced the notion of a weak point. In particular, we will show that the point $x \in W$ in the above example is a weak point, and that $\mathscr{K}(W)$ is homotopically trivial.

### 5.9.2. Weak points.

**Definition 5.9.5.** Let $X$ be an $F$-space. We say that $x \in X$ is an *up weak point* if $\hat{F}_x$ is contractible and a *down weak point* or $\hat{U}_x$ is contractible. A point is a *weak point* if it is either an up weak point or a down weak point.

Note that a beat point is necessarily a weak point, since for any beat point $x$, either $\hat{U}_x$ has a maximum or $\hat{F}_x$ has a minimum, which makes at least one of these two sets contractible. To lighten the notation, we make the following definitions.

**Definition 5.9.6.** Given an $F$-space $X$, the *link* of $x \in X$ is defined as $lk(x) = \hat{C}_x = \hat{U}_x * \hat{F}_x$.

The following lemma gives us an alternative way to characterize weak points.

**Lemma 5.9.7.** *Let $X, Y$ be $F$-spaces. Then the join $X * Y$ is contractible if and only if either $X$ or $Y$ is contractible.*

PROOF. Without loss of generality, suppose that $X$ is contractible with point $\{+\}$. By Corollary 5.9.1, we can find a decreasing sequence of spaces

$$X = X_n \supset X_{n-1} \supset \dots X_1 = \{+\},$$

where we remove beat points from $X$ one by one such that each $X_i$ contains $i$ points and $x_i \in X_i$ is a beat point. Note that $x_i$ is also a beat point of $X_i * Y$, so $X * Y$ inductively deformation retracts to $\{+\} * Y$, which has a minimum and is therefore contractible. The argument where $Y$ is contractible is exactly analogous if one replaces minimum by maximum at the end.
Conversely, suppose that $X * Y$ is contractible. Again by Corollary 5.9.1, there exists a decreasing sequence of spaces

$$X * Y = (X * Y)_n \supset (X * Y)_{n-1} \supset \dots (X * Y)_1 = \{+\},$$

where $(X * Y)_i = \{z_1, z_2, \dots, z_i\}$ such that $z_i$ is a beat point of $(X * Y)_i$.
Fix some $2 \leq i \leq n$, and suppose that $z_i \in X_i$. Then $z_i$ is a beat point of $X_i$ unless it is a maximal point of $X_i$, $Y_i$ has a minimum, and $X_i \backslash \{z_i\}$ has no maximum. Similarly, if $z_i \in Y_i$, then either $z_i$ is a beat point of $Y_i$ or $X_i$ has a maximum and $Y_i \backslash \{z_i\}$ has no minimum. Thus for every $i$, at least one of the following statements

is true: (1) either $X_{i-1} \hookrightarrow X_i$ or $Y_{i-1} \hookrightarrow Y_i$ is a deformation retract, and (2) one of $X_i$ and $Y_i$ is contractible. Hence $X$ or $Y$ is contractible, as desired. □

**Proposition 5.9.8.** *Let $X$ be an $F$-space. Then $x \in X$ is a weak point if and only if $lk(x) = \hat{C}_x$ is contractible.*

As shown, if $x$ is a beat point of $X$, then $X\backslash\{x\}$ is homotopy equivalent to $X$. This is no longer true if we replace beat points with weak points. Nevertheless, a weaker version of this result holds.

**Proposition 5.9.9.** *Let $X$ be an $F$-space, and let $x \in X$ be a weak point. Then the inclusion $i : X\backslash\{x\} \hookrightarrow X$ is a weak homotopy equivalence.*

The proof of this proposition makes use of Theorem 3.3.1. Note that for any $F$-space $X$, the minimal basis $\{U_x\}_{x\in X}$ is a basis like open cover.

Proof. Without loss of generality, suppose that $x$ is a up weak point. Then $\hat{F}_x$ is contractible. Let $y \in X$. Then the set $i^{-1}(F_y) = F_y\backslash\{x\}$ has a minimum if $y \neq x$, and is contractible if $y = x$. Hence the restricted map

$$i\big|_{i^{-1}(F_y)} = i^{-1}(F^y) \to F_y,$$

is a weak homotopy equivalence, since the map $\pi_n(i^{-1}(F_y), y) \to \pi_n(F_y, y)$ is an isomorphism for all $n$. As remarked above, the minimal basis of $X$ is a basis like open cover of $X$. Now applying Theorem 3.15 to the minimal basis of $X$ shows that the restricted inclusion is a weak homotopy equivalence.
The case where $x$ is a down weak point follows immediately by applying the above argument to $X^{op}$, noting that $\mathscr{K}(X^{op}) = \mathscr{K}(X)$. □

To illustrate this proposition, let us return to Example 5.9.4, as promised. To see that the point $x$ is a weak point, we draw out the subspace $\hat{U}_x$ as follows.

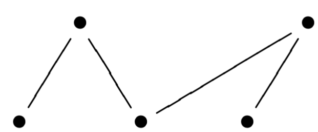


Figure 2. $\hat{U}_x$

Clearly, $\hat{U}_x$ is contractible, so $x$ is a weak point. Hence Proposition 5.9.9 tells us that $W$ is weak homotopy equivalent to $W\backslash\{x\}$, whose Hasse diagram looks like the following:
$W\backslash\{x\}$ is contractible because we can remove beat points one by one (starting with the point $y$ as labeled in the diagram, then proceed to $z$, and so on), eventually obtaining a space consisting of a single point. This motivates the following definition.

**Definition 5.9.10.** Let $X$ be an $F$-space and $Y \subset X$ a subspace. We say that $X$ collapses to $Y$ by an *elementary collapse* (or that $Y$ expands to $X$ by an *elementary expansion*) if $Y$ is obtained from $X$ by removing a weak point. In this case, we denote $X \searrow^e Y$ or $Y \nearrow_e X$.

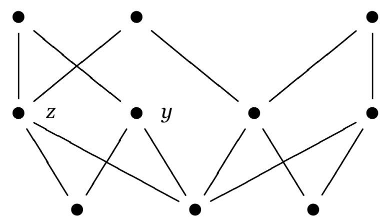



FIGURE 3. $W \backslash \{x\}$

In general, given two $F$-spaces $X$ and $Y$, we say that $X$ collapses to $Y$ (or $Y$ expands to $X$) if there is a sequence of $F$-spaces $X = X_1, X_2, \ldots, X_n = Y$ such that for each $1 \leq i < n$, $X_i \searrow^e X_{i+1}$. In this case, we write $X \searrow Y$or $Y \nearrow X$. Two $F$-spaces $X$ and $Y$ are *simply equivalent* if one can be obtained from another via a sequence of elementary collapses and expansions.

Before stating the following corollary from Proposition 5.9.9, we make a quick note on convention: adopting the terminology of Barmak and Minian, we will say that two $F$-spaces are simply equivalent and two simplicial complexes are simple homtopy equivalent (or have the same simple homotopy type). The crux of Theorem 5.9.12 is that these two definitions are really describing the same relation for two kinds of objects.

**Corollary 5.9.11.** *Let $X, Y$ be two simply equivalent $F$-spaces. Then they are weakly equivalent.*

The next theorem, which was proved by Barmak and Minian [**7**] as the main result of simple homotopy theory of finite spaces and simplicial complexes, essentially says that weak points do exactly what we want them to do. That is, removal of weak points is the $F$-space counterpart to an elementary simplicial collapse in simplicial complexes.

**Theorem 5.9.12** (Barmak and Minian)**.**

(1) *Let $X$ and $Y$ be $F$-spaces. Then $X$ and $Y$ are simply equivalent if and only if $\mathscr{K}(X)$ and $\mathscr{K}(Y)$ have the same simple homotopy type. In particular, if $X \searrow Y$, then $\mathscr{K}(X) \searrow \mathscr{K}(Y)$.*

(2) *Let $K$ and $L$ be finite simplicial complexes. Then $K$ and $L$ are simple homotopy equivalent if and only if $\mathscr{X}(K)$ and $\mathscr{X}(L)$ are simply equivalent. In particular, if $K \searrow L$, then $\mathscr{X}(K) \searrow \mathscr{X}(L)$.*

We say that an $F$-space is *collapsible* if it collapses to a point. Similarly, a simplicial complexes is said to be *collapsible* if it simplicially collapses to a single point. Since every beat point is a weak point, the set of contractible $F$-spaces is a proper subset of collapsible spaces. For example, the wallet $W$ as constructed above is a collapsible space that is not contractible.

**5.9.3. $\gamma$-points.** Recall that our goal is to define a formal class of homotopy equivalences of simplicial complexes that are not simple homotopy equivalences. Having seen that removing weak points induces simple homotopy equivalences in simplicial complexes, we want to relax the condition even further. This motivates the definition of a $\gamma$-point.

**Definition 5.9.13.** Let $X$ be an $F$-space. Then $x \in X$ is a $\gamma$-point if $\hat{C}_x$ is homotopically trivial. That is, $\pi_n(\hat{C}_x) = 0$ for all $n \geq 0$.

This definition gives us a new method of reduction of finite spaces.

**Definition 5.9.14.** We say that $X$ $\gamma$-collapses to $X \backslash \{x\}$ by an *elementary $\gamma$-collapse* if $x \in X$ is a $\gamma$-point. More generally, an $F$-space $X$ *$\gamma$-collapses* to a subspace $Y \subset X$ if there is a sequence of spaces

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_k = Y \quad (n > k)$$

such that $X_i$ $\gamma$-collapses to $X_{i-1}$ via an elementary $\gamma$-collapse for all $k \leq i \leq n$. In this case, we write $X \nearrow^{\gamma} Y$ . If $X$ $\gamma$-collapses to a point, we say that $X$ is *$\gamma$-collapsible.*

Note that every weak point is a $\gamma$-point, since a contractible space necessarily has all trivial homotopy groups. To see what kind of homotopy equivalence a $\gamma$-point reduction will induce on simplicial complexes, we first consider the relationship between $X \backslash \{x\}$ and $X$ where $x \in X$ is a $\gamma$-point.

**Proposition 5.9.15.** *If $x \in X$ is a $\gamma$-point, then the inclusion $i : X \backslash \{x\} \to X$ is a weak homotopy equivalence.*

The proof for Proposition 5.9.9 does not apply directly because neither $\hat{F}_x$ nor $\hat{U}_x$ is necessarily contractible. Nevertheless, the following pushout diagram still holds:

$$\begin{array}{ccc} |\mathscr{K}(\hat{C}_x)| & \xrightarrow{\varphi} & |\mathscr{K}(C_x)| \\ \downarrow \psi & & \downarrow \\ |\mathscr{K}(X \backslash \{x\})| & \longrightarrow & |\mathscr{K}(X)| \end{array}$$

Note that $\varphi : |\mathscr{K}(\hat{C}_x)| \to |\mathscr{K}(C_x)|$ is a homotopy equivalence, and that $\psi : |\mathscr{K}(\hat{C}_x)| \to |\mathscr{K}(X \backslash \{x\})|$ satisfies the homotopy extension property. Hence the map $|\mathscr{K}(X \backslash \{x\})| \to |\mathscr{K}(X)|$ is a homotopy equivalence. This implies that $i : X \backslash \{x\} \to X$ is a weak homotopy equivalence. The converse to this proposition, however, is true only when $x$ is neither maximal nor minimal (Theorem 3.13 in [**7**]).

If $x \in X$ is a $\gamma$-point, one can show that the map $\mathscr{K}(X \backslash \{x\}) \to \mathscr{K}(X)$ is a simple homotopy equivalence (the proof uses the relativity principle of simple homotopy theory; see [**19**]). In fact, Barmak and Minian [**7**] proved the following more general result, which says that this is the case whenever we have a weak homotopy equivalence between finite spaces.

**Theorem 5.9.16.** *Let $X$ be an $F$-space, and let $x \in X$. Suppose that the inclusion $i : X \backslash \{x\} \to X$ is a weak homotopy equivalence. Then the induced simplicial map $\mathscr{K}(X \backslash \{x\}) \to \mathscr{K}(X)$ is a simple homotopy equivalence.*

This theorem essentially shows that one-point reductions do not generate all weak homotopy types of finite spaces. We might then look beyond one-point reductions, a discussion for the following section. Before proceeding, we briefly discuss how some of the previous results can be generalized to a broader class of topological spaces.

While we cannot directly take these results for granted in general CW complexes, we can consider them on subsets called *regular* and *h-regular* CW complexes.

**Definition 5.9.17.** Let $K$ be a CW complex. We say that $K$ is *regular* if, for each open cell $e^n$, the characteristic map $D^n \to e^n$ is a homeomorphism. Equivalently, the attaching map $S^{n-1} \to K$ is a homeomorphism onto its image $\partial e^n$.

For a regular CW complex $K$, the closure $\overline{e^n}$ of each cell is a subcomplex of $K$. There is also a more general notion of $h$-regular CW complex, where one only requires the attaching map of each cell to be a homotopy equivalence with its image and that the closed cells $\overline{e^n}$ are subcomplexes of $K$.

Theorem 5.9.12 fails even when we consider only regular CW complexes (see page 60 of [7] for a counterexample). Nevertheless, a weaker version of the second part of Theorem 5.9.12, as proved in the same book, shows that simplicial collapses of $h$-regular CW complexes do induce $\gamma$-collapses in the corresponding finite spaces.

## 5.10. Remarks on an old list of problems

Revisit

We give a few problems that spring immediately to mind. To the best of my knowledge, these have not been studied, at least not thoroughly. The original 2003 list was considerably longer, but a number of people around the world have since solved many of its problems. Some of their solutions are sprinkled through the book.

**Problem 5.10.1.** *For small $n$, determine all homotopy types and weak homotopy types of spaces with at most $n$ elements.*

**Addendum 5.10.1.** We have given the answer or left it as an exercise when $n \leq 6$. Most finite spaces with so few points are disjoint unions of (weakly) contractible spaces, but we have seen several more interesting examples. I'd like to see the answer for larger $n$.

**Problem 5.10.2.** *Is there an effective algorithm for computing the homotopy groups of $X$ in low degrees in terms of the increasing chains in $X$? An REU paper of Weng described in §20 elaborated on the computation of the fundamental group by Barmak [7].*

Not written yet

**Remark 5.10.3.** The dimension of the simplicial complex $\mathscr{K}(X)$ is the maximal length of a sequence $x_0 < \cdots < x_n$ in $X$. A map $g\colon K \longrightarrow L$ of simplicial complexes of dimension less than $n$ is a homotopy equivalence if and only if it induces an isomorphism of homotopy groups in dimension less than $n$ and an epimorphism of homotopy groups in dimension $n$.

**Problem 5.10.4.** *Let $X$ be a minimal finite space. Give a descriptive interpretation of what this says about $|\mathscr{K}(X)|$.*

**Addendum 5.10.2.** There is a nice paper of Osaki [55] that interprets Stong's process of passing from an $F$-space to its core $Y$. He shows that $\mathscr{K}(Y)$ is obtained from $\mathscr{K}(X)$ by a sequence of elementary simplicial collapses, so that $|\mathscr{K}(X)|$ and $|\mathscr{K}(Y)|$ have the same "simple" homotopy type. It follows that if $X$ and $Y$ are homotopy equivalent $F$-spaces, then $\mathscr{K}(X)$ and $\mathscr{K}(Y)$ have the same simple homotopy type. If $K$ is not collapsible, then $\mathscr{X}(K)$ is a minimal finite space. As

Osaki points out and is clear from Example 3.4.15, there are non-collapsible triangulations $K_1$ and $K_2$ of $S^1$ such that $\mathscr{X}(K_1)$ and $\mathscr{X}(K_2)$ are not homeomorphic and therefore, being minimal, not homotopy equivalent. Barmak and Minian [**10**] went further and proved that two finite spaces $X$ and $Y$ are homotopy equivalent if and only if $|\mathscr{K}(X)|$ and $|\mathscr{K}(Y)|$ have the same simple homotopy type. reference to Barmak's book already here?

Finite spaces can be weak homotopy equivalent but not homotopy equivalent, as we have seen in Examples 3.4.14 and 3.4.15. The following problems are far more difficult than their analogues for homotopy equivalence, which we have treated in §2.5, following the REU paper of Fix and Patrias. Note that the work of Fix and Patrias implicitly addresses the problem of finding a computationally effective algorithm for enumerating the homotopy types of finite spaces. Not written yet

**Problem 5.10.5.** *Are there computationally effective algorithms for enumerating the weak homotopy types of finite spaces for small $n$? What is the asymptotic behavior of the number of weak homotopy types of spaces with at most $n$ elements?*

**Addendum 5.10.3.** Osaki [**55**] has given two theorems that describe when one can shrink an $F$-space, possibly minimal, to a smaller weakly homotopy equivalent $F$-space. He asks whether all weak homotopy equivalences are generated by the simple kinds that he describes. The question has since been answered in the negative, by Barmak and Minian [**8**]. Barmak's thesis, which was inspired by my 2003 REU notes and has now become the book [**7**], goes a good deal further. There is much more to be done on this problem, which is still not well understood.

**Problem 5.10.6.** *Is there a combinatorial way of determining when a weak homotopy equivalence of finite spaces is a homotopy equivalence?*

**Problem 5.10.7.** *Rather than restricting to finite simplicial complexes, can we model the world of finite CW complexes, or at least the world of finite regular CW complexes, in the world of finite spaces. The discussion of spheres and cones in §3.4 gives a possible starting point. This is related to the combinatorially interesting question of relating finite topological spaces to discrete Morse theory.*

CHAPTER 6

# A concise introduction to categories

## 6.1. Categories

**Definition 6.1.1.** A *category* $\mathscr{C}$ is a collection of objects $(X, Y, Z, ...)$, denoted $Ob(\mathscr{C})$, together with, for each pair $(X, Y)$ of objects of $\mathscr{C}$, a set of morphisms (alias maps) $f : X \longrightarrow Y$ , denoted $\mathscr{C}(X, Y)$, satisfying the following: For each object $X$ of $\mathscr{C}$ there is a given identity morphism $1_X : X \longrightarrow X$ and for each triple $(X, Y, Z)$ of objects of $\mathscr{C}$ and pair of morphisms $f : X \longrightarrow Y$ , $g : Y \longrightarrow Z$ there is given a morphism $g \circ f : X \longrightarrow Z$. This is viewed as a composition law

$$\circ : \mathscr{C}(Y, Z) \times \mathscr{C}(X, Y) \longrightarrow \mathscr{C}(X, Z).$$

We require $1_Y \circ f = f = f \circ 1_X$ and $h \circ (g \circ f) = (h \circ g) \circ f$ for any morphism $h$ with domain $Z$.

**Remark 6.1.2.** We do not require that $Ob(\mathscr{C})$ be a set; it may be a proper class. If it is a set, we say that the category is small.

**Example 6.1.3.** The collection of all sets is a category denoted $\mathscr{S}et$. Its morphisms are functions.

**Example 6.1.4.** The collection of all groups is a category denoted $\mathscr{G}rp$. Its morphisms are group homomorphisms.

**Example 6.1.5.** The collection of all topological spaces is a category denoted $\mathscr{T}op$. Its morphisms are continuous functions.

**Example 6.1.6.** A *monoid* is a set $\mathscr{M}$ with an associative binary operation and an identity element. Note that in a category $\mathscr{C}$ the composition law $\circ$ on the set $\mathscr{C}(X, X)$ is just such a binary operation with identity element $1_X$. Therefore a monoid is a category with one object. A category can be thought of as a "monoid with many objects".

In any category, there is a notion of isomorphism. It answers the sensible version of the question "when are two things the same". The nonsensical version would have the answer "when they are equal". The sensible version interprets "things" to mean objects of a category" and the sensible answer is that we think of two objects as essentially the same when they are isomorphic.

**Definition 6.1.7.** A morphism $f : X \longrightarrow Y$ in a category $\mathscr{C}$ is called an *isomorphism* if there is a morphism $g : Y \longrightarrow X$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$ .

If a morphism $f$ has a left inverse and a right inverse then it is an isomorphism and the left and right inverses coincide.

**Definition 6.1.8.** A *groupoid* is a category in which every morphism is an isomorphism. Just as a monoid can be defined to be a category with just one object, a *group* can be defined to be a groupoid with just one object. Similarly, a *groupoid* can be thought of as a "group with many objects".

## 6.2. Functors

A morphism of categories is called a functor.

**Definition 6.2.1.** Let $\mathscr{C}$, $\mathscr{D}$ be categories. A *functor* $F : \mathscr{C} \longrightarrow \mathscr{D}$ consists of a rule that assigns to each object $X$ of $\mathscr{C}$ an object $FX$ of $\mathscr{D}$, together with, for each pair $(X, Y)$ of objects of $\mathscr{C}$, a function

$$F : \mathscr{C}(X, Y) \longrightarrow \mathscr{D}(FX, FY),$$

written $f \mapsto Ff$, such that $F(1_X) = 1_{FX}$ and $F(g \circ f) = Fg \circ Ff$.

If $f$ is an isomorphism in $\mathscr{C}$, then $Ff$ is an isomorphism in $\mathscr{D}$.

**Example 6.2.2.** The collection of all small categories is a category denoted $\mathscr{C}at$. Its morphisms $F : \mathscr{C} \longrightarrow \mathscr{D}$ are the functors.

**Remark 6.2.3.** We insist that categories be small for the purposes of this definition to ensure that we have a well-defined set and not just a proper class of functors between any two categories.

**Example 6.2.4.** The *abelianization of a group* $G$ is the group $G/[G, G]$ where $[G, G]$ is the commutator subgroup, that is, the subgroup generated by the set $\{ghg^{-1}h^{-1} | g, h \in G\}$. Abelianization defines a functor $A : \mathscr{G}rp \longrightarrow \mathscr{A}b$ where $\mathscr{A}b$ is the category of abelian groups.

**Definition 6.2.5.** A functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is said to be *faithful* if the function

$$F : \mathscr{C}(X, Y) \longrightarrow \mathscr{D}(FX, FY)$$

is injective for every pair $(X, Y)$ of objects of $\mathscr{C}$.

**Definition 6.2.6.** A functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is said to be *full* if the function

$$F : \mathscr{C}(X, Y) \longrightarrow \mathscr{D}(FX, FY)$$

is surjective for every pair $(X, Y)$ of objects of $\mathscr{C}$.

**Definition 6.2.7.** A functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is said to be an *isomorphism of categories* if there is a functor $G : \mathscr{D} \longrightarrow \mathscr{C}$ such that $FG$ is the identity functor on $\mathscr{D}$ and $GF$ is the identity functor on $\mathscr{C}$.

**Definition 6.2.8.** A functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is said to be *essentially surjective* if, for every object $Y$ of $\mathscr{D}$, there is an object $X$ of $\mathscr{C}$ and an isomorphism $FX \cong Y$ .

**Definition 6.2.9.** A functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is said to be an *equivalence of categories* if it is full, faithful, and essentially surjective.

**Definition 6.2.10.** A *subcategory* of a category $\mathscr{C}$ is a category that consists of some of the objects and some of the morphisms of $\mathscr{C}$; it is a full subcategory if it contains all of the morphisms in $\mathscr{C}$ between any two of its objects. The *skeleton* of a category $\mathscr{C}$ is a a full subcategory which contains exactly one object from each isomorphism class of objects of $\mathscr{C}$.

**Proposition 6.2.11.** *The inclusion of a skeleton of $\mathscr{C}$ in $\mathscr{C}$ is an equivalence of categories.*

PROOF. We understand a skeleton to be a full subcategory, so the inclusion is full and faithful, and it is essentially surjective by definition. □

## 6.3. Natural Transformations

Naturally, there are also morphisms of functors. Let $F, F' : \mathscr{C} \longrightarrow \mathscr{D}$ be functors. A natural transformation $\eta : F \longrightarrow F'$ is a collection of maps $\eta_X : FX \longrightarrow F'X$, one for each object $X$ of $\mathscr{C}$, such that the following diagram commutes for each map $f : X \longrightarrow Y$ in $\mathscr{C}$:

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_Y} \\ F'X & \xrightarrow{F'f} & F'Y. \end{array}$$

**Definition 6.3.1.** A natural transformation $\eta$ is said to be a *natural isomorphism* if each of the maps $\eta_X$ is an isomorphism.

**Example 6.3.2.** A finite dimensional vector space $V$ over $K$ is naturally isomorphic to its double dual $DDV$, where $DV = Hom(V, K)$. That is, there is a natural isomorphism $\mathrm{Id} \longrightarrow DD$ on the category of finite dimensional vector spaces over $K$.

**Definition 6.3.3.** A functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is said to be an *equivalence of categories* if there is a functor $G : \mathscr{D} \longrightarrow \mathscr{C}$ and there are natural isomorphisms $FG \longrightarrow Id_{\mathscr{D}}$ and $GF \longrightarrow Id_{\mathscr{C}}$. Note that an isomorphism of categories is an equivalence, but not conversely.

The senior author has a liking for theorems of the following form.

**Proposition 6.3.4.** *An equivalence of categories is an equivalence of categories.*

That is, the two definitions of what it means for a functor to be an equivalence of categories are equivalent. It is easy to show that if $F$ is an equivalence of categories in our second sense, then $F$ is certainly full, faithful, and essentially surjective. The converse requires a little work and a use of the axiom of choice that the fastidious set-theoretically minded reader may find distasteful: the first step is to choose an object $G(D)$ in $\mathscr{C}$ such that $FG(D)$ is isomorphic to $D$ for each object $D$ of $\mathscr{D}$. The second is to choose an isomorphism $\eta : FG(D) \longrightarrow D$ for each $D$. We then define $G$ on morphisms so as to make $\eta$ a natural isomorphism by definition, using that

$$F : \mathscr{C}(G(D), G(D')) \longrightarrow \mathscr{D}(FG(D), FG(D'))$$

is a bijection. For a morphism $g : D \longrightarrow D'$ in $\mathscr{D}$, we define $Gg : G(D) \longrightarrow G(D')$ to be $F^{-1}$ of the composite

$$FG(D) \xrightarrow{\eta} D \xrightarrow{f} D' \xrightarrow{\eta^{-1}} FG(D').$$

The reader can see how composition must be defined in order to complete the proof.

Note that the proof of Proposition 6.2.11 is easy using our first definition of an equivalence of categories, but not so easy using the second. Proposition 6.3.4 has real force: it makes it easy to recognize equivalences of categories (in the second sense) when we see them.

## 6.4. The Fundamental Groupoid of a Space

We illustrate the idea of translating topology into algebra by explaining the fundamental groupoid. This brief section will leave the drawing of relevant diagrams to the reader.

We construct a functor $\Pi : \mathscr{T}op \longrightarrow \mathscr{G}pd$, where $\mathscr{G}pd$ is the full subcategory of $\mathscr{C}at$ whose objects are groupoids. For a topological space $X$, the objects of the category $\Pi X$ are the points of the space $X$. Let $I = [0,1]$ be the unit interval. A path $p : x \longrightarrow y$ is a continuous map $p : I \longrightarrow X$ such that $p(0) = x$ and $p(1) = y$. Two paths $p$ and $p'$ from $s$ to $y$ are said to be equivalent if there is a map $h : I \times I \longrightarrow X$ such that, for all $t \in I$,

$$h(t,0) = x, \quad h(t,1) = y, \quad h(0,t) = p(t), \quad \text{and } h(1,t) = p'(t).$$

$h$ is said to be a homotopy from $p$ to $p'$ through paths from $x$ to $y$. The set of morphisms $x \longrightarrow y$ in $\prod X$ is the set of equivalence classes of paths $x \longrightarrow y$. For a path $q : y \longrightarrow z$, the composite $q \circ p$ is defined by

$$(q \circ p)(t) = \begin{cases} p(2t) & \text{if } 0 \le t \le 1/2 \\ q(2t-1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

Define $\mathrm{id}_x$ to be the constant path at $x$, $\mathrm{id}_x(t) = x$. Define $p^{-1}(t) = p(1t)$. Composition is not associative or unital, but it becomes so after passage to equivalence classes. Verifications that we leave to the reader (or the first chapter of [48]) show that $\Phi X$ is a well-defined groupoid. For a map $f : X \longrightarrow Y$, we define $\Pi f$ on objects by sending $x$ to $f(x)$ and on morphisms by sending the equivalence class $[p]$ to the equivalence class $[f \circ p]$. Then $\Pi$ is a well-defined functor.

If we fix basepoints, we get a functor that is perhaps more familiar. The fundamental group of $X$ at the basepoint $x$ is the group $\pi_1(X,x)$ given by the morphisms $x \longrightarrow x$ in the groupoid $\Pi X$. If we define $\mathscr{T}op_*$ to be the category of spaces $X$ with a chosen basepoint $x$ and maps $f : X \longrightarrow Y$ that preserve basepoints, $f(x) = y$, then $\pi_1$ gives a functor from based spaces to groups, called the fundamental group functor. Its construction is the first step towards algebraic topology.

By definition, $\pi_1(X,x)$, regarded as a category with a single object $x$, is a full subcategory of $\prod X$. Show that if $X$ is path connected, then $\pi_1(X,x)$ is a skeleton of $\Pi X$. Thus the essential information in $\Phi X$ is captured by the fundamental group.

CHAPTER 7

# Group actions and finite groups

We shall explain some of the results and questions in a beautiful 1978 paper [**57**] by Daniel Quillen. He relates properties of groups to homotopy properties of the simplicial complexes of certain posets constructed from the group. He does not explicitly think of these posets as finite topological spaces. He seems to have been unaware of the earlier papers of McCord [**50**] and Stong [**65**] that we have studied, and it is interesting to look at his work from their perspective. Stong himself first looked at Quillen's work this way [**66**], and we will include his results on the topic. We usually work with a finite group $G$, but the basic definitions apply more generally.

## 7.1. Equivariance and finite spaces

We begin with some general observations about equivariance and $F$-spaces, largely following Stong [**66**].

A topological group $G$ is a group and a space whose product $G \times G \longrightarrow G$ and inverse map $G \longrightarrow G$ are continuous. An action of $G$ on a topological space $X$ is a continuous map $G \times X \longrightarrow X$, written $(g, x) \mapsto gx$, such that $g(hx) = (gh)x$ and $ex = x$, where $e$ is the identity element of $G$. A map $f\colon X \longrightarrow Y$ of $G$-spaces is a continuous map $f$ such that $f(gx) = gf(x)$ for $g \in G$ and $x \in X$.

For a space $X$, the automorphism group $\operatorname{Aut} X$ is the topological group of homeomorphisms $X \longrightarrow X$. The group operation is composition, and $\operatorname{Aut} X$ is topologized as a subspace of the space of maps $X \longrightarrow X$ with the compact open topology. Suppose a topological group $G$ acts on $X$. Then the action of $g$ on $X$ gives a homeomorphism $g\colon X \longrightarrow X$. This gives a group homomorphism $G \longrightarrow \operatorname{Aut} X$. At least if $X$ is first countable, this map is also continuous. That is, it is a map of topological groups.

We say that $G$ acts trivially on $X$ if $gx = x$ for all $g$ and $x$. We let $G$ act diagonally on products $X \times Y$, $g(x, y) = (gx, gy)$. In particular, with $G$ acting trivially on $I$, we have the notion of a $G$-homotopy, namely a $G$-map $h\colon X \times I \longrightarrow Y$. There is a large subject of equivariant algebraic topology, in which one studies the algebraic invariants of $G$-spaces.

We begin with some basic ideas of equivalence in this context. We say that a $G$-map $f\colon X \longrightarrow Y$ is a $G$-homotopy equivalence if there is a $G$-map $f'\colon Y \longrightarrow X$ and there are $G$-homotopies $f \circ f' \simeq \mathrm{id}$ and $f' \circ f \simeq \mathrm{id}$. For a subgroup $H$ of $G$, define the $H$-fixed point space $X^H$ of $X$ to be $\{x | hx = x \ \text{ for } \ h \in H\}$. Say that a $G$-map $f$ is an $H$-equivalence if $f^H\colon X^H \longrightarrow Y^H$ is a nonequivariant homotopy equivalence. For nice $G$-spaces, the sort one usually encounters in classical algebraic topology, which are called $G$-CW complexes, a map $f$ is a $G$-homotopy equivalence if and only if it is an $H$-equivalence for all subgroups $H$. Note that we have the much weaker notion of an $e$-equivalence, namely a $G$-map which is a homotopy equivalence of underlying spaces, forgetting the action of $G$.

We also have weak notions. A $G$-map $f$ is a weak $G$-homotopy equivalence if each $f^H\colon X^H \longrightarrow Y^H$ is a weak homotopy equivalence in the nonequivariant sense. We also have the notion of a weak $e$-equivalence, meaning a $G$-map that is a weak homotopy equivalence of underlying spaces, forgetting the action of $G$.

In general, the notions of $G$-equivalence are very much stronger than the notions of $e$-equivalence. There are lots of $G$ maps that are $e$-equivalences but are not $G$-equivalences. We show that cannot happen when $G$ acts on a finite space. We start with some general observations.

**Lemma 7.1.1.** *If an $A$-space $G$ is a topological group, then it is discrete.*

Proof. If $h \leq g$, then, by the continuity of the inverse map, $h^{-1} \leq g^{-1}$. By the continuity of left multiplication by $h$, $e \leq hg^{-1}$, and then, by the continuity of right multiplication by $g$, $g \leq h$. Since $G$ is $T_0$, $g = h$. Thus $U_g = \{g\} is open for all g and therefore every subset is open.$ □

We have observed that if a topological group $G$ acts on a space $X$, then we can view the action as given by a map of topological groups $G \longrightarrow \operatorname{Aut} X$. This homomorphism has a kernel $K$, and the action factors through the quotient group $G/K$, which is a topological group with the quotient topology. When $X$ is an $F$-space, $\operatorname{Aut} X$ is finite since there are only finitely many functions $X \longrightarrow X$. But then $G/K$ is finite and therefore discrete. Thus we lose no generality if we restrict our attention to finite discrete groups $G$ acting on $F$-spaces. Therefore $G$ will be finite from now on.

Recall the notion of upbeat and downbeat points in an $F$-space $X$. Note that if $x$ is upbeat, so that there is a $y > x$ such that $z > x$ implies $z \geq y$, then $y$ is uniquely determined by $x$.

**Theorem 7.1.2.** *Let $X$ be an $F$-space with an action by a group $G$. Then there is a core $C \subset X$ such that $C$ is a sub $G$-space and equivariant deformation retract of $X$. We call $C$ an equivariant core of $X$.*

Proof. The orbit $Gx$ of an element $x$ is $\{gx | g \in G\}$. If $x$ is upbeat, then $gx$ is also upbeat, with $gy$ playing the role of $y$. The inclusion $X - Gx \subset X$ is the inclusion of a sub $G$-space. Define $f\colon X \longrightarrow X - Gx \subset X$ by $f(z) = z$ if $z \notin Gx$ and $f(gx) = gy$, where $y > x$ is such that $z > x$ implies $z \geq y$. Clearly $f \geq \mathrm{id}$ and thus $f \simeq \mathrm{id}$. An explicit homotopy used to show this is given by $h(z,t) = z$ if $t < 1$ and $h(z,1) = f(z)$, and this homotopy is a $G$-map. Removing upbeat and downbeat orbits successively until none are left, we reach an equivariant core. □

**Corollary 7.1.3.** *If $X$ is a contractible $F$-space with an action by a group $G$, then $X$ is equivariantly contractible to a $G$-fixed point.*

Proof. A core of $X$ is a point, so an equivariant core must be a point with the trivial action by $G$. □

**Corollary 7.1.4.** *If $X$ is a contractible $F$-space, then $X$ has a point that is fixed by every homeomorphism of $X$.*

Proof. The finite group $G$ of homeomorphisms of $X$ acts on $X$, and an equivariant core is a fixed point. □

**Theorem 7.1.5.** *Let $X$ and $Y$ be $F$-spaces with actions by $G$ and $f\colon X \longrightarrow Y$ be a $G$-map. If $f$ is an $e$-homotopy equivalence, then $f$ is a $G$-homotopy equivalence.*

Proof. Let $C$ and $D$ be equivariant cores of $X$ and $Y$. Let $i_X\colon C \longrightarrow X$ and $r_X\colon X \longrightarrow C$ be the inclusion and retraction, and similarly for $Y$. Let $p$ be the composite

$$C \xrightarrow{i_X} X \xrightarrow{f} Y \xrightarrow{r_Y} D, \quad p = r_Y \circ f \circ i_X.$$

Then $p$ is a $G$-map and a homotopy equivalence between minimal finite spaces. The latter property implies that $p$ is a homeomorphism, and $p^{-1}$ is necessarily also a $G$-map. Define $g\colon Y \longrightarrow X$ to be the composite

$$Y \xrightarrow{r_Y} D \xrightarrow{p^{-1}} C \xrightarrow{i_X} X, \quad g = i_X \circ p^{-1} \circ r_Y.$$

Then $g \circ f$ and $f \circ g$ are equivariantly homotopic to the respective identity maps. Indeed, we have the homotopies

$$gf = gf\, id_X \simeq gf\, i_X r_X = i_X p^{-1} r_Y f i_X r_X = i_X p^{-1} p r_X = i_X r_X \simeq id_X$$

and

$$fg = id_Y fg \simeq i_Y r_Y fg = i_Y r_Y f i_X p^{-1} r_Y = i_Y p p^{-1} r_Y = i_Y r_Y \simeq id_Y. \qquad \square$$

## 7.2. The basic posets and Quillen's conjecture

Fix a finite group $G$ and a prime $p$. We define two posets.

Compare with Barmak's book. Anything interesting further in there?

**Definition 7.2.1.** Let $\mathscr{S}_p(G)$ be the poset of non-trivial $p$-subgroups of $G$, ordered by inclusion. An abelian $p$-group is *elementary abelian* if every element has order 1 or $p$. This means that it is a vector space over the field of $p$ elements. Define $\mathscr{A}_p(G)$ to be the poset of non-trivial elementary abelian $p$-subgroups of $G$, ordered by inclusion and let $i\colon \mathscr{A}_p(G) \longrightarrow \mathscr{S}_p(G)$ be the inclusion.

**Remark 7.2.2.** Quillen calls a non-trivial elementary abelian $p$-group a *$p$-torus*, and he defines its rank to be its dimension as a vector space.

The reason these posets are interesting is that $G$ acts on them in such a way that their topological properties relate nicely to algebraic properties of $G$. The action of $G$ is by conjugation. If $H$ is a subgroup of $G$ and $g \in G$, write $H^g = gHg^{-1}$. The function $f_g$ that sends $P$ to $P^g$ gives an automorphism of the posets $\mathscr{A}_p(G)$ and $\mathscr{S}_p(G)$. Clearly $f_e = \mathrm{id}$, where $e$ is the identity element of $G$, and $f_{g'g} = f_{g'} \circ f_g$. These automorphisms are what give these posets their interest: the poset together with its group action describe how the different $p$-subgroups are related under subconjugation in $G$.

In particular, a point $P$ in $\mathscr{A}_p(G)$ is fixed under the action of $G$ if and only if $P^g = P$ for all $g \in G$, and this means that $P$ is a normal subgroup of $G$. Thus the poset $(\mathscr{A}_p(G))^G$ of fixed points is the poset of normal $p$-tori of $G$. We can therefore relate algebraic questions about the presence of normal subgroups to topological questions about the existence of fixed points. Of course, we may regard these posets as $F$-spaces with $G$ actions, and the theory of the previous section applies.

**Remark 7.2.3.** Some of Quillen's language for studying these posets is similar to the language we have been using, but it can be quite confusing. For example, he says that a subset $S$ of a poset $X$ is *closed* if $x \in S$ and $y \leq x$ implies $y \in S$. In our language, this means that $x \in S$ implies $U_x \subset S$, which says that $S$ is *open*.

The posets $\mathscr{S}_p(G)$ and $\mathscr{A}_p(G)$ are both empty if $p$ does not divide the order of $G$. At first sight, it might seem that $\mathscr{S}_p(G)$ is a lot more interesting and complicated than $\mathscr{A}_p(G)$, but that is not the case. To understand the discussion to follow, it is helpful to keep the following commutative diagram of spaces in mind, remembering that its vertical arrows are weak homotopy equivalences.

$$\begin{array}{ccc} |\mathscr{K}\mathscr{A}_p(G)| & \xrightarrow{|\mathscr{K}(i)|} & |\mathscr{K}\mathscr{S}_p(G)| \\ \psi \downarrow & & \downarrow \psi \\ \mathscr{A}_p(G) & \xrightarrow[i]{} & \mathscr{S}_p(G) \end{array}$$

We first consider $p$-groups.

**Proposition 7.2.4.** *If $P$ is a non-trivial $p$-group, then $\mathscr{A}_p(P)$ and $\mathscr{S}_p(P)$ are both contractible.*

PROOF. There is a central subgroup $B$ of $P$ of order $p$. We will be accepting as known some basic facts in the theory of finite groups, such as this one. But the proof is just an easy counting argument. We think of $P$ as a $P$-set, with $P$ acting on itself by conjugation. As is true for any finite $P$-set, $P$ is isomorphic to a disjoint union of orbits, each isomorphic to some orbit $P/Q$. Unless the orbit consists of a single point, its number of elements is divisible by $p$, and the total number of elements is the order of $P$. Since the identity element is an orbit with a single point, there must be at least $p-1$ other orbits with a single point, and such a point is a non-identity element in the center of $P$.

For any subgroup $A$ of $P$, we have $A \subset AB \supset B$ where $AB$ is the subgroup of $P$ generated by $A$ and $B$. If $A$ is a $p$-torus, then so is $AB$ since $B$ is central. Define three maps $\mathscr{A}_p(P) \longrightarrow \mathscr{A}_p(P)$: the identity map id, the map $f$ that sends $A$ to $AB$, and the constant map $c_B$ that sends $A$ to $B$. These are all continuous, and our inclusions say that $\mathrm{id} \leq f \geq c_B$. This implies that $\mathrm{id} \simeq f \simeq c_B$. Since the identity is homotopic to the constant map, $\mathscr{A}_p(G)$ is contractible. The proof for $\mathscr{S}_p(G)$ is the same. □

Quillen calls a poset $X$ *conically contractible* if there is an $x_0 \in X$ and a map of posets $f\colon X \longrightarrow X$ such that $x \leq f(x) \geq x_0$ for all $x$. He was thinking in terms of associated simplicial complexes, but we are thinking in terms of $F$-spaces. The previous proof says that the $F$-spaces $\mathscr{A}_p(P)$ and $\mathscr{S}_p(P)$ are conically contractible. It is to be emphasized that conically contractible finite spaces are genuinely and not just weakly contractible. As we shall see, the difference is profound in the case at hand. In contrast with the previous result, we emphasize the word "weak" in the following result.

**Theorem 7.2.5.** *The inclusion $i\colon \mathscr{A}_p(G) \longrightarrow \mathscr{S}_p(G)$ is a weak homotopy equivalence. Therefore the induced map $|\mathscr{K} i|\colon |\mathscr{K}\mathscr{A}_p(G)| \longrightarrow |\mathscr{K}\mathscr{S}_p(G)|$ is a weak homotopy equivalence and hence an actual homotopy equivalence.*

PROOF. We have the open cover of $\mathscr{S}_p(G)$ given by the $U_P$, where $P$ is a non-trivial finite $p$-group. Clearly $i^{-1}U_P$ is the poset of $p$-tori of $G$ that are contained in $P$, and this is the contractible space $\mathscr{A}_p(P)$. Our general theorem that weak homotopy equivalence is a local notion applies. □

**Definition 7.2.6.** Define the $p$-rank of $G$, denoted $r_p(G)$, to be the maximal rank of a $p$-torus in $G$. Observe that this is one greater than the dimension of the simplicial complex $\mathscr{K}\mathscr{A}_p(G)$. (We interpret the dimension of the empty complex to be $-1$).

**Example 7.2.7.** If the $p$-Sylow subgroups of $G$ are cyclic of order $p$ and there are $q$ of them, then $\mathscr{A}_p(G)$ is a discrete space with $q$ points. For example, this holds for some $q$ if $G$ is the symmetric group on $n$ letters, where $p$ is a prime and $p \leq n < 2p$.

**Remark 7.2.8.** Sylow's third theorem is relevant. The number of Sylow $p$-subgroups of $G$ is congruent to 1 mod $p$ and divides the order of $G$.

**Theorem 7.2.9.** *The following statements are equivalent.*

*(i) $G$ has a non-trivial normal $p$-subgroup.*
*(ii) $G$ has a non-trivial normal elementary abelian subgroup.*
*(iii) $\mathscr{S}_p(G)$ is contractible.*

*Moreover, they are implied by the statement*

*(iv) $\mathscr{A}_p(G)$ is contractible.*

Proof. Obviously (ii) implies (i). Conversely, as a matter of algebra, (i) implies (ii). To see that, let $P$ be a non-trivial normal $p$-subgroup of $G$ and let $C$ be its center. For $g \in G$, $c \in C$, and $p \in P$,

$$gcg^{-1}p = gcg^{-1}pgg^{-1} = gg^{-1}pgcg^{-1} = pgcg^{-1}$$

since $g^{-1}pg$ is in $P$ and therefore commutes with $c$. This shows that any conjugate of an element of $C$ commutes with any element of $P$ and is therefore in $C$, showing that $C$ is normal in $G$. Now let $B$ be the set of elements $b \in C$ such that $b^p = e$. Any conjugate of an element of $B$ is in $C$ and has $p$th power $e$, hence is in $B$. Therefore $B$ is a non-trivial normal elementary abelian subgroup of $G$.

To see that (i) implies (iii), let $P$ be a non-trivial normal $p$-subgroup of $G$. For any nontrivial $p$-subgroup $Q$ of $G$, $Q \subset QP \supset P$, where $QP$ denotes the subgroup of $G$ generated by $P$ and $Q$. Since $P$ is normal in $G$, $QP = \{qp | q \in Q \text{ and } p \in P\}$. This implies that $\mathrm{id} \leq f \geq c_P$, where $f(Q) = QP$ and $c_P(Q) = P$, hence $\mathscr{S}_p(G)$ is conically contractible, hence contractible. The same argument does not apply to show that (ii) implies (iv) since $QP$ need not be abelian when $Q$ and $P$ are abelian.

Conversely, to see that (iii) implies (i) and (iv) implies (ii), we use Corollary 7.1.3, which states that contractibility implies $G$-contractibility to a fixed point. A fixed point of $\mathscr{S}_p(G)$ is a normal $p$-subgroup and a fixed point of $\mathscr{A}_p(G)$ is a normal elementary abelian $p$-subgroup. □

The inclusion $i\colon \mathscr{A}_p(G) \longrightarrow \mathscr{S}_p(G)$ is not generally a homotopy equivalence. To see this, we use the following observation.

**Lemma 7.2.10.** *Let $\mathscr{Q}_p(G) \subset \mathscr{S}_p(G)$ be the subposet of nontrivial intersections of Sylow $p$-subgroups. Then $\mathscr{Q}_p(G)$ is a $G$-equivariant deformation retract of $\mathscr{S}_p(G)$.*

Proof. For $P \in \mathscr{S}_p(G)$, let $f(P)$ be the intersection of the Sylow $p$-subgroups that contain $P$. Then $f\colon \mathscr{S}_p(G) \longrightarrow \mathscr{Q}_p(G)$ is continuous and $G$-equivariant. Moreover, $f(P) = P$ if $P$ is itself a $p$-Sylow subgroup. Let $j\colon \mathscr{Q}_p(G) \longrightarrow \mathscr{S}_p(G)$ be the inclusion. Then $fj = \mathrm{id}$. Since $P \leq f(P)$, $\mathrm{id} \simeq jf$ via an equivariant homotopy. □

**Example 7.2.11.** Let $G = \Sigma_5$ be the symmetric group on five letters. Then $\mathscr{A}_2(G)$ and $\mathscr{S}_2(G)$ are not homotopy equivalent. There are 6 conjugacy classes of 2-subgroups of $G$, as follows.

(i) Dihedral groups $D_4$ of order 8, the Sylow 2-subgroups.
(ii) Cyclic groups $C_4$ of order 4.
(iii) Elementary 2-groups $C_2 \times C_2$ generated by transpositions $(ab)$ and $(cd)$.
(iv) Elementary 2-groups $C_2 \times C_2$ generated by products of disjoint transpositions $(ab)(cd)$, $(ac)(bd)$, whose product in either order is $(ad)(bc)$.
(v) Cyclic groups $C_2$ generated by a transposition.
(vi) Cyclic groups $C_2$ generated by a product of two disjoint transpositions.

Of course, each $C_2 \times C_2$ contains three $C_2$'s. Each $C_2$ of type (v) is contained in three $C_2 \times C_2$'s of type (iii) and each $C_2$ of type (vi) is contained in one $C_2 \times C_2$ of type (iii) and one $C_2 \times C_2$ of type (iv). This information shows that $\mathscr{A}_2(G)$ is minimal, hence not homotopy equivalent to any space with fewer points. The intersections of Sylow 2-subgroups of $G$ are the dihedral groups in (i), the groups $C_2 \times C_2$ of type (iv) and the subgroups $C_2$ of type (v). In fact, $\mathscr{Q}_2(G)$ is a core of $\mathscr{S}_2(G)$). Counting, one sees that there are fewer points in $\mathscr{Q}_2(G)$ than there are in the minimal $F$-space $\mathscr{A}_2(G)$, so these two $F$-spaces cannot be homotopy equivalent.

Quillen conjectured the following stronger version of the implication (iii) implies (i) of Theorem 7.2.9, and he proved the conjecture for solvable groups.

**Conjecture 7.2.12** (Quillen)**.** *If $\mathscr{A}_p(G)$ or equivalently $\mathscr{S}_p(G)$ is weakly contractible, then $G$ contains a non-trivial normal $p$-subgroup.*

The hypothesis holds if and only if $|\mathscr{K}\mathscr{A}_p(G)|$ or equivalently $|\mathscr{K}\mathscr{S}_p(G)|$ is weakly contractible and therefore contractible. We have seen that if $G$ has a non-trivial normal $p$-subgroup, then $\mathscr{A}_p(G)$ is contractible and therefore weakly contractible. Quillen's conjecture is that, conversely, if $\mathscr{A}_p(G)$ is weakly contractible, then it is contractible and thus $G$ has a non-trivial normal $p$-subgroup. In this form, we see that the conjecture can be thought of as a problem in the equivariant homotopy theory of $F$-spaces.

In particular, if $G$ is simple and not isomorphic to $C_p$, then it has no non-trivial normal subgroups and the conjecture implies that $\mathscr{A}_p(G)$ cannot be weakly contractible. This consequence of the conjecture has been verified for many but not all finite simple groups, using the classification theorem and proving that the space $\mathscr{A}_p(G)$ has non-trivial homology. A conceptual proof would be a wonderful achievement!

## 7.3. Some exploration of the posets $\mathscr{A}_p(G)$

As an illustration of the translation of algebra to topology, we show how to compute $\mathscr{A}_p(G \times H)$ in terms of joins for finite groups $G$ and $H$. We then see how the computation appears in Quillen's analysis of the poset $\mathscr{A}_p(\Sigma_{2p})$.

**Proposition 7.3.1.** *The poset $\mathscr{A}_p(G \times H)$ is homotopy equivalent to the poset $C^-\mathscr{A}_p(G) \times C^-\mathscr{A}_p(H) - \{(c_G, c_H)\}$.*

Proof. Let $T$ be the subposet of $\mathscr{A}_p(G \times H)$ whose points are the $p$-tori in $G = G \times e$, the $p$-tori in $H = e \times H$, and the products $A \times B$ of $p$-tori $A$ in $G$ and $B$ in $H$. (Remember that $p$-tori are non-trivial elementary abelian $p$-groups). Visibly, thinking of trivial groups as conepoints and therefore $<$ non-trivial subgroups, $T$

is isomorphic to $C^{-}\mathscr{A}_p(G) \times C^{-}\mathscr{A}_p(H) - \{(c_G, c_H)\}$. Let $i\colon T \longrightarrow \mathscr{A}_p(G \times H)$ be the inclusion. The projections $\pi_1\colon G \times H \longrightarrow G$ and $\pi_2\colon G \times H \longrightarrow H$ induce a map $r\colon \mathscr{A}_p(G \times H) \longrightarrow T$ such that $r \circ i = \mathrm{id}$. Explicitly, for $C \in \mathscr{A}_p(G \times H)$, $r(C) = \pi_1(C) \times \pi_2(C)$. Then $i(r(C)) \supset C$, which means that $i \circ r \geq \mathrm{id}$ and thus $i \circ r \simeq \mathrm{id}$. □

In view of Proposition 5.7.9, this has the following immediate consequence.

**Corollary 7.3.2.** *The space $|\mathscr{K}(\mathscr{A}_p(G \times H))|$ is homotopy equivalent to the space $|\mathscr{K}(\mathscr{A}_p(G))| * |\mathscr{K}(\mathscr{A}_p(H))|$.*

**Proposition 7.3.3.** *Quillen's conjecture holds if $r_p(G) \leq 2$.*

Proof. The hypothesis cannot hold if $r_p(G) = 0$, since $\mathscr{A}_p(G)$ is then empty and hence not weakly contractible. If $r_p(G) = 1$, then the space $\mathscr{A}_p(G)$ is discrete since there are no proper inclusions. It is weakly contractible if and only if it consists of a single point, and then its single point must be fixed by the action of $G$. This means that there is a unique $p$-torus in $G$, and it is a normal subgroup of order $p$. If $r_p(G) = 2$, then $|\mathscr{K}(\mathscr{A}_p(G))|$ is one dimensional and contractible, which means that it is a tree. According to Quillen, "one knows (Serre) that a finite group acting on a tree always has a fixed point". This means that $G$ has a normal $p$-torus. The trees here are of a particularly elementary sort, but the conclusion is still not altogether obvious. The following problem gives a way of thinking about it. □

**Problem 7.3.4.** *Consider an $F$-space $X$ such that $|\mathscr{K}(X)|$ is a tree (a contractible graph). Clearly $X$ is weakly contractible. Prove that $X$ is contractible. (Search for upbeat or downbeat points). It follows that if a finite group $G$ acts on $X$, then $X$ is $G$-contractible and therefore has a $G$-fixed point.*

Much of Quillen's paper is devoted to proving that the conjecture holds for *solvable* groups $G$. This means that there is a decreasing chain of subgroups of $G$, each normal in the next, such that the subquotients are cyclic of prime order. We shall not repeat the proof.

However, following Quillen, we shall work out the structure of $\mathscr{A}_p(G)$ when $G = \Sigma_{2p}$ is the symmetric group on $2p$ letters for an odd prime $p$. This is a first interesting case since $\mathscr{A}_p(\Sigma_n)$ is empty if $n < p$ and is a discrete space with one element for each cyclic subgroup of order $p$ if $p \leq n < 2p$. (In fact, there are $n!/(n-p)!p(p-1)$ such subgroups.) The analysis shows just how non-trivial the posets $\mathscr{A}_p(G)$ are.

Let $g \in G = \Sigma_{2p}$ have order $p$. The group $\langle g \rangle$ it generates has order $p$, and its action on the set $S = \{1, \cdots, 2p\}$ partitions $S$ into two disjoint subsets, one given by the orbit generated by an element $s$ such that $gs \neq s$ and the other given by its complement, on which $\langle g \rangle$ acts either freely or trivially. If $A \cong \mathbb{Z}/p \times \mathbb{Z}/p$ is a maximal elementary abelian $p$-subgroup of $G$ with generators $g$ and $g'$, then since $g$ and $g'$ commute we can see that they give the same partition of $S$, so that each such $A$ gives a unique partition of the set $S$ into two $A$-invariant subsets, each with $p$ elements. The set of such partitions of $S$ into two subsets with $p$ elements gives a corresponding decomposition of $\mathscr{A}_p(G)$ into disjoint subposets, each consisting of those $A$ which partition $S$ in the prescribed way.

Under the action of $G$, these partitions are permuted transitively, meaning that, given two partitions, there is an element of $G$ that permutes one into the other. Consider for definiteness the partition into the first $p$ and last $p$ elements of $S$. Let

$H$ be the subgroup of those elements of $G$ that fix this partition. The corresponding subposet of $\mathscr{A}_p(G)$ is $\mathscr{A}_p(H)$. Here $H$ is the *wreath product* $\Sigma_2 \int \Sigma_p$, which is the semi-direct product of $\Sigma_2$ with $\Sigma_p \times \Sigma_p$ determined by the permutation action of $\Sigma_2$ on $\Sigma_p \times \Sigma_p$.

Since $p$ is odd, $\mathscr{A}_p(H) = \mathscr{A}_p(\Sigma_p \times \Sigma_p)$, which, after passage to realizations of simplicial complexes, is the join $\mathscr{A}_p(\Sigma_p) * \mathscr{A}_p(\Sigma_p)$. Since $\Sigma_p$ has $(p-2)!$ Sylow subgroups, each of order $p$, $\mathscr{A}_p(\Sigma_p)$ is the disjoint union of $(p-2)!$ points. After counting the number of partitions and inspecting the join of our two discrete spaces $\mathscr{A}_p(\Sigma_p)$, Quillen informs us, and we can work out for ourselves, that $|\mathscr{A}_p(\Sigma_{2p})|$ is a disconnected graph with $(2p)!/2(p!)^2$ components, each of which is homotopy equivalent to a one-point union of $((p-2)!-1)^2$ circles. For example, for $p = 5$, there are 25 circles. The same analysis applies to the alternating groups $A_n$ for $n \leq 2p$ since $\mathscr{A}_p(A_n) = \mathscr{A}_p(\Sigma_n)$. Of course, these $\mathscr{A}_p(G)$ are not weakly contractible.

## 7.4. The components of $\mathscr{S}_p(G)$

Let $p$ be a prime which divides the order of $G$. We describe the set of components $\pi_0(\mathscr{S}_p(G))$, which of course is the same as $\pi_0(\mathscr{A}_p(G))$. Recall that two elements of a poset are in the same component if they can be connected by a chain of elements, each either $\leq$ or $\geq$ the next. In the poset $\pi_0(\mathscr{S}_p(G))$, each element is a $p$-group and is contained in a Sylow subgroup. Therefore there is at least one Sylow subgroup in each component. Since any one Sylow subgroup $P$ generates all the others by conjugation by elements of $G$, $G$ acts transitively on $\pi_0(\mathscr{S}_p(G))$, in the sense that there is a single orbit. If $N = N_P$ denotes the subgroup of $G$ that fixes the component $[P]$ of $P$, then $G/N$ is isomorphic to the $G$-set $\pi_0(\mathscr{S}_p(G))$ via $gN \mapsto [P^g]$. We want to determine the subgroup $N$. Let $\mathrm{Syl}_p(G)$ denote the set of $p$-Sylow subgroups of $G$ and let $N_G H$ denote the normalizer in $G$ of a subgroup $H$. Recall that $H^g = gHg^{-1}$.

**Proposition 7.4.1.** *The following conditions on a subgroup $M$ of $G$ are equivalent.*

*(i) For some $P \in Syl_p(G)$, $M \supset N_P$.*
*(ii) For some $P \in Syl_p(G)$, $M \supset N_G H$ for all $H \in \mathscr{S}_p(P)$.*
*(iii) For some $P \in Syl_p(G)$, $M \supset N_G P$ and $K \subset M$ whenever $K$ is a $p$-subgroup of $G$ that intersects $M$ non-trivially.*
*(iv) $p$ divides the order of $M$ and $M \cap M^g$ is of order prime to $p$ for all $g \notin M$.*

*Moreover, $\mathscr{S}_p(G)$ is connected if and only if there is no proper subgroup $M$ which satisfies these equivalent conditions.*

PROOF. The last statement holds since $G$ is connected if and only if $G = N_P$ for all $P \in \mathrm{Syl}_p(G)$, in which case no proper subgroup can satisfy the stated conditions.
(i) $\Longrightarrow$ (ii): If $g \in N_G H$ with $H \subset P$, then $H^g = H$ is contained in both $P$ and $P^g$, so that $[P] = [P^g] = g[P]$. This means that $g \in N_P \subset M$.
(ii) $\Longrightarrow$ (iii): Obviously $M \supset N_G P$. Since $P$ is a $p$-Sylow subgroup of $G$, it is also a $p$-Sylow subgroup of $M$. Thus if $H$ is a non-trivial $p$-subgroup of $M$, then $H$ is conjugate in $M$ to a subgroup, $H^m$ say, of $P$. Since $M \supset N_G(H^m)$ and $(N_G H)^m = N_G(H^m)$, $M \supset N_G H$. Let $K$ be a $p$-subgroup of $G$ such that $K \cap M$ is non-trivial. We have

$$K \cap M \subset N_K(K \cap M) = K \cap N_G(K \cap M) \subset K \cap M.$$

Since $K$ is a $p$-group, the first inclusion is proper if $K \cap M$ is a proper subgroup of $K$. Since this is a contradiction, we must have $K \cap M = K$ and $K \subset M$.
(iii) $\Longrightarrow$ (iv): Since $M \supset P$, $p$ divides the order of $M$. Assume that $p$ divides the order of $M \cap M^g$ for some $g \in G$. Then there is a non-trivial $p$-subgroup $H \subset M \cap M^g$. Let $H \subset Q$ for $Q \in \mathrm{Syl}_p(G)$. Since $Q \cap M$ is non-trivial, we have $Q \subset M$. Since $H^{g^{-1}} \subset Q^{g^{-1}}$ and $H^{g^{-1}} \subset M$, we also have $Q^{g^{-1}} \subset M$. Since $P$, $Q$, and $Q^{g^{-1}}$ are $p$-Sylow subgroups of $M$, they are conjugate in $M$, say $Q^m = P$ and $Q^{g^{-1}} = P^n$ for $m, n \in M$. Then a quick check shows that $mgn \in N_G P \subset M$ and therefore $g \in M$, proving (iv).
(iv) $\Longrightarrow$ (i): Writing $G$ as the disjoint union of double cosets $MgM$, one calculates that the index of $M$ in $G$ is the sum over double coset representatives $g$ of the indices of $M \cap M^g$ in $M$. Since $p$ divides the order of $M$ and does not divide the order of $M \cap M^g$ if $g \notin M$, these indices are divisible by $p$ except for the double coset represented by $e$. Thus the index of $M$ in $G$ is congruent to 1 mod $p$, hence $M$ must contain some $p$-sylow subgroup $P$. Let $N = N_P$. For $n \in N$, $P$ and $P^n$ are in the same component. Considering $p$-Sylow subgroups containing groups in a chain connecting them, we see that there is a sequence of $p$-Sylow subgroups $P = P_0, P_1, \ldots, P_q = P^n$ such that $P_i \cap P_{i+1} \neq \{e\}$. There are elements $g_i$ such that $P_{i-1}^{g_i} = P_i$, and we can choose $g_q$ so that $g_q \cdots g_1 = n$. We have $P \subset M$, and we assume inductively that $P_{i-1} \subset M$. Then $P_{i-1} \cap P_i \subset M \cap M^{g_i}$, so this intersection contains a $p$-group and, by (iv), $g_i \in M$. This implies that $P_i \subset M$ and, inductively, we conclude that $n \in M$, so that $N \subset M$. □

**Corollary 7.4.2.** *$N_P$ is generated by the groups $N_G H$ for $H \in \mathscr{S}_p(P)$.*

PROOF. $N_P$ contains all of these $N_G H$, so it contains the subgroup they generate, and it is the smallest such subgroup by the equivalence of (i) and (ii). □

By the contrapositive, $G$ is not connected if and only if there is a proper subgroup $M$ of $G$ that satisfies the equivalent properties of the proposition. For example, if $r_p(G) = 1$ and $G$ has no non-trivial normal $p$-subgroup, then $\mathscr{A}_p(G)$ is discrete and not contractible, and is therefore not connected. Quillen gives a condition on $G$ under which these are the only examples.

**Proposition 7.4.3.** *Let $H$ $(= O_{p'}(G))$ be the largest normal subgroup of $G$ of order prime to $p$ and let $K$ $(= O_{p',p}(G))$ be specified by requiring $K/H$ to be the largest normal $p$-subgroup of the quotient group $G/H$. If $K/H$ is non-trivial and $\mathscr{S}_p(G)$ is not connected, then $r_p(G) = 1$.*

PROOF. If $Q$ is a $p$-Sylow subgroup of $K$, then $K = QH$ since $H$ is a $p'$-group and $K/H$ is a $p$-group. This implies that $H$ acts transitively on $\pi_0(\mathscr{S}_p(K))$ since it implies that any two $p$-Sylow subgroups are conjugate by the action of some $h \in H$. The intersection with $K$ of a $p$-Sylow subgroup $P$ of $G$ is a $p$-Sylow subgroup of $K$. A $p$-subgroup of $K$ is a $p$-subgroup of $G$, and the induced map $\pi_0(\mathscr{S}_p(K)) \longrightarrow \pi_0(\mathscr{S}_p(G))$ is surjective since $P \cap K \subset P$ implies that $[P]$ is the image of $[P \cap K]$. Therefore $H$ also acts transitively on $\pi_0(\mathscr{S}_p(G))$. Let $A$ be a maximal $p$-torus of $G$. The map $\pi_0(\mathscr{S}_p(AH)) \longrightarrow \pi_0(\mathscr{S}_p(G))$ is also surjective since $H$ acts transitively on the target and the map is $H$-equivariant. Therefore $\mathscr{S}_p(AH)$ is not connected. The component $[A]$ is fixed by the centralizers $C_H(B)$ for all non-trivial subgroups $B$ of $A$ since $B^h = B \subset A$ for $h \in C_H(B)$. By [28, 6.2.4], if $A$ is not cyclic (= rank one), then $H$ is generated by these centralizers, which contradicts the fact that $\mathscr{S}_p(AH)$ is not connected. Therefore $A$ is cyclic. $\square$

CHAPTER 8

# Really finite $H$-spaces

The circle is a *topological group*. If we regard it as the subspace of the complex plane consisting of points of norm one, then complex multiplication gives the product $S^1 \times S^1 \longrightarrow S^1$. How can we model such a basic structure in terms of a map of finite spaces?

Stong proved a rather amazing *negative* result about this problem. We will not go into the combinatorial details of his proof, contenting ourselves with the statement.

Expository REU paper? Research: Alexandroff H-spaces?

**8.0.1. Topological Groups.** The interaction of group multiplication with a space's topology is captured in the following definition.

**Definition 8.0.1.** A *topological group* is a group that is also a $T_0$ topological space in which the multiplication map given by $(x, y) \mapsto x \cdot y$ and the inverse map given by $x \mapsto x^{-1}$ are continuous.

**Proposition 8.0.2.** *Let $H$ be a group that is also a $T_0$ topological space. Then $H$ is a topological group if and only if the map $\rho : H \times H \longrightarrow H$ given by $(x, y) \mapsto x \cdot y^{-1}$ is continuous.*

Proof. Suppose $H$ is a topological group. The functions $f : H \times H \longrightarrow H \times H$ where $(x, y) \mapsto (x, y^{-1})$ and $g : H \times H \longrightarrow H$ sending $(a, b) \mapsto a \cdot b$ are then continuous, and so $\rho = g \circ f$ is as well.

Conversely, suppose $\rho$ is continuous. First, the map $v$ taking $x$ to $x^{-1}$ is equal to the composition of the continuous maps $\rho$ and $h : H \longrightarrow H \times H$ defined by $x \mapsto (e, x)$, and is therefore itself continuous. Second, the product map $g$ is continuous because it equals the composition of the continuous functions $\rho$ and $f$. □

**Example 8.0.3.** $(\mathbb{Z}, +)$
When equipped with the order topology, this is a $T_1$ space. Consider the open interval $(a, b)$, an arbitrary basis element for $(\mathbb{Z}, +)$. Define $\rho : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ by $(x, y) \mapsto x - y$. For the pre-image, we have $\rho^{-1}(a, b) = \{(x, y) | a < x - y < b\} = \{(x, y) | a+y < x < b+y\}$. This pre-image is the union over all $y$ of the corresponding open sets $(a + y, b + y) \times (y - 1, y + 1)$, and is therefore open. Thus $\rho$ is continuous.

**Example 8.0.4.** $(\mathbb{R}, +)$
The continuity of $\rho : \mathbb{R} \longrightarrow \mathbb{R}$ where $(x, y) \mapsto x - y$ in the usual topology is a standard fact of analysis.

**Example 8.0.5.** $(\mathbb{R}_+, \times)$
The continuity of the quotient operation $q : \mathbb{R} \times (\mathbb{R} - \{0\}) \longrightarrow \mathbb{R}$ is a standard fact of analysis. For $\mathbb{R}_+$, construct the continuous $\rho$ by restricting $q$'s domain to $\mathbb{R}_+ \times \mathbb{R}_+$ and its range to $\mathbb{R}_+$.

**Example 8.0.6.** $(S^1, \times)$
The beauty of the algebra of these numbers is that their multiplication is the same as the addition of real numbers (complex numbers on the unit circle are written as exponentials, and their multiplication is given by the addition of the exponents). That $(S^1, \times)$ is a topological group follows from the fact that $(\mathbb{R}, +)$ is.

We take $S^1$ as our main example. We are interested in finite models of $S^1$ that can be equipped with continuous multiplication.

**8.0.2. Failure of the non-Hausdorff Suspension of $S^1$.** Our standard four-point model of $S^1$, the non-Hausdorff suspension, is incompatible with continuous complex multiplication. The model in the complex numbers is pictured in the following diagram. An arrow pointing from one element to another says that the element being pointed to is greater than the other. The far-right and far-left points are identical.

$$i \longleftarrow -1 \longrightarrow -i \longleftarrow 1 \longrightarrow i$$

**Proposition 8.0.7.** *In the complex numbers, the non-Hausdorff suspension $\mathbb{S}^1$ of the zero-sphere gives discontinuous multiplication.*

Proof. We have $(i, i) > (-1, i)$,but $i \cdot i = -1 < -1 \cdot i$. □

**8.0.3. Finite $H$-Spaces.**

**Definition 8.0.8.** Let $(X, e)$ be a finite space with a basepoint $e$ and let $\phi\colon X \times X \longrightarrow X$ be a map We say that $X$ is an $H$-space of type I if multiplication by $e$ on either the right or the left is homotopic to the identity. That is, the maps $x \longrightarrow \phi(e, x)$ and $x \longrightarrow \phi(x, e)$ are each homotopic to the identity. Say that $X$ is an $H$-space of type II if the *shearing maps* $X \times X \longrightarrow X \times X$ defined by sending $(x, y)$ to either $(x, \phi(x, y))$ or $(y, \phi(x, y))$ are homotopy equivalences.

A topological group is an $H$-space of both types, but it is much less restrictive for a space to be an $H$-space than for a space to be a topological group. In particular, a topological group is an $H$-space in which multiplication by $e$ is the identity map, so that $e$ is an algebraic identity element. The definition of a type I $H$-space is often presented as the standard definition of an $H$-space. Henceforth, it will be the focus of the chapter.

By definition, the notion of $H$-space is homotopy invariant in the sense that if one defines an $H$-space structure on $(X, e)$ to be a homotopy class of products $\phi$, then one has the following result.

**Proposition 8.0.9.** *If $(X, e)$ and $(Y, f)$ are homotopy equivalent, then $H$-space structures on $(X, e)$ correspond bijectively to $H$-space structures on $(Y, f)$.*

This motivated Stong [65] to study $H$-space structures on minimal finite spaces covered in the following sections.

**8.0.4. A combinatorial result.** Before proving propositions about $H$-spaces, we modify the definitions of minimal finite spaces and cores to respect basepoints. Recall the given definition (2.4.2) of a beat point of a finite space. We present the analogous notions for based spaces.

**Definition 8.0.10.** A based finite space $(X, x)$ is *minimal* if it satisfies the $T_0$ axiom and has no beat points except possibly $x$. A *core* of a finite space $(X, x)$ is a subspace $(Y, x)$ that is minimal and a deformation retract of $X$.

This modified definition ensures that when a based space is reduced to its core, the basepoint is not deleted.

The following fact will prove useful in the proof of following results.

**Proposition 8.0.11.** *Let $(X, e)$ be a minimal finite $H$-space. Then $\theta_1, \theta_2 : X \longrightarrow X$ given by $\theta_1(x) = xe$ and $\theta_2(x) = ex$ are equal to the identity map.*

PROOF. Since $(X, e)$ is a minimal finite space, any map from $X$ to itself that is homotopic to the identity is the identity. □

The following proposition provides the structure for the proof of Stong's major result, Theorem 8.0.17. Recall that in a poset, an upbeat point $x$ under $y$ implies that $y$ is the immediate successor of $x$. The opposite holds for downbeat points.

**Proposition 8.0.12.** *Let $(X, e)$ be a minimal finite space, $x \in X$. Then*

*(i) $x$ is less than each of two distinct maximal points, or*
*(ii) $x$ is maximal, or*
*(iii) $x$ is upbeat under a maximal point (so $x = e$)*

*and*

*(i') $x$ is greater than each of two distinct maximal points, or*
*(ii') $x$ is minimal, or*
*(iii') $x$ is downbeat over a minimal point (so $x = e$)*

PROOF. Suppose by way of contradiction that the set $A$ of points that do not satisfy any of (i), (ii), (iii) is nonempty, and let $a$ be a maximal element of $A$. Since $a$ is not maximal in $X$, there exists $z \in X$ such that $z > a$. Let $B = \{x | x > a\} \subset X$. If $B$ contains a point $z'$ other than $z$, then $a$ satisfies (i). If not, then $a$ satisfies (iii). Either way, we have arrived at a contradiction with the fact that $a$ is in $A$. We conclude that $A$ must be empty. Similarly, every point must satisfy one of (i'), (ii'), (iii'). □

We present the following combinatorial result on the relationship of general points of $X$ to the point $e$, which leads Stong to his main result.

**Proposition 8.0.13.** *Let $(X, e)$ be a minimal finite $H$-space. Then $e$ is both maximal and minimal under the associated order $\leq$.*

PROOF. Knowing that $e$ must satisfy one of the conditions (i), (ii), (iii) and one of (i'), (ii'), (iii'), we proceed to eliminate from possibility all pairs of conditions except the pair consisting of (ii) and (ii').

**Remark 8.0.14.** From the definition of the order on a finite $T_0$ space we can deduce the appropriate order on a product of two finite spaces. Let $(a, b), (c, d) \in X \times Y$ where $X$ and $Y$ are finite $T_0$ spaces. Then $(a, b) \leq (c, d)$ if and only if $a \leq c$ and $b \leq d$. If any of the two inequalities in the factor spaces is strict, then the inequality in the product space is strict as well.

**The point $e$ does not satisfy (i).**

**Lemma 8.0.15.** *Let $m$ and $m'$ be maximal points in $X$ with $m, m' > e$. Then $m = m'$.*

Proof. Since $m' > e$ and $m > e$, we have

$$(m, m') > (m, e) \text{ and } (m, m') > (e, m').$$

□

Applying the continuous (order-preserving) product $\phi$ to each, we obtain

$$mm' \geq me = m \text{ and } mm' \geq em' = m'$$

Because the right-hand sides of both inequalities above are maximal, we deduce that $m = mm' = m'$.

**The point $e$ does not satisfy (i').**
This is true for perfectly symmetric reasons.

**The point $e$ does not satisfy both (ii') and (iii).**
We show that if it did, then $X$ would have infinitely many subsets, in contradiction to the fact that $X$ is finite.

Suppose by way of contradiction that $e$ satisfies (ii') and (iii), i.e., $e$ is both minimal and upbeat under a maximal point.

Our claim is that for every integer $r \geq 0$, $X$ contains a subset

$$D_r = \{e = u_0, u_1, \ldots, u_r; m_0, \ldots, m_{r-1}\} \tag{8.0.16}$$

with all $u_i$ minimal in $X$ and all $m_i$ maximal in $X$, and such that all of the following conditions hold:

(a) For $i$ between 0 and $r-1$ (inclusive), the only points in $X$ less than $m_i$ are $u_i$ and $u_{i+1}$.
(b) $m_0$ is the only point in $X$ that is greater than $u_0$.
(c) For $i$ between 1 and $r-1$ (inclusive), the only points in $X$ greater than $u_i$ are $m_{i-1}$ and $m_i$.
(d) For $i$ between 0 and $r-1$ (inclusive), $xm_i = m_ix = m_i$ if $x$ is $m_k$ or $u_k$ with $k \leq i$.
(e) For $i$ between 0 and $r$ (inclusive), $xu_i = u_ix = u_i$ if $x = m_k$ with $k < i$ or $x = u_k$ with $k \leq i$.
(f) For every $x \in X$ not in $D_r$, $xm_i = x = xu_i$ and $m_ix = x = u_ix$.

For $r = 0$, we have the set $D_0 = \{e = u_0\}$. It contains no $m_i$. Conditions (a) – (d) and (f) are vacuously satisfied. For condition (e), the first option (involving $m_i$) is vacuously satisfied, and the second demands only that we check $ee = e$. That equation is true in the minimal space $(X, e)$ because multiplication by $e$ is homotopic to the identity, but in fact it is true for any $H$-space. Multiplication by $e$ is homotopic to the identity through maps from $(X, e)$ to $(X, e)$. That is, the fact that $e$ is the space's basepoint means the only allowed intermediate maps take $e$ to itself.

Now assume $X$ contains a set $D_k$ of the form in **(7.1.15)**. We show that there are an additional maximal point $m_k$ and an additional minimal point $u_{k+1}$ such that $D_{k+1}$ (of the form **(7.1.15)**) satisfies (a) through (f).

First we show that it satisfies (a), (b), (c).

For $k = 0$, the assumption that $e$ is upbeat under a maximal point gives a unique $m_k$, namely the point under which $e$ is upbeat.

For $k > 0$, we know that $u_k \neq e$ and that $u_k$ is not maximal (being less than $m_{k-1}$). So $u_k$ is less than each of two distinct maximal points. By (a), only one of those is in $D_k$.

In order for $D_{k+1}$ to satisfy (c), we cannot have a choice of multiple maximal points to call $m_k$. In the case $k = 0$, this needed uniqueness property has already been shown. In the case $k > 0$, in which we know that there exists a maximal point outside $D_k$, the uniqueness follows from the combination of (f) with the procedure of the previous subsection.

Existence and uniqueness of $u_{k+1}$ now follow by the analogous argument, using $m_k \neq e$.

One can now see that $D_{k+1}$ satisfies (a), (b), (c). We finally show that it satisfies (d), (e), (f).

To verify (d) for $D_{k+1}$, we substitute $x = m_k$ in the assumption (f) for $D_k$. Likewise, to verify (e), we substitute $x = u_{k+1}$.

Finally, let us verify (f) for $D_{k+1}$. We will show $xm_k = x = xu_k$. The derivation of the other identity uses the analogous argument.

Suppose $x$ is not in $D_{k+1}$. We obtain immediately $xm_k \geq xu_k = x$. We now proceed by induction.

In the base case, where $x$ is maximal, we find from the above that $xm_k = x$.

Now, for the inductive step, consider the point $w$, supposing that for every $y > w, ym_k = y$.

For any such $y$, by continuity of $\phi$, we have $y = ym_k \geq wm_k$.

Thus, either $w$ is upbeat or $wm_k = w$. The former is false because $X$ is a minimal space (no point other than $e$ can be upbeat) with $xe \neq e$ for $x \neq e$. So $wm_k = w = wu_k$. This completes the verification of (f).

We now see that if $e$ satisfied (ii') and (iii), we would be able to construct infinitely many distinct subsets of $X$, contradicting the fact that $X$ is finite.

**The point $e$ does not satisfy both (ii) and (iii').**
This possibility is ruled out in the same way as the possibility (ii') and (iii).

**The point $e$ does not satisfy both (iii) and (iii').**
This possibility is conceptually similar to the last, because it just replaces maximality by the situation of being upbeat under a maximal point. The technicalities of the demonstration are slightly different, but offer negligible additional insight.

**The point e satisfies (ii) and (ii').**
This is the only remaining possible pair of conditions. The proof of **Proposition 7.2** is now complete. ☐

This means that $e$ is a component of $X$. Stong shows that this implies the following conclusions for general finite $H$-spaces.

### 8.0.5. Inviability of finite $H$-space models of non-contractible connected spaces.

**Theorem 8.0.17.** *Let $X$ be a finite space and let $e \in X$. Then there is a product $\phi$ making $(X, e)$ an $H$-space of type I if and only if $e$ is a deformation retract of* *its component in $X$. Therefore $X$ is an $H$-space for some basepoint $e$ if and only if some component of $X$ is contractible.*

Proof. Since $(X, e)$ is homotopy equivalent to its core $(Y, e)$, **Proposition 4.3** says that there is an $H$-space structure on $(X, e)$ only if there is one on $(Y, e)$.

Because $(Y, e)$ is a minimal finite space, it is an $H$-space only if $e$ is both maximal and minimal in $Y$ under the associated order $\leq$, i.e., $\{e\}$ is a path component of $Y$.

In finite spaces, path components are the same as connected components. So, $\{e\}$ is a path component of $Y$ only if it is a component of $Y$.

If $\{e\}$ is a component of $Y$ (the core), then $\{e\}$ is the core of $e$'s component in $X$.

A core of a component is a deformation retract of the component. Thus the result is established. □

**Theorem 8.0.18.** *Let $X$ be a finite space. Then there is a product $\phi$ making $X$ an $H$-space of type II if and only if every component of $X$ is contractible.*

**Corollary 8.0.19.** *A connected finite space $X$ is an $H$-space of either type if and only if $X$ is contractible.*

So there is no way that we can model the product on $S^1$ by means of an $H$-space structure on some finite space $X$. Our standard model $\mathbb{T} = \mathbb{S}S^0$ of $S^1$ can be embedded in $\mathbb{C}$ as the four point subgroup $\{\pm 1, \pm i\}$, but then the complex multiplication is not continuous. However, the multiplication can be realized as a map $(\mathbb{T} \times \mathbb{T})^{(n)} \longrightarrow \mathbb{T}$ for some finite $n$, by the simplicial approximation theorem for finite spaces. Explicitly, it is implied that for an $H$-space $X$ with product $\phi$ and finite model $Y$, there exist an integer $n$ and a continuous map $\mu : (Y \times Y)^{(n)} \longrightarrow Y$ such that $|\mathcal{K}(\mu)| \simeq \phi$. It is natural to expect that some small $n$ works here.

The following result is proven in [**30**].

**Theorem 8.0.20.** *Choosing minimal points $e$ in $\mathbb{T}$ and $f \in \mathbb{T}'$ as basepoints, there is a map*

$$\phi \colon \mathbb{T}' \times \mathbb{T}' \longrightarrow \mathbb{T}$$

*such that $\phi(f, f) = e$ and the maps $x \longrightarrow \phi(x, f)$ and $x \longrightarrow \phi(f, x)$ from $\mathbb{T}'$ to $\mathbb{T}$ are weak homotopy equivalences.*

That is, we can realize a kind of $H$-space structure after barycentric subdivision. The proof is horribly unilluminating. The space $\mathbb{T}'$ has eight elements, the space $\mathbb{T}$ has four elements. One writes down an $8 \times 8$ matrix with values in $\mathbb{T}$, choosing it most carefully so that when the 8 point and 4 point spaces are given the appropriate partial order, and the 64 point product space the product order, the function represented by the matrix is order preserving. Then one checks the row and column corresponding to multiplication by the basepoint.

Several other interesting spaces and maps are modelled similarly in the cited paper, for example $\mathbb{R}P^2$ and $\mathbb{C}P^2$.

# Part 2

# Topological spaces, Simplicial sets, and categories

CHAPTER 9

# Simplicial sets

## 9.1. Motivation for the introduction of simplicial sets

Simplicial sets, and more generally simplicial objects in a given category, are central to modern mathematics. While I am not a mathematical historian, I thought I would describe in conceptual outline how naturally simplicial sets arise from the classical study of simplicial complexes. I suspect that something like this recapitulates the historical development.

We have described simplicial complexes in several different forms: abstract simplicial complexes, ordered simplicial complexes, geometric simplicial complexes, ordered geometric simplicial complexes and realizations of geometric simplicial complexes. It is possible to go directly from abstract simplicial complexes to realizations without passing through geometric simplicial complexes, but the construction is perhaps not as intuitive and will not be included.

An abstract simplicial complex is equivalent to a geometric simplicial complex, and neither of these notions involves anything about ordering the vertices. If one has a simplicial complex of either type, one can choose a partial ordering of the vertices that restricts to a linear ordering of the vertices of each simplex, and this gives the notion of an ordered simplicial complex. This can be done most simply, but not most generally, just by choosing a total ordering of the set of all vertices and restricting that ordering to simplices. However, there is no canonical choice.

We have seen in studying products of simplicial complexes that geometric realization behaves especially nicely only in the ordered setting. Both the category $\mathscr{S}\mathscr{C}$ of simplicial complexes and the category $\mathscr{O}\mathscr{S}\mathscr{C}$ of ordered simplicial complexes have categorical products. Geometric realization preserves products when defined on $\mathscr{O}\mathscr{S}\mathscr{C}$, but it does not preserve products when defined on $\mathscr{S}\mathscr{C}$. The functor $\mathscr{K}$ is best viewed as a functor from the category $\mathscr{P}$ of partially ordered sets to the category $\mathscr{O}\mathscr{S}\mathscr{C}$ rather than just to the category $\mathscr{S}\mathscr{C}$. Observe that there are generally many different ordered simplicial complexes with the same poset of vertices. The functor $\mathscr{K}$ picks out the largest choice, the one in which every finite totally ordered subset of the set of vertices is a simplex.

The functor $\mathscr{X}$, on the other hand, starts in $\mathscr{S}\mathscr{C}$ and lands in $\mathscr{P}$, which can be identified with the category of $A$-spaces. The composite $\mathscr{K}\mathscr{X}$ is the barycentric subdivision functor $Sd\colon \mathscr{S}\mathscr{C} \longrightarrow \mathscr{O}\mathscr{S}\mathscr{C}$. It can be viewed as the construction of a canonical ordered simplicial complex $SdK$ starting from a given unordered simplicial complex $K$, at the price of subdividing. Since the geometric realization functor gives a space $|SdK|$ that can be identified with $|K|$ there is no loss of topological generality working in $\mathscr{O}\mathscr{S}\mathscr{C}$ instead of $\mathscr{S}\mathscr{C}$.

The most important motivation for working with ordered rather than unordered simplicial complexes is that the ordering leads to the definition of an associated promise

chain complex and thus to a quick definition of homology. I'll explain that in the talks and add it to the notes if I have time.

As noted earlier, a topological space $X$ is called a polytope if it is homeomorphic to $|K|$ for a (given) simplicial complex $K$. Such a homeomorphism $|K| \longrightarrow X$ is called a triangulation of $X$, and $X$ is said to be triangulable if it admits a triangulation. Then we can define the homology of $X$ to be the homology of $K$. This is a quick definition, and useful where it applies, but it raises many questions and is quite unsatisfactory conceptually. Not every space is triangulable, and triangulable spaces can admit many different triangulations. It is far from obvious that the homology is independent of the choice of triangulation.

Simplicial sets abstract the notion of ordered simplicial complexes, retaining enough of the combinatorial structure that homology can be defined with equal ease. The generalization allow myriads of examples that do not come from simplicial complexes. The original motivating example gives a functor from topological spaces to simplicial sets. Composing with the functor from simplicial sets to homology groups gives the quickest way of defining the homology groups of a space and leads to the proof that these groups depend only on the weak homotopy type of the space, not on any triangulation, and to the proofs that different triangulations, when they exist, give canonically isomorphic homology groups.

Perhaps the quickest and most intuitive way to motivate the definition of simplicial sets is to start from structure clearly visible in the case of ordered simplicial complexes. Let $X$ denote the partially ordered set $V(K)$ of vertices of an ordered simplicial complex $K$. The reader might prefer to start with an ordered simplicial complex of the form $\mathscr{K}(X)$, where $X$ is a poset. The reader may also want to insist that $X$ is finite, but that is not necessary to the construction, and we later want to allow infinite sets.

Then an $n$-simplex $\sigma$ of $K$ is a totally ordered $n+1$-tuple of elements of $X$. Write such a tuple as $(x_0, \cdots, x_n)$. When studying products, we saw that it can become essential to consider tuples $(x_0, \cdots, x_n)$, where $x_0 \leq x_1 \leq \cdots \leq x_n$. Of course, $(x_0, \cdots, x_n)$ is no longer a simplex, but one can obtain a simplex from it by deleting repeated entries. When there are repeated entries, we think of $(x_0, \cdots, x_n)$ as a "degenerate" $n$-simplex. Let $K_n$ denote the set of such generalized $n$-simplices, degenerate or not. For $0 \leq i \leq n$, define functions

$$d_i \colon K_n \longrightarrow K_{n-1} \quad \text{and} \quad s_i \colon K_n \longrightarrow K_{n+1},$$

called face and degeneracy operators, by

$$d_i(x_0, \cdots, x_n) = (x_0, \cdots x_{i-1}, x_{i+1}, \cdots, x_n)$$

and

$$s_i(x_0, \cdots, x_n) = (x_0, \cdots x_i, x_i, \cdots, x_n).$$

Of course, the $d_i$ and $s_i$ just defined also depend on $n$, but it is standard not to indicate that in the notation. In words, $d_i$ deletes the $i^{\text{th}}$ entry and $s_i$ repeats the $i^{\text{th}}$ entry. If $i < j$ and we first delete the $j^{\text{th}}$ entry and then the $i^{\text{th}}$ entry, we get the same thing as if we first delete the $i^{\text{th}}$ entry and then delete the (new) $(j-1)^{\text{st}}$ entry. Similarly, elementary inspections give commutation relations between the $d_i$ and $s_j$ and between the $s_i$. Here is a list of all such relations:

$$d_i \circ d_j = d_{j-1} \circ d_i \quad \text{if} \quad i < j$$

$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & \text{if } \ i < j \\ \mathrm{id} & \text{if } \ i = j \ \text{ or } \ i = j+1 \\ s_j \circ d_{i-1} & \text{if } \ i > j+1 \end{cases}$$

$$s_i \circ s_j = s_{j+1} \circ s_i \ \text{ if } \ i \leq j$$

The reader can easily check that these identities really do follow immediately from the definition of the $K_n$, $d_i$, and $s_i$ above.

The $K_n$ are defined in terms of the partially ordered vertex set $V(K)$ of $K$, but there are many examples of precisely similar structure that arise differently.

## 9.2. The definition of simplicial sets

We obtain our first definition of simplicial sets by formalizing structure that, as we have just seen, is implicit in the definition of an ordered simplicial complex.

**Definition 9.2.1.** A simplicial set $K$ is a sequence of sets $K_n$, $n \geq 0$, and functions $d_i \colon K_n \longrightarrow K_{n-1}$ and $s_i \colon K_n \longrightarrow K_{n+1}$ for $0 \leq i \leq n$ that satisfy the identities just displayed. The elements of the set $K_n$ are called $n$-simplices, following the historic precedent of simplicial complexes. Just as if $K$ were a simplicial complex, a map $f \colon K \longrightarrow L$ of simplicial sets is a sequence of functions $f_n \colon K_n \longrightarrow L_n$ such that $f_{n-1} \circ d_i = d_i \circ f_n$ and $f_{n+1} \circ s_i = s_i \circ f_n$. With these objects and morphisms, we have the category $s\mathscr{S}et$ of simplicial sets.

Now our motivating example can be recapitulated in the following statement.

**Proposition 9.2.2.** *There is a canonical functor $i \colon \mathscr{O}\mathscr{S}\mathscr{C} \longrightarrow s\mathscr{S}et$ from the category of ordered simplicial complexes to the category of simplicial sets. It assigns to an ordered simplicial complex $K$ the simplicial set $K^s$ given by the sequence of sets $K^s_n$ and the functions $d_i$ and $s_i$ defined above. It assigns to a map $f \colon K \longrightarrow L$ of ordered simplicial complexes the map $f^s \colon K^s \longrightarrow L^s$ induced by its map of vertex sets:*

$$f^s_n(x_0, \cdots, x_n) = (f(x_0), \cdots, f(x_n)).$$

*It is a full embedding, meaning that the maps $K \longrightarrow L$ of ordered simplicial complexes map bijectively to the maps $K^s \longrightarrow L^s$ of simplicial sets.*

The identities listed above are hard to remember and do not appear to be very conceptual. The definition admits a conceptual reformulation that may or may not make things clearer, depending on personal taste, but definitely allows many arguments and constructions to be described more clearly and conceptually than would be possible without it. We define the category $\Delta$ of finite ordered sets.

**Definition 9.2.3.** The objects of $\Delta$ are the finite ordered sets $[n]$ with $n+1$ elements $0 < 1 < \cdots < n$. Its morphisms are the monotonic functions $\mu \colon [m] \leq [n]$. This means that $i < j$ implies $\mu(i) \leq \mu(j)$. Define particular monotonic functions

$$\delta_i \colon [n-1] \longrightarrow [n] \quad \text{and} \quad \sigma_i \colon [n+1] \longrightarrow [n]$$

for $0 \leq i \leq n$ by

$$\delta_i(j) = j \ \text{ if } \ j < i \quad \text{and} \quad \delta_i(j) = j+1 \ \text{ if } \ j \geq i$$

and

$$\sigma_i(j) = j \ \text{ if } \ j \leq i \quad \text{and} \quad \sigma_i(j) = j-1 \ \text{ if } \ j > i.$$

In words, $\delta_i$ skips $i$ and $\sigma_i$ repeats $i$.

There are identities for composing the $\delta_i$ and $\sigma_i$ that are "dual" to those for composing the $d_i$ and $s_i$ that appear in the definition of a simplicial set. Precisely, the duality amounts to reversing the direction of arrows. The following pair of commutative diagrams should make clear how to interpret this, where $i < j$.

$$\begin{array}{ccc} K_n & \xrightarrow{d_j} & K_{n-1} \\ {\scriptstyle d_i}\downarrow & & \downarrow{\scriptstyle d_i} \\ K_{n-1} & \xrightarrow[d_{j-1}]{} & K_{n-2} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} [n] & \xleftarrow{\delta_j} & [n-1] \\ {\scriptstyle \delta_i}\uparrow & & \uparrow{\scriptstyle \delta_i} \\ [n-1] & \xleftarrow[\delta_{j-1}]{} & [n-2] \end{array}$$

A moment's reflection should convince the reader that every monotonic function $\mu\colon [m] \longrightarrow [n]$ can be written as a composite of monotonic functions $\delta_i$ and $\sigma_j$ for varying $i$ and $j$. That is, $\mu$ can be obtained by omitting some of the $i$'s and repeating some of the $j$'s. Just as a group can be defined by specifying a set of generators and relations, so a category can often be specified by a set of generating morphisms and relations between their composites. The category $\Delta$ is generated by the $\delta_i$ and $\sigma_i$ subject to our "dual" relations. This leads to the proof of the following reformulation of the notion of a simplicial set. Recall that a contravariant functor $F$ assigns a morphism $FY \longrightarrow FX$ of the target category to each morphism $X \longrightarrow Y$ of the source category.

**Proposition 9.2.4.** *The category of simplicial sets can be identified with the category of contravariant functors $K\colon \Delta \longrightarrow \mathscr{S}et$ and natural transformations between them.*

Proof. The correspondence is given by viewing the functions $d_i$ and $s_i$ that define a simplicial set as the morphisms of sets induced by the morphisms $\delta_i$ and $\sigma_i$ of the corresponding functor $\Delta \longrightarrow \mathscr{S}et$. It is convenient to write $\mu^*\colon K_n \longrightarrow K_m$ for the function induced by contravariance from a morphism $\mu\colon [m] \longrightarrow [n]$, and then $d_i = \delta_i^*$ and $s_i = \sigma_i^*$. For a map $f$, the corresponding natural transformation is given on the object $[n]$ by the function $f_n$. $\square$

While we do not want to emphasize abstraction in the first instance, we nevertheless cannot resist the temptation to generalize the definition of simplicial sets to simplicial objects in a perfectly arbitrary category. The generalization has a huge number of applications throughout mathematics, and we shall use it when defining homology.

**Definition 9.2.5.** A simplicial object in a category $\mathscr{C}$ is a contravariant functor $K\colon \Delta \longrightarrow \mathscr{C}$. A map $f\colon K \longrightarrow L$ of simplicial objects in $\mathscr{C}$ is a natural tranformation $K \longrightarrow L$; it is given by morphisms $f_n\colon K_n \longrightarrow L_n$ in $\mathscr{C}$. We have the category $s\mathscr{C}$ of simplicial objects in $\mathscr{C}$. By composition of functors and natural transformations, any functor $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ induces a functor $sF\colon s\mathscr{C} \longrightarrow s\mathscr{D}$. By duality, a *covariant* functor $\Delta \longrightarrow \mathscr{C}$ is called a cosimplicial object in $\mathscr{C}$.

## 9.3. Standard simplices and their role

We explain a general conceptual way to relate simplicial sets to "standard simplices". Standard simplices exist in many categories. We have standard simplices in topological spaces, simplicial sets, and even posets and categories. In general, fixing a category $\mathscr{V}$, we often have a standard cosimplicial object in $\mathscr{V}$, that is a certain covariant functor $\Delta[\bullet]^v : \Delta \longrightarrow \mathscr{V}$. The superscript $v$ is meant as a reminder that the functor is assigning objects in $\mathscr{V}$ to objects in $\Delta$; it should also help to distinguish the functor $\Delta[\bullet]^v$ from the category $\Delta$. On objects, we write the functor $\Delta[\bullet]^v$ as $[n] \mapsto \Delta[n]^v$, but we agree to write $\mu_*$ rather than $\Delta[\mu]^v$ for the map $\Delta[m]^v \longrightarrow \Delta[n]^v$ in $\mathscr{V}$ obtained by applying our functor to a morphism $\mu$ in $\Delta$. For each object $V$ of $\mathscr{V}$ we obtain a *contravariant* functor, denoted $SV : \Delta \longrightarrow \mathscr{S}et$, by letting the set $S_nV$ of $n$-simplices be the set $\mathscr{V}(\Delta[n]^v, V)$ of morphisms $\Delta[n]^v \longrightarrow V$ in the category $\mathscr{V}$. The faces and degeneracies are induced by precomposition with the maps

$$\delta_i : \Delta[n-1]^v \longrightarrow \Delta[n]^v \quad \text{and} \quad \sigma_i : \Delta[n+1]^v \longrightarrow \Delta[n]^v$$

obtained by applying the functor $\Delta[\bullet]^v$ to the generating morphisms $\delta_i$ and $\sigma_i$ of $\Delta$. That is, for a morphism $\nu : \Delta[n]^v \longrightarrow V$ in $\mathscr{V}$,

$$d_i(\nu) = \nu \circ \delta_i \quad \text{and} \quad s_i(\nu) = \nu \circ \sigma_i.$$

Before turning to the motivating examples, in which $\mathscr{V}$ is the category $\mathscr{U}$ of topological spaces or the category $\mathscr{C}at$ of small categories, we apply this construction to the case $\mathscr{V} = s\mathscr{S}et$.

**Definition 9.3.1.** Define the standard simplicial $n$-simplex $\Delta[n]^s$ to be the contravariant functor $\Delta \longrightarrow s\mathscr{S}et$ represented by $[n]$. This means that the set $\Delta[n]^s_q$ of $q$-simplices is the set of all morphisms $\phi : [q] \longrightarrow [n]$ in $\Delta$. For a morphism $\nu : [p] \longrightarrow [q]$ in $\Delta$, the function $\nu^* : \Delta[n]^s_q \longrightarrow \Delta[n]^s_p$ is given by composition, $\nu^*(\phi) = \phi \circ \nu : [p] \longrightarrow [q]$.

**Definition 9.3.2.** We define a covariant functor $\Delta[\bullet]^s$ from $\Delta$ to the category $s\mathscr{S}et$ of simplicial sets. On objects, the functor sends $[n]$ to the standard simplicial $n$-simplex $\Delta[n]^s$. On morphisms $\mu : [m] \longrightarrow [n]$ in $\Delta$, define $\mu_* : \Delta[m]^s_q \longrightarrow \Delta[n]^s_q$ by $\mu_*(\psi) = \mu \circ \psi : [q] \longrightarrow [m] \longrightarrow [n]$. Thus the simplicial set $\Delta[n]^s$ is defined using pre-composition with morphisms of $\Delta$, and then the covariant functoriality of $\Delta[\bullet]^s$ is defined using post-composition with morphisms of $\Delta$. The object $\Delta[\bullet]^v$ is a cosimplicial simplicial set, that is, a cosimplicial object in the category of simplicial sets.

We may identify the set of all non-degenerate simplices of $\Delta[n]^s$ with the poset of non-empty subsets of the set $[n]$ of $n+1$ elements, ordered by inclusion. In other words, $\Delta[n]^s = (\mathscr{K}([n]))^s$ is the ordered simplicial set determined by the simplicial complex $\mathscr{K}([n])$.

Although we shall give a direct proof, the following result is an application of the Yoneda lemma. Let $\iota_n \in \Delta[n]^s_n$ be the identity map $\mathrm{id} : [n] \longrightarrow [n]$.

**Proposition 9.3.3.** *Let $K$ be a simplicial set. For $x \in K_n$, there is a unique map of simplicial sets $Y(x) : \Delta[n]^s \longrightarrow K$ such that $Y(x)(\iota_n) = x$. Therefore $K$ is naturally isomorphic to the simplicial set whose $n$-simplices are the maps of simplicial sets $\Delta[n]^s \longrightarrow K$.*

Proof. The map $Y(x)$ is a natural transformation from the contravariant functor $\Delta[n]^s$ to the contravariant functor $K$ from $\Delta$ to $\mathscr{S}et$. Since a $q$-simplex $\phi : [q] \longrightarrow [n]$ is $\phi^*(\iota_n)$, we can and must specify $Y(x)$ at the object $[q] \in \Delta$ by the function $\Delta[n]^s_q \longrightarrow K_q$ that sends $\phi$ to the $q$-simplex $\phi^*(x)$. □

We can vary the construction in a way that may look unnatural but that will lend itself to generalization to other examples. We show how to reconstruct $K$ directly from the $\Delta[n]^s$.

**Construction 9.3.4.** For a set $J$ and a simplicial set $L$, one can form a new simplicial set $J \times L$ by setting $(J \times L)_q = J \times L_q$ and letting the faces and degeneracies be induced by those of $L$. Said another way, we think of $J$ as a "discrete" simplicial set with each $J_q = J$ and all faces and degeneracies the identity map of $J$, and we then take the product $J \times L$ of simplicial sets. We apply this with $J = K_n$ and $L = \Delta[n]^s$ as $n$ varies to obtain a simplicial set

$$\overline{K} = \coprod_{n \geq 0} K_n \times \Delta[n]^s.$$

We define an equivalence relation $\simeq$ on $\overline{K}$ by requiring

$$(\alpha^*(k), \sigma) \simeq (k, \alpha_*(\sigma)) \tag{9.3.5}$$

for $k \in K_n$, $\sigma \in \Delta[m]^s_q$, and $\alpha\colon [m] \longrightarrow [n]$ in $\Delta$. Here $\alpha^*(k) \in K_m$ is given by the fact that $K$ is a contravariant functor from $\Delta$ to sets and $\alpha_*(\sigma) \in \Delta[n]^s_q$ is given by the fact that $\Delta[-]^s$ is a covariant functor from $\Delta$ to simplicial sets. With the simplicial structure induced from the simplicial structure on the $\Delta[n]^s$, passage to equivalence classes gives us a new simplicial set that we shall denote by $T^sK$ for the moment. Then $T^s$ is a functor from simplicial sets to simplicial sets.

**Proposition 9.3.6.** *The simplicial set $T^sK$ is naturally isomorphic to $K$.*

Proof. We claim that an arbitrary pair $(k, \tau)$ in $K_n \times \Delta[n]^s_q$ is equivalent to the pair $(\tau(k), \iota_q)$ in $K_q \times \Delta[q]^s_q$ where, as above, $\iota_q\colon [q] \longrightarrow [q]$ is the identity map viewed as a canonical $q$-simplex in $\Delta[q]^s$. Viewing $\tau\colon [q] \longrightarrow [n]$ as a morphism of $\Delta$, we have $\tau = \tau_*(\iota_q)$, and the claim follows. Identifying equivalence classes of $q$-simplices with elements of $K_q$ in this fashion, we find that the faces and degeneracies agree. Indeed, for $\xi\colon [p] \longrightarrow [q]$, $\xi \circ \iota_p = \iota_q \circ \xi$ and

$$(k, \xi^*(\iota_q)) = (k, \xi_*(\iota_p)) \simeq (\xi^*(k), \iota_p). \qquad \square$$

## 9.4. The total singular complex $SX$ and the nerve $N\mathscr{C}$

We turn to the historical motivating example $\mathscr{V} = \mathscr{U}$ by constructing the total singular complex $SX$ of a topological space $X$. We need a covariant functor $\Delta[\bullet]^t\colon \Delta \longrightarrow \mathscr{U}$, and that is given by the standard topological simplices $\Delta[n]^t$.

**Definition 9.4.1.** Recall that the standard topological $n$-simplex $\Delta[n]^t$ is the subspace

$$\{(t_0, \cdots, t_n) \,|\, 0 \leq t_i \leq 1 \ \text{ and } \ \Sigma_i t_i = 1\}$$

of $\mathbb{R}^{n+1}$. Define

$$\delta_i\colon \Delta[n-1]^t \longrightarrow \Delta[n]^t \ \text{ and } \ \sigma_i\colon \Delta[n+1]^t \longrightarrow \Delta[n]^t$$

by

$$\delta_i(t_0, \cdots, t_{n-1}) = (t_0, \cdots, t_{i-1}, 0, t_i, \cdots, t_n)$$

and

$$\sigma_i(t_0, \cdots, t_{n+1}) = (t_0, \cdots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \cdots, t_{n+1}).$$

Then the $\delta_i$ and $\sigma_i$ satisfy the commutation relations required to specify a covariant functor $\Delta[\bullet]^t$ from $\Delta$ to the category $\mathscr{U}$ of topological spaces, that is, a cosimplicial object in the category of topological spaces.

**Definition 9.4.2.** The *total singular complex* $SX$ of a space $X$ is the simplicial set whose set $S_nX$ of $n$-simplices is the set of continuous maps $\Delta[n]^t \longrightarrow X$ and whose faces $d_i$ and degeneracies $s_i$ induced by precomposition with $\delta_i$ and $\sigma_i$. By composition of continuous maps, a map $f\colon X \longrightarrow Y$ induces the map $f_* = Sf\colon SX \longrightarrow SY$ of simplicial sets that sends an $n$-simplex $s\colon \Delta[n]^t \longrightarrow X$ to the $n$-simplex $f \circ s$. This defines the total singular complex functor $S$ from topological spaces to simplicial sets.

We shall return to this example after giving an analogue that may seem astonishing at first sight. Although it has become a standard and commonplace construction, its importance and utility were only gradually recognized. Recall that a poset can be viewed as a category with at most one arrow between any pair of objects: either $x \leq y$, and then there is a unique arrow $x \longrightarrow y$, or $x \nleq y$, and then there is no arrow $x \longrightarrow y$. Composition is defined in the only possible way. By definition $[n]$ is a totally ordered set, hence of course it is a partially ordered set. We can view it as a category and then the monotonic functions $\mu\colon [m] \longrightarrow [n]$ are precisely the functors $[m] \longrightarrow [n]$: monotonicity says that if there is an arrow $i \longrightarrow j$, then there is an arrow $i \leq j$, which must be the value of the functor $\mu$ on that arrow.

**Definition 9.4.3.** Let $\mathscr{C}at$ denote the category whose objects are small categories and whose morphisms are the functors between them. Define a covariant functor $\Delta[\bullet]^c\colon \Delta \longrightarrow \mathscr{C}at$ by sending the ordered set $[n]$ to the corresponding category $[n]$ and sending a morphism $\mu\colon [m] \longrightarrow [n]$ to the corresponding functor $\mu_*\colon [m] \longrightarrow [n]$. Thus $\Delta[\bullet]^c$ is a cosimplicial category. When necessary for clarity, we write $[n]^c$ for the ordered set $[n]$ regarded as a category.

It is consistent with our previous notations to write $\Delta[n]^c$ for the poset $[n]$ regarded as a category. With that notation, the analogy with the definition of the total singular complex becomes especially obvious.

**Definition 9.4.4.** Let $\mathscr{C}$ be a small category. We define a simplicial set $N\mathscr{C}$, called the nerve of $\mathscr{C}$. Its set $N_n\mathscr{C}$ of $n$-simplices is the set of covariant functors $\phi\colon [n]^c \longrightarrow \mathscr{C}$. The function $\mu^*\colon N_n\mathscr{C} \longrightarrow N_m\mathscr{C}$ induced by $\mu\colon [m] \longrightarrow [n]$ is given by $\mu^*(\phi) = \phi \circ \mu$, where $\mu$ is viewed as a functor $[m]^c \longrightarrow [n]^c$. A functor $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ induces a function $F_n = N_nF\colon N_n \longrightarrow N_n\mathscr{D}$ by composition of functors, $F_n(\phi) = F \circ \phi$. These functions specify a map $F_* = NF\colon N\mathscr{C} \longrightarrow N\mathscr{D}$ of simplicial sets. Thus we the nerve functor $N$ from $\mathscr{C}at$ to the category of simplicial sets.

The definition can easily be unravelled. The category $[0]^c$ has one object and its identity morphism, hence a functor $\phi\colon [0]^c \longrightarrow \mathscr{C}$ is just a choice of an object of $\mathscr{C}$. That is, if we write $\mathscr{O}\mathscr{C}$ for the set of objects of $\mathscr{C}$, then $N_0\mathscr{C} = \mathscr{O}\mathscr{C}$. For $n \geq 1$, a functor $\phi\colon [n]^c \longrightarrow \mathscr{C}$ is a choice of $n$ composable morphisms

$$c_0 \xrightarrow{f_1} c_1 \longrightarrow \cdots \longrightarrow c_{n-1} \xrightarrow{f_n} c_n.$$

Denoting such a string by $(f_1, \cdots, f_n)$, the faces and degeneracies are given by

$$(9.4.5) \qquad d_i(f_1, \cdots, f_n) = \begin{cases} (f_2, \cdots, f_n) & \text{if } i = 0 \\ (f_1, \cdots, f_{i-1}, f_{i+1} \circ f_i, f_{i+2}, \cdots, f_n) & \text{if } 0 < i < n \\ (f_1, \cdots, f_{n-1}) & \text{if } i = n \end{cases}$$

$$s_i(f_1, \cdots, f_n) = (f_1, \cdots, f_{i-1}, \mathrm{id}, f_i, \cdots, f_n)$$

In words, the $0^{\text{th}}$ and $n^{\text{th}}$ faces send $(f_1, \cdots, f_n)$ to the $(n-1)$-simplex obtained by deleting $f_1$ or $f_n$; when $n = 1$ this is to be interpreted as giving the object $c_1$ or $c_0$. For $0 < i < n$, the $i^{\text{th}}$ face composes $f_{i+1}$ with $f_i$. The $i^{\text{th}}$ degeneracy operation inserts the identity morphism of $c_i$. The ordering may look unnatural, since $f_{i+1} \circ f_i$ means first $f_i$ and then $f_{i+1}$, and many authors prefer to reverse the ordering in a composable sequence so that for $n \geq 1$, a functor $\phi\colon [n]^c \longrightarrow \mathscr{C}$ is a choice of $n$ composable morphisms

$$c_0 \xleftarrow{f_1} c_1 \longleftarrow \qquad \cdots \longleftarrow c_{n-1} \xleftarrow{f_n} c_n.$$

This amounts to replacing the categories $\Delta[n]^c$ by their opposite categories. It is the choice taken in the following hugely important example.

**Example 9.4.6.** Let $G$ be a group regarded as a category with a single object $*$; the elements of the group are the morphisms $* \longrightarrow *$, and every pair of morphisms is composable. The nerve $NG$ is often written $B_*G$ and called the bar construction. It is the simplicial set with $B_nG = G^n$, with $n$-tuples of elements written $[g_1|\cdots|g_n]$ (hence the name "bar") and with faces and degeneracies specified for $0 \leq i \leq n$ by

$$d_i[g_1|\cdots|g_n] = \begin{cases} [g_2|\cdots|g_n] & \text{if } i = 0 \\ [g_1|\cdots|g_{i-1}|g_ig_{i+1}|g_{i+2}|\cdots|g_n] & \text{if } 0 < i < q \\ [g_1|\cdots|g_{n-1}] & \text{if } i = q. \end{cases}$$

$$s_i[g_1|\cdots|g_n] = [g_1|\cdots|g_{i-1}|e|g_i|\cdots|g_n]$$

However $N\mathscr{A}$ is written, in general it looks nothing like our original example of the simplicial set associated to an ordered simplicial complex! In one important case, which we will find is far more common than one might reasonably expect, it does look like that.

**Example 9.4.7.** Let $X$ be a poset. We can obtain a simplicial set by regarding $X$ as a category and taking its nerve. Alternatively, we can take the ordered simplicial complex $\mathscr{K}X$ and then take the simplicial set associated to that. It is an instructive exercise to check that we get the same simplicial set via either route. That is, $NX$ is naturally isomorphic to $(\mathscr{K}X)^s$.

## 9.5. The geometric realization of simplicial sets

We have observed that the category $\Delta$ is generated by the injections $\delta_i$ and surjections $\sigma_i$. Decomposing a morphism $\mu\colon [m] \longrightarrow [n]$ as a composite of $\delta_i$'s and $\sigma_j$'s records which elements of the target $[n]$ are not in the image of $\mu$ and which elements of the source $[m]$ have the same image under $\mu$. It is helpful to be more precise about this. Let $i_1, \cdots, i_q$ in reverse order $0 \leq i_q < \cdots < i_1 \leq n$ be the elements of $[n]$ that are not in the image $\mu([m])$. Let $j_1, \ldots, j_p$ in order $0 \leq j_1 < \cdots < j_p < m$ be the elements $j \in [m]$ such that $\mu(j) = \mu(j+1)$. With these notations, $m - p + q = n$ and

$$\mu = \delta_{i_1} \cdots \delta_{i_q} \sigma_{j_1} \cdots \sigma_{j_p}. \tag{9.5.1}$$

That is, we record duplications in such a manner that the indices record the repeated and skipped elements in a sensible canonical order. The sequences of $i$'s and $j$'s in this description of $\mu$ are uniquely determined.

Using this canonical decomposition implicitly, we can be precise about the definition and description of the geometric realization of a simplicial set $K$. The construction is precisely analogous to Construction 10.7.4 and might well be denoted by $T^tK$.

**Construction 9.5.2.** For a set $J$ and a space $L$, we regard $J$ as a discrete topological space and obtain the space $J \times L$. Applying this with $J = K_n$ and $L = \Delta[n]^t$ for $n \geq 0$, we obtain the space

$$\bar{K} = \coprod_{n \geq 0} K_n \times \Delta[n]^t$$

with the topology of the union. That is, we take the union of one topological simplex for each $n$-simplex $k \in K_n$. Say that an $n$-simplex $k$ is degenerate if $k = s_i \ell$ for some $(n-1)$-simplex $\ell$ and some $i$ and nondegenerate otherwise. We shall glue the simplices together in such a way that we obtain a space with one "$n$-cell" for each nondegenerate $n$-simplex of $K$. That means in particular that in the resulting space every point will be the interior point of the image of exactly one simplex $\{k\} \times \Delta[n]^t$, where $k$ is nondegenerate. Note that the unique point of $\Delta[0]$ is an interior point. We say that a point $(k, u)$ of $\bar{K}$ is nondegenerate if $k$ is nondegenerate and $u$ is interior.

Define an equivalence relation $\approx$ on $\bar{K}$ by letting

$$(\mu^* k, u) \approx (k, \mu_* u)$$

for each $k \in K_n$, $u \in \Delta[m]$, and $\mu \colon [m] \longrightarrow [n]$. This equivalence relation is generated by the relations obtained by specializing to $\mu = \delta_i$ or $\mu = \sigma_i$. These can be rewritten as

$$(d_i k, u) \approx (k, \delta_i u) \quad \text{and} \quad (s_i k, u) \approx (k, \sigma_i u).$$

Each $n$-simplex $k_n$ can be written uniquely in the form $k_n = s_{j_p} \cdots s_{j_1} k_{n-p}$, where $k_{n-p}$ is nondegenerate and $0 \leq j_1 < \cdots < j_p < n$. Define a function $\lambda \colon \bar{K} \longrightarrow \bar{K}$ by

$$\lambda(k_n, u_n) = (k_{n-p}, \sigma_{j_1} \cdots \sigma_{j_p} u_n)$$

where $u_n \in \Delta[n]^t$. Similarly, every $u_n \in \Delta[n]^t$ can be written uniquely in the form $u_n = \delta_{i_q} \cdots \delta_{i_1} u_{n-q}$, where $u_{n-q}$ is interior and $0 \leq i_q < \cdots < i_1 \leq n$. Define a function $\rho \colon \bar{K} \longrightarrow \bar{K}$ by

$$\rho(k_n, u_n) = (d_{i_q} \cdots d_{i_1} k_n, u_{n-q}).$$

**Lemma 9.5.3.** *The composite $\lambda \circ \rho$ carries each point of $\bar{K}$ into the unique nondegenerate point that is equivalent to it.*

Define the geometric realization of $K$, which is usually denoted $|K|$ but which we shall usually denote by $TK$, to be the set of equivalence classes $\bar{K}/(\approx)$. Define $F_pTK$ to be the image of $\coprod_{0 \leq n \leq p} K_n \times \Delta[n]$ in $TK$ and give it the quotient space topology. Then topologize $TK$ by giving it the topology of the union of the $F_pTK$. This means that a subset $C$ is closed if and only if it intersects each $F_pTK$ in a closed subset. We shall shortly give an equivalent description of this topology.

## 9.6. CW complexes

We explain the nature of the space $TK$ by introducing two equivalent definitions of a CW complex. We start with the original 1949 definition of J.H.C. Whitehead [68], which explains the name. We then observe that $TK$ satisfies the specifications of that definition. Finally, we give the more modern and now standard definition of a CW complex. Let $D^n$ be the disc $\{x||x| \leq 1\} \subset \mathbb{R}^n$.

**Definition 9.6.1.** A *cell complex* is a Hausdorff space $X$ such that $X$ is a disjoint union of subspaces $e^n$, called "open cells", each of which is homeomorphic to an open disc $\mathring{D}^n$. The closure of $e^n$ in $X$ is denoted $\bar{e}^n$, and it is *not* required to be homeomorphic to the closed disc $D^n$. Rather, for each open cell $e^n$, there must be a map $\bar{j}\colon \Delta[n] \longrightarrow \bar{e}^n$ such that

(i) The restriction of $\bar{j}$ maps $\mathring{\Delta}[n]$ homeomorphically onto $e^n$.
(ii) The restriction of $\bar{j}$ maps the boundary $\partial\Delta[n]$ into the union of the cells of dimension less than $n$.

A subcomplex $A$ of $X$ is a union of some of the cells of $X$ such that if $e^n \subset A$, then $\bar{e}^n \subset A$. A cell complex is a CW complex if

(i) $X$ is Closure finite, meaning that each $\bar{e}^n$ is contained in a finite subcomplex.
(ii) $X$ has the Weak topology, meaning that a subset is closed if and only if its intersection with each $\bar{e}^n$ is a closed subspace.

The capitalized C and W are the source of the name "CW complex", but this form of the definition is so rarely used nowadays that younger experts often have no idea where the name came from. However, it is convenient for describing $TK$.

**Theorem 9.6.2.** *The space $TK$ is a CW complex with one $n$-cell for each nondegenerate $n$-simplex $k_n \in K_n$.*

Proof. The $n$-cells $e^n$ of $TK$ are the images of the subspaces $\{k_n\} \times \mathring{\Delta}[n]$, and the map $j\colon \Delta[n] \longrightarrow \bar{e}^n$ is the restriction of the map $\bar{K} \longrightarrow TK$ to $\{k_n\} \times \mathring{\Delta}[n]$. The topology of the union we prescribed before is in fact the "weak topology". It is "weak" in the sense that in general it has more open sets than the quotient space topology, but the novice may not want to worry about the verification, preferring to simply accept that our original definition of the topology gives what once upon a time was called the weak topology. □

Here is the modern redefinition of a CW complex.

**Definition 9.6.3.** A CW complex is a space $X$ that is the union of an expanding sequence of subspaces $X^n$, where $X^n$ is called the $n$-skeleton of $X$. It is required inductively that

(1) $X^0$ is a set with the discrete topology.
(2) $X^{n+1}$ is constructed from $X^n$ as a "pushout"

$$\begin{array}{ccc} \coprod S^n & \xrightarrow{\ j\ } & X^n \\ {\scriptstyle\cap}\downarrow & & \downarrow \\ \coprod D^{n+1} & \xrightarrow[\ \bar{j}\ ]{} & X^{n+1}. \end{array}$$

This means that $X^{n+1}$ is the quotient space

$$X^n \cup_{\amalg S^n} (\amalg D^{n+1}) \equiv X^n \amalg (\amalg D^{n+1})/(\approx)$$

specified by the equivalence relation $s \approx j(x)$ for $s \in S^n \subset D^{n+1}$.

The space $X$ is given the topology of the union; equivalently, a subset is closed if its intersection with each closed cell $\bar{j}(D^n)$ is closed.

We leave it as an exercise for the reader to see that the two definitions of a CW complex give exactly the same spaces. The compactness of the spheres that are the domains of attaching maps ensures that a CW complex with the second definition is closure finite, as required in the first definition.

The intuition is that we glue discs $D^{n+1}$ to $X^n$ as dictated by attaching maps defined on their boundaries $S^n$. The attaching maps can be quite badly behaved. For an ordered simplicial complex $K$, the classical geometric realization $|K|$ is homeomorphic to the geometric realization $T(K^s)$ of its associated simplicial set $K^s$. This is visually apparent since each has an $n$-cell for each $n$-simplex of $K$. Remember that the $n$-simplices of $K$ itself are of the form $\{x_0 < \cdots < x_n\}$ whereas the elements of $K_n$ are of the form $\{x_0 \leq \cdots \leq x_n\}$. The degeneracy identifications in the construction of $TK^s$ serve to eliminate the degenerate elements in which some of the vertices are repeated.

In $T(K^s)$ the closed cells are homeomorphic to $\Delta[n]$ and the attaching maps are homeomorphisms on boundaries. Spaces can be "triangulated" as CW complexes using many fewer cells than are required for polyhedral triangulations. For example, we can triangulate the $n$-sphere $S^n$ as a CW complex with just two cells. Clearly $S^0$ is a CW complex with two 0-cells, or vertices. For $n > 0$, we start with a single 0-cell $*$, take $(S^n)^{n-1} = *$ and attach a single $n$-cell with attaching map the trivial map $S^{n-1} \longrightarrow *$. Then the $n$-skeleton is $* \cup_{S^{n-1}} D^n = D^n/S^{n-1}$, which is already homeomorphic to $S^n$.

There is a natural half-way house between simplicial complexes and CW complexes that will later play a role in our study.

**Definition 9.6.4.** A CW complex is *regular* if each of its attaching maps $S^n \longrightarrow X^n$ is a homeomorphism onto its image.

**Remark 9.6.5.** Earlier we neglected to give a precise definition of $|K|$ for a geometric simplicial complex with a possibly infinite number of vertices and thus with possibly infinite dimension: while every simplex has a finite dimension, simplices of all finite dimensions can occur. When $K$ is ordered, we now have such a definition. We just take the geometric realization of the associated simplicial set; the result is a functor from the category of ordered simplicial sets to the category of spaces. When $K$ is finite, $TK^s$ is homeomorphic to $|K|$ as we defined it originally. We can also start with $A$-spaces, alias posets $X$. Then $T\mathscr{K}(X)^s$ gives a composite functor from the category of posets to the category of spaces.

Remember that the product $K \times L$ of ordered simplicial complexes $K$ and $L$ has simplices all subsets of products $\sigma \times \tau$ of simplices, where the ordering on vertices is given by $(x, y) \leq (x', y')$ if $x \leq x'$ and $y \leq y'$.

**Definition 9.6.6.** Define the product $K \times L$ of simplicial sets $K$ and $L$ by letting $(K \times L)_n = K_n \times L_n$, with $d_i = (d_i, d_i)$ and $s_i = (s_i, s_i)$, which implies that $\mu^* = (\mu^*, \mu^*)$ for all morphisms $\mu$ in $\Delta$.

This definition is forced by two considerations. First, it ensures the consistency statement $(K\times L)^s \cong K^s\times L^s$. That is, if we start with ordered simplicial complexes $K$ and $L$, then the simplicial set $(K \times L)^s$ is naturally isomorphic to the product simplicial set $K^s \times L^s$. Second, the definition is dictated by the universal property that we require of products in any category. Recall that the $n$-simplices of $K \times L$ involve repeated vertices of $K$ and $L$. These correspond to the use of degeneracy operators in the factors $K^s$ and $L^s$ of the associated simplicial set. It clarifies matters to be precise about this. We state the following lemma for general simplicial sets $K$ and $L$, but the reader should think about what it is saying when we apply it to $K^s$ and $L^s$ for ordered simplicial complexes $K$ and $L$.

**Lemma 9.6.7.** *Let $K$ and $L$ be simplicial sets. The nondegenerate $n$-simplices of $K \times L$ can be written uniquely in the form*

$$(s_{i_p}\cdots s_{i_1}k, s_{j_q}\cdots s_{j_1}\ell),$$

*where $k$ is a nondegenerate $(n-p)$-simplex of $K$, $\ell$ is a nondegenerate $(n-q)$-simplex of $L$, $i_1 < \cdots < i_p$, $j_1 < \cdots < j_q$, and the sets $\{i_a\}$ and $\{j_b\}$ are disjoint.*

The set $\{i_a\} \cup \{j_b\}$ has $p + q$ elements and corresponds to a $(p, q)$ shuffle permutation of a set with $p+q$ elements. The term "shuffle" comes from thinking of a permutation of a deck of $p+q$ cards that starts with a cut into $p$ cards and $q$ cards, which are kept in order by the permutation. The reader will easily see that when we started with posets $X$ and $Y$ and showed that $\mathscr{K}(X \times Y)$ is a subdivision of $\mathscr{K}(X) \times \mathscr{K}(L)$, we were actually verifying an instance of essentially this lemma. From here, the reader will have no trouble believing the following result, the proof of which amounts to appropriately subdividing topological simplices $\Delta[n]^t$.

**Theorem 9.6.8.** *For simplicial sets $K$ and $L$, the map*

$$T(K \times L) \longrightarrow TK \times TL$$

*whose coordinates are the maps $T\pi_1$ and $T\pi_2$ induced by the projections of $K \times L$ on $K$ and $L$ is a homeomorphism.*

We shall not repeat the proof, which adds precision and decreases intuition, referring the reader, for example, to [**45**, 14.3] or [**26**, 4.3.15] for details. The latter book is especially recommended as a very good and relatively recent treatment of CW complexes, simplicial complexes, and simplicial sets.

CHAPTER 10

# Simplicial sets again

### 10.1. The adjoint relationship between $S$ and $T$

now out of order due to last year reorganization

It has long been known that we can use simplicial sets pretty much interchangeably with topological spaces when studying homotopy theory. We sketch how this is seen through the categorical eyes of an adjunction. For a simplicial set $K$, we have defined a space $|K| = TK$, called the geometric realization of $K$. We write $|k, u|$ for the image of $(k, u)$ in $TK$, where $k \in K_n$ and $u \in \Delta[n]$. For a space $X$, we have defined a simplicial set $SX$, called the total singular complex of $X$, whose $n$-simplices are the continuous maps $f\colon \Delta[n]^t \longrightarrow X$. The homotopical behavior is studied through an adjunction: $T$ and $S$ are left and right adjoint functors in the sense that we have just defined. That is, there is a bijection, natural in both variables, between morphism sets

$$\mathscr{U}(TK, X) \cong s\mathscr{S}et(K, SX).$$

It is specified by letting $f\colon TK \longrightarrow X$ correspond to $g\colon K \longrightarrow SX$ if

$$f(|k, u|) = g(k)(u).$$

There is an equivalent way of saying this. Define $\gamma : TSX \longrightarrow X$ by

$$\gamma|f, u| = f(u) \quad \text{for} \quad f : \Delta_n \longrightarrow X \quad \text{and} \quad u \in \Delta_n.$$

It is a fact that $\gamma$ is a weak homotopy equivalence for every space $X$, although we shall not prove that here. There is also a map $\iota\colon K \longrightarrow STK$ of simplicial sets specified by $\iota(k)(u) = |k, u|$ for $k \in K_n$ and $u \in \Delta_n$. Again, as we also shall not prove, $|\iota|\colon |K| \longrightarrow |STK|$ is a homotopy equivalence. These facts are proven, for example, in [45]. The natural composite

$$SX \xrightarrow{\iota S} STSX \xrightarrow{S\gamma} SX$$

is the identity map of $SX$. The natural composite

$$TK \xrightarrow{T\iota} TSTK \xrightarrow{\gamma T} TK$$

is the identity map of $TK$. Here $\iota S$ means first apply the functor $S$ and then the natural map $\gamma$, and similarly for $\gamma T$. The natural maps $\iota$ and $\gamma$ are the unit and the counit of the adjunction. This means that, in the correspondence above, $f = \gamma \circ Tg$ and $g = Sf \circ \iota$.

### 10.2. The fundamental category functor $\Pi$

It is also known, although this is more recent, that we can use categories pretty much interchangeably with topological spaces when studying homotopy theory. We are going to say quite a lot about this later. This comparison again starts with an adjunction. We have constructed a simplicial set $N\mathscr{C}$ called the nerve of $\mathscr{C}$. We define $B\mathscr{C} = TN\mathscr{C}$. This is called the *classifying space* of the category $\mathscr{C}$. When $G$ is a group regarded as a category with a single object, $BG$ is called the classsifying space of the group $G$. The space $BG$ is often written as $K(G,1)$. It is called an Eilenberg-Mac Lane space. It is characterized (up to homotopy type) as a connected space with $\pi_1(K(G,1)) = G$ and with all higher homotopy groups $\pi_q(K(G,1)) = 0$. A concise summary of how that works is in [**48**, §16.5]. More generally, a detailed study of the classifying spaces of topological groups and what they classify is in [**46**]. These are fundamentally important constructions in topology and its applications.

The nerve functor $N$ is accompanied by a functor $\Pi\colon s\mathscr{S}et \longrightarrow \mathscr{C}at$, called the "fundamental category" functor.[1] It is left adjoint to $N$, meaning that

$$\mathscr{C}at(\Pi K, \mathscr{C}) \cong s\mathscr{S}et(K, N\mathscr{C}).$$

This means that it is conceptually sensible, but, in contrast to such functors as $S$ and $T$, it does not have good homotopical properties, as we shall see.

For a simplicial set $K$, the objects of the category $\Pi K$ are the vertices (that is, the 0-simplices) of $K$. To construct the morphisms, one starts by thinking of the 1-simplices $y$ as maps $d_1 y \longrightarrow d_0 y$. One forms all words (formal composites) that make sense, that is, whose targets and sources match up. One then imposes the relations on morphisms determined by

$$s_0 x = \mathrm{id}_x \quad \text{for} \quad x \in K_0 \quad \text{and} \quad d_1 z = d_0 z \circ d_2 z \quad \text{for} \quad z \in K_2.$$

We use the relations $d_i d_j = d_{j-1} d_i$ for $i < j$ when $(i,j)$ is $(0,1)$, $(1,2)$, and $(0,2)$ to see that sources and targets match up. This makes good sense since if $K = N\mathscr{C}$, then a 0-simplex is an object $x$ of $\mathscr{C}$, a 1-simplex $y$ is a map $d_1 y \longrightarrow d_0 y$, the 1-simplex $s_0 x$ is $\mathrm{id}_x$, and a 2-simplex $z$ is given by a pair of composable morphisms $d_2 z$ and $d_0 z$ together with their composite $d_1 z$.

Therefore there is a natural map $\varepsilon\colon \Pi N\mathscr{C} \longrightarrow \mathscr{C}$ that is the identity on objects (the zero simplices of $N\mathscr{C}$) and is induced by the identity map from the generating morphisms of $\Pi\mathscr{N}\mathscr{C}$ (the 1-simplices on $N\mathscr{C}$) to the morphisms of $\mathscr{C}$. In fact, $\varepsilon$ is an isomorphism of categories: it is the identity on objects, and it presents the category in terms of generators given by the morphism sets modulo relations determined by the category axioms.

For the adjunction, a functor $F\colon \Pi K \longrightarrow \mathscr{C}$ is constructed from a map of simplicial sets $g\colon K \longrightarrow N\mathscr{C}$ by letting $F$ be the unique functor that agrees with $g$ on objects (= 0-simplices) and equivalence classes of morphisms (= 1-simplices). Applying the adjunction to the identity map of $\Pi K$, we obtain a natural map $\eta\colon K \longrightarrow N\Pi K$, which is the unit of the adjunction, and the counit is the isomorphism $\varepsilon$.

### 10.3. The Yoneda lemma and the structure of simplicial sets

We give a construction that is a precise categorical analogue of the geometric realization of a simplicial set, and we use the Yoneda lemma to prove that it gives an amusing way of reconstructing $K$ categorically. This kind of result is actually very useful in algebraic geometry, but we use it both to illustrate categorical ideas and to prepare for a later conceptual construction of the subdivision functor on simplicial sets.

[1] There is no fully standard notation for this category. I've seen it denoted $\tau_1$, $\pi_1$, $\pi$, and $C$.

Recall that we defined the standard simplicial $n$-simplex $\Delta[n]^s$ to be the simplicial set whose $q$-simplices are the monotonic functions $\sigma\colon [q] \longrightarrow [n]$; precomposition with monotonic functions $\xi\colon [p] \longrightarrow [q]$ gives the required contravariant functoriality on $\Delta$. The nondegenerate $q$-simplices in $\Delta[n]^s$ are the monomorphisms (= strictly monotonic functions) $[q] \longrightarrow [n]$, and there is one for each subset of $[n]$ of cardinality $q+1$. We may identify the set of all non-degenerate simplices with the poset of non-empty subsets of the set $[n]$ of $n+1$ elements, ordered by inclusion. In other words, $\Delta[n]^s = (\mathscr{K}([n]))^s$ is the ordered simplicial set determined by the simplicial complex $\mathscr{K}([n])$. A monotonic function $\alpha\colon [m] \longrightarrow [n]$ gives a map $\alpha\colon \Delta[m]^s \longrightarrow \Delta[n]^s$ of simplicial sets that sends $\sigma\colon [q] \longrightarrow [m]$ to $\alpha \circ \sigma$. Thus $\Delta[-]^s$ is a covariant functor from $\Delta$ to simplicial sets.

earlier

For a set $C$ and a simplicial set $L$, one can form a new simplicial set $C \times L$ by letting $(C \times L)_q = C \times L_q$, and similarly letting the faces and degeneracies be induced by those of $L$. A simplicial set $K$ can be reconstructed from the disjoint union over $n$ of the simplicial sets $K_n \times \Delta[n]$ for $n \geq 0$ by taking equivalence classes under the equivalence relation generated by

$$(\alpha^*(k), \sigma) \simeq (k, \alpha_*(\sigma)) \tag{10.3.1}$$

for $k \in K_n$, $\sigma \in \Delta[m]^s_q$, and $\alpha\colon [m] \longrightarrow [n]$ in $\Delta$. Here $\alpha^*(k) \in K_m$ is given by the fact that $K$ is a contravariant functor from $\Delta$ to sets and $\alpha_*(\sigma) \in \Delta[n]_q$ is given by the fact that $\Delta[-]$ is a covariant functor from $\Delta$ to simplicial sets. The simplicial structure is induced from the simplicial structure on the $\Delta[n]$. The point is that an arbitrary pair $(k, \tau)$ in $K_n \times \Delta[n]_q$ is equivalent to the pair $(\tau(k), \iota_q)$ in $K_q \times \Delta[q]_q$, where $\iota_q\colon [q] \longrightarrow [q]$ is the identity map viewed as a canonical $q$-simplex in $\Delta[q]$, and $\tau\colon [q] \longrightarrow [n]$ is viewed as a morphism of $\Delta$, so that $\tau = \tau_*(\iota_q)$. Identifying equivalence classes of $q$-simplices with elements of $K_q$ in this faction, we find that the faces and degeneracies agree. Indeed, for $\xi\colon [p] \longrightarrow [q]$, $\xi \circ \iota_p = \iota_q \circ \xi$ and

$$(k, \xi^*(\iota_q)) = (k, \xi_*(\iota_p)) \simeq (\xi^*(k), \iota_p).$$

MORE TO COME: $K \cong K \otimes_\Delta \Delta^s$

### 10.4. Tensor products of functors?

Give the idea, relate to geometric realization of simplicial spaces and $K \cong K \otimes_\Delta \Delta^s$. Motivate by coming analogy with subdivision.

QUESTION: Does the canonical map $\Delta' \longrightarrow \Delta$ define a map of cosimplicial simplicial sets. For each $n$, it is a map of simplicial sets, and it is natural, so surely yes! See Definition 4.4.12, Proposition 4.4.16, Remark 4.4.17.

Add in: see §9 examples

### 10.5. Motivation for the introduction of simplicial sets

Simplicial sets, and more generally simplicial objects in a given category, are central to modern mathematics. While I am not a mathematical historian, I thought I would describe in conceptual outline how naturally simplicial sets arise from the classical study of simplicial complexes. I suspect that something like this recapitulates the historical development.

We have described simplicial complexes in several different forms: abstract simplicial complexes, ordered simplicial complexes, geometric simplicial complexes, ordered geometric simplicial complexes and realizations of geometric simplicial complexes. It is possible to go directly from abstract simplicial complexes to realizations without passing through geometric simplicial complexes, but the construction is perhaps not as intuitive and will not be included.

An abstract simplicial complex is equivalent to a geometric simplicial complex, and neither of these notions involves anything about ordering the vertices. If one has a simplicial complex of either type, one can choose a partial ordering of the vertices that restricts to a linear ordering of the vertices of each simplex, and this gives the notion of an ordered simplicial complex. This can be done most simply, but not most generally, just by choosing a total ordering of the set of all vertices and restricting that ordering to simplices. However, there is no canonical choice.

We have seen in studying products of simplicial complexes that geometric realization behaves especially nicely only in the ordered setting. Both the category $\mathscr{SC}$ of simplicial complexes and the category $\mathscr{OSC}$ of ordered simplicial complexes have categorical products. Geometric realization preserves products when defined on $\mathscr{OSC}$, but it does not preserve products when defined on $\mathscr{SC}$. The functor $\mathscr{K}$ is best viewed as a functor from the category $\mathscr{P}$ of partially ordered sets to the category $\mathscr{OSC}$ rather than just to the category $\mathscr{SC}$. Observe that there are generally many different ordered simplicial complexes with the same poset of vertices. The functor $\mathscr{K}$ picks out the largest choice, the one in which every finite totally ordered subset of the set of vertices is a simplex.

The functor $\mathscr{X}$, on the other hand, starts in $\mathscr{SC}$ and lands in $\mathscr{P}$, which can be identified with the category of $A$-spaces. The composite $\mathscr{K}\mathscr{X}$ is the barycentric subdivision functor $Sd\colon \mathscr{SC} \longrightarrow \mathscr{OSC}$. It can be viewed as the construction of a canonical ordered simplicial complex $SdK$ starting from a given unordered simplicial complex $K$, at the price of subdividing. Since the geometric realization functor gives a space $|SdK|$ that can be identified with $|K|$ there is no loss of topological generality working in $\mathscr{OSC}$ instead of $\mathscr{SC}$.

The most important motivation for working with ordered rather than unordered simplicial complexes is that the ordering leads to the definition of an associated chain complex and thus to a quick definition of homology. I'll explain that in the talks and add it to the notes if I have time. promise

As noted earlier, a topological space $X$ is called a polytope if it is homeomorphic to $|K|$ for a (given) simplicial complex $K$. Such a homeomorphism $|K| \longrightarrow X$ is called a triangulation of $X$, and $X$ is said to be triangulable if it admits a triangulation. Then we can define the homology of $X$ to be the homology of $K$. This is a quick definition, and useful where it applies, but it raises many questions and is quite unsatisfactory conceptually. Not every space is triangulable, and triangulable spaces can admit many different triangulations. It is far from obvious that the homology is independent of the choice of triangulation.

Simplicial sets abstract the notion of ordered simplicial complexes, retaining enough of the combinatorial structure that homology can be defined with equal ease. The generalization allow myriads of examples that do not come from simplicial complexes. The original motivating example gives a functor from topological spaces to simplicial sets. Composing with the functor from simplicial sets to homology groups gives the quickest way of defining the homology groups of a space and leads to the proof that these groups depend only on the weak homotopy type of the space, not on any triangulation, and to the proofs that different triangulations, when they exist, give canonically isomorphic homology groups.

Perhaps the quickest and most intuitive way to motivate the definition of simplicial sets is to start from structure clearly visible in the case of ordered simplicial complexes. Let $X$ denote the partially ordered set $V(K)$ of vertices of an ordered simplicial complex $K$. The reader might prefer to start with an ordered simplicial complex of the form $\mathscr{K}(X)$, where $X$ is a poset. The reader may also want to insist that $X$ is finite, but that is not necessary to the construction, and we later want to allow infinite sets.

Then an $n$-simplex $\sigma$ of $K$ is a totally ordered $n+1$-tuple of elements of $X$. Write such a tuple as $(x_0, \cdots, x_n)$. When studying products, we saw that it can become essential to consider tuples $(x_0, \cdots, x_n)$, where $x_0 \leq x_1 \leq \cdots \leq x_n$. Of course, $(x_0, \cdots, x_n)$ is no longer a simplex, but one can obtain a simplex from it by deleting repeated entries. When there are repeated entries, we think of $(x_0, \cdots, x_n)$ as a "degenerate" $n$-simplex. Let $K_n$ denote the set of such generalized $n$-simplices, degenerate or not. For $0 \leq i \leq n$, define functions

$$d_i \colon K_n \longrightarrow K_{n-1} \quad \text{and} \quad s_i \colon K_n \longrightarrow K_{n+1},$$

called face and degeneracy operators, by

$$d_i(x_0, \cdots, x_n) = (x_0, \cdots x_{i-1}, x_{i+1}, \cdots, x_n)$$

and

$$s_i(x_0, \cdots, x_n) = (x_0, \cdots x_i, x_i, \cdots, x_n).$$

Of course, the $d_i$ and $s_i$ just defined also depend on $n$, but it is standard not to indicate that in the notation. In words, $d_i$ deletes the $i^{\text{th}}$ entry and $s_i$ repeats the $i^{\text{th}}$ entry. If $i < j$ and we first delete the $j^{\text{th}}$ entry and then the $i^{\text{th}}$ entry, we get the same thing as if we first delete the $i^{\text{th}}$ entry and then delete the (new) $(j-1)^{\text{st}}$ entry. Similarly, elementary inspections give commutation relations between the $d_i$ and $s_j$ and between the $s_i$. Here is a list of all such relations:

$$d_i \circ d_j = d_{j-1} \circ d_i \quad \text{if} \ \ i < j$$

$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & \text{if} \ \ i < j \\ \text{id} & \text{if} \ \ i = j \ \ \text{or} \ \ i = j+1 \\ s_j \circ d_{i-1} & \text{if} \ \ i > j+1 \end{cases}$$

$$s_i \circ s_j = s_{j+1} \circ s_i \quad \text{if} \ \ i \leq j$$

The reader can easily check that these identities really do follow immediately from the definition of the $K_n$, $d_i$, and $s_i$ above.

The $K_n$ are defined in terms of the partially ordered vertex set $V(K)$ of $K$, but there are many examples of precisely similar structure that arise differently.

## 10.6. The definition of simplicial sets

We obtain our first definition of simplicial sets by formalizing structure that, as we have just seen, is implicit in the definition of an ordered simplicial complex.

**Definition 10.6.1.** A simplicial set $K$ is a sequence of sets $K_n$, $n \geq 0$, and functions $d_i \colon K_n \longrightarrow K_{n-1}$ and $s_i \colon K_n \longrightarrow K_{n+1}$ for $0 \leq i \leq n$ that satisfy the identities just displayed. The elements of the set $K_n$ are called $n$-simplices, following the historic precedent of simplicial complexes. Just as if $K$ were a simplicial complex, a map $f \colon K \longrightarrow L$ of simplicial sets is a sequence of functions $f_n \colon K_n \longrightarrow L_n$ such that $f_{n-1} \circ d_i = d_i \circ f_n$ and $f_{n+1} \circ s_i = s_i \circ f_n$. With these objects and morphisms, we have the category $s\mathscr{S}et$ of simplicial sets.

Now our motivating example can be recapitulated in the following statement.

**Proposition 10.6.2.** *There is a canonical functor $i\colon \mathscr{OSC} \longrightarrow s\mathscr{S}et$ from the category of ordered simplicial complexes to the category of simplicial sets. It assigns to an ordered simplicial complex $K$ the simplicial set $K^s$ given by the sequence of sets $K^s_n$ and the functions $d_i$ and $s_i$ defined above. It assigns to a map $f\colon K \longrightarrow L$ of ordered simplicial complexes the map $f^s\colon K^s \longrightarrow L^s$ induced by its map of vertex sets:*

$$f^s_n(x_0, \cdots, x_n) = (f(x_0), \cdots, f(x_n)).$$

*It is a full embedding, meaning that the maps $K \longrightarrow L$ of ordered simplicial complexes map bijectively to the maps $K^s \longrightarrow L^s$ of simplicial sets.*

The identities listed above are hard to remember and do not appear to be very conceptual. The definition admits a conceptual reformulation that may or may not make things clearer, depending on personal taste, but definitely allows many arguments and constructions to be described more clearly and conceptually than would be possible without it. We define the category $\Delta$ of finite ordered sets.

**Definition 10.6.3.** The objects of $\Delta$ are the finite ordered sets $[n]$ with $n+1$ elements $0 < 1 < \cdots < n$. Its morphisms are the monotonic functions $\mu\colon [m] \leq [n]$. This means that $i < j$ implies $\mu(i) \leq \mu(j)$. Define particular monotonic functions

$$\delta_i\colon [n-1] \longrightarrow [n] \quad \text{and} \quad \sigma_i\colon [n+1] \longrightarrow [n]$$

for $0 \leq i \leq n$ by

$$\delta_i(j) = j \ \text{ if } \ j < i \quad \text{and} \quad \delta_i(j) = j+1 \ \text{ if } \ j \geq i$$

and

$$\sigma_i(j) = j \ \text{ if } \ j \leq i \quad \text{and} \quad \sigma_i(j) = j-1 \ \text{ if } \ j > i.$$

In words, $\delta_i$ skips $i$ and $\sigma_i$ repeats $i$.

There are identities for composing the $\delta_i$ and $\sigma_i$ that are "dual" to those for composing the $d_i$ and $s_i$ that appear in the definition of a simplicial set. Precisely, the duality amounts to reversing the direction of arrows. The following pair of commutative diagrams should make clear how to interpret this, where $i < j$.

$$\begin{array}{ccc} K_n & \xrightarrow{d_j} & K_{n-1} \\ {\scriptstyle d_i}\downarrow & & \downarrow{\scriptstyle d_i} \\ K_{n-1} & \xrightarrow[d_{j-1}]{} & K_{n-2} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} [n] & \xleftarrow{\delta_j} & [n-1] \\ {\scriptstyle \delta_i}\uparrow & & \uparrow{\scriptstyle \delta_i} \\ [n-1] & \xleftarrow[\delta_{j-1}]{} & [n-2] \end{array}$$

A moment's reflection should convince the reader that every monotonic function $\mu\colon [m] \longrightarrow [n]$ can be written as a composite of monotonic functions $\delta_i$ and $\sigma_j$ for varying $i$ and $j$. That is, $\mu$ can be obtained by omitting some of the $i$'s and repeating some of the $j$'s. Just as a group can be defined by specifying a set of generators and relations, so a category can often be specified by a set of generating morphisms and relations between their composites. The category $\Delta$ is generated by the $\delta_i$ and $\sigma_i$ subject to our "dual" relations. This leads to the proof of the following reformulation of the notion of a simplicial set. Recall that a contravariant functor $F$ assigns a morphism $FY \longrightarrow FX$ of the target category to each morphism $X \longrightarrow Y$ of the source category.

**Proposition 10.6.4.** *The category of simplicial sets can be identified with the category of contravariant functors $K\colon \Delta \longrightarrow \mathscr{S}et$ and natural transformations between them.*

PROOF. The correspondence is given by viewing the functions $d_i$ and $s_i$ that define a simplicial set as the morphisms of sets induced by the morphisms $\delta_i$ and $\sigma_i$ of the corresponding functor $\Delta \longrightarrow \mathscr{S}et$. It is convenient to write $\mu^*\colon K_n \longrightarrow K_m$ for the function induced by contravariance from a morphism $\mu\colon [m] \longrightarrow [n]$, and then $d_i = \delta_i^*$ and $s_i = \sigma_i^*$. For a map $f$, the corresponding natural transformation is given on the object $[n]$ by the function $f_n$. □

While we do not want to emphasize abstraction in the first instance, we nevertheless cannot resist the temptation to generalize the definition of simplicial sets to simplicial objects in a perfectly arbitrary category. The generalization has a huge number of applications throughout mathematics, and we shall use it when defining homology.

**Definition 10.6.5.** A simplicial object in a category $\mathscr{C}$ is a contravariant functor $K\colon \Delta \longrightarrow \mathscr{C}$. A map $f\colon K \longrightarrow L$ of simplicial objects in $\mathscr{C}$ is a natural tranformation $K \longrightarrow L$; it is given by morphisms $f_n\colon K_n \longrightarrow L_n$ in $\mathscr{C}$. We have the category $s\mathscr{C}$ of simplicial objects in $\mathscr{C}$. By composition of functors and natural transformations, any functor $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ induces a functor $sF\colon s\mathscr{C} \longrightarrow s\mathscr{D}$. By duality, a *covariant* functor $\Delta \longrightarrow \mathscr{C}$ is called a cosimplicial object in $\mathscr{C}$.

## 10.7. Standard simplices and their role

We explain a general conceptual way to relate simplicial sets to "standard simplices". Standard simplices exist in many categories. We have standard simplices in topological spaces, simplicial sets, and even posets and categories. In general, fixing a category $\mathscr{V}$, we often have a standard cosimplicial object in $\mathscr{V}$, that is a certain covariant functor $\Delta[\bullet]^v\colon \Delta \longrightarrow \mathscr{V}$. The superscript $v$ is meant as a reminder that the functor is assigning objects in $\mathscr{V}$ to objects in $\Delta$; it should also help to distinguish the functor $\Delta[\bullet]^v$ from the category $\Delta$. On objects, we write the functor $\Delta[\bullet]^v$ as $[n] \mapsto \Delta[n]^v$, but we agree to write $\mu_*$ rather than $\Delta[\mu]^v$ for the map $\Delta[m]^v \longrightarrow \Delta[n]^v$ in $\mathscr{V}$ obtained by applying our functor to a morphism $\mu$ in $\Delta$. For each object $V$ of $\mathscr{V}$ we obtain a *contravariant* functor, denoted $SV\colon \Delta \longrightarrow \mathscr{S}et$, by letting the set $S_nV$ of $n$-simplices be the set $\mathscr{V}(\Delta[n]^v, V)$ of morphisms $\Delta[n]^v \longrightarrow V$ in the category $\mathscr{V}$. The faces and degeneracies are induced by precomposition with the maps

$$\delta_i\colon \Delta[n-1]^v \longrightarrow \Delta[n]^v \quad \text{and} \quad \sigma_i\colon \Delta[n+1]^v \longrightarrow \Delta[n]^v$$

obtained by applying the functor $\Delta[\bullet]^v$ to the generating morphisms $\delta_i$ and $\sigma_i$ of $\Delta$. That is, for a morphism $\nu\colon \Delta[n]^v \longrightarrow V$ in $\mathscr{V}$,

$$d_i(\nu) = \nu \circ \delta_i \quad \text{and} \quad s_i(\nu) = \nu \circ \sigma_i.$$

Before turning to the motivating examples, in which $\mathscr{V}$ is the category $\mathscr{U}$ of topological spaces or the category $\mathscr{C}at$ of small categories, we apply this construction to the case $\mathscr{V} = s\mathscr{S}et$.

**Definition 10.7.1.** Define the standard simplicial $n$-simplex $\Delta[n]^s$ to be the contravariant functor $\Delta \longrightarrow s\mathscr{S}et$ represented by $[n]$. This means that the set $\Delta[n]^s_q$ of $q$-simplices is the set of all morphisms $\phi\colon [q] \longrightarrow [n]$ in $\Delta$. For a morphism

$\nu\colon [p] \longrightarrow [q]$ in $\Delta$, the function $\nu^*\colon \Delta[n]^s_q \longrightarrow \Delta[n]^s_p$ is given by composition, $\nu^*(\phi) = \phi \circ \nu\colon [p] \longrightarrow [q]$.

**Definition 10.7.2.** We define a covariant functor $\Delta[\bullet]^s$ from $\Delta$ to the category $s\mathscr{S}et$ of simplicial sets. On objects, the functor sends $[n]$ to the standard simplicial $n$-simplex $\Delta[n]^s$. On morphisms $\mu\colon [m] \longrightarrow [n]$ in $\Delta$, define $\mu_*\colon \Delta[m]^s_q \longrightarrow \Delta[n]^s_q$ by $\mu_*(\psi) = \mu \circ \psi\colon [q] \longrightarrow [m] \longrightarrow [n]$. Thus the simplicial set $\Delta[n]^s$ is defined using pre-composition with morphisms of $\Delta$, and then the covariant functoriality of $\Delta[\bullet]^s$ is defined using post-composition with morphisms of $\Delta$. The object $\Delta[\bullet]^v$ is a cosimplicial simplicial set, that is, a cosimplicial object in the category of simplicial sets.

We may identify the set of all non-degenerate simplices of $\Delta[n]^s$ with the poset of non-empty subsets of the set $[n]$ of $n+1$ elements, ordered by inclusion. In other words, $\Delta[n]^s = (\mathscr{K}([n]))^s$ is the ordered simplicial set determined by the simplicial complex $\mathscr{K}([n])$.

Although we shall give a direct proof, the following result is an application of the Yoneda lemma. Let $\iota_n \in \Delta[n]^s_n$ be the identity map $\mathrm{id}\colon [n] \longrightarrow [n]$.

**Proposition 10.7.3.** *Let $K$ be a simplicial set. For $x \in K_n$, there is a unique map of simplicial sets $Y(x)\colon \Delta[n]^s \longrightarrow K$ such that $Y(x)(\iota_n) = x$. Therefore $K$ is naturally isomorphic to the simplicial set whose $n$-simplices are the maps of simplicial sets $\Delta[n]^s \longrightarrow K$.*

Proof. The map $Y(x)$ is a natural transformation from the contravariant functor $\Delta[n]^s$ to the contravariant functor $K$ from $\Delta$ to $\mathscr{S}et$. Since a $q$-simplex $\phi\colon [q] \longrightarrow [n]$ is $\phi^*(\iota_n)$, we can and must specify $Y(x)$ at the object $[q] \in \Delta$ by the function $\Delta[n]^s_q \longrightarrow K_q$ that sends $\phi$ to the $q$-simplex $\phi^*(x)$. □

We can vary the construction in a way that may look unnatural but that will lend itself to generalization to other examples. We show how to reconstruct $K$ directly from the $\Delta[n]^s$.

**Construction 10.7.4.** For a set $J$ and a simplicial set $L$, one can form a new simplicial set $J \times L$ by setting $(J \times L)_q = J \times L_q$ and letting the faces and degeneracies be induced by those of $L$. Said another way, we think of $J$ as a "discrete" simplicial set with each $J_q = J$ and all faces and degeneracies the identity map of $J$, and we then take the product $J \times L$ of simplicial sets. We apply this with $J = K_n$ and $L = \Delta[n]^s$ as $n$ varies to obtain a simplicial set

$$\overline{K} = \coprod_{n\geq 0} K_n \times \Delta[n]^s.$$

We define an equivalence relation $\simeq$ on $\overline{K}$ by requiring

$$(\alpha^*(k), \sigma) \simeq (k, \alpha_*(\sigma)) \tag{10.7.5}$$

for $k \in K_n$, $\sigma \in \Delta[m]^s_q$, and $\alpha\colon [m] \longrightarrow [n]$ in $\Delta$. Here $\alpha^*(k) \in K_m$ is given by the fact that $K$ is a contravariant functor from $\Delta$ to sets and $\alpha_*(\sigma) \in \Delta[n]^s_q$ is given by the fact that $\Delta[-]^s$ is a covariant functor from $\Delta$ to simplicial sets. With the simplicial structure induced from the simplicial structure on the $\Delta[n]^s$, passage to equivalence classes gives us a new simplicial set that we shall denote by $T^sK$ for the moment. Then $T^s$ is a functor from simplicial sets to simplicial sets.

**Proposition 10.7.6.** *The simplicial set $T^sK$ is naturally isomorphic to $K$.*

PROOF. We claim that an arbitrary pair $(k,\tau)$ in $K_n \times \Delta[n]^s_q$ is equivalent to the pair $(\tau(k),\iota_q)$ in $K_q \times \Delta[q]^s_q$ where, as above, $\iota_q\colon [q] \longrightarrow [q]$ is the identity map viewed as a canonical $q$-simplex in $\Delta[q]^s$. Viewing $\tau\colon [q] \longrightarrow [n]$ as a morphism of $\Delta$, we have $\tau = \tau_*(\iota_q)$, and the claim follows. Identifying equivalence classes of $q$-simplices with elements of $K_q$ in this fashion, we find that the faces and degeneracies agree. Indeed, for $\xi\colon [p] \longrightarrow [q]$, $\xi \circ \iota_p = \iota_q \circ \xi$ and

$$(k, \xi^*(\iota_q)) = (k, \xi_*(\iota_p)) \simeq (\xi^*(k), \iota_p). \qquad \square$$

## 10.8. The total singular complex $SX$ and the nerve $N\mathscr{C}$

We turn to the historical motivating example $\mathscr{V} = \mathscr{U}$ by constructing the total singular complex $SX$ of a topological space $X$. We need a covariant functor $\Delta[\bullet]^t\colon \Delta \longrightarrow \mathscr{U}$, and that is given by the standard topological simplices $\Delta[n]^t$.

**Definition 10.8.1.** Recall that the standard topological $n$-simplex $\Delta[n]^t$ is the subspace

$$\{(t_0, \cdots, t_n) \,|\, 0 \le t_i \le 1 \text{ and } \Sigma_i t_i = 1\}$$

of $\mathbb{R}^{n+1}$. Define

$$\delta_i\colon \Delta[n-1]^t \longrightarrow \Delta[n]^t \text{ and } \sigma_i\colon \Delta[n+1]^t \longrightarrow \Delta[n]^t$$

by

$$\delta_i(t_0, \cdots, t_{n-1}) = (t_0, \cdots, t_{i-1}, 0, t_i, \cdots, t_n)$$

and

$$\sigma_i(t_0, \cdots, t_{n+1}) = (t_0, \cdots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \cdots, t_{n+1}).$$

Then the $\delta_i$ and $\sigma_i$ satisfy the commutation relations required to specify a covariant functor $\Delta[\bullet]^t$ from $\Delta$ to the category $\mathscr{U}$ of topological spaces, that is, a cosimplicial object in the category of topological spaces.

**Definition 10.8.2.** The *total singular complex* $SX$ of a space $X$ is the simplicial set whose set $S_nX$ of $n$-simplices is the set of continuous maps $\Delta[n]^t \longrightarrow X$ and whose faces $d_i$ and degeneracies $s_i$ induced by precomposition with $\delta_i$ and $\sigma_i$. By composition of continuous maps, a map $f\colon X \longrightarrow Y$ induces the map $f_* = Sf\colon SX \longrightarrow SY$ of simplicial sets that sends an $n$-simplex $s\colon \Delta[n]^t \longrightarrow X$ to the $n$-simplex $f \circ s$. This defines the total singular complex functor $S$ from topological spaces to simplicial sets.

We shall return to this example after giving an analogue that may seem astonishing at first sight. Although it has become a standard and commonplace construction, its importance and utility were only gradually recognized. Recall that a poset can be viewed as a category with at most one arrow between any pair of objects: either $x \le y$, and then there is a unique arrow $x \longrightarrow y$, or $x \nleq y$, and then there is no arrow $x \longrightarrow y$. Composition is defined in the only possible way. By definition $[n]$ is a totally ordered set, hence of course it is a partially ordered set. We can view it as a category and then the monotonic functions $\mu\colon [m] \longrightarrow [n]$ are precisely the functors $[m] \longrightarrow [n]$: monotonicity says that if there is an arrow $i \longrightarrow j$, then there is an arrow $i \le j$, which must be the value of the functor $\mu$ on that arrow.

**Definition 10.8.3.** Let $\mathscr{C}at$ denote the category whose objects are small categories and whose morphisms are the functors between them. Define a covariant functor $\Delta[\bullet]^c\colon \Delta \longrightarrow \mathscr{C}at$ by sending the ordered set $[n]$ to the corresponding category $[n]$ and sending a morphism $\mu\colon [m] \longrightarrow [n]$ to the corresponding functor $\mu_*\colon [m] \longrightarrow [n]$. Thus $\Delta[\bullet]^c$ is a cosimplicial category. When necessary for clarity, we write $[n]^c$ for the ordered set $[n]$ regarded as a category.

It is consistent with our previous notations to write $\Delta[n]^c$ for the poset $[n]$ regarded as a category. With that notation, the analogy with the definition of the total singular complex becomes especially obvious.

**Definition 10.8.4.** Let $\mathscr{C}$ be a small category. We define a simplicial set $N\mathscr{C}$, called the nerve of $\mathscr{C}$. Its set $N_n\mathscr{C}$ of $n$-simplices is the set of covariant functors $\phi\colon [n]^c \longrightarrow \mathscr{C}$. The function $\mu^*\colon N_n\mathscr{C} \longrightarrow N_m\mathscr{C}$ induced by $\mu\colon [m] \longrightarrow [n]$ is given by $\mu^*(\phi) = \phi \circ \mu$, where $\mu$ is viewed as a functor $[m]^c \longrightarrow [n]^c$. A functor $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ induces a function $F_n = N_nF\colon N_n \longrightarrow N_n\mathscr{D}$ by composition of functors, $F_n(\phi) = F \circ \phi$. These functions specify a map $F_* = NF\colon N\mathscr{C} \longrightarrow N\mathscr{D}$ of simplicial sets. Thus we the nerve functor $N$ from $\mathscr{C}at$ to the category of simplicial sets.

The definition can easily be unravelled. The category $[0]^c$ has one object and its identity morphism, hence a functor $\phi\colon [0]^c \longrightarrow \mathscr{C}$ is just a choice of an object of $\mathscr{C}$. That is, if we write $\mathscr{O}\mathscr{C}$ for the set of objects of $\mathscr{C}$, then $N_0\mathscr{C} = \mathscr{O}\mathscr{C}$. For $n \geq 1$, a functor $\phi\colon [n]^c \longrightarrow \mathscr{C}$ is a choice of $n$ composable morphisms

$$c_0 \xrightarrow{f_1} c_1 \longrightarrow \cdots \longrightarrow c_{n-1} \xrightarrow{f_n} c_n.$$

Denoting such a string by $(f_1, \cdots, f_n)$, the faces and degeneracies are given by

$$(10.8.5) \quad d_i(f_1,\cdots,f_n) = \begin{cases} (f_2,\cdots,f_n) & \text{if } i = 0 \\ (f_1,\cdots,f_{i-1}, f_{i+1}\circ f_i, f_{i+2},\cdots,f_n) & \text{if } 0 < i < n \\ (f_1,\cdots,f_{n-1}) & \text{if } i = n \end{cases}$$

$$s_i(f_1,\cdots,f_n) = (f_1,\cdots,f_{i-1},\mathrm{id},f_i,\cdots,f_n)$$

In words, the $0^{\text{th}}$ and $n^{\text{th}}$ faces send $(f_1, \cdots, f_n)$ to the $(n-1)$-simplex obtained by deleting $f_1$ or $f_n$; when $n = 1$ this is to be interpreted as giving the object $c_1$ or $c_0$. For $0 < i < n$, the $i^{\text{th}}$ face composes $f_{i+1}$ with $f_i$. The $i^{\text{th}}$ degeneracy operation inserts the identity morphism of $c_i$. The ordering may look unnatural, since $f_{i+1} \circ f_i$ means first $f_i$ and then $f_{i+1}$, and many authors prefer to reverse the ordering in a composable sequence so that for $n \geq 1$, a functor $\phi\colon [n]^c \longrightarrow \mathscr{C}$ is a choice of $n$ composable morphisms

$$c_0 \xleftarrow{f_1} c_1 \longleftarrow \qquad \cdots \longleftarrow c_{n-1} \xleftarrow{f_n} c_n.$$

This amounts to replacing the categories $\Delta[n]^c$ by their opposite categories. It is the choice taken in the following hugely important example.

**Example 10.8.6.** Let $G$ be a group regarded as a category with a single object $*$; the elements of the group are the morphisms $* \longrightarrow *$, and every pair of morphisms is composable. The nerve $NG$ is often written $B_*G$ and called the bar construction.

It is the simplicial set with $B_nG = G^n$, with $n$-tuples of elements written $[g_1|\cdots|g_n]$ (hence the name "bar") and with faces and degeneracies specified for $0 \le i \le n$ by

$$d_i[g_1|\cdots|g_n] = \begin{cases} [g_2|\cdots|g_n] & \text{if } i=0 \\ [g_1|\cdots|g_{i-1}|g_ig_{i+1}|g_{i+2}|\cdots|g_n] & \text{if } 0<i<q \\ [g_1|\cdots|g_{n-1}] & \text{if } i=q. \end{cases}$$

$$s_i[g_1|\cdots|g_n] = [g_1|\cdots|g_{i-1}|e|g_i|\cdots|g_n]$$

However $N\mathscr{A}$ is written, in general it looks nothing like our original example of the simplicial set associated to an ordered simplicial complex! In one important case, which we will find is far more common than one might reasonably expect, it does look like that.

**Example 10.8.7.** Let $X$ be a poset. We can obtain a simplicial set by regarding $X$ as a category and taking its nerve. Alternatively, we can take the ordered simplicial complex $\mathscr{K}X$ and then take the simplicial set associated to that. It is an instructive exercise to check that we get the same simplicial set via either route. That is, $NX$ is naturally isomorphic to $(\mathscr{K}X)^s$.

## 10.9. The geometric realization of simplicial sets

We have observed that the category $\Delta$ is generated by the injections $\delta_i$ and surjections $\sigma_i$. Decomposing a morphism $\mu\colon [m] \longrightarrow [n]$ as a composite of $\delta_i$'s and $\sigma_j$'s records which elements of the target $[n]$ are not in the image of $\mu$ and which elements of the source $[m]$ have the same image under $\mu$. It is helpful to be more precise about this. Let $i_1, \cdots, i_q$ in reverse order $0 \le i_q < \cdots < i_1 \le n$ be the elements of $[n]$ that are not in the image $\mu([m])$. Let $j_1, \ldots, j_p$ in order $0 \le j_1 < \cdots < j_p < m$ be the elements $j \in [m]$ such that $\mu(j) = \mu(j+1)$. With these notations, $m - p + q = n$ and

$$\mu = \delta_{i_1}\cdots\delta_{i_q}\sigma_{j_1}\cdots\sigma_{j_p}. \tag{10.9.1}$$

That is, we record duplications in such a manner that the indices record the repeated and skipped elements in a sensible canonical order. The sequences of $i$'s and $j$'s in this description of $\mu$ are uniquely determined.

Using this canonical decomposition implicitly, we can be precise about the definition and description of the geometric realization of a simplicial set $K$. The construction is precisely analogous to Construction 10.7.4 and might well be denoted by $T^tK$.

**Construction 10.9.2.** For a set $J$ and a space $L$, we regard $J$ as a discrete topological space and obtain the space $J \times L$. Applying this with $J = K_n$ and $L = \Delta[n]^t$ for $n \ge 0$, we obtain the space

$$\bar{K} = \coprod_{n \ge 0} K_n \times \Delta[n]^t$$

with the topology of the union. That is, we take the union of one topological simplex for each $n$-simplex $k \in K_n$. Say that an $n$-simplex $k$ is degenerate if $k = s_i\ell$ for some $(n-1)$-simplex $\ell$ and some $i$ and nondegenerate otherwise. We shall glue the simplices together in such a way that we obtain a space with one "$n$-cell" for each nondegenerate $n$-simplex of $K$. That means in particular that in the resulting space every point will be the interior point of the image of exactly one simplex $\{k\} \times \Delta[n]^t$, where $k$ is nondegenerate. Note that the unique point of

$\Delta[0]$ is an interior point. We say that a point $(k,u)$ of $\bar{K}$ is nondegenerate if $k$ is nondegenerate and $u$ is interior.

Define an equivalence relation $\approx$ on $\bar{K}$ by letting

$$(\mu^* k, u) \approx (k, \mu_* u)$$

for each $k \in K_n$, $u \in \Delta[m]$, and $\mu\colon [m] \longrightarrow [n]$. This equivalence relation is generated by the relations obtained by specializing to $\mu = \delta_i$ or $\mu = \sigma_i$. These can be rewritten as

$$(d_i k, u) \approx (k, \delta_i u) \quad \text{and} \quad (s_i k, u) \approx (k, \sigma_i u).$$

Each $n$-simplex $k_n$ can be written uniquely in the form $k_n = s_{j_p} \cdots s_{j_1} k_{n-p}$, where $k_{n-p}$ is nondegenerate and $0 \leq j_1 < \cdots < j_p < n$. Define a function $\lambda\colon \bar{K} \longrightarrow \bar{K}$ by

$$\lambda(k_n, u_n) = (k_{n-p}, \sigma_{j_1} \cdots \sigma_{j_p} u_n)$$

where $u_n \in \Delta[n]^t$. Similarly, every $u_n \in \Delta[n]^t$ can be written uniquely in the form $u_n = \delta_{i_q} \cdots \delta_{i_1} u_{n-q}$, where $u_{n-q}$ is interior and $0 \leq i_q < \cdots < i_1 \leq n$. Define a function $\rho\colon \bar{K} \longrightarrow \bar{K}$ by

$$\rho(k_n, u_n) = (d_{i_q} \cdots d_{i_1} k_n, u_{n-q}).$$

**Lemma 10.9.3.** *The composite $\lambda \circ \rho$ carries each point of $\bar{K}$ into the unique nondegenerate point that is equivalent to it.*

Define the geometric realization of $K$, which is usually denoted $|K|$ but which we shall usually denote by $TK$, to be the set of equivalence classes $\bar{K}/(\approx)$. Define $F_pTK$ to be the image of $\coprod_{0 \leq n \leq p} K_n \times \Delta[n]$ in $TK$ and give it the quotient space topology. Then topologize $TK$ by giving it the topology of the union of the $F_pTK$. This means that a subset $C$ is closed if and only if it intersects each $F_pTK$ in a closed subset. We shall shortly give an equivalent description of this topology.

## 10.10. CW complexes

We explain the nature of the space $TK$ by introducing two equivalent definitions of a CW complex. We start with the original 1949 definition of J.H.C. Whitehead [68], which explains the name. We then observe that $TK$ satisfies the specifications of that definition. Finally, we give the more modern and now standard definition of a CW complex. Let $D^n$ be the disc $\{x \mid |x| \leq 1\} \subset \mathbb{R}^n$.

**Definition 10.10.1.** A *cell complex* is a Hausdorff space $X$ such that $X$ is a disjoint union of subspaces $e^n$, called "open cells", each of which is homeomorphic to an open disc $\mathring{D}^n$. The closure of $e^n$ in $X$ is denoted $\bar{e}^n$, and it is *not* required to be homeomorphic to the closed disc $D^n$. Rather, for each open cell $e^n$, there must be a map $\bar{j}\colon \Delta[n] \longrightarrow \bar{e}^n$ such that

(i) The restriction of $\bar{j}$ maps $\mathring{\Delta}[n]$ homeomorphically onto $e^n$.
(ii) The restriction of $\bar{j}$ maps the boundary $\partial\Delta[n]$ into the union of the cells of dimension less than $n$.

A subcomplex $A$ of $X$ is a union of some of the cells of $X$ such that if $e^n \subset A$, then $\bar{e}^n \subset A$. A cell complex is a CW complex if

(i) $X$ is Closure finite, meaning that each $\bar{e}^n$ is contained in a finite subcomplex.
(ii) $X$ has the Weak topology, meaning that a subset is closed if and only if its intersection with each $\bar{e}^n$ is a closed subspace.

The capitalized C and W are the source of the name "CW complex", but this form of the definition is so rarely used nowadays that younger experts often have no idea where the name came from. However, it is convenient for describing $TK$.

**Theorem 10.10.2.** *The space $TK$ is a CW complex with one $n$-cell for each nondegenerate $n$-simplex $k_n \in K_n$.*

PROOF. The $n$-cells $e^n$ of $TK$ are the images of the subspaces $\{k_n\} \times \mathring{\Delta}[n]$, and the map $j\colon \Delta[n] \longrightarrow \bar{e}^n$ is the restriction of the map $\bar{K} \longrightarrow TK$ to $\{k_n\} \times \mathring{\Delta}[n]$. The topology of the union we prescribed before is in fact the "weak topology". It is "weak" in the sense that in general it has more open sets than the quotient space topology, but the novice may not want to worry about the verification, preferring to simply accept that our original definition of the topology gives what once upon a time was called the weak topology. □

Here is the modern redefinition of a CW complex.

**Definition 10.10.3.** A CW complex is a space $X$ that is the union of an expanding sequence of subspaces $X^n$, where $X^n$ is called the $n$-skeleton of $X$. It is required inductively that

(1) $X^0$ is a set with the discrete topology.
(2) $X^{n+1}$ is constructed from $X^n$ as a "pushout"

$$\begin{array}{ccc} \coprod S^n & \xrightarrow{\ j\ } & X^n \\ \cap\downarrow & & \downarrow \\ \coprod D^{n+1} & \xrightarrow[\ \bar{j}\ ]{} & X^{n+1}. \end{array}$$

This means that $X^{n+1}$ is the quotient space

$$X^n \cup_{\amalg S^n} (\amalg D^{n+1}) \equiv X^n \amalg (\amalg D^{n+1})/(\approx)$$

specified by the equivalence relation $s \approx j(x)$ for $s \in S^n \subset D^{n+1}$.

The space $X$ is given the topology of the union; equivalently, a subset is closed if its intersection with each closed cell $\bar{j}(D^n)$ is closed.

We leave it as an exercise for the reader to see that the two definitions of a CW complex give exactly the same spaces. The compactness of the spheres that are the domains of attaching maps ensures that a CW complex with the second definition is closure finite, as required in the first definition.

The intuition is that we glue discs $D^{n+1}$ to $X^n$ as dictated by attaching maps defined on their boundaries $S^n$. The attaching maps can be quite badly behaved. For an ordered simplicial complex $K$, the classical geometric realization $|K|$ is homeomorphic to the geometric realization $T(K^s)$ of its associated simplicial set $K^s$. This is visually apparent since each has an $n$-cell for each $n$-simplex of $K$. Remember that the $n$-simplices of $K$ itself are of the form $\{x_0 < \cdots < x_n\}$ whereas the elements of $K_n$ are of the form $\{x_0 \le \cdots \le x_n\}$. The degeneracy identifications in the construction of $TK^s$ serve to eliminate the degenerate elements in which some of the vertices are repeated.

In $T(K^s)$ the closed cells are homeomorphic to $\Delta[n]$ and the attaching maps are homeomorphisms on boundaries. Spaces can be "triangulated" as CW complexes using many fewer cells than are required for polyhedral triangulations. For example, we can triangulate the $n$-sphere $S^n$ as a CW complex with just two cells. Clearly $S^0$ is a CW complex with two 0-cells, or vertices. For $n > 0$, we start with a single 0-cell $*$, take $(S^n)^{n-1} = *$ and attach a single $n$-cell with attaching map the trivial map $S^{n-1} \longrightarrow *$. Then the $n$-skeleton is $* \cup_{S^{n-1}} D^n = D^n/S^{n-1}$, which is already homeomorphic to $S^n$.

There is a natural half-way house between simplicial complexes and CW complexes that will later play a role in our study.

**Definition 10.10.4.** A CW complex is *regular* if each of its attaching maps $S^n \longrightarrow X^n$ is a homeomorphism onto its image.

**Remark 10.10.5.** Earlier we neglected to give a precise definition of $|K|$ for a geometric simplicial complex with a possibly infinite number of vertices and thus with possibly infinite dimension: while every simplex has a finite dimension, simplices of all finite dimensions can occur. When $K$ is ordered, we now have such a definition. We just take the geometric realization of the associated simplicial set; the result is a functor from the category of ordered simplicial sets to the category of spaces. When $K$ is finite, $TK^s$ is homeomorphic to $|K|$ as we defined it originally. We can also start with $A$-spaces, alias posets $X$. Then $T\mathscr{K}(X)^s$ gives a composite functor from the category of posets to the category of spaces.

Remember that the product $K \times L$ of ordered simplicial complexes $K$ and $L$ has simplices all subsets of products $\sigma \times \tau$ of simplices, where the ordering on vertices is given by $(x, y) \leq (x', y')$ if $x \leq x'$ and $y \leq y'$.

**Definition 10.10.6.** Define the product $K \times L$ of simplicial sets $K$ and $L$ by letting $(K \times L)_n = K_n \times L_n$, with $d_i = (d_i, d_i)$ and $s_i = (s_i, s_i)$, which implies that $\mu^* = (\mu^*, \mu^*)$ for all morphisms $\mu$ in $\Delta$.

This definition is forced by two considerations. First, it ensures the consistency statement $(K \times L)^s \cong K^s \times L^s$. That is, if we start with ordered simplicial complexes $K$ and $L$, then the simplicial set $(K \times L)^s$ is naturally isomorphic to the product simplicial set $K^s \times L^s$. Second, the definition is dictated by the universal property that we require of products in any category. Recall that the $n$-simplices of $K \times L$ involve repeated vertices of $K$ and $L$. These correspond to the use of degeneracy operators in the factors $K^s$ and $L^s$ of the associated simplicial set. It clarifies matters to be precise about this. We state the following lemma for general simplicial sets $K$ and $L$, but the reader should think about what it is saying when we apply it to $K^s$ and $L^s$ for ordered simplicial complexes $K$ and $L$.

**Lemma 10.10.7.** *Let $K$ and $L$ be simplicial sets. The nondegenerate $n$-simplices of $K \times L$ can be written uniquely in the form*

$$(s_{i_p} \cdots s_{i_1} k, s_{j_q} \cdots s_{j_1} \ell),$$

*where $k$ is a nondegenerate $(n-p)$-simplex of $K$, $\ell$ is a nondegenerate $(n-q)$-simplex of $L$, $i_1 < \cdots < i_p$, $j_1 < \cdots < j_q$, and the sets $\{i_a\}$ and $\{j_b\}$ are disjoint.*

The set $\{i_a\} \cup \{j_b\}$ has $p + q$ elements and corresponds to a $(p, q)$ shuffle permutation of a set with $p + q$ elements. The term "shuffle" comes from thinking of a permutation of a deck of $p + q$ cards that starts with a cut into $p$ cards and $q$ cards, which are kept in order by the permutation. The reader will easily see that when we started with posets $X$ and $Y$ and showed that $\mathscr{K}(X \times Y)$ is a subdivision of $\mathscr{K}(X) \times \mathscr{K}(L)$, we were actually verifying an instance of essentially this lemma.

From here, the reader will have no trouble believing the following result, the proof of which amounts to appropriately subdividing topological simplices $\Delta[n]^t$.

**Theorem 10.10.8.** *For simplicial sets $K$ and $L$, the map*

$$T(K \times L) \longrightarrow TK \times TL$$

*whose coordinates are the maps $T\pi_1$ and $T\pi_2$ induced by the projections of $K \times L$ on $K$ and $L$ is a homeomorphism.*

We shall not repeat the proof, which adds precision and decreases intuition, referring the reader, for example, to [**45**, 14.3] or [**26**, 4.3.15] for details. The latter book is especially recommended as a very good and relatively recent treatment of CW complexes, simplicial complexes, and simplicial sets.

CHAPTER 11

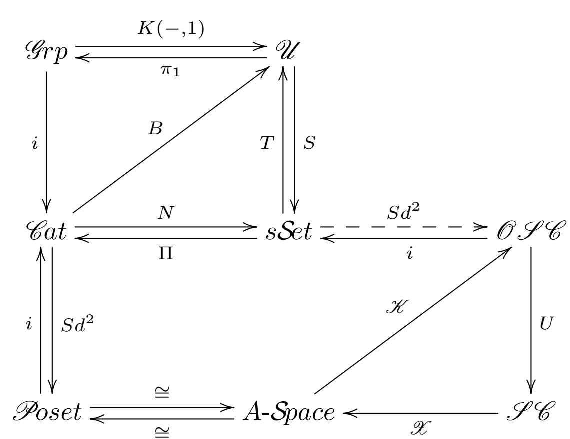
# The big picture: a schematic diagram and the role of subdivision

The $n$-skeleton $K^n$ of a simplicial set $K$ is the subsimplicial set generated by the $q$-simplices for all $q \leq n$. Visibly, $\Pi K$ depends only on the 2-skeleton $K^2$. Therefore the inclusion $K^2 \longrightarrow K$ of simplicial sets induces an isomorphism of categories $\Pi K^2 \longrightarrow \Pi K$ for any $K$. In particular, $\Pi$ takes the inclusion $\iota\colon \partial\Delta[n]^s \longrightarrow \Delta[n]^s$ of the boundary of the $n$-simplex to the identity functor when $n > 2$. Thus $\Pi$ loses homotopical information: upon realization, $|\iota|$ is equivalent to the inclusion $S^{n-1} \longrightarrow D^n$. What is amazing is that this extreme loss of information disappears after subdividing twice. This is something I have been trying to better understand for quite some time.

The reader will find it easy to believe that there is a subdivision functor on simplicial sets that generalizes the subdivision functor $Sd$ on simplicial complexes in the sense that $(SdK)^s \cong Sd(K^s)$ for a simplicial complex $K$. This allows one to define a subdivision functor on categories by setting $Sd\mathscr{C} = \Pi Sd N\mathscr{C}$. One can iterate subdivision, forming functors $Sd^2$ on both simplicial sets and categories. What is mind blowing at first is that the iterated subdivision $Sd^2\mathscr{C}$ is actually a poset whose classifying space $BSd^2\mathscr{C}$ is homotopy equivalent to $B\mathscr{C}$. I will start from a more combinatorial definition of $Sd\mathscr{C}$, and I will use it to give what I hope the reader will find an easy combinatorial proof that $Sd^2\mathscr{C}$ is indeed a poset.

However, before heading for that, let us summarize a schematic and technically oversimplified global picture of all of the big categories that we are constructing and comparing by functors. This is the same diagram as in the introduction, and it gives an interesting picture of lots of kinds of mathematics that come together with a focus on simplicial sets.

Add left adjoint to $i$, from Cat to Poset?



Our earlier work focused on finite spaces, but the basic theory generalizes with the finiteness removed, provided we understand simplicial complexes to mean abstract simplicial complexes. As noted above, we didn't define geometric realization in general earlier, but we have done so now. The equivalence of posets with $A$-spaces and the constructions $\mathscr{K}$ and $\mathscr{X}$ that we worked out in detail for finite spaces work in exactly the same way when we no longer restrict ourselves to the finite case. The functors $i$ in the diagram are thought of as inclusions of categories. Remember that we write $i(K) = K^s$ for the simplicial set associated to an ordered simplicial complex. We have defined all of the categories and functors exhibited in the diagram except for $Sd^2$, which is second subdivision.

Describe features of the diagram: posets vs ordered simplicial complexes (latter: some but not all totally ordered subsets of the poset of vertices. [Said earlier]) Remember no canonical ordering, $u$ cannot be a right adjoint, etc.

CHAPTER 12

# Subdivision and Properties $A$, $B$, and $C$ in $s\mathscr{S}et$

We shall define three properties of a simplicial set, called Properties $A$, $B$, and $C$. We say that a category satisfies property $A$, $B$, or $C$ if its nerve satisfies that property. Remember that the nerve functor $N$ is a right adjoint whose left adjoint is the fundamental category functor $\Pi$. We shall define the subdivision of a simplicial set in such a way as to generalize the subdivision of simplicial complexes that plays such a fundamental role in our study of finite spaces. We shall define the companion notion of the subdivision of a category in the next chapter. We write $\mathrm{Sd}^s$ for the subdivision functor on simplicial sets and $\mathrm{Sd}^c$ for the subdivision functor on categories when necessary for clarity. These are the main characters in our story. We want to understand the relationships between these functors and the rest of the categories and functors in our big picture. There are a number of surprising and interesting implications.

## 12.1. Properties $A$, $B$, and $C$ of simplicial sets

**Definition 12.1.1.** We define and name three properties that a simplicial set might have.

(A) Property $A$, the *nondegenerate simplex property*: $K$ has property $A$ if every face of a nondegenerate simplex $x$ of $K$ is nondegenerate.
(B) Property $B$, the *distinct vertex property*: $K$ has property $B$ if the $n+1$ vertices of any nondegenerate $n$-simplex $x$ of $K$ are distinct.
(C) Property $C$, the *unique simplex property*: $K$ has property $C$ if for any set of $n+1$ distinct vertices of $K$, there is at most one nondegenerate $n$-simplex of $K$ whose vertices are the elements of that set.

**Remark 12.1.2.** In Property $A$, we mean that all faces $d_i x$ are nondegenerate. But then all faces of all $d_i x$ are also nondegenerate. Iterating, all of the face $q$-simplices of $x$ for $q < n$ are nondegenerate.

In line with this remark, there is a less succinct but useful characterization of Property $B$. We express it with a notation that we shall use frequently later.

**Notation 12.1.3.** For a simplex $x \in K_n$ and a (nonempty) subset $S$ of the set $[n] = \{0, 1, \cdots, n\}$, let $S^*x$ denote the simplex $\mu^*x \in K_m$, where $\mu\colon [m] \longrightarrow [n]$ is the unique injection in $\Delta$ with image $S$. Then the cardinality of $S$, which we write as $|S|$, is $m+1$.

**Proposition 12.1.4.** *A simplicial set $K$ has Property $B$ if and only if for every $n$ and every nondegenerate simplex $x \in K_n$, $\mu^*x$ and $\nu^*x$ are distinct simplices of $K$ for every pair $\mu$ and $\nu$ of distinct injections with target $[n]$ in $\Delta$; equivalently, $S^*x \neq T^*x$ for every pair of distinct subsets $S$ and $T$ of $[n]$.*

Proof. Property $B$ is the case when $\mu$ and $\nu$ have source $[0]$, so it is clear that the new property implies Property $B$. For the converse, suppose that $K$ satisfies Property $B$ and that $S^*x = T^*x$ for a nondegenerate simplex $x \in K_n$ and nonempty subsets $S$ and $T$ of $[n]$. This clearly implies that $|S| = |T| = m+1$, say, where $0 \leq m \leq n$. Write $S = \{s_0, \cdots, s_m\}$ and $T = \{t_0, \cdots, t_m\}$, each in strictly increasing order. Consider the singleton subsets $\{i\} \subset [m]$, $\{s_i\} \subset [n]$, and $\{t_i\} \subset [n]$, where $0 \leq i \leq m$. Using the language of Notation 12.1.3, we have

$$\{s_i\}^*x = \{i\}^*S^*x = \{i\}^*T^*x = \{t_i\}^*x.$$

Since these are vertices of $x$, they are equal by Property $B$. This implies that $s_i = t_i$ and thus $S = T$. □

It is natural to ask if there are implications among Properties $A$, $B$, and $C$.

**Theorem 12.1.5.** *Property $B$ implies Property $A$, but there are no other implications between these properties.*

Proof. Suppose that $K$ does not have Property $A$. There is an $n \geq 1$ and a nondegenerate $n$-simplex with a degenerate face. Using the commutation relations between faces and degeneracies, we see that any degenerate simplex has a degenerate 1-simplex as one of its 1-faces. Since both vertices of a degenerate 1-simplex $s_0x$ are $x$, our original nondegenerate $n$-simplex cannot have distinct vertices. The non-implications are proven by exhibiting counterexamples. We choose nerves of categories, so that these non-implications will also be clear for categories. □

**Example 12.1.6.** Here are some examples which exhibit various non-implications.
(i) Let $K = N\mathscr{C}$ where $\mathscr{C}$ is the category with one object $x$ and one non-identity morphism $p$, with $p \circ p = p$. Then $K$ satisfies Property $A$ but not Property $B$.
(ii) Let $K = N\mathscr{C}$, where $\mathscr{C}$ is the category with two vertices $x$ and $y$, two non-identity morphisms $x \longrightarrow y$, and no morphisms $y \longrightarrow x$. Then $K$ satisfies Properties $A$ and $B$ but not Property $C$.
(iii) Let $K = NC_2$, where $C_2$ is the cyclic group of order 2 regarded as a category with one object. Then $K$ satisfies Property $C$ but not Properties $A$ or $B$. For each $q$, $K$ has a unique nondegenerate $q$-simplex $(g, \cdots, g)$, where $g$ is the generator of $C_2$. Since $g^2 = e$, that simplex has a degenerate face when $q \geq 2$.
(iv) More generally, if $K = NC_n$, where $C_n$ is the cyclic group of order $n > 2$ with generator $g$, the simplices $x = (g, \cdots, g) \in K_q$ have all faces $d_ix$ nondegenerate, but iterated face operations reach degenerate simplices when $q \geq n$.

Suspect

Here is a thought exercise. Consider the simplicial set $K^s$ associated to an ordered simplicial complex $K$. Clearly it has all three properties. What about a converse? Recall that there is a natural order on the set of vertices of the standard $n$-simplex $\Delta[n]^s$. After all, they are the $i$ with $0 \leq i \leq n$. Since the set $K_n$ can be identified with the set of simplicial maps $\Delta[n]^s \longrightarrow K_n$, each simplex has an induced ordering of its vertices. It need not be consistent as the simplices vary. We can try to give the set of vertices a partial order that restricts to a total order on each simplex by setting $v \leq w$ if [and only if] $v$ and $w$ are vertices of some simplex $x$ in some $K_n$ and $v \leq w$ in the ordering of the vertices of that simplex [and taking the partial order generated by this relation [to get transitivity]?]

This is not so easy! Transitivity? FALSE!

**Exercise 12.1.7.** Suppose that a simplicial set $K$ satisfies Properties $B$ and $C$. Then $\leq$ is a well-defined partial order on the set $V = K_0$ that restricts to a total order on the vertices of each non-degenerate simplex of $K$. With simplices those finite sets of vertices that are the vertices of some nondegenerate $x \in K_n$, we obtain a simplicial complex $L$, and $K$ is isomorphic to $L^s$. Conversely, if $K$ does not satisfy either Property $B$ or Property $C$, then it cannot be isomorphic to $L^s$ for any simplicial complex $L$.

By abuse of language, we say that a simplicial set is a simplicial complex if it is isomorphic to $L^s$ for some ordered simplicial complex $L$. In fact, $L$ is canonically determined by $K$ in the manner that we have described. The exercise proves the following result.

**Theorem 12.1.8.** *A simplicial set is a simplicial complex if and only if it satisfies Properties $B$ and $C$.*

## 12.2. The definition of the subdivision of a simplicial set

For both simplicial sets and categories, there is both a conceptual definition and an equivalent combinatorial definition. For simplicial sets, we begin with the perhaps ugly looking and hard to grasp combinatorial definition and then show that it is equivalent to a conceptual definition that is closely analogous to the definition of geometric realization.

**Definition 12.2.1.** We define the subdivision $\mathrm{Sd}K = \mathrm{Sd}^s K$ of a simplicial set $K$. The $q$-simplices of $\mathrm{Sd}K_q$ are the equivalence classes of tuples

$$(x; S_0, \cdots, S_q),$$

where, for some $n \geq 0$, $x \in K_n$, each $S_i$ is a subset of $[n]$, and $S_i \subset S_{i+1}$ for $0 \leq i < q$. The equivalence relation is specified by

$$(\mu^* x; S_0, \cdots, S_q) \ \sim \ (x; \mu_*(S_0, \cdots, S_q))$$

for a morphism $\mu\colon [m] \longrightarrow [n]$ in $\Delta$, where $x \in K_n$, hence $\mu^* x \in K_m$; here $\{S_i\}$ is an increasing sequence of subsets of $[m]$ and

$$\mu_*(S_0, \cdots, S_q) = (\mu(S_0), \cdots, \mu(S_q)).$$

The simplicial operations are induced by

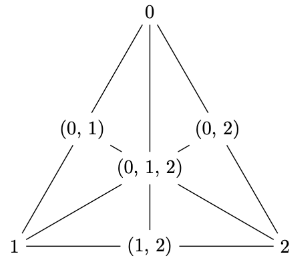
$$\nu^*(x; S_0, \cdots, S_q) = (x; S_{\nu(0)}, \cdots, S_{\nu(p)})$$

for a map $\nu\colon [p] \longrightarrow [q]$ in $\Delta$, where $x \in K_n$ and $\{S_i\}$ is an increasing sequence of subsets of $[n]$ for some $n$. Subdivision is functorial. For a map $f\colon K \longrightarrow L$ of simplicial sets, $f_* = \mathrm{Sd}f\colon \mathrm{Sd}K \longrightarrow \mathrm{Sd}L$ is induced by

$$f_*(x; S_0, \cdots, S_q) = (f(x); S_0, \cdots, S_q).$$

This definition is convenient for doing combinatorics and is directly motivated by the following comparison, which we will prove in §12.3.

**Example 12.2.2.** The following illustrates the subdivision of a 2-simplex.




**Theorem 12.2.3.** *If $K$ is an ordered simplicial complex, then the simplicial sets $\mathrm{Sd}(K^s)$ and $(\mathrm{Sd}K)^s$ are naturally isomorphic.*

However, it obscures the idea behind the definition, which we now elucidate. The conceptual definition parallels Constructions 10.7.4 and 10.9.2. The parallel with the geometric realization functor is particularly useful, but the parallel with the reconstruction functor $T^sK$ is especially illuminating.

Recall that $\Delta[n]^s$ is the represented simplicial set with $q$-simplices the maps $\alpha\colon [q] \longrightarrow [n]$ in $\Delta$. Its nondegenerate simplices are the injections. It is a simplicial complex. That is, it can be viewed as $(\mathscr{K}[n])^s$. As a simplicial complex it has the subdivision studied earlier, which we now regard as a simplicial set and denote by $Sd\Delta[n]^s$. Then the nondegenerate $q$-simplices of $Sd\Delta[n]^s$ are the ordered $q$-tuples $\underline{\alpha} = \{\alpha_0, \cdots, \alpha_q\}$ of $\Delta[n]^s$, where $\alpha_i$ is a face of $\alpha_{i+1}$, so that $\alpha_i$ is obtained from $\alpha_{i+1}$ by precomposition with an injection in $\Delta$. For a map $\nu\colon [p] \longrightarrow [q]$ in $\Delta$, the simplicial operation $\nu^*$ on $Sd\Delta[n]$ is given by

$$\nu^*(\underline{\alpha}) = (\alpha_{\nu(0)}, \cdots, \alpha_{\nu(p)}).$$

As $n$ varies, the subdivisions $Sd\Delta[n]$ define a covariant functor

$$Sd\Delta[\bullet]^s\colon \Delta \longrightarrow s\mathscr{S}et,$$

that is, a cosimplicial simplicial set. For $\mu\colon [m] \longrightarrow [n]$, $\mu_*\colon Sd\Delta[m]^s \longrightarrow Sd\Delta[n]^s$ is given by

$$\mu_*\underline{\alpha} = (\mu \circ \alpha_0, \cdots, \mu \circ \alpha_q).$$

Strictly speaking, to write simplices in terms of injections only, we must interpret $\mu\circ\alpha_i$ as the injective part $\delta$ of the canonical decomposition of $\mu\circ\alpha_i$ as the composite $\delta\sigma$ of a surjection $\sigma$ and an injection $\delta$. Here is the conceptual definition of $\mathrm{Sd}K$.

**Construction 12.2.4.** As in the construction of $T^sK$ given in Construction 10.7.4, regard each set $K_n$ as just a set, or as a discrete simplicial set with each $(K_n)_q = K$ and all faces and degeneracies the identity map. Then form the product simplicial sets $K_n \times Sd\Delta[n]^s$ and take their disjoint union to obtain the simplicial set

$$\overline{SdK} = \coprod_{n\geq 0} K_n \times Sd\Delta[n].$$

Again as in the construction of $T^sK$, define an equivalence relation on $\overline{SdK}$. For $\mu\colon [m] \longrightarrow [n]$ in $\Delta$, we let

$$(\mu^*x, \underline{\alpha}) \sim (x, \mu_*\underline{\alpha}).$$

where $x \in K_n$ and $\underline{\alpha} \in Sd\Delta[m]^s$. We suppress from the notation that this defines an equivalence relation on $q$-simplices for each $q$. Now $(\mathrm{Sd}K)_q$ is the set of equivalence classses of $q$-simplices. The simplicial operations on the simplicial sets $K_n \times Sd\Delta[n]^s$ are of the form id $\times \nu^*$. They induce the simplicial operations on $\mathrm{Sd}K$.

**Remark 12.2.5** (Categorical remark)**.** The definitions of $T^sK$, $\mathrm{Sd}K$ and $TK$ are all examples of "tensor products of functors", often written $K \otimes_\Delta L$ for a contravariant functor $K$ and a covariant functor $L$ defined on $\Delta$ (which could be replaced by any other small category) but we shall not go into the general categorical framework. However, as a specialization of a general categorical result about such categorical tensor products, there is an associativity isomorphism of simplicial sets Will we?

$$(K \otimes_\Delta L) \otimes_\Delta M \cong K \otimes_\Delta (L \otimes_\Delta M)$$

where $K$ is a simplicial set and $L$ and $M$ are cosimplicial simplicial sets. Inductively, this implies that

$$Sd^n K \cong K \otimes_\Delta Sd^n \Delta[-] = \coprod_n K_n \times Sd^n\Delta[n]/(\sim),$$

where the equivalence relation is defined exactly as in Construction 12.2.4. This gives a good hold on these functors, since $Sd^n\Delta[-] = (\mathscr{K}^{(n)}\Delta[-])^s$ is just the classical iterated barycentric subdivision, regarded as a simplicial set.

To reconcile the combinatorial and conceptual definitions of $\mathrm{Sd}K$, observe that injective maps $\alpha$ in $\Delta$ are uniquely determined by their images. The $q$-tuples $(\alpha_0, \cdots, \alpha_q)$ of injections above can just as well be viewed as the $q$-tuples $(S_0, \cdots, S_q)$ of the images of the $\alpha_i$, which are increasing sequences of subsets of $[n]$ for some $n$. After this replacement, the two definitions coincide. Observe that the degenerate simplices of $Sd\Delta[n]^s$ are those for which $S_i = S_{i+1}$ for some $i$.

The conceptual definition is the one best suited for the proof of the following basic result.

**Theorem 12.2.6.** *The geometric realization of a simplicial set $K$ is homeomorphic to the geometric realization of* $\mathrm{Sd}K$*, but there is no natural simplicial map between the two that realizes the homeomorphism. There is a natural map of simplicial sets* $\mathrm{Sd}K \longrightarrow K$ *that induces a homotopy equivalence* $T\mathrm{Sd}K \longrightarrow TK$*.*

Proof. We compare $SdK$ with the simplicial set isomorphic to $K$ given by Proposition 10.7.6. That simplicial set is constructed from $K$ and the $\Delta[n]$ rather than from $K$ and the $Sd\Delta[n]$. The standard homeomorphisms between the $|\Delta[n]|$ and the $|Sd\Delta[n]|$ induce the claimed homeomorphism between $|K|$ and $|\mathrm{Sd}K|$.

The standard maps of simplicial sets $\xi\colon Sd\Delta[n]^s \longrightarrow \Delta[n]^s$ given by Definition 4.4.12 together specify a map $\xi\colon Sd\Delta[\bullet]^s \longrightarrow \Delta[\bullet]^s$ of cosimplicial simplicial sets since they are natural, as observed in Remark 4.4.17. Using the conceptual definition of $\mathrm{Sd}K$ and the description of $K$ as $T^sK$ in Proposition 10.7.6, we see that $\xi$ induces a natural map of simplicial sets $\xi\colon SdK \longrightarrow K$. The geometric realization of the maps $\xi\colon Sd\Delta[n]^s \longrightarrow \Delta[n]^s$ are homotopy equivalences by Proposition 4.4.11. It follows that the induced map $T\xi\colon T\mathrm{Sd}K \longrightarrow TK$ is a homotopy equivalence. The proof of the implication is just a bit beyond the scope of this book; an old reference is [**?**, A.4(ii)]. The idea is that application of the maps $\xi$ gives a map that by inspection of the filtrations of $T\mathrm{Sd}K$ and $TK$ can be proven to be a local weak homotopy equivalence, so that Theorem 3.3.1 gives that $T\xi$ is a weak homotopy equivalence. Since it is a map between CW complexes, it is a homotopy equivalence. □

## 12.3. Combinatorial properties of subdivision

We use the combinatorial definition to derive some basic combinatorial properties of subdivision.

**Definition 12.3.1.** A $q$-simplex $(x; S_0, \cdots, S_q)$ of $SdK$ is in *minimal form* if $x \in K_n$ is nondegenerate and $S_q = [n]$.

**Proposition 12.3.2.** *Every simplex of* $\mathrm{Sd}K$ *is equivalent to a unique simplex in minimal form. When so written, a simplex is degenerate if and only if* $S_i = S_{i+1}$ *for some* $i$*.*

Proof. Conceptually, this is analogous to the description of the points of the geometric realization $TK$ in nondegenerate form. We think of $q$-simplices of $Sd\Delta[n]^s$ as "interior" if $S_q = [n]$, and we then use the same canonical form for morphisms of $\Delta$ as composites of $\sigma$'s and $\delta$'s that we used to prove the analogue for realization. If we start with an element $(y; T_1, \cdots, T_q)$ with $y \in K_p$, $T_i \subset [p]$ and $|T_q| = m+1$, we have a unique injection $\delta\colon [m] \longrightarrow [p]$ such that $\delta([m]) = T_q$. There are unique subsets $R_i$ of $[m]$ such that $\delta(R_i) = T_i$, and $(y; T_1, \cdots, T_q)$ is equivalent to $(\delta^* y; R_1, \cdots, R_q)$, where $R_q = [m]$. Now there is a surjection $\sigma\colon [m] \longrightarrow [n]$ and a nondegenerate simplex $x$ of $K_n$ such that $\sigma^* x = \delta^* y$. Then $(\delta^* y; R_1, \cdots, R_q)$ is equivalent to $(x; S_1, \cdots, S_q)$ where $S_i = \sigma^*(R_i)$. By the surjectivity of $\sigma$, $S_q = [n]$. It is left as an exercise to check that this process reaches the unique minimal element equivalent to the element we started with.

Now suppose that $z = (x; S_1, \cdots, S_q)$ is in minimal form. If $S_i = S_{i+1}$, then $z$ is certainly degenerate. We must show that if $z$ is degenerate, then some $S_i = S_{i+1}$. The assumption means that $z$ is equivalent to $z' = (y; T_0, \cdots, T_q)$, where $T_j = T_{j+1}$ for some $j$. However, unlike $z$, $z'$ might not be in minimal form. Just as above, let $y \in K_p$, so that the $T_i$ are contained in $[p]$. Let $|T_q| = m+1$ and choose an injection $\delta\colon [m] \longrightarrow [p]$ such that $\delta([m]) = T_q$. Define $R_i = \delta^{-1}(T_i)$ for all $i$ and note that $R_q = [m]$. Then $z'$ is equivalent to $z'' = (\delta^* y; R_0, \cdots, R_q)$. Now let $\delta^* y = \sigma^* w$ where $\sigma$ is a surjection and $w \in K_n$ is nondegenerate. Then $z''$ is equivalent to $(w; \sigma(R_0), \cdots, \sigma(R_q))$. This simplex is in minimal form since $\sigma([m]) = [n]$, so it must be $z$. Thus $x = w$ and $S_i = \sigma(R_i) = \sigma_i \delta^{-1}(T_i)$. Since $T_j = T_{j+1}$, $S_j = S_{j+1}$. This proves the result. □

**Corollary 12.3.3.** *Let* $x \in K_n$ *be nondegenerate. Then there is a nondegenerate* $q$*-simplex* $y_q$ *in* $\mathrm{Sd}K$ *with* $q$*th vertex* $(x; [n])$ *if and only if* $q \le n$*.*

Proof. If $q \le n$, set $y_q = (x; [n-q], [n-q+1], \cdots, [n])$. Then $y_q$ is in minimal form and nondegenerate, and its $q$th vertex is $(x; [n])$. Conversely, if we have a nondegenerate $y_q$ with $q$th vertex $(x; [n])$, then, in minimal form, we must have $y_q = (x; S_0, \cdots, S_{q-1}, S_q)$ with $S_i$ strictly contained in $S_{i+1}$ for $0 \le i < n$ and $S_q = [n]$. Clearly that implies $q \le n$. □

Proof of Theorem 12.2.3. The nondegenerate $q$ simplices of the barycentric subdivision $\mathrm{Sd}K$ are the strictly increasing chains $\sigma_0 \subset \cdots \subset \sigma_q$ of faces of a simplex. If $\sigma_q$ has cardinality $n+1$, its elements specify a nondegenerate $n$-simplex $x$ of $K^s$. Viewing $x$ as a map $\Delta[n] \longrightarrow K^s$ via Proposition 10.7.3, the inverse images of the $\sigma_i$ specify an increasing sequence of subsets $S_i$ of $[n]$ with $S_q = [n]$. The rest is an exercise about the description of elements of $Sd^s(K^s)$ in minimal form. □

## 12.4. Subdivision and Properties $A$, $B$, and $C$ of simplicial sets

Here is how subdivision relates to Properties $A$, $B$, and $C$.

**Theorem 12.4.1.** *Subdivision of simplicial sets has the following properties.*

*(i) $K$ has Property $A$ if and only if* $\mathrm{Sd}K$ *has Property $A$.*
*(ii) $K$ has Property $A$ if and only if* $\mathrm{Sd}K$ *has Property $B$.*
*(iii) $K$ has Property $B$ if and only if* $\mathrm{Sd}K$ *has Property $C$.*

The following two corollaries are immediate.

**Corollary 12.4.2.** *If $K$ does not have Property $A$, then* $\mathrm{Sd}^n K$ *does not have any of the three properties for any $n \geq 1$. If $K$ does have property $A$, then* $\mathrm{Sd}^n K$ *has all three properties for all $n \geq 2$.*

**Corollary 12.4.3.** *$K$ has Property $A$ if and only if* $\mathrm{Sd}^2 K$ *has Property $C$, and then* $\mathrm{Sd}^2 K$ *also has Property $B$.*

Now the following very satisfactory theorem follows directly from Theorem 12.1.8.

**Theorem 12.4.4.** *A simplicial set $K$ satisfies Property $A$ if and only* $\mathrm{Sd}^2 K$ *is a simplicial complex.*

We might also ask whether our properties shed light on the question of whether or not a simplicial complex is the nerve of a category. We have the following complement to the previous result. It is an analogue of the fact that the subdivision of a simplicial complex is a poset. We will prove it later, in §13.6.

**Theorem 12.4.5.** *A simplicial set satisfies Property $A$ if and only if* $\mathrm{Sd}K$ *is the nerve of a category, namely the category* $\Pi\mathrm{Sd}K$.

The last clause is a consequence of the following general observation.

**Proposition 12.4.6.** *If a simplicial set $K$ is isomorphic to $N\mathscr{C}$ for some category $\mathscr{C}$, then the category $\mathscr{C}$ is isomorphic to $\Pi K$.*

Proof. If $K \cong N\mathscr{C}$, then $\Pi K \cong \Pi N\mathscr{C} \cong \mathscr{C}$. □

Since ordered simplicial complexes satisfy Property $A$ when regarded as simplicial sets, Theorem 12.4.5 has the following result as a special case. It says that the subdivision of a simplicial complex is the nerve of a category. Remarkably, this appears to be a new result.

**Theorem 12.4.7.** *If $K$ is an ordered simplicial complex, then* $\mathrm{Sd}(K^s)$ *is isomorphic to $N\Pi Sd(K^s)$.*

## 12.5. The proof of Theorem 12.4.1

Since Property $B$ implies Property $A$, by Theorem 12.1.5, the following two implications prove both (i) and (ii) of Theorem 12.4.1.

Proof that if $\mathrm{Sd}K$ has Property $A$, then so does $K$. Suppose for a contradiction that we have a nondegenerate $x \in K_n$ with a degenerate face $d_i x = s_j z$, where $z \in K_{n-2}$. Recall that $d_j s_j = \mathrm{id}$. In $\mathrm{Sd}K$, we have the 2-simplex[1]

$$(x; \delta_i\delta_j[n-2], \delta_i[n-1], [n]).$$

[1]Here and below, we write $\alpha[n]$ to denote the set $\alpha([n])$.

It is written in minimal form and is nondegenerate. Its last face is the 1-simplex

$$(x;\delta_i\delta_j[n-2],\delta_i[n-1]) \sim (d_i x;\delta_j[n-2],[n-1]) = (s_j z;\delta_j[n-2],[n-1]) \sim (z;[n-2],[n-2])$$

since $\sigma_j\delta_j = \mathrm{id}$ and $\sigma_j\colon [n-2] \longrightarrow [n-2]$ is a surjection. This simplex is in minimal form and degenerate, which contradicts the assumption that Sd$K$ has Property $A$. □

Proof that if $K$ has Property $A$, then Sd$K$ has Property $B$. Consider a nondegenerate $q$-simplex $y = (x;S_0,\cdots,S_q)$ written in minimal form. For some $n$, $x \in K_n$ is nondegenerate and the $S_i$ give a strictly increasing sequence of subsets of $[n]$, with $S_q = [n]$. The vertices of $y$ are the $(x;S_i)$. Suppose that $(x;S_i) \sim (x;S_j)$ where $0 \le i < j \le q$. Let $\mu\colon [m_i] \longrightarrow [n]$ and $\nu\colon [m_j] \longrightarrow [n]$ be injective maps in $\Delta$ with images $S_i$ and $S_j$, respectively. Then

$$(\mu^*x;[m_i]) \sim (x;S_i) \sim (x;S_j) \sim (\nu^*x;[m_j]).$$

Since $K$ has Property $A$, the faces $\mu^*x$ and $\nu^*x$ are nondegenerate. Therefore, by the uniqueness of the minimal form, we must have $m_i = m_j$. Since $S_i \subset S_j$, this implies that $S_i = S_j$. The contradiction proves that Sd$K$ has Property $B$. □

Finally, the following two implications prove (iii) of Theorem 12.4.1.

Proof that if $K$ has Property $B$, then Sd$K$ has Property $C$. Let

$$z_1 = (x;S_0,\cdots,S_q) \text{ and } z_2 = (y;T_0,\cdots,T_q)$$

be nondegenerate $q$-simplices of Sd$K$ that have the same set of $q+1$ distinct vertices. We must show that $z_1 = z_2$. We may assume without loss of generality that $z_1$ and $z_2$ are in minimal form, with $x \in K_m$, $S_q = [m]$, $y \in K_n$, and $T_q = [n]$ for some $m$ and $n$. Let $m_i + 1 = |S_i|$ and $n_i + 1 = |T_i|$ and note that $m_0 < \cdots < m_q = m$ and $n_0 < \cdots < n_q = n$. Using Proposition 12.1.4, we see that the vertices of $z_1$ and $z_2$, in minimal form, are the $(S_i^*x;[m_i])$ and the $(T_i^*x;[n_i])$, respectively.

We are assuming that these two sets of vertices are the same. We claim that they are the same as *ordered* sets. That is, $(S_i^*x;[m_i]) = (T_i^*y;[n_i])$ for $0 \le i \le q$. Suppose not. Then $(S_i^*x;[m_i]) = (T_j^*y;[n_j])$ for some $i \ne j$, and we may assume $i < j$. Since these are both in minimal form, $m_i = n_j$. By the pigeonhole principle, we must have some $j' < j$ and $i' > i$ such that $m_{i'} = n_{j'}$. But then we have $m_i < m_{i'} = n_{j'} < n_j = m_i$, which is a contradiction.

Thus $m_i = n_i$ and $S_i^*x = T_i^*y$ for all $i$. Since $S_q = [m] = [n] = T_q$, we have $x = S_q^*x = T_q^*y = y$. Then, by Proposition 12.1.4 again, $S_i$ and $T_i$ must be defined by the same injection and so must be equal. Therefore $z_1 = z_2$ and Sd$K$ has Property $C$. □

Proof that if Sd$K$ has Property $C$, then $K$ has Property $B$. Suppose that $K$ does not have Property $B$. Let $x \in K_n$, $n > 0$, be nondegenerate with repeated vertices $\alpha^*x$ and $\beta^*x$ for injections $\alpha,\beta\colon [0] \longrightarrow [n]$. By the uniqueness of the minimal form, $(x;\alpha[0],[n])$ and $(x;\beta[0],[n])$ are distinct 1-simplices of Sd$K$. However, these 1-simplices have the same vertex sets since one of the vertices of each is $(x;[n])$ and the other is

$$(x;\alpha[0]) \sim (\alpha^*x;[0]) = (\beta^*x;[0]) \sim (x;\beta[0]).$$

Thus Sd$K$ does not have Property $C$. □

## 12.6. Isomorphisms of subdivisions

Not worth a section?

We saw in **??** that if $X$ and $Y$ are posets, then the subdivisions of $X * Y$ and $(X * Y)^-$ are isomorphic, hence so are their associated simplicial sets. However, the posets $X * Y$ and $(X * Y)^-$ are not isomorphic, and neither are their associated simplicial sets. We round out the picture with the following rather strange looking result, which puts this example in a more general context.

**Proposition 12.6.1.** *If $K$ and $L$ are simplicial sets such that $\mathrm{Sd}K$ and $\mathrm{Sd}L$ are isomorphic, then although $K$ and $L$ need not be isomorphic, for each $n$ there is a bijection of sets $f_n \colon K_n \cong L_n$ such that the faces of a simplex $x \in K_n$ correspond bijectively under $f_{n-1}$ to the faces of $f(x)$.*

Proof. Let $g \colon \mathrm{Sd}K \longrightarrow \mathrm{Sd}L$ be an isomorphism of simplicial sets. For a nondegenerate $n$-simplex $x \in K_n$, we have the vertex $(x;[n])$ in $\mathrm{Sd}K$. Write $g(x;[n]) = (y;[m])$ in minimal form. Using Corollary 12.3.3, we see that $m = n$, and we define $f_n(x) = y$. If $x \in K_n$ is degenerate, there is a unique surjection $\sigma$ and nondegenerate simplex $z$ such that $x = \sigma^* z$. Define $f_n(x) = \sigma^* f(z)$. If we apply the same construction starting from $g^{-1} \colon \mathrm{Sd}L \longrightarrow \mathrm{Sd}K$, we obtain an inverse function $f_n^{-1}$ to $f_n$. The $(n+1)$ faces $d_i x$ of a nondegenerate $x \in K_n$ correspond to the $(n+1)$ 1-simplices $y_i = (x;\delta_i[n-1],[n])$ of $\mathrm{Sd}K$, counted with multiplicities in case of repetitions. The vertices of $y_i$ are $d_0 y_i = (x;\delta_i[n-1]) \sim (d_i x;[n-1])$ and $d_1 y_i = (x;[n])$ in minimal form. Since the nondegenerate faces of $L$ admit a similar description, we see that these faces correspond under $f_{n-1}$ to the faces of $f_n(x)$. The following example shows that $K$ and $L$ need not be isomorphic. □

Missing!

## 12.7. Regularity and non-singularity of simplicial sets and CW complexes

Property $A$ of a simplicial set is an analogue of the classical notion of regularity for a CW complex $X$. The results of this section are peripheral to our main interests here, but they help contrast simplicial sets with CW complexes.

Incomplete section, see Piccinini?
Or expository REU paper project

**Definition 12.7.1.** A CW complex $X$ is *regular* if its closed cells are homeomorphisms onto their images so that each cell map $(D^n, S^{n-1}) \longrightarrow (e^n, \partial e^n)$ is a homeomorphism.

**Definition 12.7.2.** A nondegenerate simplex $x \in K_n$ is *regular* if the following diagram is a pushout, where $[x]$ denotes the subsimplicial set generated by $x$.

$$\begin{array}{ccc} \Delta[n-1] & \xrightarrow{d_n x} & [d_n x] \\ {\scriptstyle \delta^n}\downarrow & & \downarrow \\ \Delta[n] & \xrightarrow[x]{} & [x]; \end{array}$$

$K$ is *regular* if all of its nondegenerate simplices are regular.

**Theorem 12.7.3.** *For any $K$, $\mathrm{Sd}K$ is regular.*

**Theorem 12.7.4.** *If $K$ is a regular simplicial set, then $|K|$ is a regular CW complex.*

**Theorem 12.7.5.** *If $X$ is a regular CW complex, then $X$ is triangulable; that is $X$ is homeomorphic to $|K^s|$ for some simplicial complex $K$.*

CHAPTER 13

# Subdivision and Properties $A$, $B$, and $C$ in $\mathscr{C}at$

## 13.1. Properties $A$, $B$, and $C$ of categories

Categories are implicitly small unless they are obviously large, like the categories of spaces, simplicial sets, or (small) categories. We may interpret properties A, B, and C of the simplicial set $N\mathscr{C}$ as properties of a category $\mathscr{C}$.

**Definition 13.1.1.** A (small) category $\mathscr{C}$ has Property $A$, $B$, or $C$ if the simplicial set $N\mathscr{C}$ has Property $A$, $B$, or $C$.

**Theorem 13.1.2.** *Let $\mathscr{C}$ be a category. The following statements hold.*

*(i) $N\mathscr{C}$ has property A if and only if $\mathscr{C}$ has the* no retracts property*, meaning that retractions are identity maps: if we have morphisms $i\colon a \longrightarrow b$ and $r\colon b \longrightarrow a$ in $\mathscr{C}$ such that $r \circ i = \mathrm{id}_a$, then $a = b$ and $i = r = \mathrm{id}$.*

*(ii) $N\mathscr{C}$ has property B if and only if $\mathscr{C}$ has the* no loops property*, meaning that loops are identity maps: if we have morphisms $f\colon a \longrightarrow b$ and $g\colon b \longrightarrow a$ in $\mathscr{C}$, then $a = b$ and $f = g = id$.*

*(iii) $N\mathscr{C}$ has property $C$ if and only if $\mathscr{C}$ has the* one way property*: there is at most one sequence of nonidentity morphisms $f_i\colon C_i \longrightarrow C_{i+1}$ connecting any finite ordered set of objects $\{C_i\}$.*

*(iv) $\mathscr{C}$ is a poset if and only if $N\mathscr{C}$ has properties $B$ and $C$.*

Proof. A nondegenerate $n$-simplex of $N\mathscr{C}$ is a composable sequence

$$c_0 \xrightarrow{f_1} c_1 \longrightarrow \cdots \longrightarrow c_{n-1} \xrightarrow{f_n} c_n$$

of nonidentity morphisms. It has a degenerate face if and only if one of the composites $f_{i+1} \circ f_i$ is an identity map. This proves (i).

For (ii), Property $B$ says that the objects $c_i$ of a nondegenerate $n$-simplex are distinct, which clearly implies the no loops property. Conversely, if $c_i = c_j$ for some $i < j$, the composite of $f$'s from $c_i$ to $c_j$ is a loop $c_i \longrightarrow c_i$. We can write the composite as $g \circ f_i$. The no loops property implies that $f_i$ and $g$ are identity maps, so that our simplex is degenerate. This proves (ii)

Statement (iii) is immediate from the definition of Property C.

For (iv), it is immediate from (ii) and (iii) that $\mathscr{C}$ satisfies Properties $A$ and $B$ if and only if there is at most one morphism between any pair of objects of $\mathscr{C}$. That is precisely the characterization of posets regarded as categories. □

## 13.2. The definition of the subdivision of a category

Let $\mathscr{C}$ be a category. We start with a combinatorical definition of $\mathrm{Sd}\mathscr{C} = \mathrm{Sd}^c\mathscr{C}$. It may be hard to assimilate, but it is the right definition to start with. We will eventually see that Sd is actually nothing but the composite functor $\Pi\mathrm{Sd}^s N$, but that will require a fair amount of proof.

The intuition is that $\mathrm{Sd}\mathscr{C}$ has objects all chains of non-identity maps, and the set of morphisms from $(f_i, n)$ to $(g_i, m)$ is the set of all ways that $(f_i, n)$ can be mapped injectively to a subchain of $(g_i, m)$. These ways are to be distinct after accounting for degeneracies, which motivates the definition of the equivalence relation in the following definition.

To define $\mathrm{Sd}\mathscr{C}$ rigorously, we first define a category $\mathscr{D}\mathscr{C}$. The objects of $\mathscr{D}\mathscr{C}$ are the chains of composable arrows in $\mathscr{C}$. To abbreviate notation, we sometimes write $A = (f_i, m)$ as shorthand for a chain

$$a_0 \xrightarrow{f_1} a_1 \longrightarrow \cdots \longrightarrow a_{m-1} \xrightarrow{f_m} a_m.$$

We may think of such a chain as an $m$-simplex of $N\mathscr{C}$.

The morphisms from $(f_i, m)$ to $(g_i, n)$ are the equivalence classes of maps $\mu\colon [m] \longrightarrow [n]$ in $\Delta$ such that $\mu^*(g_i, n) = (f_i, m)$ in $N\mathscr{C}$. The equivalence relation is generated under composition by the following basic equivalences. For a surjective map $\sigma\colon [q] \longrightarrow [p]$ in $\Delta$ and for right inverses $\alpha, \beta\colon [p] \longrightarrow [q]$ to $\sigma$, so that $\sigma\alpha$ and $\sigma\beta$ are both the identity morphism of $[p]$, set $\alpha \sim \beta\colon (h_i, p) \longrightarrow \sigma^*(h_i, p)$ for any object $(h_i, p)$. This makes sense since $\alpha^*\sigma^* = \mathrm{id} = \beta^*\sigma^*$. Composition in $\mathscr{D}\mathscr{C}$ is induced by composition in $\Delta$. Then define $\mathrm{Sd}\mathscr{C}$ to be the full subcategory of $\mathscr{D}\mathscr{C}$ whose objects are the non-degenerate chains. A functor $F\colon \mathscr{C} \longrightarrow \mathscr{C}'$ induces a functor $NF\colon N\mathscr{C} \longrightarrow N\mathscr{C}'$, which in turn induces a functor $\mathrm{Sd}F\colon \mathrm{Sd}\mathscr{C} \longrightarrow \mathrm{Sd}\mathscr{C}'$. With these definitions, Sd is a functor $\mathscr{C}at \longrightarrow \mathscr{C}at$.

There is another way to view the definition, which may be easier to grasp. The letter $\mathscr{D}$ above is meant to indicate that we allow degenerate chains as objects of the category $\mathscr{D}\mathscr{C}$. We can instead start with the smaller category $\mathscr{C}\mathscr{C}$ whose objects $(f_i, m)$ are the nondegenerate chains, so that no $f_i$ is an identity map. The maps from $(f_i, m)$ to $(g_i, n)$ in $\mathscr{C}\mathscr{C}$ are the maps $\nu\colon [m] \longrightarrow [n]$ in $\Delta$ such that $\nu^*(g_i, n) = (f_i, m)$. Notice that such a map $\nu$ must be an injection since $(f_i, m)$ is nondegenerate. Now define $\mathrm{Sd}\mathscr{C}$ to be the quotient category of $\mathscr{C}\mathscr{C}$ with the same objects but with equivalence classes of morphisms under the equivalence relation generated by setting $\nu\alpha \sim \nu\beta$ when

$$\nu^*(g_i, n) = (f_i, m) = \sigma^*(h_i, q)$$

for some surjection $\sigma\colon [m] \longrightarrow [q]$ with right inverses $\alpha, \beta\colon [q] \longrightarrow [m]$.

The difference is whether we choose to first restrict to nondegenerate simplices and then impose an equivalence relation or to first impose an equivalence relation and then restrict to nondegenerate simplices. We get the same category either way.

**Remark 13.2.1.** It is useful to observe that if $\mathscr{C}$ has Property $A$, then no $\nu^*(g_i, n)$ can be degenerate and therefore $\mathscr{C}\mathscr{C} = \mathrm{Sd}\mathscr{C}$.

## 13.3. Subdivision and Properties $A$, $B$, and $C$ of categories

Despite the analogy with simplicial sets, the conclusions here read rather differently.

**Theorem 13.3.1.** *Subdivision of categories has the following properties.*

*(i) For any category $\mathscr{C}$, $\mathrm{Sd}\mathscr{C}$ has Property $B$.*
*(ii) A category $\mathscr{C}$ has Property $B$ if and only if $\mathrm{Sd}\mathscr{C}$ is a poset.*

Again, the following remarkable theorem follows directly. Since this result applies to any category $\mathscr{C}$, it does not make sense to ask for a converse.

**Theorem 13.3.2.** *For any category $\mathscr{C}$, $\mathrm{Sd}^2\mathscr{C}$ is a poset.*

**Example 13.3.3.** The nerve of a poset need not be the subdivision of a simplicial set. The poset $\mathbb{Z}$ of integers with its usual ordering provides a counterexample. If $N\mathbb{Z} \cong SdK$ and 0 corresponds to $(x;[n])$ in minimal form, then for any nondegenerate $q$-simplex $(y; S_0, \cdots, S_q)$ in minimal form that has $q$th vertex $(x;[n])$, we have $S_q = [n]$ and thus $q \leq n$. However, in $N\mathscr{C}$ there are nondegenerate simplices $(-r, -r+1, \cdots, 0)$ for arbitrarily large $r$.

Since we have subdivision functors on both categories and simplicial sets, it is natural to ask how these functors relate to the adjoint pair $(\Pi, N)$. The following result is either a theorem or a definition, depending on whether one chooses to start with the combinatorial or the conceptual definition of the subdivision of a category. We shall take it as a theorem and prove it in §13.5.

**Theorem 13.3.4.** *For any category $\mathscr{C}$, $\mathrm{Sd}^c\mathscr{C}$ is isomorphic to $\Pi\mathrm{Sd}^s N\mathscr{C}$.*

This implies another characterization of categories having Property $A$.

**Corollary 13.3.5.** *A category $\mathscr{C}$ has Property $A$ if and only if $\mathrm{Sd}^s N\mathscr{C}$ is isomorphic to $N\mathrm{Sd}^c\mathscr{C}$.*

Proof. If $\mathscr{C}$ has Property $A$, then Theorem 12.4.5 implies that $Sd^s N\mathscr{C}$ is isomorphic to $N\Pi\mathrm{Sd}^s N\mathscr{C}$. By Theorem 13.3.4, the latter is isomorphic to $N\mathrm{Sd}^c\mathscr{C}$. For the converse, $N\mathrm{Sd}^c\mathscr{C}$ has Property $B$ and therefore Property $A$ by Theorems 13.3.1(i) and 12.4.1(ii). If $\mathrm{Sd}^s N\mathscr{C} \cong N\mathrm{Sd}^c\mathscr{C}$, then $\mathscr{C}$ has Property $A$ by Theorem 12.4.1(i). □

**Remark 13.3.6.** For posets $X$, we obtain naturally isomorphic simplicial sets if we regard $X$ as a category and take its nerve or if we regard $X$ as the simplicial complex $\mathscr{K}X$ and take the associated simplicial set $(\mathscr{K}X)^s$. It is natural to ask whether $NSd^cX$ is isomorphic to $\mathrm{Sd}^s(\mathscr{K}X)^s$. Since $X$ satisfies Property $A$ (and $B$ and $C$), the previous result gives that

$$N\mathrm{Sd}^c X \cong \mathrm{Sd}^s NX \cong Sd^s(\mathscr{K}X)^s.$$

Remarkably, Theorem 13.3.4 also implies that the categorical analogue of Theorem 12.2.6 is a direct implication of that result.

**Theorem 13.3.7.** *There is a njk on passage to classifying spaces.*

gap? Does $\Pi$ preserve the property in question?

Proof. We apply the natural map of simplicial sets of Theorem 12.2.6 and the fact that the composite $\Pi N$ is isomorphic to the identity functor to obtain the required map as the composite

$$\mathrm{Sd}^c\mathscr{C} \cong \Pi\mathrm{Sd}^s N\mathscr{C} \longrightarrow \Pi N\mathscr{C} \cong \mathscr{C}.$$

□

## 13.4. The proof of Theorem 13.3.1

We have three implications to prove.

Proof that $\mathrm{Sd}\mathscr{C}$ has Property $B$. We first prove that $\mathscr{C}\mathscr{C}$ has Property $B$. Let $A = (f_i, m)$ and $B = (g_i, n)$ be objects of $\mathscr{C}\mathscr{C}$ and suppose that we have morphisms $\mu\colon A \longrightarrow B$ and $\nu\colon B \longrightarrow A$. Since these morphisms are given by injections in $\Delta$, $m = n$. Since the only injection $[n] \longrightarrow [n]$ is the identity map, we have $A = B$ and $\mu = \mathrm{id} = \nu$. Thus $\mathscr{C}\mathscr{C}$ has the no loops property, which is equivalent to Property $B$. This property is inherited by the quotient category $\mathrm{Sd}\mathscr{C}$. If we have maps $\overline{\mu}\colon A \longrightarrow B$ and $\overline{\nu}\colon B \longrightarrow A$ in $\mathrm{Sd}\mathscr{C}$, they must be represented by maps $\mu$ and $\nu$ in $\mathscr{C}\mathscr{C}$, but these maps are identity maps by what we have just shown, hence $\overline{\mu}$ and $\overline{\nu}$ are identity maps. □

Proof that if $\mathscr{C}$ has Property $B$, then $\mathrm{Sd}\mathscr{C}$ is a poset. Since Property $B$ implies Property $A$, $\mathscr{C}\mathscr{C} = \mathrm{Sd}\mathscr{C}$ by Remark 13.2.1. We must show that $\mathscr{C}\mathscr{C}$ is a poset. Let $A$ and $B$ be objects of $\mathscr{C}\mathscr{C}$. We must show that there is at most one morphism between $A$ and $B$. Suppose there is a morphism $\mu\colon A$ and $B$. Since we have just shown that $\mathscr{C}\mathscr{C}$ has the no loops property, there is no morphism $B \longrightarrow A$ unless $A = B$ and $\mu = \mathrm{id}$. Suppose there is another morphism $\nu\colon A \longrightarrow B$. We must show that $\mu = \nu$. Since $A = \mu^*B = \nu^*B$, we have $a_i = b_{\mu(i)} = b_{\nu(i)}$ for all $i$, where the $a_i$ and $b_j$ are the objects appearing in the chains $A$ and $B$. Since $B$ must be nondegenerate when thought of as an element of $N\mathscr{C}$ and $\mathscr{C}$ has the no loops property, we have $b_i \neq b_j$ for $i \neq j$. Therefore $\mu(i) = \nu(i)$ for all $i$ and $\mu = \nu$. □

Proof that if $\mathrm{Sd}\mathscr{C}$ is a poset, then $\mathscr{C}$ has Property $B$. Suppose that $\mathscr{C}$ does not have Property $B$. Then there are objects $A$ and $B$ (possibly the same) and non-identity maps $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow A$. Consider the objects $A \xrightarrow{f} B \xrightarrow{g} A$ and $A$ in $\mathrm{Sd}\mathscr{C}$. Let $\alpha, \gamma\colon [0] \longrightarrow [2]$ be the maps with images $\{0\}$ and $\{2\}$, respectively. Then

$$\alpha^*(A \xrightarrow{f} B \xrightarrow{g} A) = A = \gamma^*(A \xrightarrow{f} B \xrightarrow{g} A).$$

Since no degeneracy operator on $A$ is a face of $A \xrightarrow{f} B \xrightarrow{g} A$, we cannot have $\alpha \sim \gamma$; that is, they represent distinct morphisms of $\mathrm{Sd}\mathscr{C}$. But that contradicts the assumption that $\mathrm{Sd}\mathscr{C}$ is a poset. □

## 13.5. Relations among $\mathrm{Sd}^s$, $\mathrm{Sd}^c$, $N$, and $\Pi$

We are heading towards the proof of Theorem 13.3.4. We recall that $\Pi K$ has objects the vertices $x \in K$, morphisms generated by the 1-simplices $y \in K$, and relations dictated by the 2-simplices $z$. For a vertex $x$, $s_0x$ is the identity map of $x$. For a 1-simplex $y$, $d_1y$ is the source of $y$ and $d_0y$ is the target of $y$. For a 2-simplex $z$, $d_1z = d_0z \circ d_2z$. The functor $\Pi$ is left adjoint to $N$, and the counit of the adjunction is a natural isomorphism $\Pi N\mathscr{C} \cong \mathscr{C}$. We start work with the following understanding of the category $\Pi\mathrm{Sd}^sK$ for simplicial sets $K$.

**Proposition 13.5.1.** *Every morphism of the category $\Pi\mathrm{Sd}^sK$ can be represented by a 1-simplex in $\mathrm{Sd}^sK$, and the category $\Pi\mathrm{Sd}^sK$ has Property $B$.*

Proof. By definition, every morphism is a formal composite of 1-simplices, say $y_q \circ \cdots \circ y_1$. Since $y_{i+1} \circ y_i$ is defined, the target $d_0y_i$ is equal to the source $d_1y_{i+1}$. We will show that such a formal composite of length $q$ is equivalent to a formal composite of length $q-1$. By induction, it must be equivalent to a formal composite of length 1, which is just a 1-simplex.

Write $y_i$ in minimal form $(x_i; S_i, [n_i])$, where $x_i \in K_{n_i}$ is nondegenerate. Let $|S_i| = m_i \leq n_i$ and let $\alpha_i\colon [m_i] \longrightarrow [n_i]$ be the injection with image $S_i$. Since

$$(x_q; S_q) = d_1(x_q; S_q, [n_q]) = d_0(x_{q-1}; S_{q-1}, [n_{q-1}]) = (x_{q-1}; [n_{q-1}]),$$

there must be some surjection $\sigma\colon [m_q] \longrightarrow [n_{q-1}]$ in $\Delta$ such that $\alpha_q^* x_q = \sigma^*[x_{q-1}]$. Let $\beta\colon [n_{q-1}] \longrightarrow [m_q]$ be a right inverse to $\sigma$. Then

$$(x_q; \alpha_q\beta[n_{q-1}], S_q) \sim (\sigma^* x_{q-1}; \beta[n_{q-1}], [m_q]) \sim (x_{q-1}; [n_{q-1}], [n_{q-1}]),$$

which is degenerate and thus an identity morphism in $\Pi\mathrm{Sd}K$. Consider the 2-simplex $z = (x_q; \alpha_q\beta[n_{q-1}], S_q, [n_q])$. The relation $d_1 z = d_0 z \circ d_2 z$ gives that

$$(x_q; \alpha_q\beta[n_{q-1}], [n_q]) = (x_q; S_q, [n_q]) = y_q$$

as morphisms in $\Pi\mathrm{Sd}K$. Now use that $\beta^*\sigma^* = \mathrm{id}$ on $[n_{q-1}]$ to see that

$$y_{q-1} = (x_{q-1}; S_{q-1}, [n_{q-1}]) \sim (x_q; \alpha_q\beta S_{q-1}, \alpha_q\beta[n_{q-1}]).$$

Finally, consider the 2-simplex $w = (x_q; \alpha_q\beta S_{q-1}, \alpha_q\beta[n_{q-1}], [n_q])$. The relation $d_1 w = d_0 w \circ d_2 w$ gives that $(x_q; \alpha_q\beta S_{q-1}, [n_q]) = y_q \circ y_{q-1}$ in $\Pi\mathrm{Sd}K$. This gives the claimed reduction from word length $q$ to word length $q-1$.

To prove that $\Pi\mathrm{Sd}^s K$ has Property $B$, we must verify the no loop condition. Thus suppose that $f\colon (x; [m]) \longrightarrow (y; [n])$ and $g\colon (y; [n]) \longrightarrow (x; [m])$ are morphisms in $\Pi\mathrm{Sd}^s K$, where $x \in K_m$ and $y \in K_n$ are nondegenerate simplexes. We have just shown that $f$ and $g$ can be represented by 1-simplices. It suffices to show that both are degenerate, so that they are identity morphisms in $\Pi\mathrm{Sd}^s K$. We have

$$d_0 f = d_1 g = (y; [n]) \quad\text{and}\quad d_0 g = d_1 f = (x; [m]).$$

By the conditions on $d_0$, we can write $f = (y; T, [n])$ and $g = (x; S, [m])$ in minimal form. By the conditions on $d_1$, we then have $(y; T) \sim (x; [m])$ and $(x; S) \sim (y; [n])$. Choose injections $\alpha\colon [p] \longrightarrow [m]$ and $\beta\colon [q] \longrightarrow [n]$ with images $S$ and $T$. We then have

$$(x; [m]) \sim (y; T) \sim (\beta^* y; [p]) \quad\text{and}\quad (y; [n]) \sim (x; S) \sim (\alpha^* x; [q]).$$

Write $\alpha^* x = \sigma^* u$ where $u \in K_j$ is nondegenerate and $\sigma\colon [q] \longrightarrow [j]$ is a surjection. Then

$$(y; [n]) \sim (\alpha^* x; [q]) = (\sigma^* u; [q]) \sim (u; [j]).$$

Since these are both in minimal form, $n = j \leq q$. Similarly $m \leq p$. Since $\alpha$ and $\beta$ are injections, $n = q$, $m = p$, and $\alpha$ and $\beta$ are identity maps. Thus $S = [m]$ and $T = [n]$, showing that $f$ and $g$ are degenerate. □

Proof of Theorem 13.3.4. We shall prove that the categories $Sd^c\mathscr{C}$ and $\Pi Sd^s N\mathscr{C}$ are isomorphic by exhibiting inverse functors between these categories. Moreover, these inverse isomorphisms of categories will be natural in $\mathscr{C}$.

We first define $F\colon \mathrm{Sd}\mathscr{C} \longrightarrow \Pi Sd^s N\mathscr{C}$ and its inverse $G$ on objects. The objects $A = (f_i, m)$ of $\mathrm{Sd}\mathscr{C}$ are the nondegenerate simplices of $N\mathscr{C}$. The objects of $\Pi Sd^s N\mathscr{C}$ are the vertices of $Sd^s N\mathscr{C}$. We may write these in minimal form as $(A; [m])$, where $A$ is an object of $Sd^c\mathscr{C}$. We define $F$ and $G$ on objects by

$$F(A) = (A; [m]) \quad\text{and}\quad G(A; [m]) = A.$$

Visibly, $FG = \mathrm{Id}$ and $GF = \mathrm{Id}$ on objects.

We next define $F$ on morphisms and we first define it on the morphisms of $\mathscr{C}\mathscr{C}$, which has the same objects as $\mathrm{Sd}\mathscr{C}$. For objects $A = (f_i, m)$ and $B = (g_i, n)$, a morphism $\nu\colon A \longrightarrow B$ is an injection $\nu\colon [m] \longrightarrow [n]$ such that $\nu^* B = A$. We let $F(\nu)$ be the morphism of $\Pi Sd^s N\mathscr{C}$ represented by the 1-simplex $\overline{\nu} = (B; \nu[m], [n])$ of $Sd^s N\mathscr{C}$. It is straightforward and left to the reader to check that $F$ is indeed a functor, respecting composition and identities.

To see that $F$ induces a functor $Sd^c\mathscr{C} \longrightarrow \Pi Sd^s N\mathscr{C}$, we must show that $F$ respects the equivalence relation used to define morphisms in $Sd^c\mathscr{C}$ from morphisms in $\mathscr{C}\mathscr{C}$. Thus suppose that we have an injection $\nu\colon [m] \longrightarrow [n]$ and a surjection $\sigma\colon [m] \longrightarrow [q]$ such that $\nu^*B = A = \sigma^*C$ for some object $C$. Let $\alpha, \beta\colon [q] \longrightarrow [m]$ be right inverses to $\sigma$. Then $\nu\alpha \sim \nu\beta$ and we must show that $\overline{\nu\alpha} = \overline{\nu\beta}$ in $\Pi Sd^s N\mathscr{C}$. Observe first that

$$(B; \nu\alpha[q], \nu[q]) \sim (\sigma^*A; \alpha[q], [m]) \sim (A; [q], [q]) \sim (\sigma^*A; \beta[q], [m]) \sim (B; \nu\beta[q], \nu[q])$$

are degenerate 1-simplices of $SdN\mathscr{C}$. Therefore they are identity morphisms of $\Pi SdN\mathscr{C}$. We now use the definition of $\Pi$ to see that

$$\overline{\nu\alpha} = (B; \nu\alpha[q], [n]) = (B; \nu\beta[q], [n]) = \overline{\nu\beta}$$

$\Pi \mathrm{Sd}^s N\mathscr{C}$. In fact, both are equivalent to $(B; \nu[m], [n])$, as we see by considering the relations of the form $d_1 z = d_0 z d_2 z$ induced by the 2-simplices

$$(B; \nu\alpha[q], \nu[m], [n]) \quad \text{and} \quad (B; \nu\beta[q], \nu[m], [n])$$

of $NSd^s\mathscr{C}$. Therefore $F$ induces a well-defined functor $Sd^c\mathscr{C} \longrightarrow \Pi Sd^s N\mathscr{C}$.

We next define $G\colon \Pi Sd^s N\mathscr{C} \longrightarrow Sd^c\mathscr{C}$ on morphisms. We claim that every morphism $(A; [m]) \longrightarrow (B; [n]$ in $\Pi Sd^s N\mathscr{C}$ is of the form $\overline{\nu}$, and we define $G(\overline{\nu}) = \nu$. Visibly this will ensure that $FG = \mathrm{Id}$ and $GF = \mathrm{Id}$ on morphisms. By Proposition 13.5.1, a morphism $(A; [m]) \longrightarrow (B; [n])$ in $\Pi Sd^s N\mathscr{C}$ can be represented by some 1-simplex $(D; S, [r])$ in $Sd^s N\mathscr{C}$. Inspection of source and target shows that we must have

$$d_1(D; S, [r]) = (D; S) \sim (A; [m]) \quad \text{and} \quad d_0(D; S, [r]) = (D; [r]) \sim (B; [n]).$$

By the uniqueness in minimal form $r = n$ and $D = B$. Then $(B; S) \sim (A; [m])$. Let $S$ be the image of an injection $\nu\colon [p] \longrightarrow [n]$, and note that $\nu$ is uniquely determined by $S$. Then $(B; S) \sim (\nu^*B; [p])$. By the uniqueness in minimal form, $[p] = [m]$ and $\nu^*B = A$. Thus our morphism is given in minimal form by the 1-simplex $\overline{\nu} = (B; \nu[m], [n])$, where $\nu^*B = A$. We have effectively used the defining relations for $\Pi Sd^s N\mathscr{C}$ in the reduction to 1-simplices of Proposition 13.5.1, and $G$ is well-defined.

We have not checked that $G$ is actually a functor, but fortunately we don't have to. It is a familiar observation that a homomorphism of groups that is a bijection of sets is an isomorphism of groups. In our situation, the same argument works to prove that $G$ preserves identity morphisms and respects composition. Indeed

$$G(\mathrm{id}_{(A;[m])}) = GF(\mathrm{id}_A) = \mathrm{id}_A$$

and, for composable morphisms $\overline{\mu}$ and $\overline{\nu}$ of $\Pi Sd^s N\mathscr{C}$,

$$G(\overline{\nu} \circ \overline{\mu}) = G(F(\nu) \circ F(\mu)) = GF(\nu \circ \mu) = \nu \circ \mu$$

and

$$G(\overline{\nu}) \circ G(\overline{\mu}) = GF(\nu) \circ GF(\mu) = \nu \circ \mu. \qquad \square$$

## 13.6. Horn-filling conditions and nerves of categories

There are special kinds of simplicial sets that appear ubiquitously and are central to the applications of simplicial sets to other areas of mathematics. They are closely related to our focus on the relationship between simplicial sets and categories, and understanding them leads to several equivalent characterizations of those simplicial sets which are the nerves of categories.

Define $\Lambda^k_n$ to be the subsimplicial set of $\Delta[n]^s$ generated by the faces $d_i\iota_n$ for all $i \neq k$. The name horn comes from the picture that one sees after passage to geometric realization. The realization of $\Delta[n]^s$ is $\Delta[n]^t$, and the realization of $\Lambda^k_n$ is the "horn" that one sees after removing one of the faces of the boundary $\partial\Delta[n]^t$. If one has a map $f$ from the realization $T\Lambda^k_n$ to a space $X$, then one can extend the map to $T\Delta[n]^s = \Delta[n]^t$. In fact, the topological $n$-simplex retracts onto any of its horns, as one sees by pushing in along the missing face. Composing $f$ with such a retraction extends $f$ over the simplex. This leads to the following definition and example.[1]

**Definition 13.6.1.** A simplicial set $K$ is a *Kan complex* if every map of simplicial sets $\Lambda^k_n \longrightarrow K$ extends to a map $\Delta[n]^s \longrightarrow K$. There is a concrete combinatorial way to rephrase the condition. For every set of $n$-simplices $x_i \in K_{n-1}$, $0 \leq i \leq n$ and $i \neq k$ that satisfy the necessary compatibility condition $d_i x_j = d_{j-1} x_i$ for $i < j$ with neither $i = k$ nor $j = k$, there must exist an $n$-simplex $x \in K_n$ such that $d_i x = x_i$ for $i \neq k$.

The equivalence of the two formulations is immediate from Proposition 10.7.3.

**Proposition 13.6.2.** *For every space $X$, the simplicial set $SX$ is a Kan complex.*

One might ask whether the extensions in Definition 13.6.1 are unique. If they are, we say that $K$ has the *unique horn filling property*. Looking at the definition of the faces of the nerve of a category, (10.8.5), we see that not all horns are created equal. We say that $\Lambda^k_n$ is an *inner horn* if $0 < k < n$; the outer horns are those with $k = 0$ or $k = n$.

Looking at $N\mathscr{C}$ or at $\Pi K$, one sees that the inner horns play a special role. If we have faces $d_0 z$ and $d_2 z$, their composite is $d_1 z$. In a category, if we are given morphisms $f_0$ and $f_2$ such that the source of $f_2$ is the target of $f_0$, they define a map $\Lambda^1_2 \longrightarrow N\mathscr{C}$, and the composable pair $(f_0, f_2)$ gives a 2-simplex that extends the horn. This doesn't work if we are given $f_0$ and $f_1$ or $f_1$ and $f_2$, since we cannot compose those. We can use inverses to fill these outer horns when $\mathscr{C}$ is a groupoid. This leads to the following result whose meaning should I hope be clear. We leave some details of proof to the reader. For $1 \leq i \leq n$, let $\nu_i \colon [1] \longrightarrow [n]$ denote the injection with image $\{i-1, i\}$.

**Theorem 13.6.3.** *Let $K$ be a simplicial set. The following conditions are equivalent.*

*(i) $K$ is isomorphic to the nerve of a category.*
*(ii) Every inner horn of $K$ has a unique filler.*
*(iii) For any $n \geq 2$ and any $n$-tuple of simplices $x_i \in K_1$, $1 \leq i \leq n$, such that $d_0 x_{i-1} = d_1 x_i$ for $2 \leq i \leq n$, there is a unique $y \in K_n$ such that $\nu_i^* y = x_i$.*

*$K$ is isomorphic to the nerve of a groupoid if and only if every horn of $K$, inner or outer, has a unique filler.*

Sketch Proof. First suppose that $K \cong N\mathscr{C}$. We deduce (ii) and (iii). It helps to recall the formulas for the faces and degeneracies of $N\mathscr{C}$ as given in (10.8.5).

If we have an inner horn $\Lambda^k_n \longrightarrow K$ given by compatible $(n-1)$-simplices $x_i$ for $i \neq k$, then we can reconstruct from these simplices a unique string $(f_1, \cdots, f_n)$ of composable arrows, and they give a filler for the given inner horn. One way of

[1] These are so basic that they appear on pages 2 and 3 of my book [45].

seeing this is to look at the ordered string of $n-1$ 1-simplices obtained from $x_0$ and $x_n$ by applying all iterated face operations. Applied to $x_0$, we obtain 1-simplices in order that we denote by $f_i$, $2 \le i \le n$. Applied to $x_n$, we obtain 1-simplices that we also denote by $f_i$, but now for $1 \le i \le n-1$. The duplicate $f_i$ for $2 \le i \le n-1$ are equal by the assumed compatibility condition, and the required $y$ is the $n$-simplex $(f_1, \cdots, f_n)$. If we have simplices $x_i \in K_1$ as in (iii), they are a string of composable morphisms $(f_1, \cdots, f_n)$, and that string is the required simplex $y$.

If $\mathscr{C}$ is a groupoid, we can use inverses to modify the proof of (ii) so that it applies to outer as well as inner horns.

Conversely, assume (ii) or (iii). We claim that either suffices to prove that the unit $\eta\colon K \longrightarrow N\Pi K$ of the $(N, \Pi)$-adjunction is an isomorphism. The meaning is that the formal words of length $n$ in the 1-simplices that appear in the definition of $\Pi K$ are all realized uniquely by simplices in $K_n$. We show that $\eta$ is an isomorphism on $n$-simplices for all $n$ by induction on $n$. The induction starts with $n = 0$ and $n = 1$, where there is nothing to prove. Assume that $\eta$ is an isomorphism on $(n-1)$-simplices. Let $y$ be an $n$-simplex of $N\Pi K$. Its faces give inner horns $\Lambda^k_n$ in $K$, and they also give the data of (iii). With either hypothesis, a filler gives an $n$-simplex $x$ of $K$ such that $y$ and $\eta(x)$ have the same faces. This means $\eta(x)$ is the same composite of 1-simplices as $y$, so that $\eta(x) = y$. If also $\eta(x') = y$, then $x$ and $x'$ have the same faces and so are equal by the uniqueness assumed in (ii) or (iii).

If we have fillers for all horns, then $K \cong N\Pi K$ and the fillers for the outer horns defined on $\Lambda^0_2$ and $\Lambda^2_2$ give left and right inverses for all morphisms. Just as for groups, the left and right inverses must be equal, and $N\Pi K$ must be a groupoid. □

We use this characterization to prove Theorem 12.4.5.

Proof of Theorem 12.4.5. Suppose that $K$ has Property $A$. We show that $SdK$ satisfies condition (iii) of Theorem 13.6.3. Thus let $(x_i; S_i, [q_i])$, $1 \le i \le n$, be 1-simplices of $SdK$ in minimal form such that

$$d_0(x_{i-1}; S_{i-1}, [q_{i-1}]) = d_1(x_i; S_i, [q_i])$$

for $2 \le i \le n$. Choose an injection $\alpha_i\colon [p_i] \longrightarrow [q_i]$ with image $S_i$ for $0 \le i \le n$. Note that $p_1 = q_0$, where $q_0 = |S_0|$. The compatibility condition is equivalent to

$$(x_{i-1}, [q_{i-1}]) \sim (x_i; S_i) \sim (\alpha_i^* x_i; [p_i])$$

for $2 \le i \le n$. Since $K$ has Property $A$, the faces $\alpha_i^* x_i$ are nondegenerate. By the uniqueness in minimal form, $q_{i-1} = p_i$ and $x_{i-1} = \alpha_i^* x_i$ for $2 \le i \le n$. Letting $x_0 = \alpha_1^* x_1$, this still holds for $i = 1$. The composite $\alpha_n \cdots \alpha_1\colon [p_1] \longrightarrow [q_n]$ is defined. Let

$$y = (x_n; \alpha_n \cdots \alpha_1[p_1], \alpha_n \cdots \alpha_2[p_2], \cdots, \alpha_n[p_n], [q_n]).$$

Then $\nu_n y = (x_n; S_n, [q_n])$ and, for $1 \le i < n$,

$$\nu_i^* y = (x_n; \alpha_n \cdots \alpha_i[p_i], \alpha_n \cdots \alpha_i[p_{i+1}]) \sim (x_i; S_i, [q_i])$$

For the uniqueness, suppose that we have another extension $z = (w; T_0, \cdots, T_n)$ in minimal form such that $\nu_i z = (x_i; S_i, [q_i])$ for $1 \le i \le n$. The $n$th vertex $(w; T_n)$ of $z$ must be $(x_n; [q_n])$, so that $(w; T_n) \sim (x_n; [q_n])$. Since $K$ satisfies Property A and $w$ is nondegenerate, it follows from the uniqueness in minimal form that

$w = x_n$ and $T_n = [q_n]$. Similarly, for $0 \le i < n$, the $i$th vertex of $z$ must be the $i$th vertex of $y$, hence

$$(x_n; T_i) \sim (x_n; \alpha_n \cdots \alpha_{i+1}[p_{i+1}]).$$

Therefore $T_i$ must be $\alpha_n \cdots \alpha_{i+1}[p_{i+1}]$ and $z = y$.

We shall prove a strengthened form of the converse statement in Proposition 13.7.3 below. □

**Remark 13.6.4** (Categorical remark)**.** The functor $Sd$ is a left adjoint. Its right adjoint is denoted $Ex$. Iterating it leads to an endofunctor $Ex^\infty$ on $s\mathscr{S}et$ that assigns a Kan complex $Ex^\infty K$ to a simplicial set $K$. The composite $ST$ is another such functor. They fit into a more sophisticated context of Quillen model category theory. One recent reference is [**49**, 17.5].

## 13.7. Quasicategories, subdivision, and posets

Looking at the definition of Kan complexes and the characterization of nerves of categories, one sees that they have a natural common generalization.

**Definition 13.7.1.** A simplicial set is a *quasicategory* if and only if every inner horn has a filler, not necessarily unique.

The idea is that compositions are defined, but they need not be unique. This is a very fashionable notion, and in much current literature the rather grandiose terms "$\infty$-category" or "$(\infty,1)$-category" are used for quasicategories. To go with this, the term "$\infty$-groupoid" is then often used for Kan complexes. There is even some motivation for the terminology. In view of their importance, it seems reasonable to ask how these concepts behave with respect to subdivision and our Properties $A$, $B$, and $C$.

**Proposition 13.7.2.** *If* $\mathrm{Sd}K$ *is a Kan complex, then* $K$ *is discrete, meaning that it has no nondegenerate simplices other than vertices.*

Proof. Suppose that $K$ has a nondegenerate $n$-simplex, where $n > 0$. Let $v$ be a vertex of $x$ and let $\alpha\colon [0] \longrightarrow [n]$ be an injection such that $\alpha^* x = v$. Define an outer horn $\Lambda^2_2 \longrightarrow SdK$ by sending the vertices $0, 1, 2$ to the vertices $(x; [n])$, $(v; [0])$, $(x; [n])$ of $SdK$ and sending the 1-simplices $(1, 2)$ and $(0, 2)$ to $(x; \alpha[0], [n])$ and $(x; [n], [n])$. Since $v \in K_0$, there is clearly no 1-simplex $(y; S, [m])$ with vertices $(x; [n])$ and $(v; [0])$, so $\mathrm{Sd}K$ cannot be a Kan complex. □

**Proposition 13.7.3.** *If* $SdK$ *is a quasicategory, then* $K$ *satisfies Property* $A$*.*

Proof. Assume that $K$ does not satisfy Property $A$. We construct an inner horn $f\colon \Lambda^2_3 \longrightarrow SdK$ that cannot be extended to a map $\Delta[3] \longrightarrow K$, thus showing that $SdK$ cannot be a quasicategory. Since Property $A$ fails for $K$, we can choose a nondegenerate simplex $x \in K_n$, an injection $\alpha\colon [m] \longrightarrow [n]$, and a surjection $\sigma\colon [m] \longrightarrow [p]$, $m > p$, such that $\alpha^* x = \sigma^* y$ in $K_m$ for some nondegenerate simplex $y \in K_p$. Choose a right inverse $\beta\colon [p] \longrightarrow [m]$ to $\sigma$. The three 2-faces of $\Lambda^2_3 \subset \Delta[3]$ are $d_0\iota_3$, $d_1\iota_3$, $d_3\iota_3$, where $\iota_3$ is the identity simplex that generates $\Delta[3]$. We specify $f$ on these three 2-simplices by sending them to

$$(x; \alpha\beta[k], \alpha[m], [n]), \quad (x; \alpha[m], \alpha[m], [n]), \quad (y; [p], [p], [p])$$

respectively. It is a straightforward to check that they satisfy the required consistency on 1-faces of the horn. However, $f$ cannot be extended to the last 2-face $d_2\iota_3$.

Any possible image would have a minimal form $(x; S, T, [n])$. For consistency with the prescribed faces, we would have

$$(x; S, [n]) \sim (x; \alpha[m], [n]) \text{ and } (x; T, [n]) \sim (x; \alpha\beta[p], [n]).$$

By the uniqueness of the minimal form, $S = \alpha[m]$ and $T = \alpha\beta[p]$. Thus, since $p < m$, $T$ is a proper subset of $S$. Since $S \subset T$ by definition, $S = T$. This contradicts the choice of $\beta$ as a non-identity injection. □

**Remark 13.7.4.** There is a curious analogue for quasicategories of the result that a simplicial set is a simplicial complex if and only it satisfies Properties $B$ and $C$. If $K$ is the nerve of a poset, then it satisfies Properties $B$ and $C$ by Theorem 13.3.1, and of course it is a category and thus a quasicategory. It is reasonable to ask whether a quasicategory $K$ that satisfies Properties $B$ and $C$ is a poset. By Theorem 12.1.8, $K$ is the simplicial set associated to a simplicial complex, and we now write $K$ for the latter. The set of vertices of $K$ is a poset, and its order restricts to a total order on each simplex, so that we can write simplices in the form $\{x_0 < \cdots < x_n\}$ for vertices $x_i$. Then $K$ is isomorphic to the nerve of the poset $K_0$ if and only if every finite totally ordered set $\{x_0 < \cdots < x_n\}$ is a simplex.

The example of $\partial\Delta[1]^s$ shows that for two vertices $x_0 < x_1$, $\{x_0 < x_1\}$ need not be a simplex of $K$. However, suppose that all such sets $\{x_0 < x_1\}$ are 1-simplices. Then $K$ is a poset. To see this assume by induction that all totally ordered subsets of $K_0$ with at most $n$ elements are simplices. Suppose for a contradiction that $\{x_0 < \cdots < x_n\}$ is totally ordered but not a simplex. Since all faces of this missing simplex are simplices, it is easy to construct an inner horn $f\colon \Lambda^k_n \longrightarrow K$, in fact one for each $0 < k < n$, from all but one of the faces. A filler is an $n$-simplex of $K$, hence a totally ordered set $\{y_0, \ldots, y_n\}$; it must be totally ordered since otherwise it would have degenerate faces, which it clearly does not have; that its vertices must be the $x_i$ follows from the fact that the map $\Delta[n] \longrightarrow K$ determined by $\{y_0, \ldots, y_n\}$ extends $f$, and $f$ maps onto the vertices.

We also remark that Properties $B$ and $C$ clearly fail to imply that $K$ is a quasicategory. The inner horn $\Lambda^1_2$ is a simplicial complex, and its identity map does not extend to a simplex $\Delta[2] \longrightarrow \Lambda^1_2$.

# Part 3

# Appendices

CHAPTER 14

# Cores of Alexandroff spaces

This appendix is taken from an REU paper written by Xi (Cathy) Chen in 2015. Her paper is based on work of Kukieła[**42**]. We have made only relatively minor editorial changes. All spaces are $A$-spaces throughout.

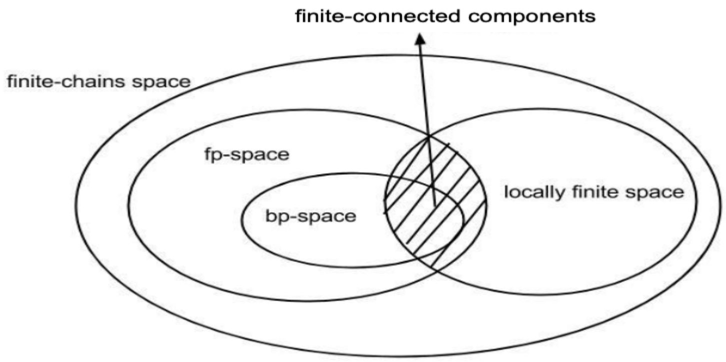
We first introduce some classes of $A$-spaces, including finite-chains spaces, locally finite spaces, finite-paths spaces, and bounded-paths spaces. Next, we present Kukieła's generalizations. If an infinite $A$- space is sufficiently well-behaved, then we get a core by recursively removing sets of beat points until no more beat points are left, just as for finite spaces. We have the following results. Every bounded-paths space or countable finite-paths space has a core, and if $X$ is a minimal finite-paths space, then the connected component of $\mathrm{id}(X)$ in the space $C(X,X)$ of self maps of $X$ is a singleton. Moreover, if $X$ and $Y$ are $fp$-spaces that both have cores, then $X$ is homotopy equivalent to $Y$ if and only if their respective cores are homeomorphic.

**Definition 14.0.1.** Given a poset $X$, we define a *chain* of $X$ to be a sequence $\{x_n\}$ of points of $X$ such that $x_i < x_{i+1}$ for all $i$.

**Definition 14.0.2.** Let $X$ be an $A$- space. A (finite or infinite) sequence $(x_n)$ of elements of $X$ is an *s-path* if $x_i \neq x_j$ for $i \neq j$ and $x_{i-1} \sim x_i$ for all $i > 0$. Given a finite s-path $k = (x_0, \dots, x_m)$, we say $m$ is the *length* of $k$ and call $k$ an *s-path from $x_0$ to $x_m$*.

**Definition 14.0.3.** An $A$- space $X$ is:
1. a *finite-chains space* if every chain in $X$ is finite,
2. a *locally finite space* if for every $x \in X$, the set $\{y \in X | y \sim x\}$ is finite,
3. a *finite-paths space (fp-space)* if every s-path of elements of $X$ is finite,
4. a *bounded-paths space(bp-space)* if there exists an $n \in \mathbb{N}$ such that every s-path of elements in $X$ has less than $n$ elements.

**Remark 14.0.4.** Bp-spaces form a strict subclass of fp-spaces and both fp-spaces and locally finite spaces are strict subclasses of finite-chains spaces. Moreover, the connected components of the spaces, which are both fp-spaces and locally finite, are finite. Finite connected components can be visualized as the intersection of the following Venn diagram.




We saw that an $F-$space can be reduced to its core through the removal of beat points. We shall see a similar notion, which the following reduction techniques help to define. For an upbeat point $x$, we write $u_x$ for the minimal point above $x$. For a downbeat point $x$, we write $d_x$ for the maximal point below $x$.

**Definition 14.0.5.** Let X be an $A$- space. A retraction $r : X \longrightarrow r(X)$ is called:
1. a *comparative retraction* if $r(x) \sim x$ for every $x \in X$.
2. an *up-retraction* if $r(x) \geq x$ for every $x \in X$.
3. a *down-retraction* if $r(x) \leq x$ for every $x \in X$.
4. a *retraction removing a beat point* if there exists an $x \in X$ that is an upbeat point under some $u_x \in X$ or a downbeat point over some $d_x \in X$ such that $r(x) = u_x$ or $r(x) = d_x$, and $r(y) = y$ for all $y \neq x$.

**Remark 14.0.6.** Every comparative retraction can be written as a composition of an up-retraction and a down-retraction. If $r : X \longrightarrow A$ is a comparative retraction, then $r = r_d \circ r_u$, where

$$r_u(x) = \begin{cases} r(x) & \text{if } r(x) \geq x \\ x & \text{if } r(x) \leq x \end{cases}$$

and

$$r_d(x) = \begin{cases} r(x) & \text{if } r(x) \leq x \\ x & \text{if } r(x) \geq x \end{cases}$$

**Definition 14.0.7.** Let $X$ be an $A$- space. Let $\mathcal{C}$ be the class of all comparative retractions and $\mathcal{I}$ be the class of {retractions removing a beat point} $\bigcup$ {identity maps}. The space $X$ is called a *$\mathcal{C}$-minimal space* (or an *$\mathcal{I}$-minimal space*) if there is no retraction $r : X \longrightarrow r(X)$ in $\mathcal{C}$ (or $\mathcal{I}$) other than $id_X$. The space $X$ is called a *$\mathcal{C}$-core* (or an $\mathcal{I}$-core) if $X$ is a $\mathcal{C}$-minimal subspace (or an $\mathcal{I}$-minimal subspace) that is a strong deformation retract of $X$.

**Proposition 14.0.8.** *A space $X$ is $\mathcal{I}$-minimal if and only if it has no beat points.*

PROOF. ($\Rightarrow$) This direction follows from the definition above. Since in the class of $\mathcal{I}$, there is no retraction removing a beat point other than $id_X$, it follows that there are no beat points in $X$.

($\Leftarrow$) If $X$ has no beat points, then the retractions of removing a beat point are the same as the identity maps. This means $id_X$ is the only retraction in $\mathcal{I}$, which implies $X$ is $\mathcal{I}$-minimal. $\square$

**Corollary 14.0.9.** *Suppose $X$ is a finite-chains space. Then $X$ is $\mathcal{C}-$minimal if and only if $X$ is $\mathcal{I}-$minimal.*

Proof. ($\Leftarrow$): Suppose $X$ is $\mathcal{I}-$minimal and that $r : X \longrightarrow r(X)$ is a $\mathcal{C}$-retraction. Factor $r$ as $r_d \circ r_u$, which gives that $r_d \leq id_X$ and $r_u \geq id_X$. Since $X$ is a finite-chains space, $X$ contains no strictly decreasing infinite sequence and we can therefore use induction. Take $y \in X$ and suppose $r_d(x) = x$ for all $x < y$. We will show that if $r_d(y) < y$, then $y$ is a downbeat point over $r_d(y)$, contradicting the $\mathcal{I}$-minimality of $X$. Hence, we must have $r_d(y) = y$ and by the induction argument, $r_d = id_X$. So, suppose $r_d(y) < y$. For any $x < y$, $x = r_d(x) \leq r_d(y) < y$ by induction and monotonicity. This means $y$ is a downbeat point over $r_d(y)$, a contradiction. By previous remarks, it follows $r_d = id_X$. A similar argument shows that if $r_d \geq id_X$, then $r_d = id_X$. Using the same arguments for $r_u$ gives that$r_u = id_X$. Therefore $X$ is $\mathcal{C}-$minimal.
($\Rightarrow$) : A retraction removing a beat point is also a comparative retraction. So if $X$ is $\mathcal{C}$-minimal, then there is no comparative retraction, and hence no $\mathcal{I}$-retraction, other than $id_X$. Therefore $X$ is $\mathcal{I}$-minimal. $\square$

**Definition 14.0.10.** [42, Defn. 5.9] Let $\gamma$ be an ordinal and $X$ be an $A$- space. Let $\{r_\alpha | X_\alpha \longrightarrow X_{\alpha+1}\}_{\alpha<\gamma}$ be a family of retractions from $\mathcal{C}$ (or $\mathcal{I}$) such that $X_0 = X$, $X_{\alpha+1} = r_\alpha(X_\alpha)$ for all $\alpha < \gamma$ and $X_\alpha = \bigcap_{\beta<\alpha} X_\beta$ for limit ordinals $\alpha < \gamma$. By transfinite recursion, we define a family of retractions $\{R_\alpha | X \longrightarrow X_\alpha\}_{\alpha\leq\gamma}$ such that:
1. $R_0 = id_X$,
2. $R_{\alpha+1} = \gamma_\alpha \circ R_\alpha$,
3. for a limit ordinal $\alpha$ and an $x \in X$, if there exists $\beta_0 < \alpha$ such that $R_\beta(x) = R_{\beta_0}(x)$ for all $\beta_0 \leq \beta < \alpha$, then $R_\alpha(x) = R_{\beta_0}(x)$, and if not, we leave $R_\alpha(x)$ undefined.

The recursion ends when $R_\gamma$ is defined or when $R_\alpha$ cannot be totally defined for some limit ordinal $\alpha$. In the first case we say the family $\{r_\alpha\}_{\alpha<\gamma}$ is *infinitely composable* and $X$ is *$\mathcal{C}$-dismantlable* (or *$\mathcal{I}$- dismantlable*) to $X_\gamma$ (*in $\gamma$ steps*). In the second case we say the family $\{r_\alpha\}_{\alpha<\gamma}$ is *not infinitely composable.*

**Definition 14.0.11.** Let $X$ be a finite-chains space. Let $u_X : X \longrightarrow X$ be given by:

$$u_X(x) = \begin{cases} u_x & \text{if } x \text{ is upbeat under } u_x \\ x & \text{otherwise} \end{cases}$$

Since $u_X(x) \geq x$ for every $x \in X$ and $X$ is a finite-chains space, it follows that for every $x \in X$ there exists an $N_x \in \mathbb{N}$ such that $(u_X)^n(x) = (u_X)^{N_x}(x)$ for every $n \geq N_x$. Let $U_X : X \longrightarrow U_X(X)$ be an up-retraction given by $U_X(x) = (u_X)^{N_x}(x)$. Similarly we define the down-retraction $D_X : X \longrightarrow D_X(X)$.

**Remark 14.0.12.** We check that $u_x$ and $U_X$ are order-preserving, as well as $d_X$ and $D_X$. Given $x, y \in X$ such that $x < y$, we will show $u_X(x) \leq u_X(y)$. Note that we can assume $x < y$ here because if $x = y$, then $u_X(x) = u_X(y)$.

- If neither $x$ nor $y$ is an upbeat point, then $u_X(x) = x < y = u_X(y)$.

- If $x$ is an upbeat point under $u_x$ and $y$ is not an upbeat point, then $u_X(x) = u_x \leq y = u_X(y)$.
- If $y$ is an upbeat point under $u_y$ and $x$ is not an upbeat point, then $u_X(x) = x < y < u_y = u_X(y)$.
- If both $x$ and $y$ are upbeat points, then $u_X(x) = u_x \leq y < u_y = u_X(y)$.

Now we check $U_X$ is order-preserving. Note that for any pair $x \leq y$, there is some $N \gg 0$ such that $U_X(x) = u_X^N(x)$ and $U_Y(y) = u_X^N(y)$. Since $u_X$ is monotone, $U_X(x) = U_X(y)$ by induction.
Similarly, we can check $d_X$ and $D_X$ are order-preserving as well.

**Definition 14.0.13.** Given an ordinal $\gamma$ and a finite-chains space $X$, we define a sequence of retractions $\{r_\alpha | X_\alpha \longrightarrow X_{\alpha+1}\}$ by transfinite recursion.
Let $X_0 = X$, $X_{\alpha+1} = r_\alpha(X)$ and $X_\alpha = \bigcap_{\beta<\alpha} X_\gamma$ if $\alpha$ is a limit ordinal. For $\alpha = 0$ or $\alpha$ a limit ordinal and $n$ a finite ordinal, let

$$r_{\alpha+n} = \begin{cases} D_{X_{\alpha+n}} & \text{if } n \text{ is even} \\ U_{X_{\alpha+n}} & \text{if } n \text{ is odd} \end{cases}$$

We call this sequence of retractions $\{r_\alpha | X_\alpha \longrightarrow X_{\alpha+1}\}_{\alpha<\gamma}$ *the standard sequence of $X$ (of length $\gamma$).*

**Theorem 14.0.14.** [**42**, Thm. 4.18] *Let $X, Y$ be $A$- spaces and $\{f_\alpha | X \longrightarrow Y\}_{\alpha \leq \gamma}$, where $\gamma$ is a countable ordinal, be a family of continuous maps such that:*

*(1) if $\alpha = \beta + 1$, then $f\alpha \sim f_\beta$,*
*(2) if $\alpha$ is a limit ordinal, then for every $x \in X$, there exists $\beta_x^\alpha < \alpha$ such that $f_\beta(x) \leq f_\alpha(x)$ for all $\beta_x^\alpha \leq \beta \leq \alpha$.*

*The $f_0$ is homotopic to $f_\gamma$.*

**Definition 14.0.15.** An $A$- space $X$ is *countably $\mathcal{C}$-dismantlable* (or *countably $\mathcal{I}$-dismantlable*) to $X' \subseteq X$ if it is $\mathcal{C}$-dismantlable (or $\mathcal{I}$- dismantlable) to $X'$ in $\gamma$ steps, where $\gamma$ is a countable ordinal.

The above theorem and definition imply that when an $A$- space $X$ is countably $\mathcal{C}$-dismantlable (or $\mathcal{I}$-dismantlable) to a $\mathcal{C}$-minimal subspace (or an $\mathcal{I}$-minimal subspace), we can build a strong deformation retraction from $X$. By Corollary 14.0.9, these two notions of minimality coincide. We call such a minimal subspace of $X$ a core of $X$.

We now present the main theorems on cores from Kukieła's paper[**42**].

**Theorem 14.0.16.** *Every bp-space or countable fp-space $X$ has a core. Moreover, if $X$ is a bp-space with path length bounded by some $n \in \mathbb{N}$, then $X$ can be $\mathcal{C}$-dismantled to a core in fewer than $2n + 2$ steps.*

Recall that in the finite case, we can construct a core by removing beat points one by one until we obtain a minimal space. Since removing a beat point is a strong deformation retract, this produces a core. However, in the infinite case, we use the standard sequence to remove many beat points at a time, and repeat. After countably many steps, $X$ is $\mathcal{C}$-dismantled to a core. The following is the sketch of the proof, and details can be found in [**42**, Thm. 5.14].

Proof. (Sketch) Assume $X$ is an infinite $A$- space. Let $\Omega$ be the first ordinal of cardinality greater that $X$. Let $\{r_\alpha | X_\alpha \longrightarrow X_{\alpha+1}\}_{\alpha<\gamma}$ be the standard sequence of $X$ of length $\Omega$.

First, we claim that if $X$ is an fp-space, then the standard sequence is infinitely composable. If not, then for some limit ordinal $\alpha$, $r_\alpha$ could not be totally defined and we could construct an infinite s-path in $X$, using a point that moves infinitely often. This would contradict that $X$ is an fp-space. Since the standard sequence of $X$ is infinitely composable, it will be constant beginning with some $\alpha_0 < \Omega$. If not, then $X$ would have cardinality at least $\Omega$, which is a contradiction. Thus we obtain an $\mathcal{I}$-minimal space at $\alpha_0$. If $X$ is countable, then $\Omega = \omega_1$, the first uncountable ordinal. Therefore $\alpha_0 < \omega_1$ is countable, and we can construct a strong deformation retract to $X_{\alpha_0}$ by Theorem 14.0.14. Thus $X_{\alpha_0}$ is a core of $X$.

If $X$ is a bp-space with path length bounded by some $n \in \mathbb{N}$, one can show that the standard sequence is constant after $2n+2$ steps. For if not, then $X$ would contain an s-path of length greater than $n$, which is a contradiction. □

Recall that $C(X, X)$ denotes the space of all continuous maps $X \longrightarrow X$ in the compact open topology, and that $W(C, U) = \{f | f(C) \subset U\}$ are the canonical subbasis elements of $C(X, Y)$. We have the following theorem [**42**, Thm. 5.16].

**Theorem 14.0.17.** *If $X$ is a $\mathcal{I}$-minimal fp-space, then the connected component of $id_X$ in $C(X, X)$ is a singleton.*

Proof. (Sketch) One first shows that for every $x \in X$, there exists a subspace $x \in A_x \subseteq X$ such that:

(1) $A_x$ is finite,
(2) if $y \in A_x$ is not maximal in $X$, then $|A_x \cap \max\{z \in X | z < y\}| \geq 2$,
(3) if $y \in A_x$ is not minimal in $X$, then $|A_x \cap \min\{z \in X | z > y\}| \geq 2$.

$A_x$ can be thought of as the image of a tree (but the order on the tree is not the same as the order on $X$). If $A_x$ is not finite, we could construct a tree $A_x$, where at each node, there are at most 4 new branches. König's Lemma [1] would imply that if $A_x$ is infinite, then $X$ has an infinite s-path, which contradicts that $X$ is an fp-space.

Since for all $y \in A_x \subseteq X, id_X(y) = y \leq y$ it follows that $id_X \in \bigcap_{y \in A_x} W(\{y\}, U_y\}$, which is an open neighborhood of $id_X$. We can show that this $\bigcap_{y \in A_x} W(\{y\}, U_y)$ is also closed. Thus $\bigcap_{y \in A_x} W(\{y\}, U_y)$ is a clopen set containing $id_X$. From point set topology, the connected component of $id_X$ is a subset of the intersection of all clopen sets $\bigcap_{y \in A_x} W(\{y\}, U_y)$ containing $id_X$, therefore the component of $id_X$ is contained in $\bigcap_{x \in X} \bigcap_{y \in A_x} W(\{y\}, U_y)$.

Next, one can show that for every $x \in X$, if $f \in \bigcap_{y \in A_x} W(\{y\}, U_y)$, then $f\big|_{A_x} = id_{A_x}$. If not, then one may inductively construct an infinite, strictly decreasing sequence in $A_x$, which is a contradiction as well. Thus the connected component of $id_X$ is contained in $\bigcap_{x \in X} \bigcap_{y \in A_x} W(\{y\}, U_y) = \{id_X\}$, and hence the connected component of $id_X$ is exactly $\{id_X\}$. □

[1] König's Lemma: Let $P$ be a well-founded poset, and $S(x) = \min\{y \in P | y > x\}$ be the set of immediate successors of $x$. If for all $x \in P$, $S(x)$ is finite, and there exists an $x \in P$ such that the set $\{y | y \geq x\}$ is infinite, then there exists an infinite ascending chain in $P$.

**Corollary 14.0.18.** *Suppose $X$ and $Y$ are fp-spaces, and suppose that they both have cores $X^C$ and $Y^C$. Then $X$ is homotopy equivalent to $Y$ if and only if $X^C$ is homeomorphic to $Y^C$.*

Lastly, we introduce the concept of chain-complete posets. Although they do not belong to one of those classes of infinite $A$- spaces considered in Definition 14.0.5, we still have similar results.

**Definition 14.0.19.** A poset $P$ is called *chain-complete* if every chain in $P$ has both a supremum and an infimum in $P$.

**Definition 14.0.20.** An *antichain* in a poset $P$ is a subset $A \subseteq P$ such that no two elements in $A$ are comparable.

**Theorem 14.0.21.** [**42**, Thm. 5.8] *Every chain-complete poset $X$ with no infinite antichains has a finite core.*

**Remark 14.0.22.** In Corollary 14.0.18, instead of requiring $X$ and $Y$ to be fp-spaces, we only need $X^C$ and $Y^C$ to be fp-spaces. Also note that if $X^C$ is a finite core, then it is an $\mathcal{I}$-minimal fp-space, so we can use Theorem 14.0.17 above. In this case, it is straightforward to prove that if any two chain-complete posets $X$, $Y$ without infinite antichains have finite cores $X^C$ and $Y^C$ respectively, then $X$ is homotopy equivalent to $Y$ if and only if $X^C$ is homeomorphic to $Y^C$.

CHAPTER 15

# The enumeration of homotopy classes of $F$-spaces

As promised in 2.5, we here give the results on the enumeration of homotopy types of $F$-spaces that appeared in the 2008 REU paper of Alex Fix and Stephen Patrias. We follow their exposition with minor edits.

**15.0.1. Constructing Posets.** Intuitively, we expect that as the number of points in a poset grows large, the number of neighbors of each point in the graph should grow large as well, and that cases where a point has exactly one neighbor should be very rare. We will examine this probabilistic reasoning rigorously in the final section, but for now, it seems a good heuristic that the large majority of graphs will be minimal once $n$ grows large enough, and that non-minimal graphs will be the exception. Thus, it makes sense to try to count the number of minimal graphs by first enumerating all posets of a given size, and then checking to see whether each such generated graph is minimal.

As a reminder, by Corollary 2.5.7 we are interested in enumerating the minimal spaces up to homeomorphism, and by Corollary 2.5.4, homeomorphism of spaces is equivalent to graph isomorphism of the constructed Hasse diagrams.

**Definition 15.0.1.** Since an isomorphism between graphs is equivalent to relabeling the vertices in a consistent fashion, an equivalence class of graphs under graph isomorphism is called an *unlabeled graph*.

Since any relabeling of a minimal graph produces another minimal graph (as it does not change the in or out degree of any of the vertices), we can treat an unlabeled minimal graph as the equivalence class of a minimal graph under graph isomorphism. This represents the same object as the equivalence class of a minimal space under homeomorphism, so our task is to produce exactly one representative for each unlabeled minimal graph.

Fortunately, a fast algorithm for producing exactly one representative of each unlabeled Hasse diagram has already been proposed by Brinkmann and McKay[**13**], and has been used to enumerate all unlabeled posets on up to 16 points. The remainder of this section will be a summary of these results.

The algorithm works by a method called the canonical construction path which, for every unlabeled poset $P$ on $n$ points, gives a canonical unlabeled poset $Q$ on $n-1$ points such that $Q$ can be obtained from $P$ by deleting a point from the top level. This essentially turns the set of all unlabeled posets into a tree, whereby each poset on $n$ points has a unique parent with $n-1$ points, turning the task of enumeration into a search on this tree.

In order for this construction to work, it is necessary to be able to reconstruct all children of a given poset, and to only construct exactly one example of each child graph (so that we do not produce two different labelings of the same graph, and consider them as different children). It is relatively straightforward to construct the set of all possible children for a graph. However, to reject possible isomorphisms between these candidates we require a device called a canonical choice function.

**Definition 15.0.2.** Let $C$ be a set of candidates, each of which is a poset on $n$ points, with vertex set $[n] = \{1, 2, \ldots, n\}$. Then a function $f : C \longrightarrow 2^{[n]}$ (from candidates to subsets of $[n]$) is a *canonical choice function* if

(1) For each candidate $G$, the set $f(G)$ is an orbit under the automorphisms of $G$ consisting of vertices on the highest level of $G$.
(2) For any two candidates $G, G'$ , if $\sigma : G \longrightarrow G'$ is an isomorphism of graphs, then $\sigma$ maps $f(G)$ onto $f(G')$.

**Definition 15.0.3.** The *parent* of a graph $G$ is the unlabeled graph formed by removing a point $v$ in $f(G)$ from the graph.

**Definition 15.0.4.** Conversely, a graph $G'$ is a *candidate child* of a graph $G$ if we can add a point $v$ to $G$ to obtain $G'$, and so that $v$ is on the highest level of $G'$.

Since the point removed will be on the highest level, we will remove only downwards pointing edges from the graph, so we cannot create any shortcuts or cycles. Thus the parent of a Hasse diagram is again a Hasse diagram.

Also, the parent of a graph is uniquely defined, regardless of which point we remove from $f(G)$ to obtain it. Since $f(G)$ is an orbit of $G$, if $v, w$ are both in $f(G)$ then there is an automorphism $\sigma$ such that $\sigma(v) = w$. But then, the two parents, $G \setminus \{v\}$ and $G \setminus \{w\}$ are isomorphic by $\sigma$, so they are actually the same unlabeled graph.

**Definition 15.0.5.** If $G'$ is a candidate child of $G$, formed by adding a point $v$, we say that $f$ *accepts* $G'$ if and only if $v$ is in $f(G')$, where $f$ is a canonical choice function. If we have fixed some $f$ beforehand, we say that $G'$ is an (actual) *child* of $G$ if $f$ accepts $G'$.

This definition allows us to use the canonical choice function to distinguish between the children of a graph so as to accept only one representative from the unlabeled children of a graph.

**Lemma 15.0.6.** *If $H$ and $H'$ are distinct children of a graph $G$,* i.e.*, both are accepted by some canonical choice function $f$, then $H$ and $H'$ are not isomorphic.*

The only remaining task is to ensure that we actually construct all possible candidate children of a graph, and accept at least one from each isomorphism class. To do this, we must consider all ways in which we can add a point to $G$ such that the new point is now on the highest level.

First, note that if $G$ has $\ell$ levels, then the new point must have an edge to some point on level $\ell - 1$ or level $\ell$, or else the new point would not be on the highest level of $G'$.

Second, the new edges we add between our new point and its neighbors cannot create any shortcuts, since $G'$ must be a Hasse diagram. So, if $x$ and $y$ are both neighbors of our new point, we cannot have $x > y$ or $y > x$. Thus, the neighbors of our new point must be pairwise incomparable. In graph theory, we call such a set an antichain. Each antichain with a point on the highest or next-highest level gives a valid set of neighbors for a new point on the top level, so these antichains describe all ways of connecting a new point to a graph to get a point at the highest level.

Finally, if we pick two antichains $A$ and $A'$ such that there is a graph automorphism $\sigma$ that sends $A$ to $A'$, then the resulting graphs formed by connecting a new point to each of $A$ and $A'$ will be isomorphic by the same permutation $\sigma$ (extended to send the new vertex to itself). Thus, it suffices to consider only one representative from each orbit of the antichains under group automorphism.

From the above considerations, we have the following algorithm:

**Theorem 15.0.7.** *To construct all children of an unlabeled poset $P$ with $\ell$ levels:*

*(1) Find a representative from each orbit of antichains that contains a point on level $\ell$ or $\ell - 1$.*
*(2) Connect a new point $v$ to each antichain computed in step (1) in turn.*
*(3) Compute the canonical choice function for each candidate constructed in step (2). A candidate is a child of $P$ if and only if the new point $v$ is in $f(P)$.*

To actually enumerate all unlabeled posets with at most $n$ points, begin with the graphs consisting of no more than $n$ points all on the first row, and then perform a depth-first search on the children of each graph that we find.

The proof of the correctness of this algorithm is due to Brinkmann and McKay [**13**], but for now, the assertion that it does generate exactly one example of each unlabeled poset should suffice to justify our modifications to count minimal graphs.

### 15.0.2. Constructing Minimal Graphs.

Since we are not in fact trying to count all posets, but only a subset of them, we really only need to generate graphs which are minimal, or some of whose children will eventually be minimal. If we can determine that a given graph will never have minimal descendants, then we can prune that node from our search, and not have to waste computation on branches which will never bear fruit. We can do this most easily by considering a slightly larger collection than the set of all minimal graphs.

**Definition 15.0.8.** We say that a graph is *non-downbeat* if there are no points with out-degree equal to 1. This is equivalent to the statement that the underlying topology has no downbeat points.

All minimal graphs are of course non-downbeat, so if we can construct all non-downbeat graphs and then check whether each one is non-upbeat as well, we will have accomplished our task of counting all minimal graphs.

The categorization of graphs as non-downbeat is useful primarily because it is a hereditary property:

**Lemma 15.0.9.** *If a graph $G'$ is non-downbeat, then its parent $G$ is non-downbeat as well.*

Proof. Let $v$ be the vertex that we remove from $G'$ to obtain $G$. Remember that $v$ is on the top level, so there cannot be any edges $w \longrightarrow v$, or else $w$ would be on a higher level; thus in removing $v$ from $G'$, we do not change the out-degree of any point $w \neq v$. Thus since no points in $G' \setminus \{v\}$ have out-degree equal to 1, no points in $G$ have out-degree 1 either. Thus $G$ is non-downbeat. □

We can also categorize which children of a non-downbeat graph will also be non-downbeat (allowing us to not construct the other children in the first place).

**Lemma 15.0.10.** *If $G$ is non-downbeat, and $G'$ is obtained from $G$ by adding a point $v$ on the highest level, then $G$ is non-downbeat if and only if $v$ has two or more neighbors.*

Proof. Again, by adding a point at the top level, we do not change the out-degree of any of the points in $G$, so $G'$ is non-downbeat if and only if $v$ is not a downbeat point. Then, it is clear that $v$ will not be a downbeat point if and only if it has two or more neighbors. □

Finally, we can identify a special case of child which will never produce any minimal descendants, even though the child itself is non-downbeat.

**Lemma 15.0.11.** *If $G$ has exactly one point on the top level $\ell$, and $G'$ is obtained from $G$ by adding a point to a new level $\ell + 1$, then no descendant of $G'$ will ever be minimal.*

Proof. We claim that all descendants of $G'$ will have exactly one point on level $\ell$, but have a highest level $\ell' > \ell$. By Proposition 2.5.11, such graphs cannot be minimal.

We proceed by structural induction on the tree of descendants of $G'$. As a base case, this is trivially true of $G'$. Now, let $H$ be a descendant of $G'$ with exactly one point on level $\ell$ and with highest level $\ell' > \ell$. Then all children of $H$ are formed by adding a point on level $\ell'$ or $\ell' + 1$, so all children of $H$ still have exactly one point on level $\ell$. □

These three Lemmas allow us to make the following changes to the above algorithm which will prune dead-ends. We call all children which are not known to be dead-ends by the above lemmas *useful children*.

**Theorem 15.0.12.** *To construct all useful children of a graph $G$ with highest level $\ell$:*

1. *Find a representative from each orbit of antichains that contains a point on level $\ell - 1$. If $G$ has more than one point on level $\ell$, also find representatives from each orbit of antichains with a point on level $\ell$.*
2. *Connect a new point $v$ to each antichain computed in step 1 whenever the antichain contains at least two vertices.*
3. *Compute the canonical choice function for each candidate constructed in step 2. A candidate is a child of $P$ if and only if the new point $v$ is in $f(P)$.*
4. *If the canonical choice function accepts, then verify that the graph is non-upbeat as well by checking that no point has in-degree 1. If the graph is non-upbeat, then increment our count of minimal graphs encountered. Even if the graph contains upbeat points, it is still a useful child of $G$ and could have minimal descendants, so we must recursively find its children as well.*

By the above Lemmas, the children which we ignore are all such that they are not minimal, and will never have minimal descendants, so we can ignore those branches and still find representatives of all minimal graphs.

**15.0.3. Computational Results.** The above algorithm was actually implemented and run to obtain the exact counts of unlabeled minimal graphs with small numbers of points. Various optimizations described in [**13**] were implemented to expedite the computation of the canonical choice function, and in the construction of antichains. Canonical labeling of graphs (needed for the canonical choice function) was achieved by the using the graph isomorphism library `nauty` [**51**]. This is the same library used by Brinkmann and McKay in their original library [**13**].

| Points | Minimal graphs | Homotopy classes | Unlabeled posets |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 2 |
| 3 | 1 | 3 | 5 |
| 4 | 2 | 5 | 16 |
| 5 | 4 | 9 | 63 |
| 6 | 11 | 20 | 318 |
| 7 | 36 | 56 | 2045 |
| 8 | 160 | 216 | 16999 |
| 9 | 954 | 1170 | 183231 |
| 10 | 7929 | 9099 | 2567284 |
| 11 | 92092 | 101191 | 46749427 |
| 12 | 1493102 | 1594293 | 1104891746 |

TABLE 1. Counts of minimal graphs and homotopy classes

To ensure the correctness of these results, we used the C preprocessor to compile two different versions of the algorithm, one with our changes as described above, and one functionally identical to the original algorithm for enumerating all unlabeled posets. The unmodified algorithm successfully reproduced the counts for all unlabeled posets up to 11 points, but could not be run on higher inputs since it takes far longer to run than the modified version (This was the purpose of pruning branches in the first place). Since the code for the two versions is 99% identical, it is much more feasible for a human to check that the changes we implemented actually produce the desired result. Furthermore, at the beginning of researching this topic, one of the authors enumerated all minimal graphs up to 8 points by hand, and these counts were verified by the algorithm.

Table 15.0.3 gives the counts for the number of unlabeled minimal graphs with up to 12 points. Since the number of homotopy classes with $n$ points is the number of unlabeled graphs with at most $n$ points, their number is simply the sum of the counts of minimal graphs with at most $n$ points. We also provide the number of unlabeled graphs (equal to the number of $F$-spaces up to homeomorphism) from [**13**] for reference.

**15.0.4. Asymptotic Enumeration.** Kleitman and Rothschild's paper [**38**] has been used to describe the asymptotic behavior of posets as consisting of graphs with exactly three levels with 'roughly' $n/4, n/2$ and $n/4$ points on each of the three levels. However, the exact statement of the result will prove much more useful in describing the asymptotic behavior of minimal graphs.

Their paper describes a set of posets on a vertex set $V$ of $n$ points which formalizes this notion of three-leveled posets. The collection, $Q(V)$ consists of the posets $P$ such that

(1) The vertices of $P$ are the disjoint union, $S_1 \amalg S_2 \amalg S_3$ where points in $S_i$ only have edges going to points in $S_{i-1}$ or $S_{i-2}$
(2) The size of the partition is such that
   (a) $||S_i| - n/4| < (n-1)^{\frac{1}{2}} \log(n-1)$
   (b) $||S_2| - n/2| < log(n-1)$
(3) For every $u \in S_1 \cup S_3$, $||N(u) \cap S_2| - n/4| < (n-1)^{7/8}$, where $N(u)$ is the set of neighbors of u.
(4) For every $u \in S_2$, $||N(u) \cap S_i| - n/8| < (n-1)^{7/8}$ for $i = 1$ or $i = 3$

By a collection of logarithmic bounds given by their lemma, they find that the number of posets on n points, $P_n$, is asymptoticaly equivalent to the number of posets in $X(V)$, and that this is asymptoticaly equivalent to the number of posets in $Q(V)$. Specifically, if $Q_n$ counts the number of posets in $Q(V)$ with $n$ points, then $P_n = (1 + O(1/n))Q_n$.

In our enumeration we have been concerned with non-isomorphic, minimal, leveled digraphs (equivalently unlabeled, minimal Hasse diagrams) as these define the homotopy classes of $F$-spaces, yet Kleitman and Rothschild's result is using *labeled* Hasse Diagrams, which gives the number of all $F$-spaces. To make use of their result, we need to know the relation between the number of unlabeled graphs and labeled graphs. For this we make use of an exceedingly general result from Prömel [**56**], which states that in any large enough collection of labeled objects, the fraction of objects with non-trivial automorphism group goes to 0, and thus asymptotically, the ratio of labeled objects to unlabeled objects approaches $\frac{1}{n!}$.

**Lemma 15.0.13.** *Let $\mathscr{C}$ be a class of finite labeled structures (* i.e., *a finite labeled set with a single binary relation) which is closed under substructures and isomorphisms. Let $C(n)$ count the number of such structures on sets with n points, and let $C^u(n)$ count the number of unlabeled structures on n points. If $(C)$ satisfies the growth condition*

$$C(n) = cn^2 + dn + o(n)$$

*where $c > 0$ and $d$ is arbitrary, then*

$$C^u(n) \sim \frac{C(n)}{n!}$$

Applied to the case of classes of posets, this lemma states that as long as our collection of labeled posets is large enough, we can directly derive asymptotic bounds on the growth of the collection of unlabeled posets. Since this condition is satisfied both by the set of all posets and by the set of posets in $Q(V)$ we have the immediate corollary:

**Corollary 15.0.14.** *The number of unlabeled posets in $Q(V)$, $Q_n^u$, is asymptoticaly equal to the number of unlabeled posets, $P_n^u$.*

PROOF. We know, by Kleitman and Rothschild's result [**38**], that the number, $P_n$, of all labeled posets, is such that $\log(P_n) = \frac{n^2}{4} + \frac{3n}{2} + O(\log(n))$ . So by the above lemma, $P_n^u \sim \frac{1}{n!}P_n$. Similarly, since $P_n \sim Q_n$, we have that $Q(V)$ satisfies the growth condition as well, so $Q_n^u \sim \frac{1}{n!}Q_n$. Also, $P_n \sim Q_n$ implies that $\frac{P_n}{n!} \sim \frac{Q_n}{n!}$ so

$$Q_n^u \sim \frac{Q_n}{n!} \sim \frac{P_n}{n!} \sim P_n^u$$

□

An asymptotic enumeration of the homotopy classes of finite $F$-spaces follows directly from this.

**Corollary 15.0.15.** *The number of homotopy classes of finite $T_0$ topological spaces is asymptotically equivalent to the number of all $T_0$ spaces up to homeomorphism.*

PROOF. By definition, graphs in $Q(V)$ have the property that

(1) For every $u \in S_1 \cup S_3$, the number of neighbors of $u$ in $S_2$ is greater than $n/4 - (n-1)^{7/8}$
(2) For every $u \in S_2$, the numbers of neighbors of $u$ in $S_1$ and $S_3$ are each greater than $n/8 - (n-1)^{7/8}$

Thus, for $n$ large enough, every point in the top row has out-degree at least 2, every point in the middle row has out-degree and in-degree at least 2, and every point in the bottom row has in-degree at least 2. Thus, every graph in $Q(V)$ with enough points is a minimal graph.

But then, every unlabeled graph in $Q(V)$ is an unlabeled minimal graph, so if we let $M_n^u$ be the number of unlabeled minimal graphs with $n$ points, then we have that $Q_n^u \leq M_n^u \leq P_n^u$. Since $Q_n^u \sim P_n^u$, by the squeeze theorem we have $M_n^u \sim P_n^u$.

But remembering that $M_n^u$ also counts the number of homotopy classes of finite spaces up to homotopy, and $P_n^u$ counts the number of finite spaces up to homeomorphism, we have that almost every unlabeled graph on n vertices is minimal and therefore the number of homotopy classes of $F$-spaces is asymptotically equal to the number of all $F$-spaces. □

Before considering the implications of this, it is worth noting that the above method is not the only way to prove this result; instead, one only needs that almost every poset has three levels and that these levels monotonically increase in size as the poset grows.

**Lemma 15.0.16.** *Almost all graphs with 3 levels are minimal.*

PROOF. Let $P = L_1 \amalg L_2 \amalg L_3$ be an unlabeled digraph with three levels, and let $|L_3| = j$, $|L_2| = k$, and $|L_1| = l$.

To determine the probability of this graph being minimal, consider that $P$ is formed by taking the complete tri-partite graph on its levels, randomly deleting some number of edges, and possibly adding edges from $L_3$ to $L_1$.

So $x \in L_3$ has between 1 and k edges leading to $L_2$, by definition of the levels of a graph; for $y \in L_2$ $y$ has between 0 and j edges to it from $L_3$. A point in $L_3$ might have edges going to $L_1$ in addition to its edges going to $L_2$, so for any $x \in L_3$ $\text{prob}(\text{outdegree}(x) > 2) \geq 1 - \frac{1}{k}$. This bound is from the fact that there are $k$ ways for $x$ to have one edge, but also $k$ ways for it to have any degree up to $k-1$ and so we get a very conservative bound by considering only one possibility for each possible degree that $x$ may have.

Each event (placing edges from a point in $L_3$ to points in $L_2$) is independent from the others, so

$$\text{prob}(\forall x \in L_3, \text{outdegree}(x) \geq 2) \geq \left(1 - \frac{1}{k}\right)^j = \left(\frac{k-1}{k}\right)^j = \frac{k^j - jk^{j-1} + \cdots - (-1)^j k + (-1)^j}{k^j}$$

Therefore, for a given $j$,

$$\lim_{k \to \infty} (\text{prob}(\forall x \in L_3, \text{outdegree}(x) \geq 2)) = 1.$$

Then, we have that for any $x \in L_3$, $\text{prob}(\text{indegree}(x) \geq 2) > (1 - \frac{1}{j})^2$ and

$$\text{prob}(\forall y \in L_2, \text{outdegree}(y) \geq 2) > \left(1 - \frac{1}{j}\right)^{2k} = \left(\frac{j-1}{j}\right)^{2k} = \frac{j^{2k} - 2kj^{2k-1} + \cdots - k + 1}{k^j}$$

Therefore, for a given $k$,

$$\lim_{j \to \infty} (\text{prob}(\forall y \in L_2, \text{outdegree}(y) \geq 2)) = 1.$$

Similarly

$$\lim_{l \to \infty} (\text{prob}(\forall y \in L_2, \text{outdegree}(y) \geq 2)) = 1$$

and

$$\lim_{k \to \infty} (\text{prob}(\forall z \in L_1, \text{outdegree}(z) \geq 2)) = 1.$$

These events are not probabilistically independent, so we cannot just multiply the individual probabilities to obtain the probability of all 4 events happening simultaneously. However, we can take the union bound on the complement of these events, giving prob($P$ is not minimal) $\leq \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4$ where

$$\epsilon_1 = \text{prob}(\exists\ x \in L_3,\ \text{outdegree}(x) < 2),$$
$$\epsilon_2 = \text{prob}(\exists\ y \in L_2,\ \text{indegree}(x) < 2),$$
$$\epsilon_3 = \text{prob}(\exists\ y \in L_2,\ \text{outdegree}(x) < 2),$$

and

$$\epsilon_4 = \text{prob}(\exists\ z \in L_1,\ \text{outdegree}(z) < 2).$$

Then almost all such graphs $P$ are minimal, provided that the size of each level increases as the graph itself grows, meaning graphs on n vertices $P = L_1 \amalg L_2 \amalg L_3$ with $|L_3| = an$ $|L_2| = bn$ $|L_1| = cn$ such that $a + b + c = 1$. □

**Remark 15.0.17.** The graphs in $Q(V)$ are of this form, but this proof is perhaps more intuitive.

Let us go back and consider this result. In some ways it is unsurprising to find this behavior; given a large space, the digraph representing it is large and thus has many more possible edges between vertices. In this way it makes sense that with enough edges on the graph, there is a good probability that every vertex has in-degree and out-degree at least 2. However, with respect to the topology, this result is startling; homotopy equivalence does not narrow down the classification of $F$-spaces any more than homeomorphism for large $F$-spaces. Nevertheless, when we look at the actual, numerical counts for number of spaces up to homotopy and homeomorphism, we see a large gap between the relative growth rates. For example, for spaces with 12 points, there are 1,104,891,746 spaces up to homeomorphism, with only 1,594,293 distinct spaces up to homotopy equivalence (a factor of 70 difference). Thus, even though the asymptotic behavior of these two numbers is the same, the convergence for small values is very slow.

CHAPTER 16

# An outline summary of point set topology

We have implicitly given a quick outline of a bare bones introduction to point set topology in Chapter 1. The focus was on basic concepts and definitions rather than on the usual examples that give substance to the subject. We thought the reader might like to see a brief summary of some of the most basic parts of point-set topology that were not discussed in Chapter 1, including but not limited to those results we that we have used in our exposition.

## 16.1. Metric spaces

The intuition for and the most important examples in point-set topology come from metric spaces, where the topology is defined in terms of a distance function.

**Definition 16.1.1.** A metric $d$ on a set $X$ is a function $d : X \times X \longrightarrow \mathbb{R}$ such that

(i) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.
(ii) $d(x, y) = d(y, x)$.
(iii) $d(x, y) + d(y, z) \geq d(x, z)$.

The basis $\mathscr{B}$ determined by a metric $d$ consists of the sets $B(x, r) = \{y | d(x, y) < r\}$. The topology generated by $\mathscr{B}$ is called the metric topology on $X$ determined by $d$. A topological space $X$ is *metrizable* if its topology is determined by a metric.

A subset $A$ of a metric space $X$ has an induced metric, and the metric and subspace topologies coincide. Any metric space is Hausdorff.

Of course, $\mathbb{R}^n$ has the standard metric

$$d(x, y) = (\sum (y_i - x_i)^2)^{1/2}.$$

The metric topology that it determines coincides with the product topology. The product of countably many copies of $\mathbb{R}$ is metrizable, but the product of uncountably many copies of $\mathbb{R}$ is not. There is a metric topology on any product of copies of $\mathbb{R}$, called the uniform topology, but it is finer than the product topology when the product is infinite.

For metric spaces, Lemma 1.5.8 leads to the familiar $\varepsilon$, $\delta$ formulation of continuity.

**Lemma 16.1.2.** *A function $f : X \longrightarrow Y$ between metric spaces is continuous if and only if for each $x \in X$ and each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$f(B(x, \delta)) \subset B(f(x), \varepsilon);$$

*that is, if the distance from $x$ to $y$ is less than $\delta$, then the distance from $f(x)$ to $f(y)$ is less than $\varepsilon$.*

Moreover, we can characterize continuity in terms of convergent sequences.

**Definition 16.1.3.** A sequence $\{x_n\}$ of points in a space $X$ *converges* to a point $x$ if every neighborhood of $x$ contains all but finitely many of the $x_n$. We then write $\{x_n\} \longrightarrow x$. If $X$ is Hausdorff, then the limit of $\{x_n\}$ is unique if it exists.

Observe that if $\{x_n\} \subset A$ and $\{x_n\} \longrightarrow x$, then $x \in \bar{A}$. The converse does not hold for general topological spaces, but it does hold for metric spaces. Actually, what is relevant is not the metric but something it implies.

defined and used earlier

**Definition 16.1.4.** A space $X$ is *first countable* if for each $x \in X$, there is a countable set of neighborhoods $U_n$ of $x$ such that any neighborhood of $x$ contains at least one of the $U_n$; $X$ is *second countable* if its topology has a countable basis.

Using the neighborhoods $B(x, 1/n)$, we see that a metric space is first countable.

**Lemma 16.1.5.** *Let $X$ be first countable. Then $x \in \bar{A}$ if and only if there is a sequence $\{x_n\} \subset A$ such that $\{x_n\} \longrightarrow x$.*

Using Lemma 1.5.2 this leads to the promised characterization of continuity.

**Proposition 16.1.6.** *Let $f : X \longrightarrow Y$ be a function, where $X$ is first countable and $Y$ is any space. Then $f$ is continuous if and only for every convergent sequence $\{x_n\} \longrightarrow x$ in $X$, $\{f(x_n)\} \longrightarrow f(x)$ in $Y$.*

## 16.2. Compact and locally compact spaces

**Definition 16.2.1.** A space $X$ is *compact* if every open cover contains a finite subcover. That is, if $X$ is the union of open sets $U_i$, then there are finitely many indices $i_j$, such that $X$ is the union of the $U_{i_j}$.

Using standard facts about complements, one can reformulate the notion of compactness as follows. Say that a set of subsets of $X$ has the finite intersection property if any finite subset has nonempty intersection.

defined and used earlier

**Proposition 16.2.2.** *A space $X$ is compact if and only if any set of closed subsets of $X$ with the finite intersection property has nonempty intersection. In particular, if $X$ is compact and if $\{C_n\}$ is a nested sequence of closed subsets of $X$, $C_n \supset C_{n+1}$, then $\cap C_n$ is nonempty.*

A metric space $X$ is *bounded* if $d(x, y) \leq D$ for some fixed $D$ and all $x, y \in X$; the least such $D$ is called the *diameter* of $X$. Boundedness is not a "topological" property, since it depends on the choice of metric: different metrics can define the same topology but have very different bounded subsets. With the standard Euclidean metric, we have the following result.

**Theorem 16.2.3** (Heine-Borel)**.** *A subspace of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

In general, we have the following observations about the compactness of subspaces. For a subset $A$ of a space $X$, a cover of $A$ in $X$ is a set of subsets of $X$ whose union contains $A$.

**Proposition 16.2.4.** *Let $A$ be a subspace of a space $X$. Then $A$ is compact if and only if every cover of $A$ in $X$ has a finite subcover. If $X$ is compact, then every closed subspace of $X$ is compact.*

For compact Hausdorff spaces, the second statement has a converse.

**Proposition 16.2.5.** *Every compact subspace of a compact Hausdorff space is closed.*

**Proposition 16.2.6.** *If $f : X \longrightarrow Y$ is a continuous function and $X$ is compact, then the image of $f$ is a compact subspace of $Y$. In particular, any quotient space of a compact space is compact.*

**Theorem 16.2.7.** *Let $X$ be compact and $Y$ be Hausdorff. Then a continuous bijection $f : X \longrightarrow Y$ is a homeomorphism (hence $X$ is Hausdorff and $Y$ is compact).*

Proof. If $C$ is closed in $X$, then $C$ is compact, hence $f(C)$ is compact, hence $f(C)$ is closed in $Y$. This proves that $f^{-1}$ is continuous. □

The results above give the behavior of compactness with respect to subspaces and quotient spaces. The behavior with respect to products is deeper than anything that we have stated so far.

**Theorem 16.2.8** (Tychonoff)**.** *Any product of compact spaces is compact.*

The case of finite products is not difficult, but the general case is.

For metric spaces, compactness can be characterized in terms of limit points and convergent sequences.

**Theorem 16.2.9.** *Consider the following conditions on a space $X$.*

*(i) $X$ is compact.*
*(ii) Every infinite subset of $X$ has a limit point.*
*(iii) Every sequence in $X$ has a convergent subsequence.*

*In general, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). If $X$ is a metric space, the three conditions are equivalent.*

We say that $X$ is *sequentially compact* if it satisfies (iii). The following important fact is used in proving that $(iii) \Rightarrow (i)$ when $X$ is a metric space.

**Lemma 16.2.10** (Lebesgue Lemma)**.** *Let $\mathscr{O}$ be an open cover of a sequentially compact metric space $X$. Then there is a $\delta > 0$ such that if $A \subset X$ is bounded with diameter less than $\delta$, then $A$ is contained in some $U \in \mathscr{O}$.*

Proof. If not, then for each $n$ we can choose a subset $A_n$ of diameter less than $1/n$ which is not contained in any $U \in \mathscr{O}$. Choose a point $x_n \in A_n$ for each $n$. Suppose that $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ that converges to some $x$. Certainly $x \in O$ for some $U \in \mathscr{O}$. For small enough $\varepsilon$ and large enough $n_i$, $B(x, 2\varepsilon) \subset U$, $d(x, x_{n_i}) < \varepsilon$ and $1/n_i < \varepsilon$. It follows easily that $A_{n_i} \subset U$, which is a contradiction. □

**Definition 16.2.11.** A space $X$ is locally compact if each point of $X$ has a neighborhood that is contained in a compact subspace of $X$.

Clearly $\mathbb{R}^n$ is locally compact but not compact.

**Lemma 16.2.12.** *Let $X$ be a Hausdorff space. Then $X$ is locally compact if and only if for any point $x$ and any neighborhood $U$ of $x$, there is a smaller neighborhood $V$ of $x$ such that $\bar{V}$ is compact and $\bar{V} \subset U$.*

This criterion is needed to prove the second part of the following result.

**Lemma 16.2.13.** *Let $A$ be a subspace of a locally compact subspace $X$. If $A$ is closed or if $A$ is open and $X$ is Hausdorff, then $A$ is locally compact.*

Locally compact Hausdorff spaces admit a canonical compactification, as we now make precise.

**Definition 16.2.14.** A *compactification* of a space $X$ is an inclusion of $X$ as a dense subspace in a compact Hausdorff space $Y$. Observe that a compactification of a compact Hausdorff space must be a homeomorphism. Two compactifications $Y$ and $Y'$ are equivalent if there is a homeomorphism $Y \longrightarrow Y'$ which restricts to the identity map on $X$.

Compactifications are of fundamental importance in topology and algebraic geometry. The most naive example is the one-point compactification. The construction applies to any Hausdorff space, but it only gives a Hausdorff space when $X$ is locally compact.

**Construction 16.2.15.** Let $X$ be a Hausdorff space and let $Y$ be the union of $X$ and a disjoint point denoted $\infty$. Then $Y$ is a topological space whose open sets are the open sets in $X$ together with the complements of the compact sets in $X$. The space $Y$ is called the *one point compactification of* $X$.

If $X$ is itself compact, then $\{\infty\}$ is open and closed in $Y$ and $Y$ is the union of its components $X$ and $\{\infty\}$.

**Proposition 16.2.16.** *If $X$ is a locally compact Hausdorff space that is not compact, then the one point compactification $Y$ of $X$ is in fact a compactification: $Y$ is compact Hausdorff and $X$ is a dense subspace.*

Since $X$ is itself one of the open sets in $Y$, Lemma 16.2.13 gives the following implication.

**Corollary 16.2.17.** *A space $X$ is locally compact and Hausdorff if and only if it is homeomorphic to an open subset of a compact Hausdorff space.*

## 16.3. Further separation properties

We have defined $T_0$, $T_1$ spaces and $T_2$, or Hausdorff spaces. We give three analogous definitions, and we describe various implications relating these separation properties to each other and to local compactness.

**Definition 16.3.1.** Let $X$ be a $T_1$ space (points are closed), let $x \in X$, and let $A$ and $B$ be closed subsets of $X$.

(i) $X$ is *regular* if whenever $x \notin A$, there are open subsets $U$ and $V$ such that $x \in U$ and $A \subset V$.
(ii) $X$ is *completely regular* if whenever $x \notin A$, there is a continuous function $f : X \longrightarrow [0,1]$ such that $f(x) = 0$ and $f(a) = 1$ for $a \in A$.
(iii) $X$ is *normal* if whenever $A \cap B = \emptyset$, there are open subsets $U$ and $V$ such that $A \subset U$ and $B \subset V$.

Together with Lemma 16.2.12, the following result makes clear that these separation properties are closely related to local compactness.

**Lemma 16.3.2.** *Let $X$ be a $T_1$ space.*

*(i) $X$ is regular if and only if for any point $x$ and any neighborhood $U$ of $x$, there is a smaller neighborhood $V$ of $x$ such that $\bar{V} \subset U$.*
*(ii) $X$ is normal if and only if for any closed set $A$ contained in an open set $U$, there is an open set $V$ such that $A \subset V$ and $\bar{V} \subset U$.*

Language varies. The terms regular, completely regular, and normal are often defined without assuming that $X$ is $T_1$. Then what we call regular and normal spaces are called $T_3$ and $T_4$ spaces and what we call completely regular spaces are called Tychonoff spaces. (As already noted, the $T_i$ notation goes back to a 1935 paper of Alexandroff and Hopf [**3**], but some later references confuse things further by forgetting history and using $T_i$ differently).

**Lemma 16.3.3.** *The following implications hold: A normal space is completely regular. A completely regular space is regular. A regular space is Hausdorff.*

$$\textit{normal} \Rightarrow \textit{completely regular} \Rightarrow \textit{regular} \Rightarrow \textit{Hausdorff}$$

The implications normal $\Rightarrow$ regular $\Rightarrow$ Hausdorff are obvious. The implication normal $\Rightarrow$ completely regular is a consequence of the following important result.

**Theorem 16.3.4** (Urysohn's lemma)**.** *If $X$ is normal and $A$ and $B$ are disjoint closed subsets of $X$, then there is a continuous function $f : X \longrightarrow I$ such that $f(a) = 0$ if $a \in A$ and $f(b) = 1$ if $b \in B$.*

The proof is non-trivial, and the closely analogous assertion that regular implies completely regular is false. Uryssohn's lemma can be used to prove the following equally important result.

**Theorem 16.3.5** (Tietze extension theorem)**.** *If $A$ is a closed subspace of a normal space $X$ and $f : A \longrightarrow I$ is a continuous function, then $f$ can be extended to a continuous function $X \longrightarrow I$.*

Normality is the most desirable separation property, but it is much less nicely behaved than our other separation properties.

**Proposition 16.3.6.** *A subspace of a Hausdorff, regular, or completely regular space is again Hausdorff, regular, or completely regular. A product of Hausdorff, regular, or completely regular spaces is again Hausdorff, regular, or completely regular. Neither of these assertions is true in general for normal spaces.*

For example, the product of uncountably many copies of $\mathbb{R}$ is not normal. Since $\mathbb{R}$ is homeomorphic to the open interval $(0, 1)$ and Tychonoff's theorem implies that the product of uncountably many copies of $I$ is compact Hausdorff, this example also shows that a subspace of a normal space need not be normal. Nevertheless, most spaces of interest are normal.

**Theorem 16.3.7.** *If $X$ is metrizable or compact Hausdorff, then $X$ is normal.*

Some indication of the importance of complete regularity is given by the following sequence of results, the second of which should be compared with Corollary 16.2.17.

**Theorem 16.3.8.** *If $X$ is completely regular, then it can be embedded as a subspace of a product of copies of the unit interval.*

**Corollary 16.3.9.** *The following conditions on a space $X$ are equivalent.*

*(i) $X$ is completely regular.*
*(ii) $X$ is homeomorphic to a subspace of a compact Hausdorff space.*
*(iii) $X$ is homeomorphic to a subspace of a normal space.*

**Corollary 16.3.10.** *A space $X$ admits a compactification if and only if it is completely regular.*

PROOF. If $Y$ is a compactification of $X$, then $X$ is a subspace of the compact Hausdorff space $Y$ and is thus completely regular. Conversely, if $X$ is completely regular and thus homeomorphic to a subspace of some compact Hausdorff space $Z$, then the closure of the image of $X$ in $Z$ is a compactification of $X$, called the compactification induced by the inclusion of $X$ in $Z$. □

The very definition of complete regularity leads to a canonical compactification.

**Construction 16.3.11.** Let $X$ be completely regular. Let $F = F(X)$ be the set of all continuous functions $f : X \longrightarrow I$, let $Z = Z(X)$ be the product of copies of $I$ indexed on the set $F$, and let $i : X \longrightarrow Z$ be the map whose $f$th coordinate is the map $f$. Then $i$ is an inclusion. The induced compactification is denoted $\beta X$ and called the *Stone-Čech compactification* of $X$.

The Stone-Čech compactification is characterized as the unique compactification (up to equivalence) that satisfies the following "universal property".

**Proposition 16.3.12.** *Let $X$ be a completely regular space. A map $f : X \longrightarrow Y$, where $Y$ is a compact Hausdorff space, extends uniquely to a map $\tilde{f} : \beta X \longrightarrow Y$.*

PROOF. Uniqueness holds by Lemma 1.5.3. When $Y = I$, the existence is immediate from the construction: $f$ is one of the coordinate maps, and the projection from $Z(X)$ to this coordinate restricts to $\tilde{f} : \beta X \longrightarrow I$. In general, $Y$ is homeomorphic to $\beta Y \subset Z(Y)$. The map $f_g : X \xrightarrow{f} Y \cong \beta Y \subset Z(Y) \xrightarrow{\pi_g} I$ obtained from the $g$th coordinate projection $\pi_g$, $g \in Z(Y)$, extends to a map $\tilde{f}_g : \beta X \longrightarrow I$, and $\tilde{f}_g$ is the $g$th coordinate of a map $\beta X \longrightarrow Z(Y)$. This map sends $X$ into the closed set $\beta Y$, hence it sends the closure $\beta X$ into $\beta Y \cong Y$, giving $\tilde{f}$. □

## 16.4. Metrization theorems and paracompactness

Since we are much more comfortable with metric spaces than with general spaces, it is important to be able to recognize when the topology on a given space is that induced by some metric. The simplest criterion is the following. Metrization theorems are proven by embedding a given space as a subspace of a space that is known to be metrizable. Let $I^\omega$ denote the product of countably many copies of $I$. It is a metric space, which would be false for an uncountable product.

**Theorem 16.4.1** (Urysohn metrization theorem)**.** *The following conditions on a $T_1$ space $X$ are equivalent.*

*(1) $X$ is regular and second countable.*
*(2) $X$ is homeomorphic to a subspace of $I^\omega$.*
*(3) $X$ is metrizable and has a countable dense subset.*

Remember that second countable means that there is a countable basis for the topology. This ensures the following analogue of compactness.

**Lemma 16.4.2.** *If $X$ is second countable, then any open cover of $X$ has a countable subcover and $X$ has a countable dense subset.*

Second countability is a strong condition, and a weaker countability condition, plus regularity, is necessary and sufficient for metrizability.

**Definition 16.4.3.** A set $\mathscr{V}$ of subsets of $X$ is *locally finite* if each $x \in X$ has a neighborhood that intersects at most finitely many subsets of $\mathscr{V}$. A cover $\mathscr{O}$ of $X$ is *$\sigma$-locally finite* if it is the union of countably many locally finite subsets.

**Theorem 16.4.4** (Nagata-Smirnov metrization theorem)**.** *A space is metrizable if and only if it is regular and has a $\sigma$-locally finite basis.*

The "$\sigma$" here is essential: if a Hausdorff space has a locally finite cover, then it is discrete.

There is another characterization of metrizability that is perhaps more intuitive.

**Definition 16.4.5.** A space $X$ is locally metrizable if every point $x \in X$ has a neighborhood $U$ such that $U$ (with its subspace topology) is metrizable.

Clearly any metric space is locally metrizable. There is a property, called paracompactness, that is very often used to patch local conditions to obtain a global condition, and Stone proved that any metric space is paracompact.

**Theorem 16.4.6** (Smirnov metrization theorem)**.** *A space is metrizable if and only if it is paracompact and locally metrizable.*

We explain paracompactness. A *refinement* of a cover $\mathscr{O}$ of $X$ is a collection of subspaces each of which is contained in at least one of the spaces in $\mathscr{O}$.

**Definition 16.4.7.** A space $X$ is paracompact if every open cover of $X$ has a locally finite refinement that is again an open cover of $X$.

Clearly a compact Hausdorff space is paracompact. The following sharpening of part of Theorem 16.3.7 holds.

**Theorem 16.4.8.** *A paracompact space $X$ is normal.*

Like normality, paracompactness is not preserved by standard constructions. For this reason, Stone's theorem that metrizable $\Rightarrow$ paracompact seems more useful than the converse implication of Smirnov's metrization theorem.

**Proposition 16.4.9.** *A closed subspace of a paracompact space is paracompact. In general, subspaces of paracompact spaces and products of paracompact spaces need not be paracompact.*

The point of paracompactness is that it ensures the existence of particularly convenient open covers. This is very important in the theory of fiber bundles in algebraic topology.

**Definition 16.4.10.** An open cover $\mathscr{O}$ of $X$ is *numerable* if it is locally finite and for each $U \in \mathscr{O}$ there is a continuous function $\phi_U : X \longrightarrow I$ such that $\phi_U(x) > 0$ only if $x \in U$. A numerable cover $\mathscr{U}$ is a *partition of unity* if $\sum_U \phi_U(x) = 1$ for each $x \in X$.

Given a numerable cover $\mathscr{O}$, we can define $\phi(x) = \sum_U \phi_U(x)$ and $\psi_U(x) = \phi_U(x)/\phi(x)$, thereby obtaining a partition of unity.

**Proposition 16.4.11.** *If $X$ is paracompact, then any open cover of $X$ has a numerable refinement.*

**Definition 16.4.12.** An *$n$-manifold* $M$ is a second countable Hausdorff space each point of which has a neighborhood homeomorphic to $\mathbb{R}^n$.

By the Urysohn metrization theorem, an $n$-manifold is metrizable. By Stone's theorem, it is therefore paracompact. The following theorem can be proven by use of a numerable cover of $M$.

**Theorem 16.4.13.** *Any $n$-manifold $M$ can be embedded as a subspace of $\mathbb{R}^N$ for a sufficiently large $N$.*

FINITE METRIC SPACES AND THEIR EMBEDDING INTO LEBESGUE SPACES

Abstract. The properties of the metric topology on infinite and finite sets are analyzed. We answer whether finite metric spaces hold interest in algebraic topology, and how this result is generalized to pseudometric spaces through the Kolmogorov quotient. Embedding into Lebesgue spaces is analyzed, with special attention for Hilbert spaces, $\ell^p$, and $\mathbb{E}^N$.

## 16.5. Introduction

A finite metric space is a finite collection of points with a real distance defined between each pair. From the perspective of algebraic topology, they have no interest as discrete spaces. Although relaxing metrics to pseudometrics appears to provide finite metric spaces with more interest, pseudometric spaces are homotopically equivalent to the discrete space formed when they are passed through the Kolmogorov quotient. Despite their uninteresting topogical structure, finite metric spaces have applications to computer science. Many physical systems can be modeled with finite points and distances between them, so computer scientists are motivated to embed finite metric spaces into host spaces like $\mathbb{R}^N$ where detailed analysis can be done. Perfect embeddings cannot always be achieved, so the study of the distortion needed for embeddings and when isometric embeddings exist is a rich area.

## 16.6. Finite Metric Spaces

Finite spaces have different metrization and pseudometrization conditions and their metrics can be represented in convenient ways.

### 16.6.1. Pseudometrizing Metrics on Finite Spaces.

**Definition 16.6.1.** A *pseudometric* is a function $d : X \times X \longrightarrow \mathbb{R}$ which satisfies the following properties:

(1) $d(x, x) = 0$ for all $x \in X$
(2) $d(x, y) \geq 0$
(3) $d(x, y) = d(y, x)$ for all $x, y \in X$
(4) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$

This definition is a weakening of the standard metric. Two distinct points may have a distance of zero. Pseudometrics are sometimes referred to as *semimetrics*.

**Definition 16.6.2.** A space $X$ is *pseudometrizable* if there is a pseudometric $d$ on $X$ that induces the topology of $X$.

**Definition 16.6.3.** A space is $R_0$ if each pair of topologically distinct points (i.e. points which do not have the same set of neighborhoods) have some neighborhood not containing the other point.

**Theorem 16.6.4.** *A finite topological space is pseudometrizable iff it is $R_0$.*

PROOF. Given a topological space $X$ and points $x$ and $y$ in $X$, define $x \equiv y$ to mean that $x$ and $y$ are topologically indistinguishable.

Define the *standard discrete pseudometric* to be:

$$d(x,y) = \begin{cases} 0 & \text{if } x \equiv y \\ 1 & \text{if } x \not\equiv y \end{cases}$$

Given $x \not\equiv y$, take neighborhoods $B\left(x, \left(\frac{1}{2}\right)\right)$ and $B\left(y, \left(\frac{1}{2}\right)\right)$ of $x$ and $y$ so that

$$B\left(x, \left(\frac{1}{2}\right)\right) \bigcap B\left(y, \left(\frac{1}{2}\right)\right) = \emptyset$$

This metric induces a topology on $X$ where every topologically distinguishable pair is separated.

If a finite space is $R_0$ with its given topology, then it can be given this topology which separates topologically distinguishable points, satisfying the $R_0$ condition as well as inducing a topology which puts families of points equivalent to the given topology into the same neighborhoods.

Take a space $X$ to be pseudometrizable. Then its metric topology forms open balls around topologically distinguishable points which can be separated.

If no points in the space have distinct neighborhoods (i.e. the pseudometric outputs 0 given any two points), then there are no topologically distinguishable points, so the space is vacuously $R_0$. □

**16.6.2. Representing Metrics on Finite Spaces.** A metric on a finite space can be explicitly defined by $\binom{n}{2}$ non-negative numbers, where each number corresponds to a distance between two points. This property of finite metric spaces allows them to represented in convenient ways, most importantly with matrices and graphs.

16.6.2.1. *Matrix Representation.* Take a finite metric space $(X, d)$ with points $(x_0, x_1, \ldots, x_n)$. Construct an $n \times n$ matrix with entries $(a_{i,j})$ giving the distance between point $i$ and point $j$ in the space. Then the following characteristics can be observed.

(1) $d(x_i, x_j) \geq 0$ for all $0 \leq i, j \leq n$ so the matrix is comprised of nonnegative real numbers.
(2) $d(x_i, x_i) = 0$ for all $0 \leq i \leq n$ so the diagonal of the matrix is 0.
(3) $d(x_i, x_j) = d(x_j, x_i)$ for all $0 \leq i, j \leq n$ so the matrix equals its transpose.

Thus any finite metric space has a real, positive, symmetric matrix containing all the information of its metric.

16.6.2.2. *Graph Representation.* The matrix defined by the finite metric space can be translated to an undirected, no loop, weighted, finite graph. Given a finite metric space $(X, d)$ with points $(x_0, x_1, \ldots, x_n)$, a graph $G$ with $n$ vertices and $\binom{n}{2}$ weighted edges giving the distance between vertices can be constructed to represent it.

The distance function defines a distance between any two points of the space, so each vertex of the graph connects to every other vertex, forming a complete graph. Metrics satisfy the triangle inequality, so all edges may not be necessary if the shortest path metric is used on the graph.

**Definition 16.6.5.** Given a weighted graph $G$, the *shortest path metric* is a metric which defines the distance between two vertices to be the length of the shortest path between them. If the two vertices are not connected, the distance is said to be infinite.

**Theorem 16.6.6.** *A graph $G$ with $n$ vertices and the shortest path metric represents an $n$ point finite metric space $(X, d)$ iff it is undirected, no loop, weighted and connected.*

PROOF. Set each vertex in $G$ to represent a distinct point in the underlying set $X$. The properties of a metric give rise to the conditions necessary for the graph.

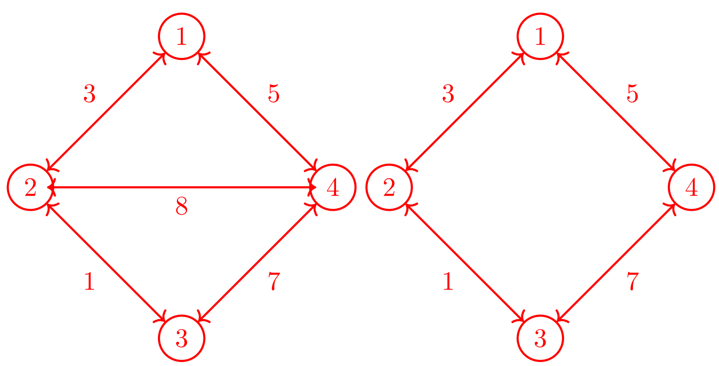
(1) $d(x_i, x_j) = d(x_j, x_i)$ for all $0 \leq i, j \leq n$ ($G$ must be undirected).
(2) $d(x_i, x_i) = 0$ for all $0 \leq i \leq n$ ($G$ must have no loops).
(3) $d(x_i, x_j) \geq 0$ for all $0 \leq i \leq n$ ($G$ must be weighted with nonnegative real values).
(4) $d(x_i, x_j) < \infty$ for all $0 \leq i, j \leq n$ ($G$ must be connected).

The triangle inequality means that the shortest path metric must be used.

Conversely, a graph fulfilling the above properties can be made into a finite metric space if the vertices are made into the underlying set and the shortest path metric is made into the metric on that set. □

**Definition 16.6.7.** It may be possible to obtain a graph with fewer than $\binom{n}{2}$ (i.e. not a complete graph) to represent the finite metric space. When all edges which do not alter the output of the shortest path metric are dropped, the *critical graph* is obtained.

**Example 16.6.8.** Where the triangle inequality is satisfied by an equality an edge can be removed. In this case a critical graph is obtained.




## 16.7. The Problem with Finite Metric Spaces

Finite metric spaces are of no interest to algebraic topologists as they induce the discrete topology on the space. This section illustrates why this is the case and how an indiscrete pseudometric space can be made into a discrete space when it is made $T_0$ through the Kolmogorov Quotient.

**16.7.1. The Discrete Topology.** Recall that the discrete topology is the finest topology possible on a set. Every subset is an open set, and therefore every subset is also a closed set. The fact that finite metric spaces have the discrete topology can be proved directly, or illustrated through Lipschitz equivalence of metrics.

**Theorem 16.7.1.** *Any metric on a finite space induces the discrete topology.*

PROOF. Take a finite metric space $(X, d)$. If every point in the space is open, then all of their possible unions are open, giving the discrete topology.

For any $x \in X$, find $r = \min_{y \in X}(d(x, y))$. This $r$ exists and is nonzero as $X$ is finite and $d(x, y) > 0$ for $x \neq y$. Then the open ball of radius $r$ about $x$ contains only $x$. Thus, the set $\{x\}$ is open. □

**Theorem 16.7.2.** *A finite space is metrizable iff it is discrete.*

PROOF. Given a finite space with the discrete topology, the discrete metric ensures that every point is in a singleton open set (any open ball of radius less than 1) and so the finite space can be metrized.

Conversely, any finite space can be metrized in order to give the discrete topology. In fact, as proved above, the discrete topology is the only possible metric topology given to a finite space. □

**16.7.2. The Kolmogorov Quotient.** Finite pseudometric spaces allow distinct points to have the same open neighborhoods in the induced topology. This seems to give them greater topological interest as they are not necessarily discrete. The Kolmogorov quotient $K(X)$ of a space $X$ identifies points with the same open neighborhoods, and allows for a way to form a $T_0$ space. In this case, the $T_0$ space would be a metric space. This process of converting a pseudometric space into a metric space through a Kolmogorov quotient is called *metric identification*.

16.7.2.1. *Metric Identification.* Suppose $(X, d)$ is a pseudometric space with $x, y \in X$, and let $x \sim y$ if $d(x, y) = 0$. Define $X^* = X/\sim$. If we construct a metric $d^*$ on $X^*$ by setting $d^*([x], [y]) = d(x, y)$, then $(X^*, d^*)$ is a metric space.

**Proposition 16.7.3.** *Metric $d^*([x], [y]) = d(x, y)$ is well-defined.*

PROOF. It is clear that $d^*$ is a metric as it inherits properties from metric $d$. We show that for $x_1, x_2 \in [x]$ and $y \in [y]$, $d^*(x_1, y) = d^*(x_2, y) = d(x, y)$. Take $d^*(x_1, y) = d(x, y)$. By the triangle inequality on $d^*$, $d^*(x_1, x_2) + d^*(x_2, y) \geq d^*(x_1, y)$. Because $x_1 \sim x_2$, $d^*(x_1, x_2) = 0$, so $d^*(x_2, y) = d^*(x_1, y)$. Thus $d^*$ is is independent of choice of representative from the equivalence class, and hence is well-defined. □

**Theorem 16.7.4.** *Metric identification preserves the metric induced topology.*

PROOF. We show the set $A \subset X$ is open iff set $[A]$ (the set of all $[x]$ where $x$ is in $A$) is open in $(X^*, d^*)$.

Take $A \subset (X, d)$, open. Then for all $x \in A$, there is an open ball around $x$ which is contained in $A$. Identify all $x, y$ such that $d(x, y) = 0$. These equivalence classes are made of points distance zero from each other, so the set of open balls $[B(x, \epsilon)]$ for a given $[x]$, all overlap. □

16.7.2.2. *Kolmogorov Quotient of Pseudometric Spaces.*

**Theorem 16.7.5.** *The topology induced by metric identification forms a quotient space that is the Kolmogorov quotient.*

PROOF. Take $(X, d)$ a pseudometric space with metric identified as above. It must be shown that the relation $\sim$ is an equivalence relation and that topology induced by $d^*$ on $X/\sim$ forms $K(X)$.

(1) The relation $\sim$ is an equivalence relation
  (a) Reflexivity: $d(x,x) = 0$ for all $x \in X$, so $x \sim x$.
  (b) Symmetry: $d(x,y) = d(y,x)$ for all $x = y \in X$, so if $d(x,y) = 0$, then $d(y,x) = 0$. Thus, if $x \sim y$, then $y \sim x$.
  (c) By the triangle inequality, $d(x,y) + d(y,z) \geq d(x,z)$ for all $x, y, z \in X$. If $x \sim y$ and $y \sim z$, then $d(x,y) + d(y,z) = 0$, $d(x,z) \geq 0$, and so $d(x,z) = 0$.

(2) For the topology induced by $d^*$ on $X/\sim$ to be $K(X)$, the equivalence classes must be comprised of topologically indistinguishable points. Take $x, y \in X$, with $x$ and $y$ topologically distinguishable. Then there is an open subset $U$ of $X$ where $x \in U$ but $y \notin U$. This means that there an open ball of some radius about $x$ that does not contain $y$, so $d(x,y) > 0$, so $x \not\sim y$.

   Conversely, if $x$ and $y$ are topologically indistinguishable, then there is no open ball containing only one of the points. Then each $B\left(x, \frac{1}{n}\right)$ must contain both $x$ and $y$, so $d(x,y)$ must be zero. This means that the topology induced by $d^*$ on $X/\sim$ is putting only topologically indistinguishable points into equivalence classes. This, taken with Theorem 16.7.5 above, shows that this quotient forms $K(X)$.

□

16.7.2.3. *Homotopy Equivalence of the Kolmogorov Quotient.* Finite pseudometric spaces (in fact all finite spaces) are homotopy equivalent to their Kolmogorov Quotient $K(X)$.

**Theorem 16.7.6.** *Every finite space is homotopically equivalent to a $T_0$ space, $K(X)$.*

**Corollary 16.7.7.** *Any finite pseudometric space $X$ is homotopically equivalent to its Kolmogorov Quotient, $K(X)$, with $K(X)$ being a finite metric space.*

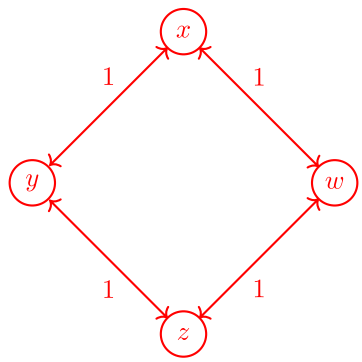
## 16.8. Embedding Finite Metric Spaces

Despite the properties explored above, finite metric spaces are of interest to fields other than algebraic topology. In fields like microbiology, large tables of numbers are generated and need to be analyzed. It can be difficult to work with large tables, meaning that a representation in Euclidean space is desirable. An embedding would offer a way to see the distribution and behavior of the points of the metric space. In addition, a metric space with $n$ points could be described in $2n$ numbers instead of $\binom{n}{2}$ numbers.

The interest in representing combinatorial objects like finite metric spaces in this way comes from a wider interest in the *geometrization of combinatorial objects*, which is a method used to transform large amounts of information into a usable form. Considering the equivalence between linear graphs and finite metric spaces given above, it would seem that all finite metric spaces could be represented in $\mathbb{R}^N$ for some finite $N$. This is not the case.

The distance metric on the weighted graph representing the finite metric space is the shortest path metric. In $\mathbb{R}^N$, the shortest path between two points is a straight line, so if equality holds in the triangle equality, those three points lie on the same line in $\mathbb{R}^N$. This fact will mean that not all finite metric spaces can be embedded without distorting the distances between points. This is illustrated in the following example.

**Example 16.8.1.** Take finite metric space $(X, d)$ with 4 points represented by the weighted graph below with distance given by the shortest path metric.




This is a simple 4 cycle with edges of uniform length. Note that

$$d(x, z) = d(x, y) + d(y, z) = 2 \text{ and } d(x, z) = d(x, w) + d(w, z) = 2$$

This fact will give a contradiction when an embedding is done. Embed this metric space in $\mathbb{R}^N$. There are then two minimal paths between $x$ and $z$ and both obtain equality with the triangle inequality. As explained above, the fact that

$$d(x, z) = d(x, y) + d(y, z) \text{ and } d(x, z) = d(x, w) + d(w, z)$$

implies that points $x, y, z$ are collinear, as are $x, w, z$. Line segments $xyz$ and $xwz$ are the same as they have the same endpoints. Because $y$ and $w$ are both distance 1 away from $x$ on the same line, they are distance zero from each other. This implies that $y = w$, contradicting the fact that $X$ has 4 points.

The graph must be *distorted* to be represented in $\mathbb{R}^N$.

**Definition 16.8.2.** Take metric spaces $(X, d_X)$ and $(Y, d_Y)$ and a function $f : X \longrightarrow Y$. Then the *distortion* of $f$ can be realized by its Lipschitz constants. The *expansion* of $f$ is defined as

$$\|f\|_{Lip} = \sup_{x,y \in X} \frac{d_Y(f(x), f(y))}{d_X(x, y)}$$

The *contraction* of $f$ is given by

$$\|f\|_{Lip}^{-1} = \sup_{x,y \in X} \frac{d_X(x, y)}{d_Y(f(x), f(y))}$$

The *distortion* of $f$ is given by

$$\text{distortion}(f) = \text{contraction}(f) * \text{expansion}(f) = \|f\|_{Lip}^{-1} * \|f\|_{Lip}$$

This is equivalent to finding the closest $a, b \in \mathbb{R}$ such that

$$a \geq \frac{d_Y(f(x), f(y))}{d_X(x, y)} \geq b$$

and defining $\text{distortion}(f) := \frac{a}{b}$.

**Remark 16.8.3.** A mapping $f : X \longrightarrow Y$ is an *isometry* if $\frac{a}{b} = 1$. That is, all distances are preserved up to scaling.

**Definition 16.8.4.** Take metric spaces $(X, d)$ and $(Y, d')$. Then $(X, d)$ is *isometrically embeddable* into $(Y, d')$ if there is a map $f : X \longrightarrow Y$ such that $d(x, y) = d'(f(x), f(y))$ for all $x$ and $y$ in $X$.

As Example 4.1 illustrates, distortion is often necessary for embedding to occur. In that particular case, the distances can be distorted by a factor of $\sqrt{2}$ in order to form the square cycle.

Embedding a metric space in $\mathbb{R}^N$ is a useful case of embedding, but embedding can be described in general settings.

**Definition 16.8.5.** For $0 < p < \infty$, $\ell_p$ *space* is the set of all real sequences $\{x_n\}$ such that $\sum_n |x_n|^p < \infty$.

The norm of this space is given by

$$\|x\|_p = \left(\sum_n |x_n|^p\right)^{\frac{1}{p}}$$

Note that when $p = 2$ this is the Euclidean norm.

**Definition 16.8.6.** A metric space $(X, d)$ is $\ell_p$ *embeddable* if $(X, d)$ is isometrically embeddable into $\ell_p^n$ for some natural number $n$. This number $n$ is the $\ell_p$ dimension of $(X, d)$.

### 16.8.1. Embedding in $\ell_2$.

Embedding in $\ell_2$ attracts special attention. To those looking to analyze large amounts of data, translating data points into a finite metric space and then into a representation can be useful. In $\ell_2$ there are extremely well developed tools in analysis and geometry to aid in the analysis of the data, so obtaining a good representation is important.

For its usefulness, $\ell_2$ is very strict in its behavior, making embeddings difficult. The general theory of Banach spaces gives additional insight into why this is the case and additional motivation to consider $\ell_2$ embeddings.

**Definition 16.8.7.** The *Banach-Mazur distance* is a measure of distance on the set of $n$-dimensional normed spaces. Take two normed spaces $X$ and $Y$ of dimension $n$ and $GL_{X,Y}$, the set of linear isomorphisms from $X$ to $Y$.

The *Banach-Mazur distance* between $X$ and $Y$ is defined to be

$$\delta(X, Y) = \log\left(\inf_{T \in GL_{X,Y}} \text{distortion}(T)\right)$$

This is a metric on the space of $n$-dimensional normed spaces. For many purposes (including ours) the *multiplicative Banach-Mazur distance*

$$d(X, Y) = e^{\delta(X,Y)} = \inf_{T \in GL_n} \text{distortion}(T)$$

will be used. Because $\delta(X, Y)$ is a metric, the multiplicative Banach-Mazur distance obeys the multiplicative triangle inequality, $d(X, Z) \leq d(X, Y)d(Y, Z)$. For convenience, this will be referred to as the *Banach-Mazur distance.*

The Banach-Mazur distance gives a sense of how close two normed spaces are to one another. If the distance is small, then the space needs little distortion for there to be a linear isomorphism between them. The following theorem, Dvoretzky's theorem, is a classical theorem which gives a quantitative sense of how close $\ell_2$ space is to arbitrary normed spaces.

**Theorem 16.8.8.** *(Dvoretzky's Theorem [43]) For every $n \in \mathbb{N}$ and $\epsilon > 0$, every $n$-dimensional normed space contains a subspace $X$ of dimension $m = \Omega(\epsilon^2 log(n))$ such that $d(X, \ell_2) \leq 1 + \epsilon$.*
*$\Omega$ denotes that $m$ is bounded asymptotically by $\epsilon^2 log(n)$ as $n \longrightarrow \infty$.*

16.8.1.1. *Bourgain's Theorem.* [12]. Motivated by this property of $\ell_2$, in 1986, Jean Bourgain developed an algorithm which describes embedding in $\ell_2$.

**Theorem 16.8.9.** *Any metric space $(X, d)$ with $n$ points can be embedded in $\ell_2$ with distortion $\leq O(log\ n)$.*

Proof. Bourgain's proof gives an efficient randomized algorithm for the embedding in $\ell_2$ with distortion $\leq O(\log n)$.
Take a metric space $(X, d)$ with $n$ points.

(1) Take $m$ and $q$ to be integers $m = \lfloor \log 2 \rfloor$ and $q = \lfloor C \log(n) \rfloor$ where $C$ is a constant.
(2) Construct an embedding into $\ell_2^{mq}$ with coordinates $i = 1, \dots, m$ and $j = 1, \dots, q$.
(3) Construct subsets of $X$, $A_{ij}$ by putting each $x \in X$ into $A_{ij}$ with probability $2^{-j}$ .
(4) Now embed with function $f(x)_{ij} = d(x, A_{ij})$.

This is an embedding in $\ell_2^{O(\log)^2 n}$. It has distortion $O(\log n)$. □

16.8.1.2. *Tightness of Bound.* The construction of this algorithm raises the question whether a better embedding can be achieved. A paper by Nathan Linial (2002) shows that this bound is tight. He considers a specific type of graph that has a shortest path metric which is as far from the $\ell_2$ metric as possible in order to guarantee a large distortion, giving a lower bound on distortion of graphs. To state his theorem, some definitions from graph theory are needed.

**Definition 16.8.10.** The *girth* of a graph is the shortest cycle contained in the graph. The girth of an acyclic graph is defined to be infinite.

**Definition 16.8.11.** An *expander graph* is a connected graph in which every "small" subset of vertices has a "large" boundary. That is, the graph cannot be disconnected without removing many edges.

This quality can be quantified in the notion of an $\epsilon$ *edge expander.* A graph with $n$ vertices is an $\epsilon$ edge expander if every set of $K$ vertices with $0 \leq K \leq \frac{n}{2}$ has $\epsilon|K|$ edges connected to $K^c$ (the set of vertices not in $K$).

**Definition 16.8.12.** A *k-regular graph* is a graph where each vertex is of degree $k$.

**Theorem 16.8.13.** *Linial's Lower Bound* [43]
*Take $G$, a $k$-regular graph, with $k \geq 3$, and girth $g$. Then every embedding $f : G \longrightarrow \ell_2$ has distortion $\Omega(\sqrt{g})$.*

Proof. Sketch. This proof uses a random walk on the graph. Knowing the girth of the graph and that all vertices are connected to $k$ other vertices, it can be proven that the walk moves away from where it started at constant speed at a time bounded asymptotically by $g$. The geometry of Euclidean space means that this class of random walks is at time $T$ expected to be $O(\sqrt{T})$ from its origin. This difference must be accounted for by a distortion in the metric if it is to be embedded in $\ell_2$. Comparing the two walks on the graph at time $O(g)$ gives a distortion of $\Omega(\sqrt{g})$. □

The triangle inequality is satisfied by equality many times, necessitating significant distortion.

16.8.1.3. *Isometric Embedding in $\ell_2$.* I. J. Schoenberg's 1937 paper [61] outlines the necessary and sufficient conditions for an isometric embedding in $\ell_2$. In particular, he addresses separable pseudometric spaces and characterizes embeddable metrics in terms of positive definite functions.

**Definition 16.8.14.** A real function $f = f(x_1, x_2, \ldots, x_n)$ is a *positive definite function* if it is defined for all real values, and if for any real numbers $x_1, x_2, \ldots, x_n$ the $n \times n$ matrix $A$ where $A = (a_{i,j})$ and $a_{i,j} = f(x_i - x_j)$ is a positive, semi-definite matrix (that is, $x^t Ax \geq 0$ for all real numbers $x$).
A similar notion of positive definite functions can be defined for real-valued functions which take as input distances on a pseudometric space $(X, d)$.
A real function $g(t)$ is positive definite if $g$ is continuous, even, defined on the range of distances in the pseudometric space, and satisfies the inequality

$$\sum_{i,j=1}^{n} g(d(x_i, x_j)) \geq 0$$

Examples of positive definite functions in $\ell_2$ are $f(t) = e^{-t^2}$, and more generally, $f(t) = e^{-\lambda t^2}$ for all $\lambda \in \mathbb{R}$.

**Theorem 16.8.15.** *Schoenberg's Embedding*
*A separable pseudometric space $(X, d)$ is isometrically embeddable in $\ell_2$ if and only if the functions $e^{-\lambda t^2}$ are positive definite in $(X, d)$.*

Proof. Sketch. The idea of this proof is to note that $e^{-\lambda t^2}$ for $(\lambda \in \mathbb{R})$ is a family of positive definite functions in $\ell_2$. It is only necessary to consider $\lambda > 0$ as $\lambda = 0$ is an accumulation point of this family and the cases where $\lambda < 0$ follow by symmetry. The proof uses ideas from analysis about positive definite functions to show that if the given characteristics of positive definite functions are preserved on embedding into $\ell_2$, then all distances must have been preserved and if the given family of functions are positive definite in the metric space, then the metric of the space will allow isometric embedding into $\ell_2$. □

**16.8.2. Embedding in $\ell_1$.** Following the formula given for $\ell_p$ space $\ell_1$ is the set of all real sequences $\{x_n\}$ such that $\sum_n |x_n| < \infty$. The distance metric on $\ell_1$ is defined to be $d_{\ell_1}(x, y) := \sum_n |x_n - y_n| < \infty$.
To consider isometric embedding in $\ell_1$, the *cut semimetric* will be used.

**Definition 16.8.16.** The *cut semimetric* is a pseudometric $d$ on a set $X$. Given partitions $A$ and $B$ of $X$, define $d(x, y) = \begin{cases} 0 & \text{if } x, y \in A \text{ or } x, y \in B \\ 1 & \text{otherwise} \end{cases}$.

Every cut semimetric is clearly isometrically embeddable in $\ell_1$.

The set of all linear combinations of semimetrics on a set forms a special class of metrics on that set. These are exactly the $\ell_1$ metrics on the set (that is, the metrics which can be isometrically embedded in $\ell_1$) [**21**].

### 16.8.3. Embedding in $\ell_\infty$.

**Definition 16.8.17.** $\ell_\infty$ space is defined to be the set of all real bounded sequences. It takes on the norm $\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$.

**Theorem 16.8.18.** [**27**, Ch 8.1.3] *Every finite metric space $(X, d)$ with $n$ points can be embedded in $\ell_\infty^n$.*

Proof. Take a finite metric space $(X, d)$ with $X = \{x_1, x_2, \ldots, x_n\}$ and define an embedding function $f : X \longrightarrow \ell_\infty^n$ by $f(x_i)_j = d(x_i, x_j)$ for all $1 \leq i$ and $j \leq n$. □

Embeddings into lower dimensional $\ell_\infty^k$ spaces exist.

**Definition 16.8.19.** Take a metric space $(X, d)$ and every subset $S \subset X$. Then define a mapping $f_S : X \longrightarrow \mathbb{R}$ for each $S$ by

$$f_S(x) = d(x, S) = \min_{s\in S}(d(x, s))$$

A *Frechet Embedding* is a map $f : X \longrightarrow \mathbb{R}^k$ where each coordinate in $\mathbb{R}^k$ is a scaled $f_S$ mapping. Then $f$ is a Frechet Embedding if, for some $\beta_S \in \mathbb{R}$

$$f(x) = \oplus_{S\subset X}\beta_S f_S(x)$$

**Proposition 16.8.20.** [**58**] *When $\beta_S = 1$ for all $S \subset V$, $\|f(x) - f(y)\|_\infty \leq d(x, y)$. That is, Frechet embeddings are contraction mappings in the $\ell_\infty$ metric.*

Proof. Let $S_x$ denote the point in $S \subset X$ closest to some point $x \in X$. Then both

$$d(x, S) - d(y, S) \leq d(x, S_y) - d(y, S_x) \leq d(x, y), \text{ and}$$

$$d(y, S) - d(x, S) \leq d(y, S_x) - d(x, S_y) \leq d(x, y)$$

This implies that $\|f(x) - f(y)\|_\infty = |d(x, S) - d(y, S)| \leq d(x, y)$. □

A 1996 paper by Jiri Matousek uses these mappings to do distorted mappings into lower dimension $\ell_\infty^k$ space.

**Theorem 16.8.21.** [**44**] *Take an $n$-point metric space $(X, d)$ and integer $D$. Then $(X, d)$ can be embedded into $\ell_\infty^{O(Dn^{2/D} log(n)}$.*

Proof. The idea of this proof is to divide X into $O(Dn^{2/D}\log(n))$ subsets, each of which will correspond to a dimension in the range $\ell_\infty$ space.

Construct the embedding function $\psi : (X, d) \longrightarrow \ell_\infty^{O(Dn^{2/D}\log(n))}$ to be a Frechet embedding with $j$th coordinate of $\psi(x)$ to be $d(x, S)$. Noting the proposition above, function $\psi$ must be a contraction mapping. The rest of the proof uses an algorithm and probability to show that its contraction is limited. □

**16.8.4. Embedding in $\mathbb{R}^N$ [53].** A paper by C.L. Morgan published in 1974 proved necessary and sufficient conditions for embedding a metric space in $\mathbb{R}^N$. His theorem applies to arbitrary metric spaces, not only finite ones. It holds special interest for embedding finite metric spaces. His theorem makes the computation necessary to determine whether embeddability is feasible. His proof also shows that for any metric space, embedding into $\mathbb{R}^N$ is a very strong condition, but it is one that is determined by a finite number of points in the metric space.

In order to state and prove the embedding theorem, some special definitions will be needed, as well as some general results about inner products, metrics, and linear algebra.

**Definition 16.8.22.** An *inner product* on a vector space $V$ over a field $\mathbb{F}$ with characteristic 0 is a bilinear map $\langle\ ,\ \rangle : V \times V \longrightarrow \mathbb{F}$. This function satisfies conjugate symmetry and positive definiteness.

For a vector space $V$ with element $x \in V$, define a norm $\|x\| = \sqrt{\langle x, x\rangle}$.

**Theorem 16.8.23.** *For a vector space $V$ over characteristic 0 field $F$ with inner product $\langle\ ,\ \rangle$, and norm $\|x\| = \sqrt{\langle x, x\rangle}$, a metric $d(x, y) = \|x - y\|$ is induced by the norm.*

**Definition 16.8.24.** Let $(X, d)$ be a metric space and for points $x, y, z \in X$ define a function from $X \times X \times X \longrightarrow \mathbb{R}$ by:

$$\langle x, y, z\rangle = \frac{1}{2}\left(d(x,z)^2 + d(y,z)^2 - d(x,y)^2\right)$$

If we define $X$ to be a subset of some vector space $V$ such that metric $d$ is *induced* by an inner product on $V$, then $\langle x, y, z\rangle$ is the inner product of $x - z$ and $y - z$.

**Definition 16.8.25.** Take metric space $(X, d)$. Then define $Y$ to be a *metric subspace* of $X$ if $Y \subset X$ and $Y$ has the distance function $d|_{Y\times Y}$ .

Finite metric subspaces of $X$ are *$n$-simplices* in $X$. In particular, a metric subspace of $n+1$ elements is an *$n$-simplex* in $X$.

If $(X, d)$ is a subspace of Euclidean space, then these simplices have a clear notion of volume. The following function will begin to generalize this idea to arbitrary metric spaces.

**Definition 16.8.26.** Define a function $D : X^{n+1} \longrightarrow \mathbb{R}$ as follows:
Construct an $n\times n$ matrix $A$ from $(x_0, x_1, \ldots, x_n)$ with real entries $(a_{i,j}) = \langle x_i, x_j, x_0\rangle$ and let $D(x_0, x_1, \ldots, x_n) = \det(A)$. This function $D$ is a real valued function on the $n$-simplices of $X$.

**Proposition 16.8.27.** *The function $D$ is symmetric.*

PROOF. In Euclidean space, the entry $(a_{i,j})$ in the above matrix is

$$\begin{aligned}
\langle x_i, x_j, x_0\rangle &= \frac{1}{2}\left(\left(\sqrt{(x_i - x_0)^2}\right)^2 + (\left(\sqrt{(x_j - x_0)^2}\right)^2 - (\left(\sqrt{(x_i - x_j)^2}\right)^2\right)\\
&= \frac{1}{2}\left((x_i - x_0)^2 + (x_j - x_0)^2 - (x_i - x_j)^2\right)\\
&= \frac{1}{2}\left(-2x_jx_0 - (-2x_0x_i) + 2x_ix_j + 2x_0^2\right)\\
&= -x_jx_0 - x_0x_i + x_jx_i + x_0^2\\
&= (x_i - x_0) * (x_j - x_0)
\end{aligned}$$

The determinant of a matrix with these entries is the square of the volume of a parallelpiped spanned by the set of $n$ vectors $(x_1, \ldots, x_n)$ based at $x_0$. $\square$

With this machinery, it is possible to find the volume of the simplex $(x_1, \ldots, x_n)$.

**Proposition 16.8.28.** *The volume of the $n-$simplex $Y = (x_1, \ldots, x_n)$ in Euclidean space is*

$$Vol_n(Y) = \frac{1}{n!}\sqrt{D(x_0, x_1, \ldots, x_n)}$$

Having computed this volume in Euclidean space, define the volume of an $n$-simplex $Y$ in any metric space to be the formula given by $Vol_n(Y)$. We can now provide two definitions which will describe which metric spaces can be embedded in $\mathbb{R}^N$.

**Definition 16.8.29.** A metric space $(X, d)$ is *flat* if for each $n$-simplex $Y$ in $X$, $Vol_n(Y)$ is real.

**Definition 16.8.30.** If $(X, d)$ is a flat metric space, the dimension of $(X, d)$ is the largest $n \in \mathbb{N}$ where there exists an $n$-simplex of $X$ with positive volume.

These characteristic of metric spaces will determine which can be isometrically embedded in $\mathbb{R}^N$. To prove Morgan's main theorem, some results from linear algebra are quickly cited.

**Lemma 16.8.31.** *Any real $n$-dimensional inner product space is linearly isometric to Euclidean $n$-space.*

**Lemma 16.8.32.** *Let $M$ be an $m \times m$ real symmetric matrix with all non-negative eigenvalues. If $D[i, j]$ is the determinant of the $m - 1 \times m - 1$ minor of $M$ obtained by deleting its $i$th row and $j$th column, then $D[i, j]^2 \leq D[i, i]D[j, j]$.*

**Theorem 16.8.33.** *Morgan's Embedding in $\mathbb{R}^N$. A metric space can be isometrically embedded in Euclidean $n$-space iff the metric space is flat and has dimension less than or equal to $n$.*

Proof. Take a metric space $(X, d)$ which can be isometrically embedded in Euclidean $n$-space. Isometries preserve volume, so the simplices must have real volume in $(X, d)$ (as they have real volume in $\mathbb{R}^N$), so $(X, d)$ is flat. Because volume is preserved, the simplices of positive volume in $(X, d)$ have positive volume in $\mathbb{R}^N$. Since there cannot be any simplices of positive volume in $\mathbb{R}^N$ with greater than $n + 1$ points, $(X, d)$ must have dimension less than or equal to $n$.

Suppose $(X, d)$ is flat and of dimension $n$ with $n$-simplex $Y = (x_0, x_1, \ldots, x_n)$ such that $Y$ has positive volume.

If a map $f : X \longrightarrow \mathbb{R}^N$ can be constructed such that $f$ embeds $X$ isometrically in $\mathbb{R}^N$ with some inner product, then $(X, d)$ can be embedded in Euclidean $n$-space because any real $n$-dimensional inner product space is linearly isometric to Euclidean $n$-space.

Let $f : X \longrightarrow \mathbb{R}^N$ be the map defined by $f(x) := (\langle x, x_1, x_0\rangle, \ldots, \langle x, x_n, x_0\rangle)$, and construct bilinear form on $\mathbb{R}^N$ as follows: Let $L$ be an $n \times n$ matrix with entries $(a_{i,j}) = \langle x_i, x_j, x_0\rangle$, and let

$$\langle u, v\rangle = u^t L^{-1} v \text{ for all } u, v \in \mathbb{R}^N$$

If the eigenvalues of matrix $L$ are positive, this bilinear form is an inner product on $\mathbb{R}^N$ and $f$ embeds $(X, d)$ isometrically into this inner-product space.

The roots of the polynomial $\det(xI+L)$ are the negatives of the eigenvalues of $L$. Thus, we can look at the coefficient of the term of degree $n-k$ in this polynomial, which is the sum of the $k*n$ minors that lie along the main diagonal. These minors are all non-negative because they are volumes of $k$-simplicial complexes (these volumes are all real, nonnegative as $(X,d)$ is flat and dimension $n$). These make the polynomial positive, so it must have no positive roots, so there cannot be negative eigenvalues of $L$. $L$ being symmetric and non-singular (as $(X,d)$ has non-zero dimension) ensures that its eigenvalues are positive.

We show the inner product given on $\mathbb{R}^N$ preserves the structure of all of the $n$-simplexes of $(X,d)$, and that therefore $f$ is an isometry, by showing that

$$\langle f(x), f(y)\rangle = \langle x, y, x_0\rangle \text{ for all } x, y \in X$$

Construct a $(n+2)\times(n+2)$ matrix $M$ with entries $\langle x_j, x_i, x_0\rangle$. By the same reasoning used on the similarly constructed matrix $L$, $M$ has all non-negative eigenvalues.

Set $D[i,j]$ to be the determinant of the $(n+1)\times(n+1)$ of the matrix obtained by deleting the $i$th row and $j$th column of $M$.

Recall the lemma stating that $D[i,j]^2 \le D[i,i]D[j,j]$. $D[i,i]$ is the determinant corresponding to the volume of a $(n+1)$-simplex squared and scaled by a factor of $(n+1)!$ and since $(X,d)$ is $n$-dimensional, the volume of any $(n+1)$-simplex must be zero, and therefore $D[i,i]=0$. By the lemma, this also means that $D[i,j]=0$.

Setting $i=n$ and $j=n+1$ shows that, in particular, $D[n,n+1]=0$. Consider the minor of $M$ with the $n$th row and $(n+1)$st columns deleted.

$$\begin{pmatrix} \langle x_1,x_1,x_0\rangle & \dots & \dots & \langle x_n,x_1,x_0\rangle & \langle x_{n+2},x_1,x_0\rangle \\ \vdots & \ddots & \ddots & \dots & \vdots \\ \langle x_1,x_{n-1},x_0\rangle & \dots & \dots & \langle x_n,x_{n-1},x_0\rangle & \langle x_{n+2},x_{n-1},x_0\rangle \\ \langle x_1,x_{n+1},x_0\rangle & \dots & \dots & \langle x_n,x_{n+1},x_0\rangle & \langle x_{n+2},x_{n+1},x_0\rangle \\ \langle x_1,x_{n+2},x_0\rangle & \dots & \dots & \langle x_n,x_{n+2},x_0\rangle & \langle x_{n+2},x_{n+2},x_0\rangle \end{pmatrix}$$

Since in $\langle f(x), f(y)\rangle = f(x)^t L^{-1} f(y)$ in general, the condition for isometry is

$$\langle f(x), f(y)\rangle = \langle x, y, x_0\rangle$$

Set $x := x_{n+1}$ and $y := x_{n+2}$ so that

$$f(x) = (\langle x_{n+1},x_1,x_0\rangle,\dots,\langle x_{n+1},x_n,x_0\rangle)\,,f(y) = (\langle x_{n+2},x_1,x_0\rangle,\dots,\langle x_{n+2},x_n,x_0\rangle)$$

Note that by deleting one row and one column from the matrix above, and dividing by the determinant of $L$, the matrix becomes the $L^{-1}$ (when assigning the correct cofactor signs).

Expand the above matrix by the last row to calculate the determinant, using the minors

$$\begin{pmatrix} \langle x_1,x_1,x_0\rangle & \dots & \dots & \langle x_n,x_1,x_0\rangle \\ \vdots & \ddots & \ddots & \dots \\ \langle x_1,x_{n-1},x_0\rangle & \dots & \dots & \langle x_n,x_{n-1},x_0\rangle \\ \langle x_1,x_{n+1},x_0\rangle & \dots & \dots & \langle x_n,x_{n+1},x_0\rangle \end{pmatrix} = \begin{pmatrix} \langle x_2,x_1,x_0\rangle & \dots & \dots & \langle x_{n+2},x_1,x_0\rangle \\ \vdots & \ddots & \ddots & \dots \\ \langle x_2,x_{n-1},x_0\rangle & \dots & \dots & \langle x_{n+2},x_{n-1},x_0\rangle \\ \langle x_2,x_{n+1},x_0\rangle & \dots & \dots & \langle x_{n+2},x_{n+1},x_0\rangle \end{pmatrix}$$

Taking the appropriate sign changes and summing their determinants gives zero (as $D[n,n+1]=0$). So dividing by $\det(L)$ still yields zero.

Continue the calculation to get that

$$\langle x_{n+1}, x_{n+2}, x_0\rangle = f(x_{n+1})^t L^{-1} f(x_{n+2})$$

This means that $\langle f(x), f(y)\rangle = \langle x, y, x_0\rangle$ for all $x, y \in X$ and thus, $f$ is an isometry. □

These characterizations of metric spaces provides a useful way to analyze examples of metric spaces.

**Theorem 16.8.34.** [53] *For $n \geq 2$, $\mathbb{R}^N$ with the $\ell^p$ metric is flat iff $p = 2$.*

Proof. Morgan gives the two examples used below for his proof of this theorem without additional argument. However, working through the process to show why these examples work illustrates why the case when $p = 2$ is special.

Given $\mathbb{R}^N$ with the $\ell^2$ metric, the previous theorem proves that it is flat (i.e. $(\mathbb{R}^N, \ell^2)$ can embed in itself). The example given in 16.8.1 of a non-embeddable metric space suggests how to construct simplices of imaginary volume in $(\mathbb{R}^N, \ell^p)$ when $p \neq 2$. It is only necessary to find examples in $\mathbb{R}^2$ as $\mathbb{R}^2 \subset \mathbb{R}^N$ for $n \geq 2$.

Consider $(\mathbb{R}^N, \ell^p)$ for $p < 2$.

If $1 \leq p$, the $\ell^p$ is induced by the norm

$$\|x\|_p = \left(\sum_n |x_n|^p\right)^{\frac{1}{p}}$$

Take the example of the 3-simplex $Y$ in $(\mathbb{R}^N, \ell_2)$ with $Y = \{(0,0), (1,0), (1,1), (0,1)\}$. Observe that for any value of $p \geq 1$, the horizontal and vertical distances on this simplex are the same.

If $p \geq 1$,

$$d((a,b),(a,c)) = \|(a,b) - (a,c)\|_p = (|(a-a)|^p + |(b-c)|^p)^{\frac{1}{p}} = |b-c|$$

The same argument applies, by symmetry, when the second coordinates are equal. This means that distortion would occur in the distance between two non-adjacent points in this simplex. By the triangle inequality, for any $p \geq 1$,

$$d((0,0),(1,1)) \leq d((0,0),(0,1)) + d((0,1),(1,1)) = 1 + 1 = 2$$
$$d((0,0),(1,1)) \leq d((0,0),(1,0)) + d((1,0),(1,1)) = 1 + 1 = 2$$
$$d((0,0),(1,1)) = \|(0,0) - (1,1)\|_p = (|(0-1|^p + |(0-1)|^p)^{\frac{1}{p}} = 2^{\frac{1}{p}}$$

As $p \longrightarrow \infty$, the quantity $d((0,0),(1,1)) \longrightarrow 1$, so this square in $(\mathbb{R}^N, \ell_2)$ collapses to a line as $p$ increases.

Now consider the matrix constructed to compute function $D(Y)$:

$$A = \begin{pmatrix} \langle(0,0),(1,0),(1,0)\rangle & \langle(0,0),(1,0),(1,1)\rangle & \langle(0,0),(1,0),(0,1)\rangle \\ \langle(0,0),(1,1),(1,0)\rangle & \langle(0,0),(1,1),(1,1)\rangle & \langle(0,0),(1,1),(0,1)\rangle \\ \langle(0,0),(0,1),(1,0)\rangle & \langle(0,0),(0,1),(1,1)\rangle & \langle(0,0),(0,1),(0,1)\rangle \end{pmatrix}$$

Notice an entry on the diagonal takes the form

$$\langle x, y, y\rangle = \frac{1}{2}\left(d(x,y)^2 + d(y,y)^2 - d(x,y)^2\right) = 0,$$

and therefore $A$ has a zero diagonal. Then since $d((0,0),(0,1)) = d((0,0),(1,0)) = d((1,0),(1,1)) = d((0,1),(1,1)) = 1$ for any $p$, matrix $A$ can be simplified to

$$A = \begin{pmatrix} 0 & \frac{1}{2}[d((0,0),(1,1))^2] & \frac{1}{2}[d((1,0),(0,1))^2] \\ 1 - \frac{1}{2}[d((0,0),(1,1))^2] & 0 & 1 - \frac{1}{2}[d((0,0),(1,1))^2] \\ \frac{1}{2}[d((0,1),(1,0))^2] & \frac{1}{2}[d((0,0),(1,1))^2] & 0 \end{pmatrix}$$

We find $D(Y) := D((0,0),(1,0),(1,1),(0,1))$ can be calculated:

$$\begin{aligned} D(Y) &= \left[\frac{1}{2}d((0,0),(1,1))^2\right]\left[\frac{1}{2}d((0,1),(1,0))^2\right]\left[1-\frac{1}{2}d((0,0),(1,1))^2\right] \\ &+ (\left[\frac{1}{2}d((1,0),(0,1))^2\right]\left[1-\frac{1}{2}d((0,0),(1,1))^2\right]\left[\frac{1}{2}d((0,0),(1,1))^2\right] \\ &= d\big((1,0),(0,1)\big)^2 d\big((0,0),(1,1)\big)^2 (\frac{1}{2}-\frac{1}{4}d((0,0),(1,1))^2) \end{aligned}$$

The term $d\big((1,0),(0,1)\big)^2 d\big((0,0),(1,1)\big)^2$ is always positive. Then this value of $D(Y)$ is negative (and so the volume of $Y$ imaginary) only when

$$\frac{1}{2} < \frac{1}{4}d\big((0,0),(1,1)\big)^2$$

Solving this inequality gives that the volume is imaginary when $\sqrt{2} < d\big((0,0),(1,1)\big)$ If $0 < p < 1$ then $\ell^p$ has the metric $d_p(x,y) = \sum_{i=1}^n |x_i - y_i|^p$ so

$$d\big((0,0),(1,1)\big) = \sum_{i=1}^{2} |0-1|^p = 1^p + 1^p = 2$$

Then since $D(Y)$ is negative for $0 < p < 1$, $Vol(Y)$ is imaginary, and therefore $(\mathbb{R}^N, \ell^p)$ is not flat for $0 < p < 1$. If $1 \le p < 2$, then this distance takes the form

$$d\big((0,0),(1,1)\big) = \|(0,0)-(1,1)\|_p = (1^p + 1^p)^{\frac{1}{p}} = 2^{\frac{1}{p}}$$

If $p < 2$, then the inequality is satisfied, meaning that $(\mathbb{R}^N, \ell^p)$ is not flat for $1 \le p < 2$.

Consider $(\mathbb{R}^N, \ell^p)$ for $p > 2$. Take example of the 3-simplex $Y$ in $(\mathbb{R}^N, \ell^p)$ with $Y = \{(0,1),(1,0),(-1,0),(0,-1)\}$. This simplex has vertical and horizontal distances of 2 which are preserved in all $(\mathbb{R}^N, \ell^p)$ for all $p$. It is the distances which are not preserved which will cause this simplex to have imaginary volume for $p > 2$. This example's invariant distances are larger than the changing distances, so by repeating the same computation as above, the inequality is reversed, giving that the volume of $Y$ is imaginary when

$$\sqrt{2} > d\big((-1,0),(0,1)\big)$$

This is an equality when $p = 2$. By the same analysis as above, as $p$ becomes greater than 2, this inequality is satisfied, showing that $Y$ has an imaginary volume when $p > 2$, and therefore $(\mathbb{R}^N, \ell^p)$ is not flat for $p > 2$. □

**16.8.5. Embeddings of the $\ell_2$ Metric.** In section 16.8.2 it was shown that $\ell_2$ is close to other normed spaces. That is, there is a linear isomorphism between them which requires little distortion of the spaces. It is then natural to ask when there is an isometric embedding from $\ell_2$ to other spaces.

16.8.5.1. *Dimension reduction in $\ell_2$.* Given a metric space $(X, \ell_2)$ in $\mathbb{R}^N$, it is useful to ask whether the dimension of the host space, $\ell_2$, can be reduced in exchange for distortion. A paper by William Johnson and Joram Lindenstrauss quantified the possible dimension reduction.

**Theorem 16.8.35.** *(Johnson and Lindenstrauss Dimension Reduction [36]) Given any n-point metric space $(X, \ell_2) \subset \mathbb{R}^N$ and $\epsilon > 0$, there is an embedding of distortion of at most $1+\epsilon$ such that*

$$(X, \ell_2) \longrightarrow \ell_2^{O(\frac{\log n}{\epsilon^2})}$$

The proof of this dimension reduction theorem and other proofs of isometric embedding from $\ell_2$ to $\ell_p$ uses a technique in theoretical computer science, *random projection.*

**Definition 16.8.36.** Take vectors $r_1 \ldots, r_k \subset \mathbb{R}^N$ which have been obtained by some random process. Then define map $\psi : \mathbb{R}^N \longrightarrow \mathbb{R}^k$ as follows:

$$\psi : v \longrightarrow (\langle v, r_1 \rangle, \ldots, \langle v, r_k \rangle)$$

The map $\psi$ is a *random projection* from $\mathbb{R}^N \longrightarrow \mathbb{R}^k$.

Random projection $\psi$ can be conveniently expressed as a $k \times n$ matrix $A$ whose rows are $r_1, \ldots, r_k$ so that $\psi(v) = Av$. This means that random projections are linear.
There are three notable examples of random process used to generate the $r_1, \ldots, r_k$. All three have been used to prove the Johnson-Lindenstrauss Theorem.

**Example 16.8.37.**
(1) Set each $r_i = (r_i^1, \ldots, r_i^n)$ and obtain values for each $r_i^j$ from a normal probability distribution centered at 0 with variance 1. This is labeled $\psi_N$ and was used to prove Johnson-Lindenstrauss [35].
(2) Set each $r_i = (r_i^1, \ldots, r_i^n)$ and obtain values for each $r_i^j$ by choosing either $+1$ or $-1$, each with probability $\frac{1}{2}$. This method is called binary coins and is labeled $\psi_B$. This is the simplest method used to prove Johnson-Lindenstrauss [1].
(3) Take $r_1, \ldots, r_k$ to be a set of $k$ orthogonal vectors from $S^{n-1}$. This is labeled $\psi_S$ and was originally used by Johnson and Lindenstrauss [36].

16.8.5.2. *Isometric Embedding from $\ell_2$ to $\ell_1$.* Two interesting cases of $\ell_p$ spaces are $\ell_2$ and $\ell_1$, so the existence of an isometric embedding of a $n$-point metric space in $\ell_2^n$ to some finite dimensional $\ell_1^k$ is an important one. In order to prove that there does exist such an embedding, the space $\ell_1^{S^{n-1}}$ will be explored. The definition of this space and the proof of an embedding theorem is given in lecture 12 of the series on finite metric spaces given at TTIC [58].

**Definition 16.8.38.** Space $\ell_1^{S^{n-1}}$ is a $\ell_1$ metric space with a coordinate for each vector in $S^{n-1}$. Each point in $\ell_1^{S^{n-1}}$ is given by a function $f : S^{n-1} \longrightarrow \mathbb{R}$. The $\ell_1$ norm is given by

$$\|f\|_1 = \int_{r \in S^{n-1}} |f(r)| dr$$

**Lemma 16.8.39.** *There exists an isometric embedding of every $n-$point metric space in $\ell_2^n$ to $\ell_1^{S^{n-1}}$.*

With this embedding lemma, it only need be shown that there is an isometric embedding from to isometric embeddings from $\ell_1^{S^{n-1}}$ into a finite dimensional $\ell_1$. This result can also be generalized to isometric embeddings from $\ell_p^{S^{n-1}}$ to finite dimensional $\ell_p$.

**Theorem 16.8.40.** [58] *Every n-point metric space in $\ell_2^n$ can be isometrically embedded in $\ell_1^{n!}$.*

Proof. Sketch. Isometrically embed space metric space $X = \{x_1, \ldots, x_n\}$ in $\ell_2^n$ by the above lemma. $S^{n-1}$ is partitioned into $n!$ regions and each region is assigned an $x_i$ and $x_j$. Each region is defined in such a way that the sign of $\langle x_i, r\rangle - \langle x_j, r\rangle$ is constant within it. It can then be shown that this produces an isometric embedding from $\ell_2^n$ to $\ell_1^{S^{n-1}}$ and into $\ell_1^{n!}$. □

CHAPTER 17

# Morse Theory

CHAPTER 18

# The Euler Characteristic and Möbius Functions of Simplicial Complexes and Finite Spaces

## 18.1. The Euler Characteristic

The notion of the Euler characteristic is one that exists for any arbitrary topological space, not necessarily finite. In particular, we have the following definition.

**Definition 18.1.1.** The *Euler characteristic of a simplicial complex* $K$ is given by

$$\chi(K) = V - E + F,$$

where $V$ is the number of vertices, $E$ is the number of edges and $F$ is the number of faces.

In fact, these notions coincide if $K$ is the CW-decomposition of the space $X$. To see this, recall the purely algebraic fact that for a short exact sequence of finitely generated abelian groups $0 \to A \to B \to C \to 0$, $rank(B) = rank(A) + rank(C)$.

**Theorem 18.1.2.** *Let $X$ be a compact CW-complex. Then $\chi(X) = \sum_{n \geq 0} (-1)^n c_n$ where $c_n$ is the number of $n$-cells contained in the complex.*

Proof. Let

$$0 \to C_k \xrightarrow{d_k} C_{k-1} \to \ldots \to C_1 \xrightarrow{d_1} 0$$

be the chain complex of chain groups of the CW-complex and the $d_i$ are the boundary maps. Letting $B_i = im(d_{i+1})$ and $Z_i = ker(d_i)$ we have following short exact sequences:

$$0 \to Z_i \overset{i}{\hookrightarrow} C_i \xrightarrow{d_i} B_{i-1} \to 0$$

$$0 \to B_i \xrightarrow{d_{i+1}} Z_i \xrightarrow{q} H_i \to 0$$

We can derive that

$$rank(C_i) = rank(Z_i) + rank(B_{i-1})$$
$$rank(Z_i) = rank(B_i) + rank(H_i)$$

By substituting the second equation into the first, multiplying the resulting equality by $(-1)^i$ and then summing over $i$, the $B_i$ terms will cancel, giving $\sum_{n \geq 0} (-1)^n c_n = \sum_{n \geq 0} (-1)^n \cdot b_i(X)$ as desired. □

By regarding simplicial complexes as special cases of CW-complexes, we may use this result to derive the Euler characteristic of a finite $T_0$ space.

## 18.2. The Euler characterisic of a finite space

**Definition 18.2.1.** The *Euler characteristic of a finite $T_0$-space* is given by

$$\chi(X) = \sum_{C \in \mathcal{C}(X)} (-1)^{\sharp C + 1}$$

where $\mathcal{C}(X)$ is the set of non-empty chains of $X$ and $\sharp C$ is the cardinality of some element of that set.

For ordinary topological spaces, the Euler characteristic is a homotopy invariant. Using this last definition we can prove for finite spaces that the Euler characteristic is also homotopy invariant.

**Theorem 18.2.2.** *Let $X$ and $Y$ be finite $T_0$-spaces that are homotopy equivalent. Then $\chi(X) = \chi(Y)$.*

Proof. Let $X_c$ and $Y_c$ be the cores of $X$ and $Y$ respectively, which must exist by 1.9. 1.10 implies that $X_c$ and $Y_c$ are homeomorphic and thus $\chi(X_c) = \chi(Y_c)$. As per 1.9, we may think of $X_c$ as part of a sequence of subspaces of $X$, where each successive element in the sequence is generated by removing a beat point. Thus, it suffices to show that removing a beat point does not affect the Euler characteristic. Let $P$ be a finite poset with beat point $p$, where there must exist some $q$ such that if $r$ is comparable with $p$ then $r$ is also comparable with $q$. We can then construct a bijection

$$\varphi : \{C \in \mathcal{C}(P) \mid p \in C, q \notin C\} \to \{C \in \mathcal{C}(P) \mid p \in C, q \notin C\}$$
$$C \mapsto C \cup \{q\}$$

We may thus write:

$$\chi(P) - \chi(P - \{p\}) = \sum_{p \in C \in \mathcal{C}(P)} (-1)^{\sharp C+1} = \sum_{q \notin C \ni p} (-1)^{\sharp C+1} + \sum_{q \in C \ni p} (-1)^{\sharp C+1} =$$
$$\sum_{q \notin C \ni p} (-1)^{\sharp C+1} + \sum_{q \notin C \ni p} (-1)^{\sharp \varphi(C)+1} = \sum_{q \notin C \ni p} (-1)^{\sharp C+1} + \sum_{q \notin C \ni p} (-1)^{\sharp C} = 0$$

□

## 18.3. The Möbius Function

The Euler characteristic of finite $T_0$-spaces is particularly interesting because of its relationship to the Möbius function of posets, a combinatorial object. To define the Möbius function we first define an *incidence algebra* $\mathfrak{A}$ on $P$. $\mathfrak{A}(P)$ is the set of functions $P \times P \to \mathbb{R}$ such that for $f \in \mathfrak{A}(P)$, $f(x,y) = 0$ if $x \not\leq y$. This forms a vector space over $\mathbb{R}$ where we have a product defined as

$$fg(x,y) = \sum_{z \in P} f(x,z)g(z,y)$$

We let $\xi_p \in \mathfrak{A}$ be the function such that $\xi_p(x,y) = 1$ whenever $x \leq y$. This function has an inverse in $\mathfrak{A}$ which we call the *Möbius function* and denote $\mu_p$. Note that $\xi_p(x,y)$ is invertible according to [7] page 26. The identity of $\mathfrak{A}$ is

$$\delta(s,t) = \begin{cases} 1 & : s = t \\ 0 & : s \neq t \end{cases}$$

Note that in equations where elements of $\mathfrak{A}$ are added to some integer that integer simply denotes a multiple of $\delta$.
It follows directly from the definition of the multiplication that:

$$\xi^2(s,u) = \sum_{s \le t \le u} 1$$

so we may deduce that $\xi^2(s,u)$ is the number of chains of length 2 between $s$ and $u$ (note that the length is given as the number of elements minus 1). Similarly

$$\xi^k(s,u) = \sum_{s=s_0 \le s_1 \le \ldots \le s_k = u} 1$$

which is the number of chains of length $k$. Observing that

$$(\xi - 1)(s,u) = \begin{cases} 1 & : s < u \\ 0 & : s = u \end{cases}$$

we can use $(\xi - 1)^k$ to count the number of strictly-increasing chains. Note furthermore that

$$(2 - \xi)(s,t) = \begin{cases} 1 & : s = t \\ -1 & : s < u \end{cases}$$

**Proposition 18.3.1.** $(2-\xi)^{-1}(s,t)$ *gives the total number of strictly increasing chains from* $s$ *to* $t$*.*

Proof. Let $\ell$ be the length of the longest chain between $s$ and $t$ so that $(\xi - 1)^{\ell+1}(u,v) = 0$ for $s \le u \le v \le t$. For such $u$ and $v$

$$(2-\xi)[1 + (\xi-1) + (\xi-1)^2 + \ldots + (\xi-1)^\ell](u,v) =$$
$$[1 - (\xi-1)][1 + (\xi-1) + (\xi-1)^2 + \ldots + (\xi-1)^\ell](u,v) =$$
$$[1 - (\xi-1)^{\ell+1}](u,v) = \delta(u,v)$$

The equality from the second line to the third comes from multiplying out so that all of the central terms cancel. Because $\delta$ is the identity, $(2-\xi)^{-1} = 1 + (\xi-1) + (\xi-1)^2 + \ldots + (\xi-1)^\ell$ when restricted to the elements between some $s$ and $t$. But as explained above, $(\xi-1)^k$ are just the chains of length $k$ between $s$ and $t$ so it follows that $(2-\xi)^{-1}$ is the total number of chains from $s$ to $t$. □

The following theorem connects the combinatorial notion of the Möbius function to the topological notion of the Euler characteristic:

**Theorem 18.3.2** (Hall's Theorem)**.** *Let* $P$ *be a finite poset and let* $\hat{P}$ *be* $P \cup \{\hat{0}, \hat{1}\}$ *where* $\hat{0}$ *and* $\hat{1}$ *are minimum and maximum elements. Let* $c_i$ *be the number of strictly increasing chains between* $\hat{0}$ *and* $\hat{1}$ *of length* $i$*. Then*

$$\mu_{\hat{P}}(\hat{0},\hat{1}) = c_0 - c_1 + c_2 - c_3 + \ldots \tag{18.3.3}$$

Proof.

$$\begin{aligned} \mu_{\hat{P}}(\hat{0},\hat{1}) &= (1 + (\xi-1))^{-1}(\hat{0},\hat{1}) \\ &= (1 - (\xi-1) + (\xi-1)^2 - \ldots)(\hat{0},\hat{1}) \\ &= 1(\hat{0},\hat{1}) - (\xi-1)(\hat{0},\hat{1}) + (\xi-1)^2(\hat{0},\hat{1}) - \ldots \\ &= c_0 - c_1 + c_2 - \ldots \end{aligned}$$

□

This expression is very close to the expression developed for the Euler characteristic. Indeed the only difference is that when computing the Euler characteristic, the empty-set is not regarded as a face of the simplicial complex whereas in this expression it is, entering the sum as $-1$. Thus by defining the reduced Euler characteristic, $\tilde{\chi}(X) = \chi(X) - 1$ we have the following remarkable fact:

**Proposition 18.3.4.** *Let $P$ be a finite poset.*

$$\mu_{\hat{P}}(\hat{0}, \hat{1}) = \tilde{\chi}(\mathscr{K}(P)).$$

For more information on Hall's theorem (18.3.2) see [**64**][p.307-8].

CHAPTER 19

# Finite Manifolds and Minimal Finite Models of Closed Surfaces

The following two chapters cover minimal models of various surfaces.

In characterizing finite manifolds, a comparatively well-understood class of finite spaces, the following definitions present useful.

**Definition 19.0.1.** If $X$ is a topological space, a *finite model* of $X$ is a finite $T_0$ space which is weak homotopy equivalent to $X$.

Since every space is weak homotopy equivalent to a regular CW complex[1], this implies that every space has a $T_0$ Alexandroff model, and if the regular CW complex is finite, so is the model.

We describe Stong minimality and absolute minimality of finite models of topological spaces, exhibit Stong minimal models of all closed surfaces, and derive several elementary lower bounds for the size of absolutely minimal models. We define the notion of a finite manifold and characterize finite surfaces, then use this characterization to show that a finite model of a closed surface is a finite surface if and only if it is induced by a regular CW structure on the surface. Finally, we use this result to deduce a better lower bound for the size of models which are finite surfaces and construct minimal finite surface models of orientable surfaces whose genera satisfy nice number- theoretic properties.

One basic example which illustrates the relationship between ordinary spaces and finite models is that of [?]. Recall that the non-Hausdorff suspension of $S^n$ produces a weakly equivalent version with $2n+2$ points for each $n$.

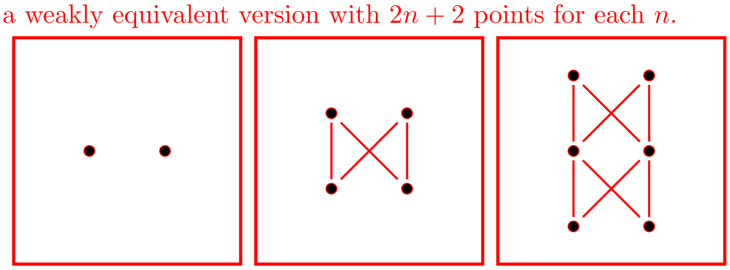


**Figure 1:** Hasse diagrams for finite models of $S^0$, $S^1$ and $S^2$.

Notice that a given space will have many finite models. For example, every finite simplicial structure gives rise to a finite model, and we can always enlarge a model by adding beat points. To avoid superfluous information, reduce complexity, and gain a better understanding of these models, it is desirable to find finite models which are minimal in one of two senses.

[1]Every space is weak homotopy equivalent to a CW complex, while every CW complex is homotopy equivalent to a simplicial complex of the same dimension; see, for example, Theorem 2C.5 in [31].

**Definition 19.0.2.** We say a finite $T_0$ space is *Stong minimal* if its cardinality is minimal in its homotopy class. We say a finite $T_0$ space is absolutely minimal if its cardinality is minimal in its weak homotopy class.[2]

Note that the second notion of minimality is stronger than the first; the sphere is a rare case where the evident Stong minimal model is absolutely minimal. It is also perhaps a more natural notion of minimality when it comes to the study of finite models, since it is equivalent to being the smallest finite model of a space. However, Stong minimality is easier to check, and Stong minimal models are easier to find: two finite $T_0$ spaces are homotopy equivalent if and only if their cores are homeomorphic, so a space is Stong minimal if and only if it has no beat points, and a Stong minimal model can be obtained from any finite model simply by removing beat points. In contrast, at the time of the writing of this paper, there is no known algorithm for reducing an arbitrary finite $T_0$ space to an absolutely minimal space, or even for determining whether a space is absolutely minimal.

For this reason, results regarding absolutely minimal models have typically involved exhibiting a particular model for a space and showing that no smaller space can have the same homotopy or homology groups. Barmak and Minian take this approach in [**8**] in which they show that the finite models described above are the unique absolutely minimal models of $S^n$ for each $n$. Having found such models for the most basic topological spaces, we turn next to another well-known countable collection of spaces with simple homology: closed surfaces. Cianci and Ottina exibit absolutely minimal models of the torus, the projective plane, and the Klein bottle in [**16**], but their methods for bounding model size below are not related to the genus of these surfaces, and hence do not generalize directly to surfaces of higher genus. In this paper, we begin by describing Stong minimal models for all closed surfaces. [3] We then find some lower bounds for the size of arbitrary finite models of closed surfaces using results from [**16**] together with some elementary combinatorial facts. Having derived some minor results for the general case, we specialize to a particularly well-behaved class of finite $T_0$ spaces called finite manifolds, characterize them in dimension 2, and conclude by deriving a much stronger bound for finite models of this type, which we use to find some finite models which are minimal among finite surfaces.

**19.0.1. Stong minimal models of closed surfaces.** In this section, we construct regular CW models for all closed surfaces and show that the associated posets have no beat points, making them Stong minimal. These models are generalizations of absolutely minimal models presented in [**16**].

Given an orientable surface $S$ of genus $g$, the usual CW structure for $S$ is a regular $4g$-gon with edge identifications represented by the word $a_1b_1a_1^{-1}b_1^{-1}\dots a_gb_ga_g^{-1}b_g^{-1}$.

However, this structure is not regular. To fix this issue, we add in the perpendicular bisectors of each edge with no new identifications, splitting each external edge into two 1-cells attached by a 0-cell. We also gain one new vertex in the center at the intersection of all the bisectors. This gives a regular CW structure for $S$

[2]These definitions are standard, but the terminology is not: the first is typically called a "minimal finite space" or simply "minimal", and the second a "minimal finite model". This nomenclature allows for such peculiar entities as spaces which are both minimal and finite models, but are not minimal finite models. We use different terminology to avoid confusion.

[3]We assume throughout that our surfaces are connected, as all our results can be immediately generalized to disconnected closed surfaces by taking coproducts.

and thus a finite model (see Figure 2). It is easy to check that this model has $14g + 2$ points. Note that every 0-cell is contained in at least two 1-cells, every 1-cell contains exactly two 0-cells and is contained in at least two 2-cells, and every 2-cell contains exactly two 1-cells. Consequently, no vertex in the Hasse diagram of the model has in-degree or out-degree 1, so there are no beat points. Thus, this model is Stong minimal.

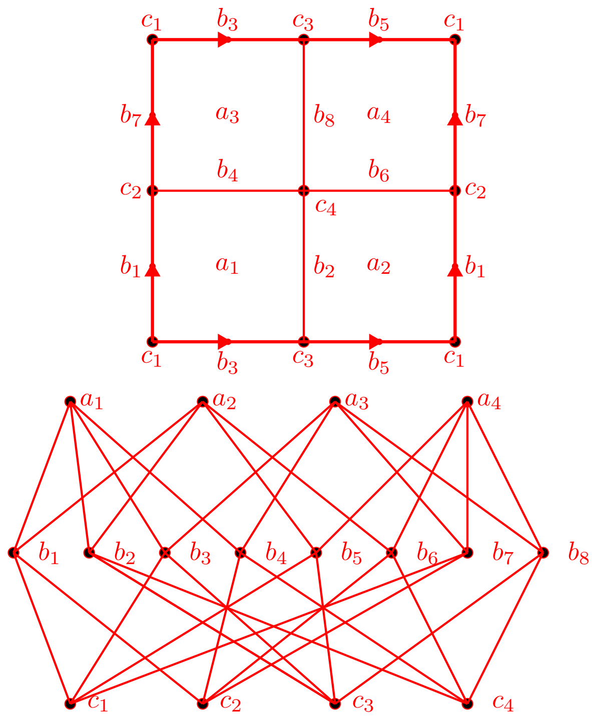



**Figure 2:** The regular CW structure and associated finite model for the orientable surface of genus 1. Taken from [**16**].

The construction of the models for nonorientable surfaces is similar. Given a nonorientable surface $S$ of genus $g$, we begin with the usual CW model: the regular $2g$-gon with edge identifications given by the word $a_1^2 \dots a_g^2$. To make this regular, we add in both the perpendicular bisectors of the edges and the line segments between opposing vertices. This yields a regular CW structure with $11g+2$ points, and the face poset is Stong minimal for the same reason as in the orientable case.

**19.0.2. Elementary bounds.** In this section, we use the weak homotopy invariance of Euler characteristic and several results from [**16**] to derive lower bounds for the size of arbitrary finite models of closed surfaces other than $S^2$ and $\mathbb{R}P^2$, whose absolutely minimal finite models are already known. We denote the Euler characteristic of a space $X$ by $\chi(X)$ and the cardinality of $X$ by $\#X$.

In [**16**], Cianci and Ottina define what they call a *splitting property* (S2) for finite posets. The details are not relevant, but the following result they derive is.

**Proposition 19.0.3.** *Let $X$ be a finite $T_0$ space which is Eilenberg-MacLane of type $(G, 1)$. If $X$ satisfies (S2) then $H_1(X)$ is free abelian and $H_n(X) = 0$ for $n > 1$.*

We obtain the following corollary.

**Proposition 19.0.4.** *No closed surface other than $S^2$ or $\mathbb{R}P^2$ can have a model satisfying (S2).*

PROOF. Let $S$ be a closed surface that is not $S^2$ or $\mathbb{R}P^2$. Then $S$ is covered by $\mathbb{R}^2$, so it is Eilenberg-MacLane of type $(\pi_1(S), 1)$. If $S$ is nonorientable, then $H_1(S)$ is not free abelian, and if $S$ is orientable, $H_2(S)$ is nontrivial. Since homology is a weak homotopy invariant, the same is true of any finite model of $S$, so the result follows by the previous proposition. $\square$

There are two relevant consequences derived in [16] of not satisfying (S2).

**Proposition 19.0.5.** *If $X$ is a finite $T_0$ space not satisfying (S2) that is connected and Stong minimal, then $\#X \geq 16$.*

**Proposition 19.0.6.** *If $X$ is a finite $T_0$ space with at most two maximal points or at most two minimal points, $X$ satisfies (S2).*

These give us our lower bounds.

**Theorem 19.0.7.** *Let $X$ be a finite model of a surface $S$ other than $S^2$ or $\mathbb{R}P^2$. Then $\#X \geq \max(16, \log_2(|\chi(S)|))$.*

PROOF. Since we can reduce any model to a Stong minimal model by removing beat points, we may assume without loss of generality that $X$ is already Stong minimal. Furthermore, because path-connectedness is detected by homology, $X$ must be path-connected and thus connected. It follows that $\#X \geq 16$.

To obtain the other bound, note that since Euler characteristic is weak homotopy invariant, $\chi(S) = \chi(\mathscr{K}(X))$, which is the alternating sum of the number of chains in $X$ of various lengths, $\sum_{k \text{ a chain}} (-1)^{\#k+1}$. By the triangle inequality, the absolute value of the Euler characteristic must be less than or equal to the total number of chains in $X$, $\sum_{k \text{ a chain}} 1$. Since chains are subsets of $X$, this is less than or equal to the number of subsets of $X$, $2^{\#X}$. The result follows. $\square$

We can improve our logarithmic bound to a square root bound in the case where $X$ has height 3.

**Proposition 19.0.8.** *Let $X$ be a height-3 finite model of a surface other than $S^2$ or $\mathbb{R}P^2$. Then $\#X \geq \sqrt{2|\chi(S) - 7|}$.*

PROOF. Let $n = \#X$. The only negative contribution to $\chi(\mathscr{K}(X))$ is from the edges, of which there are at most $\binom{n}{2}$ since they are 2-chains in $X$. We know that there are at least 6 vertices since $X$ does not satisfy (S2): there are at least three maximal points and three minimal points and no point can be both maximal and minimal since $X$ is connected. We also know there must be at least one face because $X$ is of height 3. Thus, there must be at least $|\chi(S) - 7|$ edges, so $n^2 \geq n^2 - n = 2\binom{n}{2} \geq 2|\chi(S) - 7|$, from which the result follows. $\square$

It is conceivable that this method could be extended to posets of greater height. (It is trivial from the simplicial homology of $\mathscr{K}(X)$ that any finite model must have height at least 3.)

## 19.1. Characterization of finite manifolds

We now describe a particularly well-behaved class of finite spaces and characterize them in dimension 2.

**Definition 19.1.1.** A finite $T_0$ space $X$ is a *finite $n$-manifold* if $|\mathscr{K}(X)|$ is a topological $n$-manifold.

The following definitions present useful towards this end. The familiar notions in the theory of simplicial complexes of links and pure complexes can be similarly connected to posets.

**Definition 19.1.2.** The height of $X$ is given by

$$ht(X) = \max_{C \in \mathcal{C}(X)} \{ht(C)\}.$$

Note that the height of a chain is equal to the dimension of its corresponding simplex. It is immediate from invariance of dimension that a finite $n$-manifold must be of height $n+1$. This notion can be also extended to the realm of finite spaces.

**Definition 19.1.3.** Let $X$ be a finite $T_0$ space, and let $\mathcal{C}(X)$ be the set of non-empty chains of $X$. For $C \in \mathcal{C}(X)$, the *height* of $C$ is given by

$$ht(C) = \#C - 1.$$

**Definition 19.1.4.** Let $X$ be a finite $T_0$ space, and let $x \in X$. The *level of $x$* in $X$ is given by

$$\ell_X(x) = ht(\hat{U}_x^X) + 1.$$

Equivalently, the level of $x$ is the maximum height of all chains in $X$ with $x$ as its greatest element.

**Definition 19.1.5.** Let $K$ be an abstract simplicial complex, and let $\sigma$ be a face in $K$. Then the *link* of $\sigma$ in $K$ is given by

$$lk_K(\sigma) = \{x \in X | \tau \cup \sigma \in K, \tau \cap \sigma = \emptyset\}.$$

In other words, the link consists of all faces of $K$ whose union with $\sigma$ is a face of $K$, and whose intersection with $\sigma$ is empty. Note that a link is always itself a simplicial complex.

We now define an analogous term for finite $T_0$ spaces.

**Definition 19.1.6.** Let $X$ be a finite $T_0$ space, and let $C$ be a non-empty chain of $X$. Then the *link* of $C$ in $X$ is given by

$$lk_X(C) = \{x \in X \setminus C | C \cup \{x\} \text{ is a chain}\}.$$

We can easily see that these correspond in the expected manner.

**Proposition 19.1.7.** *Let $X$ be a finite $T_0$ space, and let $C$ be a chain in $X$. Then $\mathscr{K}(lk_X(C)) = lk_{\mathscr{K}(X)}(C)$.*

Proof. If $D$ is a chain in $lk_X(C)$, then $D \cup C$ is a chain in $X$ and $D \cap C = \emptyset$. Conversely, if $v$ is a vertex in $lk_{\mathscr{K}(X)}(C)$, then $v \notin C$ and $C \cup \{v\}$ is a chain in $X$. □

Finally, we introduce a related concept for individual vertices.

**Definition 19.1.8.** Let $X$ be a finite $T_0$ space, and let $x \in X$. The *lower link* of $x$ in $X$ is given by

$$\hat{U}_x^X = \{y \in X | y < x\}$$

The *upper link* of $x$ in $X$ is given by

$$\hat{F}_x^X = \{y \in X | y > x\}$$

When it is clear from context where the lower or upper link comes from, we write simply $\hat{U}_x$ and $\hat{F}_x$.

We define lower and upper links for $\mathscr{K}(X)$ in the expected manner. Note that $\hat{U}_x \cup \hat{F}_x = lk_X(\{x\})$. Furthermore, we can extend $x$ "upwards" into a chain $C$ such that $\hat{U}_x = lk_X(C)$, and similarly "downwards" into a chain $D$ such that $\hat{F}_x = lk_X(D)$. Finally, note that

$$\hat{U}_x^X = \hat{F}_x^{X^{OP}}$$

and similarly

$$\hat{F}_x^X = \hat{U}_x^{X^{OP}}$$

## 19.2. Pure complexes

The first of the following definitions is standard, while the second extends the idea of the first to finite $T_0$ spaces.

**Definition 19.2.1.** An $n$-dimensional simplicial complex $K$ is called *pure* if every simplex in $K$ is contained in an $n$-simplex.

Equivalently, we require all maximal faces have the same dimension.

**Definition 19.2.2.** A finite $T_0$ space $X$ of height $n$ is called *pure* if every maximal chain in $X$ is of height $n$.

As suggested by the terminology, these notions are equivalent.

**Proposition 19.2.3.** *A finite $T_0$ space $X$ is pure if and only if $\mathscr{K}(X)$ is pure, and a finite simplicial complex $K$ is pure if and only if $\mathcal{X}(K)$ is pure.*

Proof. Suppose $X$ is a finite $T_0$ space of height $n$, so $\mathscr{K}(X)$ is a simplicial complex of dimension $n-1$. A $k$-simplex in $\mathscr{K}(X)$ is a chain of length $k+1$, so every simplex in $\mathscr{K}(X)$ is contained in an $(n-1)$-simplex if and only if every chain in $X$ is contained in a chain of length $n$.

Now suppose $K$ is a finite simplicial complex of dimension $n$, so $\mathcal{X}(K)$ is a poset of height $n+1$. The height of a maximal chain in $\mathcal{X}(K)$ is one greater than the dimension of the largest simplex it contains, so every maximal chain is of height $n+1$ if and only if every simplex in $K$ is contained in an $n$-simplex. □

**Proposition 19.2.4.** *If $X$ is a pure finite $T_0$ space, then for all $x \in X$, the level $\ell_X(x)$ of $x$ can be presented as follows:*

$$\ell_X(x) = ht(X) - \ell_{X^{OP}}(x).$$

Proof. Let $x \in X$ and let $C$ be a maximal chain in $X$ containing $x$. Since $X$ is pure, $ht(C) = ht(X)$. Let $C_{\leq} = \{y \in C | y \leq x\}$, and $C_{\geq} = \{y \in C | y \geq x\}$. Then $ht(C) = ht(C_{\leq}) + ht(C_{\geq})$. It must be the case that $ht(C_{\leq}) = \ell_X(x)$ (otherwise, there would be some maximal chain longer than $C$), and using that $\hat{F}_x^X = \hat{U}_x^{X^{OP}}$, we have that $ht(C_{\geq}) = \ell_{X^{OP}}(x)$ for the same reasons. Our desired result immediately follows. □

The reason for introducing pureness is that it plays an important role in the characterization of finite surfaces.

**Theorem 19.2.5.** *A finite $T_0$ space $X$ is a finite surface if and only if it satisfies the following conditions:*

*(1) $X$ is pure of height 3;*
*(2) For each height-2 point $x$, there are exactly two points greater than $x$ and two points less than $x$; and*
*(3) For each maximal point $x_m$ and each minimal point $x_n$, the set $(x_m, x_n) = \{x \in X | x_n < x < x_m\}$ contains either zero or two points.*
*(4) For each extremal point $x$, the set of points other than $x$ which are comparable to $x$ is connected.*

The bulk of the proof of this theorem is based on the corresponding result for simplicial complexes. Stating it requires the following standard definition.

**Definition 19.2.6.** If $v$ is a vertex in a simplicial complex $K$, the *link* of $v$, $Lk(v, K)$, is the undirected graph whose vertices are the 1-simplices of $X$ with $v$ as a face, and where there is an edge between two vertices if they are faces of a common 2-simplex.

**Lemma 19.2.7.** *The geometric realization of a finite simplicial complex $K$ is a surface if and only if $K$ satisfies the following conditions:*

*(1) $K$ is pure and 2-dimensional;*
*(2) Each 1-simplex of $K$ is a face of exactly two 2-simplices; and*
*(3) For each vertex $v$ of $K$, $|Lk(v, K)|$ is homeomorphic to $S^1$.*

Proof. If (1) fails, $|K|$ is not a surface by invariance of dimension. If (2) fails, removing a line from any sufficiently small connected neighborhood of a point in the edge yields three components, so it is not locally Euclidean. If (3) fails, removing $v$ from a sufficiently small connected neighborhood yields two components, so it is not locally Euclidean.

Suppose now that all three conditions hold. Then (1) guarantees that we only need to check the interior of 0-, 1-, and 2-simplices. The last is trivial. Since gluing together two polygons at an edge yields a Euclidean neighborhood for points on the edge, 1-simplices follow by (2). Finally, 0-simplices follow by (3), since it implies that at a 0-simplex $v$, the realization is locally homeomorphic to the disk obtained by gluing together triangles along their edges circularly. □

Now we can prove the theorem.

Proof. The first condition for the poset is equivalent to the first condition for the simplicial complex. Given pureness, the second and third poset conditions together are equivalent to the second simplicial complex condition, because (together with the pureness) they are equivalent to the statement that for any two comparable points $p$ and $q$, there are exactly two ways of extending the 2-chain $\{p, q\}$ to a 3- chain. Finally, the second, third and fourth poset conditions together are equivalent to the third simplicial complex condition, since a graph is a circle if and only if it is connected and each vertex has degree 2. □

There is an alternate characterization of finite surfaces which is also useful. While it is ultimately just a more compact rephrasing of Theorem 4.7, we will see that it is convenient for a number of purposes. The proof is given by point-counting together with the above criterion for a graph to be a circle, and comparing to the conditions of our original classification.

**Definition 19.2.8.** Let $X$ be a finite poset and $x \in X$. Then the *link* of $x$, $Lk(x)$, is the set of points other than $x$ which are comparable to $x$.

**Corollary 19.2.9.** *A finite $T_0$ space $X$ is a finite surface if and only if for each $x \in X$, $|Lk(x)|$ is homeomorphic to $S^1$.*

One of the reasons this statement of the theorem is advantageous is that it can more easily describe the higher-dimensional version of the theorem. Although we have written it out specifically for finite surfaces, the proof of this theorem generalizes directly to higher dimensions[4], so we obtain the following.

**Corollary 19.2.10.** *A finite $T_0$ space $X$ is a finite $n$-manifold if and only if for each $x \in X$, $|\mathscr{K}(Lk(x)|)$ is homeomorphic to $S^{n-1}$.*

Another benefit of this form of the theorem is its relationship to the following result of A. Björner in [**11**].

**Theorem 19.2.11.** *Let $P$ be a finite poset, and for each $x \in P$, denote the set of points less than $x$ by $\hat{U}_x$. Then $P$ is the face poset of a regular CW complex if and only if for each $x \in P$, $|\mathscr{K}(\hat{U}_x)|$ is homeomorphic to a sphere.* [5]

This gives us a final characterization of finite surfaces which will be crucial in obtaining our bound in the next section.

**Theorem 19.2.12.** *A finite $T_0$ space $X$ is a finite surface if and only if it is the face poset of a regular CW structure on some closed surface.*

Proof. Firstly, suppose $X = \mathcal{X}(Y)$, where $Y$ is a regular CW structure on some closed surface. Then $|\mathscr{K}(X)|$ is nothing more than the cellular subdivision of $Y$, so the two are homeomorphic.

Suppose conversely that $X$ is a finite surface, and let $x$ be some point in $X$. If $x$ is minimal, then by definition $|\mathscr{K}(\hat{U}_x)| \cong S^{-1}$. If $x$ is on the second level, there are exactly two points below it by Theorem 4.7, so $|\mathscr{K}(\hat{U}_x)| \cong S^0$. Finally, if $x$ is maximal, then $|\mathscr{K}(\hat{U}_x)| \cong S^1$ by Theorem 4.13. □

Before moving on, it is worth taking a moment to consider the common element between Corollary 4.11 (more generally Corollary 4.12) and Theorem 4.13 which allowed us to prove this relationship: subposets whose order complexes are simplicial spheres. It is generally a nontrivial problem to determine whether a finite poset has this property, although the case in dimension 1 is simple: a height-2 poset has geometric realization $S^1$ if and only if it is connected and every vertex has degree 2. Given this fact, the following theorem suggests the idea of taking an inductive approach to the problem.

[4] In two dimensions, the conditions guarantee precisely that we have triangles glued in a circular fashion, which yields a Euclidean neighborhood of every point. The higher-dimensional equivalent is for $n$-simplices to be glued so as to form a ball in the neighborhood of a vertex, which is expressed via the condition that the indicated poset has order complex homeomorphic to $S^{n-1}$.

[5] We take the empty space to be the sphere of dimension $-1$.

**Theorem 19.2.13.** *If $X$ is a finite $n$-manifold, then $|\mathscr{K}(X)|$ is homeomorphic to $S^n$ if and only if $X$ is a finite model of $S^n$.*

PROOF. One direction is obvious: if $|\mathscr{K}(X)|$ is homeomorphic to $S^n$, then $X$ is a finite model of $S^n$ by Theorem 1.4.

To prove the other direction, suppose $X$ is a finite $n$-manifold which is a finite model of $S^n$. Then $|\mathscr{K}(X)|$ is a CW space which is weak homotopy equivalent to $S^n$, and hence homotopy equivalent to it by the Whitehead theorem. Since $|\mathscr{K}(X)|$ is a closed $n$-manifold, the result follows by the Poincaré conjecture. □

**19.2.1. Bounds for finite surfaces.** Throughout this section, we will denote the number of height 1, 2, and 3 points by $\ell$, $m$, and $n$ respectively.

The problem of finding absolutely minimal finite models amounts to minimizing the sum of the number of points at each level. As the following result shows, by restricting to finite surfaces, we need consider only one number rather than three or more.

**Proposition 19.2.14.** *Let $X$ be a finite surface which is a model of a closed surface $S$ of genus $g$. If $S$ is orientable, then $\#X = 2m + 2 - 2g$. If $S$ is nonorientable, then $\#X = 2m + 2 - g$.*

PROOF. Because $X$ is a finite surface, $\#X = \ell + m + n$. But we also know by Theorem 4.14 that $X$ is the face poset of a regular CW complex structure on $S$, so $n - m + \ell = \chi(S)$. Thus, $\#X = \ell + m + n = 2m + \chi(S)$. The result follows by the standard formula for the Euler characteristic of a closed surface. □

Using the fact that any finite model of a closed surface other than $\mathbb{R}P^2$ or $S^2$ must satisfy the (S2) splitting property and thus have at least three maximal and three minimal points, we can immediately derive from this the linear lower bounds $2g+10$ and $g+10$ for the size of finite surface models of orientable and nonorientable closed surfaces respectively. However, we can do slightly better than this.

**Theorem 19.2.15.** *Let $X$ be a finite surface modelling the closed surface $S$ of genus $g$. If $S$ is orientable, then $\#X \geq 2\lceil 4\sqrt{g}\rceil + 2g + 6$. If $S$ is nonorientable, then $\#X \geq 2\lceil 2\sqrt{2g}\rceil + g + 6$.*

PROOF. Let $c_i$ denote the degree of the $i^{th}$ maximal point in the Hasse diagram of $X$. Then since each point in the middle level has up-degree 2, $\Sigma_i c_i = 2m$, and so for at least one $i$, $c_i \geq 2m/n$. Call this point $x_i$. Because $\hat{U}_{x_i}$ is a finite model of $S^1$, the number of minimal points less than $x_i$ must be equal to the number of level 2 points less than $x_i$, which is just $c_i$. Thus, we get $c_i \leq l$, so $\lceil 2m/n\rceil \leq l$. The same argument for bottom points shows that $\lceil 2m/\ell\rceil \leq n$.

Adding these inequalities (and ignoring the ceilings), we get $2m(1/n + 1/\ell) \leq n + \ell = m + \chi(S)$, since $n - m + \ell = \chi(S)$. The smallest possible value of the left side of the inequality is achieved when $n = \ell = (m + \chi(S))/2$, and we get $8m \leq (m + \chi(S))^2$. Solving, we get $m \geq 4\sqrt{g} + 2g + 2$ in the orientable case and $m \geq 2\sqrt{2g} + g + 2$ in the nonorientable case. The result follows from Proposition 5.1. □

It is not clear that these inequalities are sharp, especially because we dropped the ceilings to derive them. However, there are some cases in which we can be certain they are achieved. To show this, we perform the following construction, illustrated in Figure 3.

**Proposition 19.2.16.** *Let $n$ and $\ell$ be positive even integers and set $2m = n\ell$. Then there is a finite orientable surface with $n$, $m$, and $\ell$ points in its third, second, and first levels respectively.*

Proof. To construct this surface, take $n$ $\ell$-gons and identify them in the following way. Glue every other edge of the first $\ell$-gon to every other edge of the second with coherent orientation, then glue the remaining edges of the second $\ell$-gon to every other edge of the third (again with coherent orientation), and continue until the final $\ell$-gon is glued back to the first. Because we have an even number of polygons, the final gluing will also have coherent orientation. Explicitly, we may embed the polygons in $\mathbb{R}^3$ centered at equal intervals along a circle and with parallel top edges, and glue them together via homotopies of $\mathbb{R}^3$. Then each step of gluing switches the sides which are glued between containing and not containing the top edge, so having an even number of polygons guarantees that the first and last polygons will glue properly, so the space we have constructed admits an embedding in $\mathbb{R}^3$. This construction also guarantees that the link of every vertex will be a circle (since it is connected and every vertex in the graph has degree two) and every edge will be adjacent to exactly two faces, so this will produce a closed orientable surface with a regular CW structure consisting of $n$ faces, $m$ edges, and $\ell$ vertices. We finish the construction by taking its face poset. □

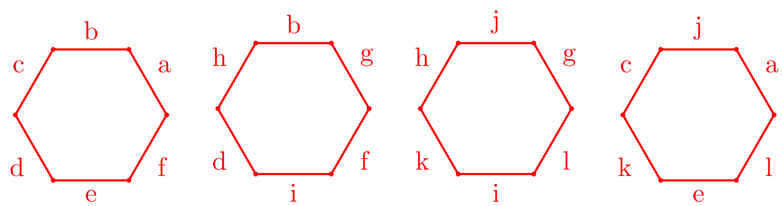



**Figure 3:** The polygons and identifications obtained by performing the construction with $\ell = 6$ and $n = 4$. All edges are oriented clockwise. Performing the gluing will yield the orientable surface of genus 2.

If we take $n = 4$, $\ell = 6$, this produces a model of the orientable surface of genus 2 with $n = 12$ (Figures 3,4). Geometrically, this is obtained by gluing together four hexagons in pairs to obtain two pairs of pants, then gluing together the pairs of pants to obtain the surface. By our bound above, this is minimal among finite orientable surfaces of genus 2.

It is unfeasible to explicitly construct every model individually to check if it achieves our bound. However, as the following theorem shows, for $g$ with particularly nice number-theoretic properties, we don't need to.

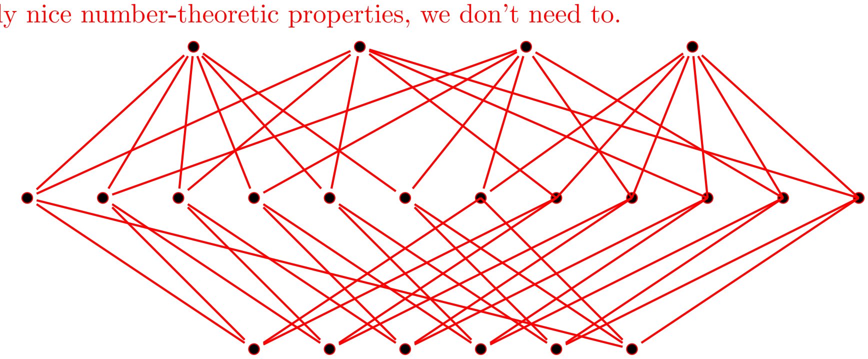


**Figure 4: A minimal finite orientable surface of genus 2.**

**Theorem 19.2.17.** *If $g$ is a perfect square, then performing this construction with $n = \ell = 2\sqrt{g} + 2$ yields a minimal finite orientable surface of genus $g$.*

Proof. The resulting space has $m = 2g + 2 + 4\sqrt{g}$, so its Euler characteristic is $n - m + \ell = 2 - 2g$, which shows that it is indeed of genus $g$. Its cardinality is $\ell + m + n = 8\sqrt{g} + 2g + 6$, and since $\sqrt{g}$ is an integer, this is precisely the lower bound derived above. □

The simplest case is when $g$ is a perfect square. However, the lower bound is more generally achieved by this construction when $g$ is a product of two integers which are sufficiently close. For example, if $g$ is of the form $(k-1)(k-2)$, then as long as $k$ is at least 3, we get $4k - 7 < 4\sqrt{g} \leq 4k - 6$, so $\lceil 4\sqrt{g} \rceil = 4k - 6 = k - 1 + 2 + 4\sqrt{g}$, and setting $n = 2k$, $\ell = 2(k-1)$ yields a surface of the desired genus which achieves the bound. To further generalize this result is a problem of number theory.

## 19.3. the Euler characteristic of finite representations of homology manifolds

Define closed homology manifold.

Though the formula of the Euler characteristic of a finite $T_0$ space $X$ has already been derived, there is more that can be said when the geometric realization of the associated simplicial complex $|(K)(X)|$ is a closed homology manifold.

Given a topological space $X$, let $\chi(X)$ be the Euler characteristic of $X$. If $K$ is a finite simplicial complex, it is clear that

$$\chi(|K|) = \sum_{\sigma \in K} (-1)^{dim(\sigma)}.$$

Let $X$ be a finite $T_0$ space. Since $|(K)(X)|$ and $X$ are weak homotopy equivalent, their homology groups are isomorphic, and hence they have the same Euler characteristic. Let $\mathcal{C}(X)$ be the set of non-empty chains of $X$. The definition of $(K)$ allows us to conclude

$$\chi(X) = \sum_{C \in \mathcal{C}(X)} (-1)^{ht(C)}$$

[7].

We can relate the Euler characteristic of a finite $T_0$ space $X$ to the Euler characteristics of lower links in $X$ with the following proposition.

**Proposition 19.3.1.** *Let $X$ be a finite $T_0$ space. Then*

$$\chi(X) = \sum_{x \in X} (1 - \chi(\hat{U}_x)).$$

Proof. Proof by induction on the cardinality $\#X$ of $X$. The case $\#X = 0$ is trivial. Assume our hypothesis is true for $\#X = k$. Let $\#X = k + 1$, and let $x_0 \in X$ be a maximal point. Since $x_0 \notin \hat{U}_y$ for all $y \neq x_0$, we have

$$\chi(X \setminus \{x_0\}) = \sum_{y \in X \setminus \{x_0\}} (1 - \chi(\hat{U}^X_y))$$

by our hypothesis. Furthermore,

$$\begin{aligned}\chi(X) &= \sum_{C\in\mathcal{C}(X)} (-1)^{ht(C)} \\ &= \sum_{C\in\mathcal{C}(X), x_0\in C} (-1)^{ht(C)} + \sum_{D\in\mathcal{C}(X), x_0\notin D} (-1)^{ht(D)} \\ &= \sum_{C\in C(X), x_0\in C} (-1)^{ht(C)} + \chi(X\setminus\{x_0\}).\end{aligned}$$

Clearly, if $x_0 \in C \subset X$, then $C \in \mathcal{C}(X)$ if and only if $C = \{x_0\}$ or $C\setminus\{x_0\} \in \mathcal{C}(\hat{U}_{x_0})$. Hence,

$$\begin{aligned}\sum_{C\in\mathcal{C}(X), x_0\in C} (-1)^{ht(C)} &= 1 - \sum_{C\in\mathcal{C}(\hat{U}_{x_0})} (-1)^{ht(C)} \\ &= 1 - \chi(\hat{U}_{x_0}).\end{aligned}$$

Our induction immediately follows. □

Of course, this proof can be altered slightly to provide an analogous result for upper links.

We now reach the main result.

**Theorem 19.3.2.** *Let $X$ be a finite $T_0$ space. If $|(K)(X)|$ is a closed homology manifold, then*

$$\chi(X) = \sum_{x\in X} (-1)^{\ell_X(x)}$$

PROOF. Recall that a compact polyhedron $M$ is a closed homology manifold if its underlying simplicial complex $K$ is such that for any simplex $\sigma$ of $K$, the homology groups of $|lk_K(\sigma)|$ are isomorphic to the homology groups of $S^{dim(M)-dim(\sigma)-1}$[6]. Note that the polyhedron condition implies that $K$ is pure.

For $x \in X$, let $C$ be a maximal chain in $X$ containing $x$, and let $C_{\geq} = \{y \in C | y \geq x\}$. Since $(K)(X)$ is pure, $X$ is pure, so $ht(C_{\geq}) = ht(\hat{F}_x^X) + 1$. Furthermore, $lk_X(C_{\geq}) = \hat{U}_x^X$ . Hence,

$$\begin{aligned}\chi(\hat{U}_x^X) &= \chi(S^{ht(X)-ht(C_{\geq})-1}) \\ &= 1 + (-1)^{ht(X)-ht(\hat{F}_x^X)} \\ &= 1 + (-1)^{ht(X)-ht(\hat{U}_x^{X^{OP}})} \\ &= 1 + (-1)^{ht(X)-\ell_{X^{OP}}(x)+1} \\ &= 1 + (-1)^{\ell_X(x)+1}\end{aligned}$$

Our result follows from the above proposition □

With this result, we can now provide another proof of a well-known fact.

**Corollary 19.3.3.** *All odd-dimensional polyhedral closed homology manifolds have Euler characteristic 0.*

[6]Note the similarity between this definition and piecewise-linear triangulations of a manifold, in which the link of a simplex is homeomorphic to a sphere of appropriate dimension.

Proof. Let $M$ be an odd-dimensional polyhedral homology manifold with underlying complex $K$. Then $K_\Delta$ is a finite $T_0$ space such that $(K)(K_\Delta)$ is a triangulation of $M$. Thus,

$$\chi(X) = \sum_{x \in K_\Delta} (-1)^{\ell_{K_\Delta}(x)}.$$

But $(K_\Delta)^{OP}$ is also a finite $T_0$ space such that $(K)((K_\Delta)^{OP})$ is a triangulation of $M$, so

$$\chi(X) = \sum_{x \in (K_\Delta)^{OP}} (-1)^{\ell_{(K_\Delta)^{OP}}(x)}.$$

Since $\ell_{K_\Delta}(x) = ht_{K_\Delta}(x) - \ell_{(K_\Delta)^{OP}}(x)$, and since $ht(K_\Delta)$ is odd, $\ell_{K_\Delta}(x)$ and $\ell_{(K_\Delta)^{OP}}(x)$ have different parities. Hence we conclude that $\chi(X) = -\chi(X) = 0$, and thus that $\chi(M)$. □

CHAPTER 20

# The Fundamental Group of a Finite Space

In the following, we will present two methods of computing the fundamental group of a finite space, and eventually prove their equivalence.

## 20.1. H-Loop Groups

The following definitions are due to Barmak and Minian [8].

**Definition 20.1.1.** For any poset $X$ with a base point $x_0$, let $H(X)$ be the associated Hasse diagram. We call an ordered pair $e = (x, y)$ an *$H$-edge* if $(x, y) \in E(H(X))$ or $(y, x) \in E(H(X))$. The point $x$ is called the *origin* of $e$, denoted by $o(e)$ and the point $y$ is called the *end* of $e$, denoted by $e(e)$. The *inverse* of an $H$-edge $e = (x, y)$ is the $H$-edge $e^{-1} = (y, x)$.

**Definition 20.1.2.** If we have a sequence of $H$-edges $e_1, e_2, \dots, e_n$ with $e(e_i) = o(e_{i+1})$ for all $1 \le i \le n-1$, we can connect them together to get an $H$-path $\xi = e_1 e_2 \dots e_n$. Typically we say the *origin* of this $H$-path is $o(\xi) = o(e_1)$ and the *end* of this $H$-path is $e(\xi) = e(e_n)$. The *inverse* of an $H$-path $\xi = e_1 e_2 \dots e_n$ is the $H$-path $\xi^{-1} = e_n^{-1} e_{n-1}^{-1} \dots e_1^{-1}$.

**Definition 20.1.3.** An $H$-path $\xi = e_1 e_2 \dots e_n$ is said to be *monotonic* if either $e_i \in E(\mathscr{H}(X))$ for all $1 \le i \le n$ or $e^{-1} \in E(\mathscr{H}(X))$ for all $1 \le i \le n$.

**Definition 20.1.4.** For two $H$-paths $\xi_1 = e_1 e_2 \dots e_n$ and $\xi_2 = f_1 f_2 \dots f_m$ with $e(\xi_1) = o(\xi_2)$, it makes sense to define a *composition* of $\xi_1$ and $\xi_2$:

$$\xi_1 \xi_2 = e_1 e_2 \dots e_n f_1 f_2 \dots f_m.$$

**Definition 20.1.5.** An *$H$-loop* at $x_0$ is an $H$-path $\xi$ such that $o(\xi) = e(\xi) = x_0$.

**Definition 20.1.6.** Two $H$-loops $\xi$ and $\xi'$ at $x_0$ are said to be *close* if there exist four $H$-paths $\xi_1, \xi_2, \xi_3$, and $\xi_4$ with $\xi_2$ and $\xi_3$ being monotonic, such that $\xi = \xi_1 \xi_4$ and $\xi' = \xi_1 \xi_2 \xi_3 \xi_4$. Denote this close relation by $\xi \simeq \xi'$.

Two $H$-loops $\xi$ and $\xi'$ at $x_0$ are said to be *$H$-equivalent* if there exists a sequence of loops at $x_0$, $\xi = \xi_0, \xi_1, \xi_2, \dots, \xi_n = \xi'$ such that $\xi_{i-1} \simeq \xi_i$ for each $1 \le i \le n$.

It is not hard to verify that $H$-equivalence is actually an equivalence relation. Therefore, we obtain the equivalence classes for $H$-loops at $x_0$. Let us denote the equivalence class of the $H$-loop $\xi$ by $\langle \xi \rangle$ and collect all the equivalence classes into the set $\mathscr{H}(X, x_0)$. Similar to the way we handle the idea of fundamental group, we can define a product on these equivalence classes by taking $\langle \xi_1 \rangle \langle \xi_2 \rangle = \langle \xi_1 \xi_2 \rangle$. It is not hard to show that this product is well defined. This gives a group structure on the set $\mathscr{H}(X, x_0)$, which is called the *$H$-loop group*.

When we apply the functor $\mathscr{K}$ to the finite poset $X$, we obtain a simplicial complex $\mathscr{K}(X)$, and there is another special kind of group called the edge-path group.

## 20.2. Edge Path Groups

Next we are going to define the edge-path group of $\mathscr{K}(X)$ and show that it is actually isomorphic to the $H$-loop group of the space $(X, x_0)$.

**Definition 20.2.1.** For a simplicial complex $K$, an *edge-path* $\xi$ is a finite sequence of vertices $v_0 v_1 v_2 \dots v_n$ such that either $\{v_{i-1}, v_i\}$ is an edge (1-dimensional subsimplex) of $K$ or $v_{i-1} = v_i$. If we write the ordered pair $(v_{i-1}, v_i) = \epsilon_i$, then an edge-path can be written as $\xi = \epsilon_1 \epsilon_2 \dots \epsilon_n$. An *edge-loop* $\xi$ at a vertex $v$ is an edge-path such that $v_0 = v_n = v$. In particular, we set the *zero edge-loop* to be $v$.

The reason that we use $\epsilon$ instead of $e$ to represent an edge here is because an edge in $\mathscr{K}(X)$ may not correspond to an $H$-edge in $X$. In fact, for an edge $\epsilon = (x, y)$ in $\mathscr{K}(X)$, $x$ is comparable to $y$, and we can always find $x_1, x_2 \dots x_n$ such that $(x, x_1)(x_1, x_2) \dots (x_n, y)$ is a monotonic $H$-path. Conversely, an $H$-edge in $X$ always corresponds to an edge in $\mathscr{K}(X)$.

**Definition 20.2.2.** Two edge-loops at $v$ are said to be *equivalent* if one can be obtained from the other by a series of the following move: for any $\{x, y, z\}$ that is a subset of a triangle (2-dimensional simplex), it is allowed to switch the edge $(x, y)$ with two consecutive edges $(x, z)(z, y)$ (note that $x, y$ and $z$ need not be distinct). Denote this equivalence relation by $\approx$.

Notice that for any edge-loop at $v$, the start and end point $v$ should never be changed under the move described above, and thus the move does not change the nature of being an edge-loop at $v$.

In fact, one can verify that the definition above gives an equivalence relation. If we denote the equivalence class of $\xi = \epsilon_1 \epsilon_2 \dots \epsilon_n$ to be $[\epsilon_1 \epsilon_2 \dots \epsilon_n]$ and put in the composition operation, this gives a group with the identity being the zero edge-loop. This group is called the *edge-path group*, and it is denoted by $E(K, v)$.

**Remark 20.2.3.** One basic fact from algebraic topology is that the edge-path group $E(K, v)$ of a simplicial complex $K$ is isomorphic to the fundamental group $\pi_1(|K|, v)$ of the geometric realization of $K$. A proof of this can be found in Spanier [**63**].

Now we are ready to prove the following:

**Theorem 20.2.4.** *If $(X, x_0)$ is a finite poset, then the edge-path group $E(\mathscr{K}(X), x_0)$ of $\mathscr{K}(X)$ is isomorphic to the $H$-loop group $\mathscr{H}(X, x_0)$.*

Proof. Define the map

$$\phi : \mathscr{H}(X, x_0) \longrightarrow E(\mathscr{K}(X), x_0)$$

$$\langle e_1 e_2 \dots e_n \rangle \mapsto [e_1 e_2 \dots e_n].$$

We first want to show this map is well defined. Suppose $\xi_1 \xi_2 \xi_3 \xi_4 \simeq \xi_1 \xi_4$ as $H$-loops at $x_0$, where $\xi_2 = e_1 e_2 \dots e_n$ and $\xi_3 = f_1 f_2 \dots f_m$ are monotonic. Then without loss of generality, we can assume that $\xi_2$ is monotonically increasing, and thus

$$o(e_1) < e(e_1) = o(e_2) < e(e_2) = o(e_3) < \cdots < e(e_n),$$

which means that any three consecutive vertices are within one triangle. Therefore, by induction on the subscript $i$ of $e_i$,

$$\begin{aligned}[\xi_1\xi_2\xi_3\xi_4] &= \xi_1(o(e_1), e(e_1))(o(e_2), e(e_2))\dots(o(e_n), e(e_n))\xi_3\xi_4] \\ &= [\xi_1(o(e_1), e(e_2))\dots(o(e_n), e(e_n))\xi_3\xi_4] \\ &= [\xi_1(o(e_1), e(e_n))\xi_3\xi_4].\end{aligned}$$

Similarly, we can replace $\xi_3$ with $(o(f_1), e(f_m))$ to get

$$[\xi_1(o(e_1), e(e_n))\xi_3\xi_4] = [\xi_1(o(e_1), e(e_n))(o(f_1), e(f_m))\xi_4].$$

But by the definition of closed $H$-loops, we know that $o(e_1) = e(f_m)$ and $e(e_n) = o(f_1)$. Therefore, we can replace the middle part by the zero edge-path and get $[\xi_1\xi_2\xi_3\xi_4] = [\xi_1\xi_4]$.

Note that the map $\phi$ is a homomorphism by construction.

In the reverse direction, we can define another map

$$\psi : E(\mathscr{K}(X), x_0) \longrightarrow \mathscr{H}(X, x_0)$$

$$[\epsilon_1\epsilon_2\dots\epsilon_n] \mapsto \langle\xi_1\xi_2\dots\xi_n\rangle,$$

where each $\xi_i$ is a monotonic $H$-path sharing the same origin point and end point with the edge $\epsilon_i$. This in fact does not depend on the choice of the monotonic $H$-paths, because if $\xi_i$ and $\xi_i'$ are two possible choices, then

$$\begin{aligned}\xi_1\xi_2\dots\xi_i\dots\xi_n &\simeq \xi_1\xi_2\dots\xi_i\xi_i^{-1}\xi_i'\dots\xi_n \\ &\simeq \xi_1\xi_2\dots\xi_i'\dots\xi_n.\end{aligned}$$

To show this map $\psi$ is well defined, it is enough to show the move in Definition 20.2.2 for equivalent edge-loops does not change the image. Suppose

$$\epsilon_1\epsilon_2\dots(x,z)(z,y)\dots\epsilon_n \approx \epsilon_1\epsilon_2\dots(x,y)\dots\epsilon_n.$$

Let $\alpha, \beta, \gamma$ be the three monotonic paths corresponding to $(x,z), (z,y)$ and $(x,y)$, respectively and let $\xi_i$ correspond to $\epsilon_1$ for the other $i$ as usual. Since $x, y$ and $z$ are the three vertices of a triangle, without loss of generality, we can assume either $x < z < y$ or $x < y < z$.

If $x < z < y$, then $\alpha\beta$ is also monotonic, and

$$\begin{aligned}\xi_1\xi_2\dots\alpha\beta\dots\xi_n &\simeq \xi_1\xi_2\dots\alpha\beta\beta^{-1}\alpha^{-1}\gamma\dots\xi_n \\ &\simeq \xi_1\xi_2\dots\gamma\dots\xi_n.\end{aligned}$$

If $x < y < z$, then $\gamma\beta^{-1}$ is also monotonic, and

$$\begin{aligned}\xi_1\xi_2\dots\alpha\beta\dots\xi_n &\simeq \xi_1\xi_2\dots\alpha\alpha^{-1}\gamma\beta^{-1}\beta\dots\xi_n \\ &\simeq \xi_1\xi_2\dots\gamma\dots\xi_n.\end{aligned}$$

Thus $\psi$ is well defined and it is a homomorphism by construction. Now we claim that $\phi$ and $\psi$ are inverses of each other.

Pick any $H$-loop class $\langle\xi\rangle = \langle e_1e_2\dots e_n\rangle$ in $\mathscr{H}(X, x_0)$ and apply $\psi\circ\phi$, we get

$$\psi\circ\phi(\langle\xi\rangle) = \psi([e_1e_2\dots e_n]) = \langle e_1e_2\dots e_n\rangle.$$

Now pick any edge-loop class $[\xi] = [\epsilon_1\epsilon_2\dots\epsilon_n]$ in $E(\mathscr{K}(X), x_0)$ and apply $\phi\circ\psi$, we get

$$\phi\circ\psi([\xi]) = \phi(\langle\xi_1\xi_2\dots\xi_n\rangle),$$

where each $\xi_i = e_{i,1}ei,2\dots e_{i,n_i}$ is a monotonic $H$-path that corresponds to the edge $\epsilon_i$. But as we showed above,

$$\begin{aligned}\phi(\langle\xi_1\xi_2\dots\xi_i\dots\xi_n\rangle) &= [\xi_1\xi_2\dots\xi_i\dots\xi_n] \\ &= [\xi_1\xi_2\dots(o(e_{i,1}), e(e_{i,n_i}))\dots\xi_n] \\ &= [\xi_1\xi_2\dots\epsilon_i\dots\xi_n] \\ &= [\xi_1\xi_2\dots\xi_n].\end{aligned}$$

Since $\psi\circ\phi$ is the identity on $\mathscr{H}(X, x_0)$ and $\phi\circ\psi$ is the identity on $E(\mathscr{K}(X), x_0)$, we conclude that $\mathscr{H}(X, x_0)$ is isomorphic to $E(\mathscr{K}(X), x_0)$. □

As we mentioned before, the edge-path group of a simplicial complex $(K, v)$ is isomorphic to the fundamental group of its geometric realization. Therefore, we have the following:

**Corollary 20.2.5.** *For a finite poset $(X, x_0)$, the following groups are isomorphic:*

*(1)* $\mathscr{H}(X, x_0)$*;*
*(2)* $E(\mathscr{K}(X), x_0)$*;*
*(3)* $\pi_1(|\mathscr{K}(X)|, x_0)$*;*
*(4)* $\pi_1(X, x_0)$*.*

**Remark 20.2.6.** The $H$-loop group in the Hasse diagram provides a way to compute the fundamental group of a topological space by just looking at its minimal finite model. As we know, a minimal finite model is weak homotopy equivalent to the original space, and hence all the information of every homotopy group is carried by its minimal finite model. However, it is not known yet whether there is an efficient way to extract the information of higher homotopy groups just from a minimal finite model.

## 20.3. Hasse Diagrams and the Fundamental Group

Finite graphs are another class of geometric objects whose minimal finite models have been completely understood. One important fact about finite graphs that makes it easier to find their minimal models is that a finite graph is a 1-dimensional CW complex, i.e. a wedge sum of circles. Therefore, the weak homotopy type of a finite graph is determined by its Euler characteristic, and from this we can work out a way to compute minimal finite models of finite graphs.

Before we go into the actual argument, we would like to study the Hasse diagram a little further. As we know from the previous section, the edge-path group of $(\mathscr{K}(X), x_0)$ is isomorphic to the fundamental group of $(X, x_0)$. But what is more about the Hasse diagram is that it can provide another way of looking at the fundamental group in terms of generators and relations. Here we first want to show how to get the generators.

**Proposition 20.3.1.** *Let $(X, x_0)$ be a poset. If $x \in X$ is neither maximal nor minimal and $x \neq x_0$, then the inclusion $i : X - \{x\} \longrightarrow X$ induces an epimorphism*

$$i_* : E(\mathscr{K}(X - \{x\}), x_0) \longrightarrow E(\mathscr{K}(X), x_0).$$

Proof. Since every edge-loop at $x_0$ in $X - \{x\}$ has a natural image as an edge-loop in $X$ under inclusion, $i_*$ is natually a homomorphism. Therefore, to show $i_*$ is an epimorphism, it is sufficient to check that every edge-loop in $\mathscr{K}(X)$ that goes through $x$ is equivalent to an edge-loop that does not go through $x$.

Suppose $\epsilon_1\epsilon_2\dots(y,x)(x,z)\dots\epsilon_n$ is an edge-loop. Then without loss of generality, we can assume either $y \le x \le z$ or $y \le x$ and $z \le x$.

If $y \le x \le z$, then $\{x, y, z\}$ is within a triangle and therefore we can apply the move from Definition 20.2.2 and deduce that

$$\epsilon_1\epsilon_2\dots(y,x)(x,z)\dots\epsilon_n \approx \epsilon_1\epsilon_2\dots(y,z)\dots\epsilon_n.$$

If $y \le x$ and $z \le x$, then since $x$ is not maximal, we can find $w \in X - \{x\}$ such that $w > x$. Then

$$\begin{aligned}\epsilon_1\epsilon_2\dots(y,x)(x,z)\dots\epsilon_n &\approx \epsilon_1\epsilon_2\dots(y,w)(w,x)(x,w)(w,z)\dots\epsilon_n\\ &\approx \epsilon_1\epsilon_2\dots(y,w)(w,z)\dots\epsilon_n.\end{aligned}$$

□

We know that for a path connected space, the fundamental group does not depend on the choice of the base point. Thus without loss of generality, we can always choose the base point $x_0$ to be one of the minimal points. Now imagine that if we eliminate all the points that are neither maximal nor minimal in $X$, then we will be left only with all the maximals and minimals. Call this subspace with only maximals and minimals $Y$. Then we have the following corollary:

**Corollary 20.3.2.** *For any finite poset $(X, x_0)$, let $(Y, x_0)$ be the subspace that consists of only maximals and minimals in $(X, x_0)$. Then the inclusion induces an epimorphism $i_* : E(\mathscr{K}(Y), x_0) \longrightarrow E(\mathscr{K}(X), x_0)$, or equivalently, $i_* : \pi_1(Y, x_0) \longrightarrow \pi_1(X, x_0)$.*

**Remark 20.3.3.** Note that since there are only maximals and minimals in $Y$ , $h(Y) \le 2$. Typically, for a non-contractible space $X$, $h(Y) = 2$. Also, if $X$ is connected, then removing middle points will not disconnect the space, i.e. $Y$ remains connected.

**Remark 20.3.4.** When $h(Y) = 2$, we know that $\mathscr{K}(Y)$ is a finite 1-dimensional simplicial complex, i.e. a finite graph. Since a finite graph is always homotopy equivalent to a wedge sum of circles, we can assume that $\mathscr{K}(Y)$ is homotopy equivalent to $\vee_{i=1}^m S^1$. Therefore, we have

$$\pi_1(Y, x_0) \cong E(\mathscr{K}(Y), x_0) \cong \pi_1\left(\bigvee_{i=1}^m S^1, s_0\right) \cong \mathbb{Z}^{*m}.$$

Now we can go into the search for minimal finite models of finite graphs:

**Theorem 20.3.5.** *If $X$ is a* minimal finite model *of $\vee_{i=1}^n S^1$, then $h(X) = 2$.*

Proof. Take the subspace of maximals and minimals $Y$ . Since $X$ is a minimal finite model of a noncontractible space, we know that $h(Y) = 2$. By Remark 20.3.4, $\pi_1(Y, x_0) = \mathbb{Z}^{*m}$.

By Proposition 20.3.1, there is an epimorphism $i_* : \pi_1(Y, x_0) \longrightarrow \pi_1(X, x_0)$. Note that since $\pi_1(X, x_0) = \mathbb{Z}^{*n}$, thus we must have $m \ge n$.

Now consider $\mathscr{K}(Y)$. Since it is a finite graph, in other words, a wedge sum of $m$ circles, there are $m$ edges that are not contained in any maximal tree of the graph. If we remove $m - n$ of these edges by forgetting the relations between the vertices, we obtain a new finite space $Z$ and $\mathscr{K}(Z)$ is homotopy equivalent to $\vee_i^n S^1$.

Note that $\#Z = \#Y \le \#X$. But since $X$ is a minimal finite model of $\vee_i^n S^1$, we also have $\#X \le \#Z$. Therefore, $\#Z = \#Y = \#X$, which implies $X = Y$. □

The following theorem will conclude our search:

**Theorem 20.3.6.** *Let $j$ be the number of maximal points in $X$ and $k$ be the number of minimal points in $X$. Then $X$ is a minimal finite model of $\vee_i^n S^1$ if and only if $h(X) = 2$, $\#X = \min\{j+k|(j-1)(k-1) \geq n\}$ and the number of edges in $\mathscr{K}(X)$ is $\#X + n - 1$.*

Proof. We have shown that if $X$ is a minimal finite model of $\vee_i^n S^1$, then $h(X) = 2$. Since $j$ is the number of maximal points and $k$ is the number of minimal points in $X$, we know that there can be at most $jk$ many edges in $\mathscr{K}(X)$. Let $E$ be the number of edges in $\mathscr{K}(X)$ and $V$ be the number of vertices, then the Euler characteristic formula tells us that

$$1 - n = V - E \geq j + k - jk.$$

Therefore, we must have $(j-1)(k-1) = jk - j - k + 1 \geq n$, and hence $\#X = j + k \geq min\{j+k|(j-1)(k-1) \geq n\}$.

Now suppose we have $j$ and $k$ such that $(j-1)(k-1) \geq n$. Then consider the finite poset $W = \{x_1, x_2, \ldots, x_j, y_1, y_2, \ldots, y_k\}$ with $xs > yt$ for any $1 \leq s \leq j$ and $1 \leq t \leq k$. As we can see, $W$ is a finite model of $\vee_{i=1}^{(j-1)(k-1)} S^1$. But then we $i = 1$ can remove $(j-1)(k-1) - n$ edges from $\mathscr{K}(W)$ by forgetting the corresponding relations, and the resulting finite poset would be a finite model of $\vee_i^n S^1$.

Now since for any $j$ and $k$ with $(j-1)(k-1) \geq n$ we can find a finite model with $j + k$ points, we conclude that $\#X = \min\{j+k|(j-1)(k-1) \geq n\}$, and the number of edges just follows from the Euler characteristic formula.

Conversely, suppose we have a finite poset $X$ with $h(X) = 2$, $\#X = \min\{j + k|(j-1)(k-1) \geq n\}$ and the number of edges in $\mathscr{K}(X)$ being $\#X + n - 1$. Note that if $X$ is connected, then we are done, for the reason that $\mathscr{K}(X)$ will also be connected, and the three conditions will determine that $\mathscr{K}(X)$ is a finite graph with the Euler characteristic $1 - n$. Therefore, the only thing we need to show here is connectedness.

Suppose $X$ is disconnected. Let $X_1, X_2, \ldots, X_l$ be distinct connected components in $X$. Let $M_i$ be the set of maximal points in $X_i$ and $m_i$ be the set of minimal points in $X_i$. Then $j = \Sigma_{i=1}^l \#M_i$ and $k = \Sigma_{i=1}^l m_i$. Since $\#X = \min\{j+k|(j-1)(k-1) \geq n\}$, we must have $(j-2)(k-1) < n$. But at the same time, $n = E - j - k + 1$ by the Euler characteristic formula. Therefore,

$$(j-2)(k-1) < E - j - k + 1$$

$$jk < E + (k-1).$$

Note that $jk$ is in fact the number of edges in the complete bipartite graph $\left(\bigcup_{i=1}^l m_i, \bigcup_{i=1}^l M_i\right)$. The inequality above shows that $\mathscr{K}(X)$ differs from the complete bipartite graph in less than $k - 1$ edges.

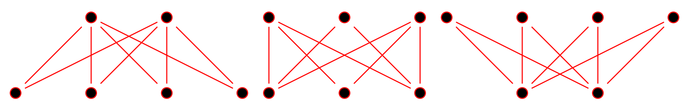
Since there are no edges between $M_i$ and $m_r$ for $i \neq r$, we have

$$k = 1 > \sum_{i=1}^{l} \#M_i(k - \#m_i) \geq \sum_{i=1}^{l}(k - \#m_i) = (l-1)k.$$

This forces $l = 1$ and hence $X$ is connected. □

This theorem gives a method to compute minimal finite models of all finite graphs. Unlike the $n$-spheres, some finite graphs have more than one minimal finite model, with the same number of points but different arrangements. For example, the following three are minimal finite models of $\vee_{i=1}^{3} S^1$:

**Example 20.3.7.**



Up to this point are the minimal finite models that have been completely understood. But we want to push the frontier a little bit further to some slightly more complicated spaces and investigate the possible size of their minimal finite models.

## 20.4. Towards Realizing Groups with Finite Presentations

One fact from algebraic topology is that any group can be realized as the fundamental group of a geometric CW complex of dimension less than or equal to two. The way to do this is by taking a presentation of that group, and gluing a 1-cell to the base point for each generator of that group and a 2-cell along the 1-cells for each relation. (Note that if the starting group is free, then we only need the 1-cells, and the resulting CW complex would just be a graph.)

This makes us wonder whether we can do the same thing with finite spaces, i.e. realizing certain groups just by finitely many points, and have a restriction on the height of the finite posets. Of course, we should point out that we can never realize a group that requires infinitely many generators, for the reason from Corollary 20.3.2 and Remark 20.3.4 that the fundamental group of any finite space is an epimorphic image from a finitely generated free group. Nevertheless, for a group with a finite presentation, we assert that we can always realize it with a finite poset, simply by subdividing the corresponding CW complex into a simplicial complex and applying the $\mathcal{X}$ functor. The resulting finite poset automatically has height no more than 3. However, we can even assure more with the following theorem:

**Theorem 20.4.1.** *For any finite poset $(X, x_0)$, there exists a finite poset $(X', x_0')$ with no more than $\#X$ many points, whose fundamental group is isomorphic to that of $X$ and $h(X') \leq 3$. In other words, among all the realizations of a certain group, we can find such a poset with the least number of elements that has height of no more than 3.*

Proof. Without loss of generality, let us assume that $x_0$ is a minimal. We are going to construct $X'$ explicitly as the follows.

First copy the subspace $Y$ of all the maximals and minimals in $X$, call it $X'$. Then for any point $x$ that is neither maximal nor minimal in $X$, put a point $x'$ in $X'$ with relations:

(1) For any maximal $\alpha \in Y$,let $\alpha'$ be the copy of $\alpha$ in $X'$. Then $x' < a'$ if $x < \alpha$ in $X$.
(2) For any minimal $\beta \in Y$,let $\beta'$ be the copy of $\beta$ in $X'$. Then $x' < \beta'$ if $x > \beta$ in $X$.
(3) If $x_1$ and $x_2$ are both neither maximal nor minimal, then $x_1'$ and $x_2'$ are incomparable in $X'$.

This construction gives a finite poset $X'$ with no more than $\#X$ many points, and the third condition restricts the height of the poset to be no bigger than 3. Now we claim that $E(\mathscr{K}(X), x_0)$ is isomorphic to $E(\mathscr{K}(X'), x_0')$.

To show this, let us define a map

$$\phi : E(\mathscr{K}(X'), x'_0) \longrightarrow E(\mathscr{K}(X), x_0)$$
$$[(x'_0, x'_1)(x'_1, x'_2) \dots (x'_{n-1}, x'_0)] \mapsto [(x_0, x_1)(x_1, x_2) \dots (x_{n-1}, x_0)].$$

This map is well defined for the following reason: if

$$\epsilon'_1\epsilon'_2 \dots (x', y')(y', z') \dots \epsilon'_n \approx \epsilon'_1\epsilon'_2 \dots (x', z') \dots \epsilon'_n$$

in $(\mathscr{K}(X'), x'_0)$, then $x', y'$ and $z'$ are within a triangle, which implies that $x, yandz$ are also within a triangle since the relation on $X'$ corresponds to a subset of the relation on $X$. Thus in $(\mathscr{K}(X), x_0)$, we also have

$$\epsilon_1\epsilon_2 \dots (x, y)(y, z) \dots \epsilon_n \approx \epsilon_1\epsilon_2 \dots (x, z) \dots \epsilon_n$$

Note that this map is a homomorphism by construction.

To show that the two groups are actually isomorphic, we want to define another map in the reverse direction:

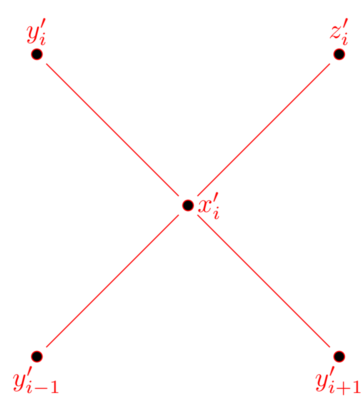
$$\psi : E(\mathscr{K}(X), x_0) \longrightarrow E(\mathscr{K}(X'), x'_0),$$

Let $\psi$ work as follows: for any edge-loop class $[\sigma] \in E(\mathscr{K}(X, x_0))$, first use the equivalence move from Definition 20.2.2 to shrink the edge-loop $\sigma$ to the stage $(x_0, x_1)(x_1, x_2) \dots (x_{n-1}, x_0)$ where $x_i$ is bigger than both $x_{i-1}$ and $x_{i+1}$ for all odd $i$'s and smaller than both $x_{i-1}$ and $x_{i+1}$ for all even $i$. Now for each $x_i$ with odd $i$, pick a maximal point $y_i \geq x_i$, and similarly pick a minimal point $y_i \leq x_i$ for all the even $i$. Collect all the $y_i$ in order into an edge-loop $\xi$ at $x_0$. It is not hard to see that $\xi$ is equivalent to $\sigma$ in $(\mathscr{K}(X), x_0)$. Then since $\xi$ is an edge-loop that consists of edges only with maximal and minimal vertices, it has a copy of it in $(\mathscr{K}(X'), x'_0)$, say $\xi'$. Now we simply set $\psi([\sigma]) = [\xi']$.

We need to show this map is well defined, i.e., the image does not depend on the maximal and minimal that are chosen nor the representative of the edge-loop class in $E(\mathscr{K}(X), x_0)$.

Without loss of generality, we are going to just look at the case when $i$ is odd. Suppose we choose the a different maximal $z_i$ instead of $y_i$. Then within $(\mathscr{K}(X'), x'_0)$:

$$\begin{aligned} &(x'_0, y'_1)(y'_1, y'_2) \dots (y'_{i-1}, y_i)(y'_i, y'_{i+1}) \dots (y'_{n-1}, x'_0) \\ \approx\ &(x'_0, y'_1)(y'_1, y'_2) \dots (y'_{i-1}, x'_i)(x'_i, y'_{i+1}) \dots (y'_{n-1}, x'_0) \\ \approx\ &(x'_0, y'_1)(y'_1, y'_2) \dots (y'_{i-1}, z'_i)(z'_i, y'_{i+1}) \dots (y'_{n-1}, x'_0). \end{aligned}$$

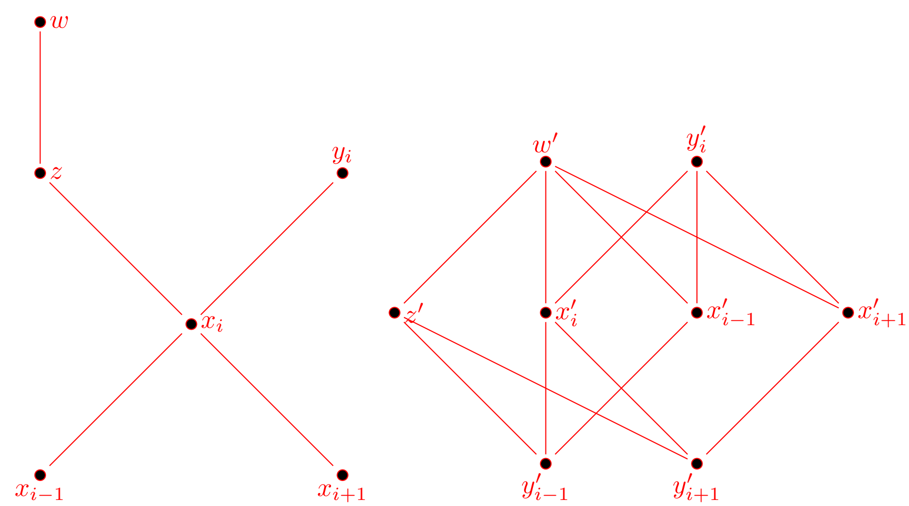


Therefore, the image does not depend on the maximal and minimal points that are chosen.

Now to show that $\psi$ does not depend on the choice of representative, we can just check that the image does not change after the equivalence move from Definition 20.2.2. Notice that $\sigma$ is already assumed to be shrunk to the least, and therefore it is impossible to use one move to combine two edges into one. On the other hand, if we replace the edge $(x_{i-1}, x_i)$ by two other edges within a triangle, say $(x_{i-1}, z)(z, x_i)$, and assume without loss of generality that $x_{i-1} < x_i$, then one the following three is going to be true: $z < x_{i-1}$, $x_{i-1} \le z \le x_i$ or $x_i < z$.

If it is the case where $x_{i-1} \le z \le x_i$, then it does not change the image at all, since after shrinking we would get back $\sigma$.

The other two cases are analogous, and we are just going to consider the case where $x_i < z$. (Note that even in this situation, we have not mess up the alternating order of maximal points and minimal points, since $z$ will be the new maximal element instead of $x_i$, and the representative, after shrinking, will be $\sigma = x_0 x_1 \dots x_{i-1} z x_{i+1} \dots x_0$.) Now if the $y_i$ that we chose before is also greater than or equal to $z$, then we are set, because we can use $y_i$ again for the maximal that extends $z$. The only "bad" situation is the left hand side of the following diagram, when $z$ is incomparable with $y_i$: □




But this is actually not bad at all, since as we can see in $(\mathscr{K}(X'), x'_0)$,

$$\begin{aligned}
&(x'_0, y'_1)(y'_1, y'_2) \dots (y'_{i-1}, y_i)(y'_i, y'_{i+1}) \dots (y'_{n-1}, x'_0)\\
\approx\ &(x'_0, y'_1)(y'_1, y'_2) \dots (y'_{i-1}, x'_i)(x'_i, y'_{i+1}) \dots (y'_{n-1}, x'_0)\\
\approx\ &(x'_0, y'_1)(y'_1, y'_2) \dots (y'_{i-1}, w'_i)(w'_i, y'_{i+1}) \dots (y'_{n-1}, x'_0).
\end{aligned}$$

Therefore, the image does not depend on the representative of the edge-loop class in $E(\mathscr{K}(X), x_0)$ either, and we conclude that the map $\psi$ is well defined.

Note that $\psi$ is also a homomorphism by construction. Moreover, if we take any edge-loop class $[\sigma] \in E(\mathscr{K}(X), x_0)$, we can choose the representative $\sigma$ to contain only maximal and minimal points, say $\sigma = (x_0, y_1)(y_1, y_2) \dots (y_{n-1}, x_0)$. Then we have

$$\phi \circ \psi([\sigma]) = \phi\left([(x'_0, y'_1)(y'_1, y'_2) \dots (y'_{n-1}, x'_0)]\right) = [((x_0, y_1)(y_1, y_2) \dots (y_{n-1}, x_0))].$$

Conversely, if we take any edge-loop class $[\xi'] \in E(\mathscr{K}(X'), x'0)$, we can also choose the representative $\xi$ to contain only maximal and minimal points, say $\xi' = (x'_0, y'_1)(y'_1, y'_2)\dots(y'_{n-1}, x'_0)$. Then we have

$$\psi \circ \phi([\xi']) = \psi\left([(x_0, y_1)(y_1, y_2)\dots(y_{n-1}, x_0)]\right) = [\left((x'_0, y'_1)(y'_1, y'_2)\dots(y'_{n-1}, x'_0)\right)].$$

From these evaluations, we see that $\phi$ and $\psi$ are actually inverses of each other, and hence $E(\mathscr{K}(X), x_0)$ is isomorphic to $E(\mathscr{K}(X'), x'_0)$. By Corollary20.3.2, we deduce immediately that $(X, x_0)$ and $(X', x'_0)$ have isomorphic fundamental group.

Having fully explored the case of finitely generated free groups, we turn attention to non-free groups with finite presentations. Note that the fundamental group of a poset with height no more than 2 is free, thus we are only focusing on finite realizations with a height of exactly 3.

A finite poset with a height of 3 is has unrelated middle points, enabling easier computation. Recall from Proposition 20.3.1 that, for a finite based poset $(X, x_0)$ with the subspace of extremals $Y$ (assume $x_0$ is minimal), there is an epimorphism from $\pi_1(Y, x_0)$ onto $\pi_1(X, x_0)$, and since $h(Y) \leq 2$, $\pi_1(Y, x_0)$ is free. Hence the subspace $Y$ gives us the generators of the fundamental group, and the middle points induce relations on $\pi_1(Y, x_0)$ making it identical to $\pi_1(X, x_0)$ (equivalently, making $E(\mathscr{K}(Y), x_0)$ into $E(\mathscr{K}(X), x_0)$).

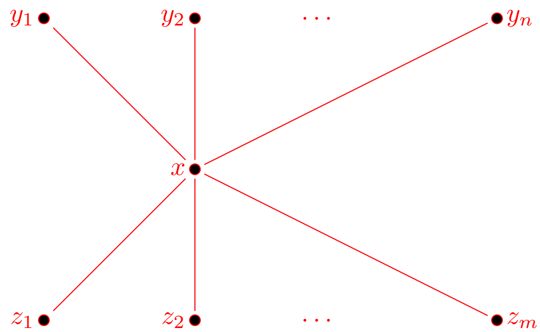
Recalling the notion of upbeat and downbeat points, a middle point that is either upbeat or downbeat can be removed without changing the weak homotopy type. Hence, in this instance, the fundamental group remains unchanged. Thus, when we try to realize certain group with as few points as possible, all middle points in the realization should be connected to at least minimal points and two maximal points. But then this implies that in $\mathscr{K}(X)$, any edge that contains a middle point must belong to a triangle, and adding a middle point is the same as gluing triangles onto $\mathscr{K}(Y)$.

Now suppose that there are two equivalent edge-loops $\xi$ and $\xi'$ in $\mathscr{K}(X)$ at $x_0$ that consist of edges between extremals only (i.e. edge-loops that are originally in $\mathscr{K}(Y)$). By definition of edge-loop equivalence, there exists a finite sequence of edge-loops $\{\xi_i | 0 \leq i \leq n\}$ at $x_0$ such that $\xi = \xi_0$ and$\xi' = \xi_n$, and $\xi_{i+1}$ is obtained by applying the equivalence move we defined in Definition 20.2.2 to $\xi_i$. Since we know that each move must take place within one triangle, thus each move, which can be viewed as a relation, is induced by only one triangle. But since each triangle only has vertices of one maximal, one minimal and one middle point $x$ only, it also exists in $Y \cup \{x\}$. Therefore, we have the following proposition:

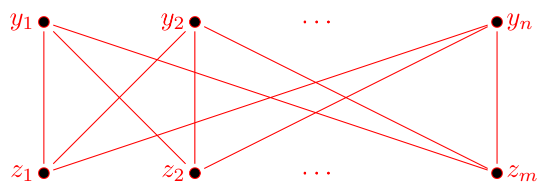
**Proposition 20.4.2.** *Let $(X, x_0)$ be a finite poset with $h(X) \leq 3$ and let $(Y, x_0)$ be the subspace of extremals (assuming $x_0$ is a minimal). Suppose $x_1, x_2, \dots, x_n$ are the middle points in $X$. Then we can look at the subspace $Y \cup \{x_i\}$ for each $1 \leq i \leq n$ and consider the relations $x_i$ induces on $\pi_1(Y, x_0)$. Let $r^i_1 r^i_2 \dots r^i_{m_i}$ be these relations. Then*

$$\pi_1(X, x_0) \cong \pi_1(Y, x_0) / \bigcup_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m_i}} \{r^i_j\}.$$

We examine the relations a single middle point induces on $\pi_1(Y, x_0)$ (equivalently, $E(\mathscr{K}(Y), x_0)$). Note that we restrict interest to middle points connected to at least two maximals, the format of which is displayed in the following Hasse diagram:




The corresponding part in the subspace $Y$ can be visualized in the Hasse diagram below:




As pictured, if we apply the $\mathscr{K}$ functor to this part of the subspace $Y$, we obtain a complete bipartite graph $(\{y_i\}, \{z_j\})$, and any two maximals with any two minimals form a loop. After we include the middle point $x$, all these loops become trivial. Therefore, the relations that the middle point x induces on $E(\mathscr{K}(Y), x_0)$ are just

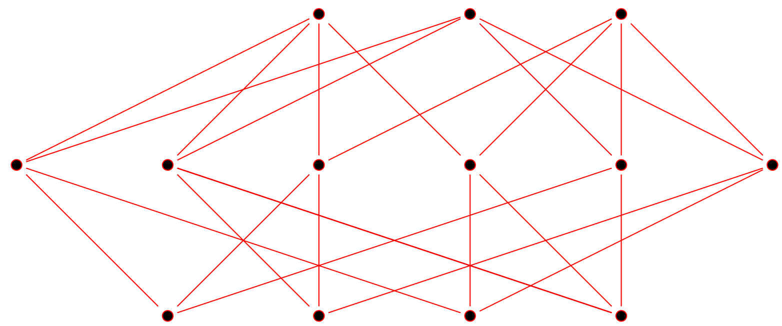
$$y_{i_1} z_{j_1} y_{i_2} z_{j_2} y_{i_1} = e, \quad 1 \leq i_1, i_2 \leq n \text{ and } 1 \leq j_1, j_2 \leq m$$

Thus, a the fundamental group of a finite space $X$ with $h(X) = 3$ can be either computed using the $H$-group or in the way outlined above. One might write a program according to this method that computes the smallest size of a height 3 poset needed to realize a certain group with finite presentation. The significance of this observation is expounded upon in the following remark.

**Remark 20.4.3.** Let $G$ be any finitely presented group. If $X$ is one of the smallest finite posets of height 3 and $\pi_(X) \simeq G$, then $X$ is a minimal finite model of $|\mathscr{K}(X)|$. This is because if $Z$ is another minimal finite model of $|\mathscr{K}(X)|$, then $Z$ can be reduced to $Z'$ according to the previous theorem. Additionally, $Z'$ also has $G$ as its fundamental group, and by assumption we know that $\#X \leq \#Z' \leq \#Z$. Therefore, $X$ is a minimal finite model of $|\mathscr{K}(X)|$.

The remark above leads to the following conjecture about minimal finite models of $\mathbb{R}P^2$.

**Conjecture 20.4.4.** *The smallest finite posets of height 3 that realize $\mathbb{Z}_2$ have cardinality 13, and the following one is a minimal finite model of $\mathbb{R}P^2$.*



CHAPTER 21

# Covers of Finite Spaces

Covering spaces are important objects in a variety of areas of mathematics. As the investigation into finite spaces and their properties continues, information about what covers of these spaces look like and how to construct them could become useful. One way of searching for this information is to look for similarities between covers of finite spaces and covers of spaces we are accustomed to working with. In this paper, we will investigate the relationship between the wedge of two circles and the 5-point space weakly homotopy equivalent to it. Later, we will suggest an intuitive way to find covers for any height-2 poset by looking at other wedges of circles.

## 21.1. Introduction

One of the problems mathematicians face as they explore the new territory of finite spaces is how to classify them. A useful way to differentiate between some spaces is to calculate their fundamental groups, which give us an understanding of the "holes" in a pointed topological space by taking as elements the equivalence classes of loops from a chosen basepoint in the space. However, it is difficult to intuitively understand what a loop in a finite space would look like, and it can be difficult to calculate the fundamental groups of finite spaces.

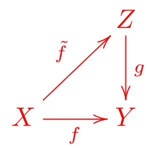
One of the reasons covering spaces are so useful is that they are deeply connected to the fundamental group. If a space $X$ is path-connected, locally path-connected, and semi-locally simply connected, there is a *Galois correspondence* between the covers of $X$ and the subgroups of its fundamental group. We will show that any connected finite space is also path-connected and locally contractible. It follows that any connected finite space is path-connected, locally path-connected, and semi-locally simply connected, so we have the Galois correspondence for finite spaces.

### 21.1.1. Covering spaces.

Given a space $X$, a cover is intuitively a larger space $\tilde{X}$ which can be projected neatly into $X$ in such a way that locally $\tilde{X}$ can be regarded as a stack of pancakes.

**Definition 21.1.1.** A *cover* of a space $X$ is a space $\tilde{X}$ and a map $p : \tilde{X} \longrightarrow X$ such that for each point $x$ in $X$, there is an open neighborhood $U$ of $x$ where $p^{-1}(U)$ is the union of disjoint open sets in $\tilde{X}$, and $p$ maps each of these sets homeomorphically onto $U$.

**Definition 21.1.2.** A *lift* of $f : X \longrightarrow Y$ along $g : Z \longrightarrow Y$ is a map $\tilde{f} : X \longrightarrow Z$ such that $g \circ \tilde{f} = f$.




**Theorem 21.1.3.** *Given a cover $p : \tilde{X} \longrightarrow X$, a homotopy $f_t : Y \longrightarrow X$, and a map $\tilde{f}_0 : Y \longrightarrow \tilde{X}$ lifting $f_0$, there exists a unique homotopy $\tilde{f}_t : Y \longrightarrow \tilde{X}$ of $\tilde{f}_0$ that lifts $f_t$.*

Proofs of this theorem and the ones below can be found in Allen Hatcher's Algebraic Topology . We will only need this theorem to lift paths, not any larger spaces, because none of the objects we will be working with in this paper will have dimension greater than 1.

Now, the Galois correspondence between the covers of a space $X$ and the subgroups of its fundamental group gives us the following:

**Theorem 21.1.4.** *Suppose $X$ is path-connected, locally path-connected, and semilocally simply-connected. Then there is a bijection between the set of basepoint-preserving isomorphism classes of path- connected covering spaces $p : (\tilde{X}, \tilde{x_0}) \longrightarrow (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$. This bijection is obtained by associating each subgroup $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ to the cover $(\tilde{X}, \tilde{x_0})$. If basepoints are ignored, this correspondence gives a bijection between isomorphism classes of path-connected covering spaces $p : \tilde{X} \longrightarrow X$ and conjugacy classes of subgroups of $\pi_1(X, x_0)$.*

Notice that if $K$ is a subgroup of $H$, then the space corresponding to $K$ will cover the space corresponding to $H$. This means that the bijection in the previous theorem is order reversing. Also note that the fundamental group of a space $X$ must have a trivial subgroup, which is a subgroup of every other subgroup, so there must be a cover of $X$ that covers every other cover. This is called the *universal cover of* $X$, and it is unique up to isomorphism. In fact, a space need only be locally path-connected and semi-locally simply connected to have a universal cover, and the proof of the previous theorem actually uses the existence of a universal cover.

This theorem is also stated as an *equivalence of categories* in Theorem which will give us a general relationship between the covers of any two weakly homotopy equivalent spaces. However, the aim of this paper is to suggest an explicit geometric relationship between covers of weakly equivalent spaces. We will describe this explicit relationship for the wedge of circles and the 5-point space weakly equivalent to it.

The following theorem will provide some intuition about the relationship between posets (which we will see are equivalent to finite spaces) and wedges of circles, if both are considered as graphs.

**Theorem 21.1.5.** *For a connected graph $X$ with maximal tree $T$, $\pi_1(X)$ is a free group with basis the classes of loops $[f_\alpha]$ corresponding to the edges $e_\alpha$ of $X - T$ .*

This makes sense intuitively because a tree has no non-trivial loops and can be retracted to a single vertex. Collapsing a maximal tree in a connected graph leaves one vertex with a bouquet of edges, forming a wedge of circles, and the fundamental group of a wedge of $\kappa$ circles is the free group on $\kappa$ generators. Because trees are contractible, collapsing a maximal tree is a homotopy equivalence. This means that the fundamental group of $X$ is the same as the fundamental group of the wedge of circles made up of the edges left over when the maximal tree is collapsed. These edges are exactly the edges $e_\alpha$ of $X - T$ .

**Proposition 21.1.6.** *For any connected finite space $X$, there is a Galois correspondence between the covers of $X$ and the subgroups of its fundamental group.*

PROOF. Since $X$ is connected, it is path-connected, and $X$ is locally path-connected because its connected components and path components coincide. By the previous lemma, $X$ is locally contractible and hence semi-locally simply connected, so theorem 21.1.4applies. □

21.1.1.1. *Formalizing our claim.* To prove that the categories of covers of weakly equivalent spaces are equivalent, we will define a functor $E : \mathcal{O}(\pi_1(X, x_0)) \longrightarrow Cov(X)$ from the *orbit category* of the fundamental group of a space $X$ to $Cov(X)$. This functor will be the same as mapping from the category of subgroups of the fundamental group of $X$ to the category of isomorphism classes of path-connected covers of $X$, which is just the Galois correspondence. Any two weakly homotopy equivalent spaces $X$ and $Y$ have isomorphic orbit categories, and we achieve the desired equivalence between $Cov(X)$ and $Cov(Y)$ because categorical equivalence is an equivalence relation.

Before defining the orbit category, we need to define the constructions that will be its objects and morphisms. A *left action* of a group $G$ on a set $X$ is a function $G \times S \longrightarrow S$ such that $ex = x$ for all $x$ in $X$ and $(gh)x = g(hx)$ for all $g$ and $h$ in $G$ and $x$ in $X$. An action is *transitive* iff or all $x$ and $y$ in $X$, there is an element $g$ in $G$ such that $gx = y$. If $H$ is a subgroup of $G$, the set $G/H$ of cosets $gH$ is a *transitive $G$-set*. An equivariant map is a map $\alpha : G/H \longrightarrow G/K$ such that $\alpha(gx) = g\alpha(x)$. This means that $\alpha$ commutes with the action of $G$. Also, if there is an equivariant map as above, then $H$ is subconjugate to $K$, meaning that there is an element $g$ in $G$ such that $g^{-1}Hg$ is a subgroup of $K$. Finally, the *orbit* generated by $x$ is $\{gx : g \in G\}$.

**Definition 21.1.7.** The category $\mathcal{O}(G)$ has as objects the canonical orbit $G$-sets $G/H$, and as morphisms $G$-equivariant maps.

Proofs of the following two theorems can be found in

**Theorem 21.1.8.** *The category $\mathcal{O}(G)$ is isomorphic to the category $G$ whose objects are the subgroups of $G$ and whose morphisms are the distinct subconjugacy relations $\gamma^{-1}H\gamma \subset K$ for $\gamma$ in $G$.*

This means that we can think of the functor $E$ between the category of covers of a space $X$ and the orbit category of its fundamental group as building off the Galois correspondence between the covers of $X$ and the subgroups of its fundamental group.

The following theorem will give us the equivalences we need to prove the desired result.

**Theorem 21.1.9.** *Choose a basepoint $b$ in $B$. There is a functor*

$$E : \mathcal{O}(\pi_1(B, b)) \longrightarrow Cov(B)$$

*that is an equivalence of categories. Let $G = \pi_1(B, b)$. For each subgroup $H$ of $G$, the cover $p : E(G/H) \longrightarrow B$ has a canonical basepoint $e$ in its fiber over $b$ such* *that*

$$p_* : (\pi_1(E(G/H), e)) = H.$$

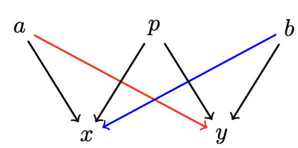
*Also,* $F_b \cong G/H$ *as a* $G$*-set, and for a* $G$*-map* $\alpha : G/H \longrightarrow G/K$ *in* $\mathcal{O}(G)$*, the restriction of* $E(\alpha) : E(G/H) \longrightarrow E(G/K)$ *to fibers over* $b$ *coincides with* $\alpha$*.*

Given any finite space weakly homotopy equivalent to a well-known space, we can get an equivalence between the isomorphism classes of covers of the finite space and of the well-known space. However, by taking this route to prove the categorical equivalence, we lose the geometric intuition behind our investigation into the connection between covers of weakly homotopy equivalent spaces. To recover the intuition motivating this high level categorical proof, we will consider the wedge of two circles and the space $W$, depicted below, which is a finite space weakly homotopy equivalent to $S^1 \vee S^1$. We will describe how to construct two other functors, Thin $: Cov(W) \longrightarrow Cov(S^1 \vee S^1)$ and $Thick : Cov(S^1 \vee S^1) \longrightarrow Cov(W)$, and use them to move between examples in $Cov(W)$ and $Cov(S^1 \vee S^1)$ to demonstrate an explicit equivalence between these two categories.

## 21.2. Finding an explicit equivalence

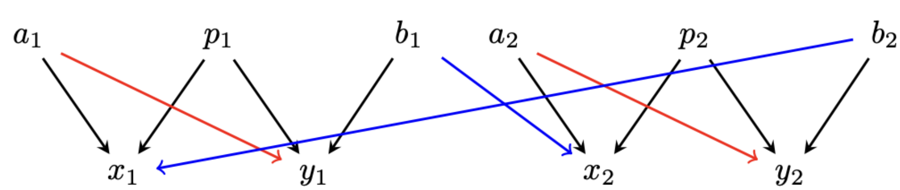
Although $Cov(W)$ and $Cov(S^1 \vee S^1)$ are equivalent by 21.1.9, it is a worthwhile exercise to reformulate the equivalence in a more intuitive way. We will suggest two pseudo-inverse functors, a thinning map and a thickening map that are constructed according to the method discussed informally below.

**21.2.1. Covers of the 5-point space.** Consider the following 5-point space, $W$, which is weakly homotopy equivalent to the wedge of two circles.



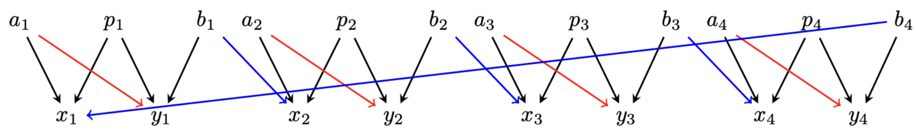

If we consider the two zigzags $p < x > a < y > p$ and $p < x > b < y > p$, then the first is a loop containing the red edge, and the second is a loop containing the blue edge.

We can construct a 2-fold cover of $W$ by connecting two copies of the space as follows.

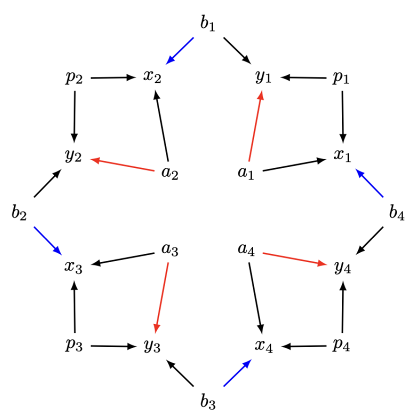



It is easy to imagine simply picking up one of the two $W$ shapes, shifting it over so that its points match up with the points in the other $W$ shape, and flattening it to get the original space.

Similarly, we can construct a 3-fold cover by taking three copies of $W$ without the blue zigzag $b < x$, and then connecting $b_i$ to $x_{i+1}$ for $i = 1, 2$ and $b_3$ to $x_1$. A 4-fold cover, shown below, can be constructed in the same way by taking four copies of $W$ without the blue zigzag and connecting them with the blue zigzags $b_i < x_{i+1}$ for $i = 1, \ldots, 3$ and $b_4 < x_1$. It is easy to see that an $n$-fold cover of this space may be constructed by stringing together $n$ of these $W$-shaped "beads" using the blue edges.

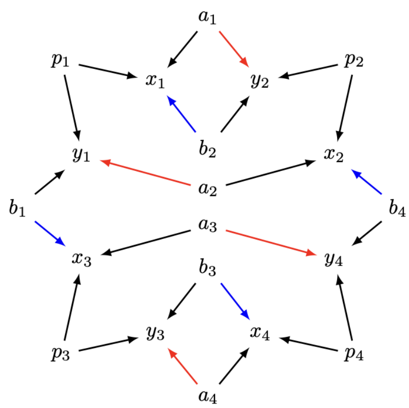



The symmetry of this 4-fold cover is clearer if it is drawn planar:

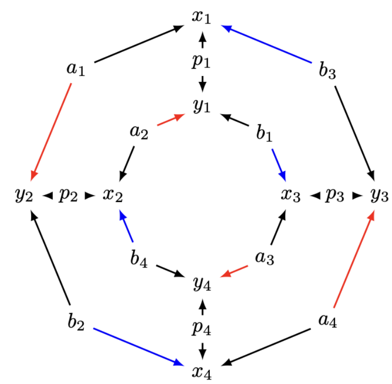



A

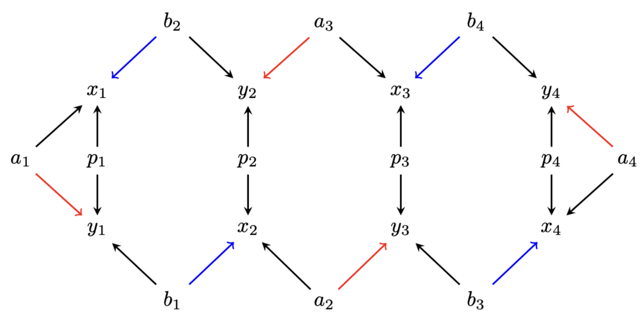
Three more 4-fold covers follow:




B



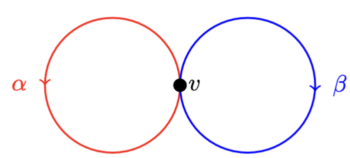

C



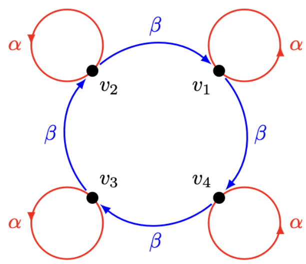

D

Now that we have found several 4-fold covers and a way to construct an $n$-fold cover of $W$ for any $n$, it makes sense to ask whether there is a simple way to construct all covers of this space. To address this question, we will look at the wedge of two circles.

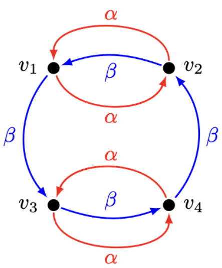
**21.2.2. Finding a relationship to the wedge of circles.** Depicted below is the wedge of two circles. Note that we have colored the edges to distinguish the two generators $\alpha$ and $\beta$ and have given them orientations.



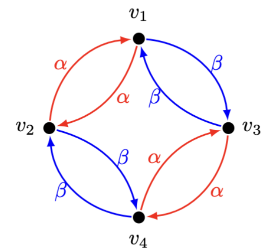

Four 4-fold covers of the wedge of two circles are shows below.



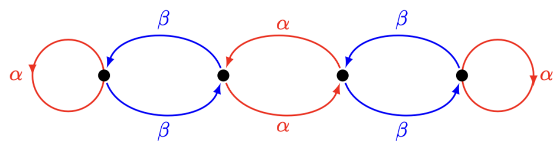

1




2




3




4

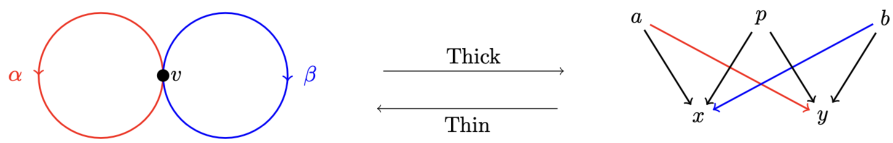
There are some immediate visual parallels between these covers and those of $W$ in the previous section. On one hand, looking at the covers of the wedge of circles as graphs, each vertex has one red edge and one blue edge going in, and one red edge and one blue edge going out. On the other hand, each zigzag $a_i < x_i > p_i < y_i > b_i$ is connected to one red edge and one blue edge pointing in, and one red edge and one blue edge pointing out.

If we think of collapsing the points in $W$ along $a < x > p < y > b$, then $a < y$ and $b < x$ would correspond to the red and blue generators of the wedge of two circles. This fits with theorem21.1.5 because $a < x > p < y > b$ is a maximal tree in $W$,and $W$ has the same fundamental group as the wedge of two circles. Similarly, we can collapse the black edges in covers $A, B, C$, and $D$ above to get 1, 2, 3, and 4, respectively.

Now, if we think of the wedge of two circles and its covers as graphs, it becomes clear that for every node, we will have a zigzag $a_i < x_i > p_i < y_i > b_i$ in the corresponding poset cover of $W$. We already know that generator $\alpha$ corresponds to the zigzag containing $a < y$ in $W$ , $\beta$ corresponds to the zigzag containing $b < x$, and the direction of each generator is preserved by which point is reached first in the zigzag. Therefore, given a cover of the wedge of two circles, we need only turn each point in the cover into five points, endow these points with the appropriate ordering, and connect the colored edges to the correct points to create an analogous cover of $W$. We shall turn to the formalism of category theory to show how this correspondence between the covers of $W$ and the covers of $S^1 \vee S^1$ can be made more precise.

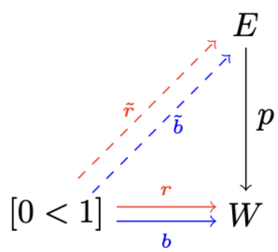
**21.2.3. Constructing thinning and thickening functors.** In this section, we will not be so concerned with proofs and rigorous definitions, as making these two functors precise is a quite lengthy and tedious process, and our aim is to give geometric intuition for what is happening in these categories.

Our thinning map will collapse the black edges in a cover of $W$ , leaving only red and blue edges and forming a corresponding cover of the wedge of circles, and our thickening map will turn each vertex in a cover of $S^1 \vee S^1$ into five points connected by four black edges.



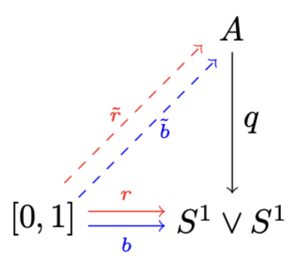

For this geometric method to work, we will have to color the edges of $S^1 \vee S^1$ and $W$ to keep track of them, and these colors will lift to color the edges of the covers of $S^1 \vee S^1$ and $W$.

**Definition 21.2.1.** Let $W$ be the 5-point space weakly homotopy equivalent to the wedge of two circles with points $a, p, b, x,$ and $y$. Of the six edges $a < x, a < y, p < x, p < y, b < x$, and $b < y$ in $W$, one is labeled by $r : \{0,1\} \longrightarrow W$, one by $b : \{0,1\} \longrightarrow W$, and the rest are simply included into $W$, just as we have marked the red and blue zigzags above. The objects in the category $Cov(W)$ are covers of $W$ with points $\{a_i, p_i, b_i, x_i, y_i : i \in \mathscr{I}\}$ for some index set $\mathscr{I}$ and edges labeled by $\tilde{r}$ and $\tilde{b}$:



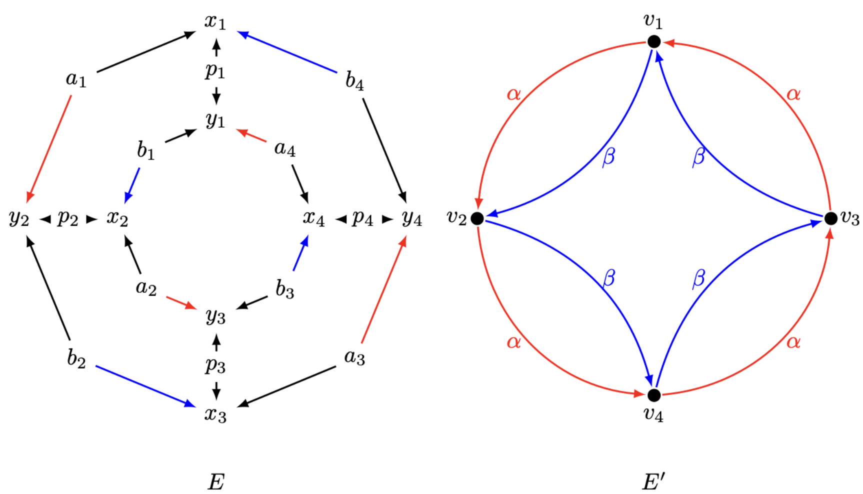

The lifts $\tilde{r}$ and $\tilde{b}$ are not unique. The number of red edges in $E$ equals the number of blue edges in $E$, which equals the degree of the cover.

Similarly, the generators of the wedge of circles are labeled with maps $r : [0,1] \longrightarrow S^1 \vee S^1$ and $b : [0,1] \longrightarrow S^1 \vee S^1$, and we will call the point of intersection $v$. The objects in $Cov(S^1 \vee S^1)$ are covers of the wedge of circles labeled by the lifts $\tilde{r}$ and $\tilde{b}$ with points of intersection $p^{-1}(v)$.



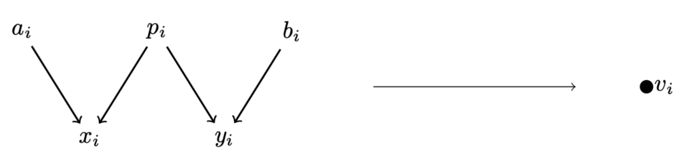

Again, there are the same number of red and blue lifts in $A$, and this is equal to the degree of the cover. The maps $r$ and $b$ into $W$ or $S^1 \vee S^1$ are the same for all the objects in each category of covers. We will call an edge or zigzag (depending on the context) red if it is a lift of $r$ and blue if it is a lift of $b$.

Now we would like to show how the thinning and thickening functors work with a particular example. We will consider $W$, $S^1 \vee S^1$, and their covers as directed graphs when working through the example. Pick a cover $p : E \longrightarrow W$ and a cover $q : E' \longrightarrow S^1 \vee S^1$ that we think will match up.



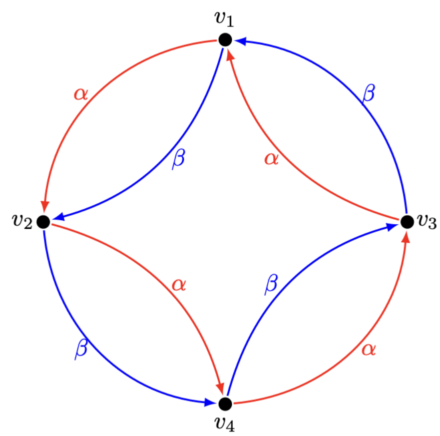

These two covers look similar to cover $C$ in section 21.2.1 and cover 3 in section 21.2.2, but notice that the points are labeled differently in $E$, and the edges are colored differently in $E'$.

First we will thin $p : E \longrightarrow W$ to get what we claim will be a cover of $S^1 \vee S^1$. The idea is to collapse the points $a_i, x_i, p_i, y_i$, and $b_i$ to a single vertex $v_i$, and to throw out the edges forming the $W$ between them. Then only colored edges will remain.




Formally, we could do this by defining the vertex set of the graph given by Thin $(p)$ using an equivalence relation that identifies two points if they are connected by a black edge, and by taking the edge set given by Thin $(p)$ to include only the edges that are red or blue in $E$. The source of each red edge $a_i \longrightarrow y_j$ in $E$ will map to $v_i$, and the target will map to $v_j$. This will ensure that the direction of the edge is preserved. Similarly, each blue edge $b_i \longrightarrow x_j$ in $E$ will have its source mapped to $v_i$ and its target to $v_j$.

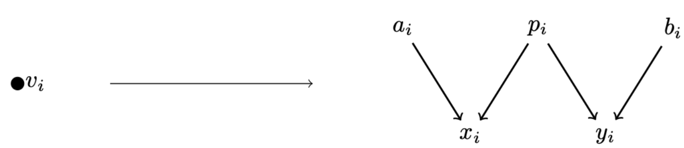
If we follow these instructions, we get that Thin$(p)$ maps the following space down to $S^1 \vee S^1$.




$E'$

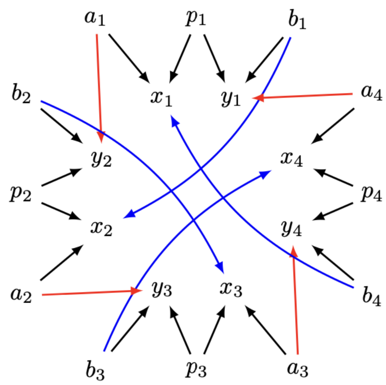
First we must check that this is indeed a cover of $S^1 \vee S^1$. Since we have transferred our work to the category of directed graphs, our computations are combinatorial instead of topological. The graph above has four vertices, and each is the source of one red and one blue edge and the target of one red and one blue edge, so it is a cover of $S^1 \vee S^1$. Furthermore, it is isomorphic to $E'$: the two edges between $v_1$ and $v_3$ have switched places and the two edges between $v_2$ and $v_4$ have also switched places.

Now we will apply Thick to $q$ to get what we claim will be a cover of $W$ isomorphic to $E$. Here, we wish to expand each vertex $v_i$ to five points $a_i, x_i, p_i, y_i$ with edges between them as follows:

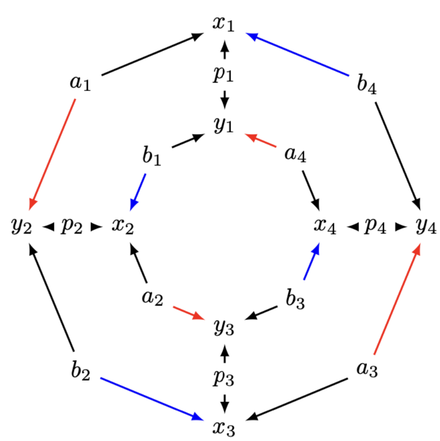



If $V$ is the vertex set of $E'$, we could do this formally by taking the vertex set of the graph given by Thick$(q)$ to be $\{a, p, b, x, y\} \times V$ . Then we would have five points $\{a_i, p_i, b_i, x_i, y_i\}$ for every vertex in $E$. To define the edge set given by T$hick(q)$, we would need to take the edge set of $E'$, call it $D$, and add in four edges in for every vertex in $E'$. If we call the four black edges in $W$ $e_1, e_2, e_3$, and $e_4$, then the edge set given by Thick$(q)$ would be $D \coprod (\{e_1, \ldots, e_4\} \times V)$. Now, we have to be careful about where we connect up our red and blue edges. If there is a red edge $v_i \longrightarrow v_j$ in $E'$, then Thick$(q)$ would take this edge to a red edge with $a_i$ as its source and $y_j$ as its target. Similarly, a blue edge $v_i \longrightarrow v_j$ in $E'$ would map to a blue edge starting at $b_i$ and ending at $x_j$.

Following these rules, Thick$(q)$ is a map sending the following space down to $W$.




Although the cover may look messy when arranged like this, it is easy to see that each black $W$ is the source of one blue and one red edge and the target of one blue and one red edge. This means that it is indeed a cover of $W$. Also, each black $W$ has a red and a blue edge coming in from another black $W$, and a red and a blue edge going out to a different black $W$. This is exactly true of $E$, and it is not too difficult to see how the cover above can be unwound to form $E$ as we depicted it above.




We have shown that $\text{Thin}(p) \cong q$ and $\text{Thick}(q) \cong p$. The process of using the functors Thick and Thin to move between corresponding covers of $W$ and $S^1 \vee S^1$ will be the same for other covers of these two spaces, and if the functors are carefully defined, they will be pseudo-inverses. This means that we could get natural isomorphisms

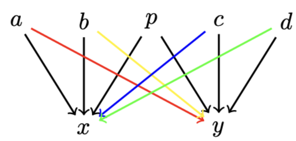
$$\eta : (\text{Thin} \circ \text{Thick}) \Rightarrow id_{Cov(S^1 \vee S^1)}$$

$$\epsilon : id_{Cov(W)} \Rightarrow (\text{Thick} \circ \text{Thin}).$$

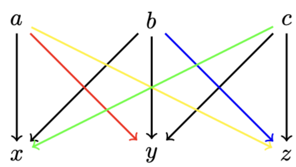
that give the desired equivalence of categories.

**21.2.4. Extension to other finite spaces.** By replacing $W$ and $S^1 \vee S^1$ with any other spaces that are weakly homotopy equivalent, we get an equivalence between the categories of their coverings spaces by Theorem 21.1.9, but the more intuitive, explicit definitions of Thick and Thin also extend to some other spaces.

We can get a finite space weakly homotopy equivalent to a wedge of any finite number of circles if we simply add a point of height 2 to the 4-point circle. For example, the following poset has two additional points, $c$ and $d$, and it is weakly homotopic to wedge of four circles. The zigzags corresponding to the four generators are colored.




However, there are multiple ways to form a poset weakly homotopy equivalent to a wedge of certain numbers circles. For example, the following 6-point space also corresponds to the wedge of four circles.




By theorem 21.1.5, if $X$ is a graph containing a subtree $T$ , then $X$ is homotopy equivalent to $X/T$ . If we consider the previous two posets as graphs, the black edges form maximal trees, and the graphs retract to wedges of four colored circles.

We claim that thinning and thickening functors can be defined using maximal trees in any height-2 poset $W'$ to associate its covers with the covers of the appropriate wedge of circles $\bigvee_\kappa S^1$. Proving that $Cov(W')$ and $Cov(\vee_\kappa S^1)$ are equivalent using the thinning and thickening functors appropriate for these categories would be a very long and involved process. However, these functors are useful purely for the lovely geometric connection they formalize between the two categories.

CHAPTER 22

# Homotopy Theory for Subdivision and $A$-Space Models

The respective works of Clader and Thibault allow us to consider arbitrary topological spaces as represented by a system of subdivided models. Formally, our first result says that given a locally finite $A$-space $X$, its realization $|\mathscr{K}(X)|$ is homotopy equivalent to the inverse limit of the directed system:

$$\cdots \xrightarrow{\inf} (\mathrm{Sd}^2 X)^{op} \xrightarrow{\inf} (\mathrm{Sd}X)^{op} \xrightarrow{\inf} X^{op} .$$

After establishing this result, we turn our attention away from topological spaces and instead focus on modeling continuous functions. While we can associate to every CW complex a partially ordered set, the category of posets suffers from a "deficiency of morphisms" that limits our ability to model functions in the category of topological spaces. To remedy this issue, we follow the work of Hardie and Vermeulen as well as Thibault to formulate an elegant relationship between $A$-spaces and their geometric realizations using the subdivision functor.

The main result establishes that, given a finite $A$-space $X$ and an arbitrary $A$-space $Y$, there is a natural bijection between the colimit of the system:

$$[X, Y] \xrightarrow{\sup^*} [\mathrm{Sd}X, Y] \xrightarrow{\sup^*} [\mathrm{Sd}^2 X, Y] \xrightarrow{\sup^*} \cdots$$

and $[|\mathscr{K}(X)|, |\mathscr{K}(Y)|]$. In plain terms, this result allows us to model continuous maps between topological spaces with order-preserving maps between posets, up to homotopy. These two results in conjunction effectively justify that the category of $A$-spaces is an appropriate place in which to study algebraic topology.

To add algebraic importance to these results, we develop the final piece of this result. In general, an isomorphism of sets is the strongest relationship we can hope for between this colimit and the homotopy classes of maps between the realizations of two $A$-spaces, since for arbitrary topological spaces $X$ and $Y$, $[X, Y]$ need not be a group. However, if we restrict our attention to basepoint-preserving homotopy classes of the form $\langle \Sigma X, Y \rangle$, where $\Sigma X$ is the reduced suspension of a space $X$, we have an important group structure. Here, we recover a *group isomorphism* between

$$\operatorname*{colim}_n \langle \mathrm{Sd}^n(\mathbb{S}^{op}X), Y\rangle \quad \text{and} \quad \langle \Sigma|\mathscr{K}(X)|, |\mathscr{K}(Y)|\rangle$$

by demonstrating an $A$-space model of the co-H-space structure on $\Sigma X$. We conclude by focusing on spheres as a particular class of suspensions that allow our system to model the group structure of homotopy groups, as well as return some results regarding the contractibility of particular spaces.

We will use $\mathbb{S}^n$ to denote the minimal finite model of the $n$-sphere, consisting of $2n+2$ points, which we construct inductively by taking the non-Hausdorff suspension of $\mathbb{S}^{n-1}$.

When describing $n$-simplices of $\mathscr{K}(X)$, we will write $\{x_0, x_1, ..., x_n\}$, where each $x_i$ and element of the underlying poset $X$ and we assume $x_0 < x_1 < ... < x_n$. Likewise, when referring to the subdivision of a partially ordered set, we will abuse the language of $n$-simplices to refer to the elements of the subdivision that correspond to totally ordered subsets with $n+1$ elements.

Recall definition 3.2.1. It is important to note that weak equivalence of topological spaces is *not* transitive, so we must refrain from a notion of weak equivalence classes unless we undergo a process to formally invert these induced maps, which will not be necessary for our purposes.

**Definition 22.0.1.** Given a topological space $X$, an $A$-space $Y$ is said to be an *$A$-space model of* $X$ if there exists a chain of weak equivalences from $X$ to $Y$. That is, there is a finite collection consisting of both topological spaces and $A$-spaces $X_i$ along with weak equivalences $f_i : X_i \to X_{i+1}$ that give the chain

$$X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-2}} X_{n-1} \xrightarrow{f_{n-1}} X_n = Y$$

which in turn induces a bijection on all homotopy groups of $X$ and $Y$.

**Proposition 22.0.2.** *If $Y$ is an $A$-space model of a CW-complex $X$, then $|\mathscr{K}(Y)| \simeq X$.*

This is a direct application of Whitehead's Theorem. In light of this result, we see that once we have found a model or, looking ahead, a collection of related models for a space, we may study them instead of our space directly, as they encode the same homotopy-invariant information of the original space after realization. Particularly, since every CW complex is homotopy equivalent to the geometric realization of some poset, these relationships are sufficient to study the homotopical properties of CW complexes. While many of the general theoretic results of this section exist independent of these facts, the results presented here would be of little use for studying recognizable topological spaces.

## 22.1. Modeling Topological Spaces

While the functors $\mathscr{K}$ and $\mathscr{X}$ give us a way to build subdivisions of posets and simplicial complexes, it will also be useful for us to define a way to go from the subdivision of an $A$-space back to the underlying space. With this in mind, we define the following map.

**Definition 22.1.1.** Given an $A$-space $X$, let $\sup : \mathrm{Sd}X \to X$ be the map such that if $\sigma = \{x_0, x_1, ..., x_k\}$ is an element of $\mathrm{Sd}X$, then $\sup(\sigma) = x_k$.

**Proposition 22.1.2.** *The map* sup *is continuous.*

Proof. Suppose $\sigma \leq \sigma'$. Since the ordering on $\mathrm{Sd}X$ is induced by simplicial inclusion in $\mathscr{K}(X)$, we have $\sigma \subseteq \sigma'$ when regarded as simplices of $\mathscr{K}(X)$. Inclusion requires $x_i \in \sigma \implies x_i \in \sigma'$, so in particular $x_k \leq x'_{k'}$, and thus sup is continuous. □

**Proposition 22.1.3.** *The map* sup *is is a weak homotopy equivalence.*

It is a straightforward check to see that $|\mathscr{K}(f)| : |\mathscr{K}(\mathrm{Sd}X)| \to |\mathscr{K}(X)|$ is a homotopy equivalence. This implies $f$ is a weak homotopy equivalence of $A$-spaces.

**Definition 22.1.4.** Let $X$ be an $A$-space. Define $\inf : (\mathrm{Sd}X)^{op} \to X^{op}$ to be the map that coincides pointwise with sup on $\mathrm{Sd}X$ to $X$.

One should note that sup returns the largest element of a given chain, and inf, the smallest. We will use this descriptive language in proofs, as it illuminates the continuity of some maps and the commutativity of some diagrams that we will introduce in the coming pages.

To establish a concrete relationship between an $A$-space and its realization that will allow us to regard $X$ as an $A$-space model of $|\mathscr{K}(X)|$, we present the following theorem.

**Definition 22.1.5.** For an $A$-space $X$, let $u$ be a point in $|\mathscr{K}(X)|$. Then $u$ is in the interior of a unique simplex $\{x_0, x_1, ..., x_k\}$ of $\mathscr{K}(X)$. Define the map $p : |\mathscr{K}(X)| \to X$ by $p(u) = x_0$.

**Theorem 22.1.6.** [50] *For an $A$-space $X$, $p : |\mathscr{K}(X)| \to X$ is a natural weak homotopy equivalence.*

We can in fact extend this result to see that there is a weak homotopy equivalence $p_n : |\mathscr{K}(X)| \to (\mathrm{Sd}^n X)^{op}$. While it may seem odd for us to pick the opposite topology, one should note that each $p_n$ as defined above returns the maximal element of the unique simplex in $\mathrm{Sd}^n X$ containing a point $u$. When composed with the inf map as above, we generate the following commutative diagram. [1]

$$\begin{array}{ccccccc} & & & |\mathscr{K}(X)| & & & \\ & \swarrow^{p_0} & & \downarrow^{p_1} & \searrow^{p_2} & & \\ X^{op} & \xleftarrow[\inf]{} & (\mathrm{Sd}X)^{op} & \xleftarrow[\inf]{} & (\mathrm{Sd}^2X)^{op} & \xleftarrow[\inf]{} & \cdots \end{array}$$

Given this diagram, we take the inverse limit of the system of the bottom row, given by

$$\tilde{X} = \prod_n (\mathrm{Sd}^n X)^{op}/(\sim)$$

where equivalence is generated by the inf map. We would like to imagine that the increasingly fine subdivisions of a simplex "converge" to a point in $|\mathscr{K}(X)|$. To formalize this intuition, we offer the following setup:

We define the map $\tilde{p} : |\mathscr{K}(X)| \to \tilde{X}$ by $\tilde{p}(a) = (p_0(a), p_1(a), ...)$, which associates to each point $a$ in the realization of $X$ the corresponding sequence of images of $a$ under each $p_n$. One may note that this is a sequence of nested simplices all containing $a$. Since the maps $p_i$ give a cone to our inverse system, so continuity is clear. In tandem, we offer the following map that will act as an inverse up to homotopy:

[1]Note the importance of the opposite topology and the substitution of the inf map to make this a commutative diagram of continuous maps. For a more explicit construction of each $p_n$, which Thibault denotes $\tilde{p}_n$, his thesis gives full details that make the checks on this diagram and the inverse limit much more concrete. We elect to omit these here, and refer the skeptical and intrigued reader to [67].

**Definition 22.1.7.** Let $x = (x_0, x_1, ...) \in \tilde{X}$, with each $x_i \in (\mathrm{Sd}^i X)^{op}$. Pick some $a_i \in \tilde{p_i}^{-1}(x_i)$ for each $x_i \in x$. Then $\{a_n\}$ converges to a point $a \in |\mathscr{K}(X)|$. Let $G : \tilde{X} \to |\mathscr{K}(X)|$ denote this map.

The restriction in the following theorem that $X$ be a *locally finite* $A$-space, i.e. an $A$-space where each element has finite closure and a finite neighborhood, is required to ensure $G$ is continuous. These complete checks of continuity and well-definedness can be found in [**67**], which additionally correct a detail on the inherited topology of $\tilde{X}$ in Clader's original proof.

**Theorem 22.1.8.** *Let $X$ be a locally finite $A$-space. Then $\tilde{X}$ and $|\mathscr{K}(X)|$ are homotopy equivalent and $|\mathscr{K}(X)|$ is a deformation retract of $\tilde{X}$.*

Proof. First, observe that $G \circ \tilde{p}$ is in fact the identity map on $|\mathscr{K}(X)|$, so we need only show $\tilde{p} \circ G \simeq id_{\tilde{X}}$. Suppose $x, x' \in \tilde{X}$ and say $x \sim x'$ if $G(x) = G(x')$. If $E$ denotes a subset of $\tilde{X}$ corresponding to one equivalence class partitioned by $(\sim)$, define a homotopy $h_E : E \times [0,1] \to E$ by

$$h_E(x,t) = \begin{cases} x & t < 1 \\ \tilde{p}(G(x)) & t = 1 \end{cases}.$$

Thibault proves that this homotopy is continuous in [**67**], so we have an explicit deformation retract of each equivalence class under $(\sim)$. These maps allow us to define globally $H : \tilde{X} \times [0,1] \to \tilde{X}$.

To show $H$ itself is continuous, consider $U \subset \tilde{X}$ open. Suppose $x \in E$, and let $y = G(x) \in |\mathscr{K}(X)|$. Because of our partition, $y$ is the same for all $x \in E$. Observe that if $\tilde{p}(y) \notin (U \cap E)$, then $h_E^{-1}(U \cap E) = (U \cap E) \times [0,1)$. Likewise, if $(U \cap E)$ does contain $\tilde{p}(y)$, then $h_E^{-1}(U \cap E) = (U \cap E) \times [0,1]$.

Define $U' = \{x \in U \mid \tilde{p}(G(x)) \notin U\}$. Then $H^{-1}(U) = (U \times [0,1]) \setminus (U' \times \{1\})$. Since $\tilde{p}$ and $G$ are continuous, $U'$ is closed in $U$. So $H$ is continuous on all of $\tilde{X}$, and gives a homotopy from the identity to $\tilde{p} \circ G(\tilde{X})$. Therefore, $\tilde{X}$ deformation retracts onto $\tilde{p} \circ G(\tilde{X})$, and since $G$ is surjective, $\tilde{p} \circ G(\tilde{X}) = \tilde{p}(|\mathscr{K}(X)|)$. $\square$

It should be surprising that the inverse limit and geometric realization act equivalently up to homotopy on a locally finite $A$-space $X$. This should not be a readily accepted result, as these functors typically do not behave nicely with one another. After all, the geometric realization functor is left adjoint, which in general does not preserve limits. While the intuition laid out in this proof makes the conclusion feel motivated, one should keep in mind the interesting mathematical relationships at play.

## 22.2. Motivation and Colimit Setup

Now, we turn our attention to another system that, while similar in style to the former, marks a departure from modeling topological spaces to instead modeling continuous functions. The benefit of a model built similarly to our inverse limit system is that with increasingly finer subdivisions of $X$, we are able to "fill in the gaps" between morphisms, so to speak.

Given the way we will construct our directed system for a finite $A$-space $X$ and an arbitrary $A$-space $Y$, its colimit is given by

$$\coprod_n [\mathrm{Sd}^n X, Y]/(\sim)$$

subject to the equivalence relation on homotopy classes generated by precomposition with the sup map. That is, elements, which for us are equivalence classes of maps, $[f] \in [\mathrm{Sd}^i X, Y]$ and $[g] \in [\mathrm{Sd}^j X, Y]$ are identified in the colimit if there exists an $N \geq 0$ such that $[f \circ \sup^{N-i}] = [f \circ \sup^{N-j}] \in [\mathrm{Sd}^N X, Y]$. This relationship based on the sup map simply consolidates extraneous information; the crux of the result will come from the Simplicial Approximation Theorem.

To prove that this equivalence relation coincides precisely with those maps whose realizations are homotopic, we introduce a basic vocabulary to understand homotopy classes in the setting of $A$-spaces, and turn to the simplicial notion of contiguity that provides us with precisely the tools necessary to prove that the above colimit is a useful model.

## 22.3. Function Spaces and Homotopies

For arbitrary $A$-spaces $X$ and $Y$, $Y^X := \mathrm{Hom}(X, Y)$ need not also be an $A$-space. However, under the restriction that $X$ is a finite $A$-space, then $Y^X$ is an $A$-space whose order is determined by that of $Y$: for maps $f, g : X \to Y$, we say $f \leq g$ in $Y^X$ if $f(x) \leq g(x)$ for all $x \in X$.

This ordering on $Y^X$ gives us a natural way to extend the notion of (path) connectivity of $A$-spaces to homotopy equivalence of maps between $A$-spaces.

**Proposition 22.3.1.** *Let $f \leq g$, where $f$ and $g$ are maps from a finite $A$-space $X$ to an $A$-space $Y$. Then $f \simeq g$.*

Proof. $f$ and $g$ are connected by the path $H : I \to Y^X$ such that

$$H(t) = \begin{cases} f & t < 1 \\ g & t = 1 \end{cases}$$

We may reformulate this result to resemble the standard formulation of a homotopy by noting that the existence and continuity of $H$ implies that of $H^* : X \times I \to Y$ where $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. Thus, $f$ and $g$ are homotopic. □

Conversely, we may say that if $f \simeq g$, there exists a finite chain of comparable maps (i.e. maps $\{f_i\}$ for $1 \leq i \leq n$ such that either $f_i \leq f_{i+1}$ or $f_i \geq f_{i+1}$ for $1 \leq i \leq n-1$) connecting $f$ and $g$. This is proven in Section 2.2 of [48] by noting that the image of $I$ in $Y^X$ is compact, and therefore contains a finite number of elements given $Y^X$ is an $A$-space.

With this understanding, we turn to a notion that is different from but related to homotopy that allows us to compare maps residing in different components of our colimit, as well as characterize the behavior of related maps under realization.

## 22.4. Contiguity

In order to create a formal relationship between maps with different subdivisions as their domains, we utilize the concept of contiguity of simplicial maps to generate a notion of equivalence of maps in distinct components in our colimit. We conclude many of our intermediate results in the setting of simplicial complexes, as the notion of contiguity does not extend perfectly to $A$-spaces. This will be sufficient for our purposes, but for a more thorough treatment of contiguity of $A$-space maps, see [**48**] and [**67**].

**Definition 22.4.1.** Let $K$ and $L$ be simplicial complexes. Two simplicial maps $u : K \to L$ and $v : \mathrm{Sd}^m K \to L$ are *contiguous* if for each simplex $\sigma \in \mathrm{Sd}^m K$, there is a simplex $\tau \in L$ such that $u(\sup^{(m)}(\sigma)) \subset \tau$ and $v(\sigma) \subset \tau$.

If we have an $A$-space $X$ such that $\mathscr{K}(X) = K$, we can somewhat extend this definition to maps between $A$-spaces. However, it should be clear by this definition that we are limited by the existence of simplices, which requires at least one subdivision of $X$ to be sensible. Thus, we say that two maps $f$ and $g$ between $A$-spaces are contiguous if $\mathscr{K}(f)$ and $\mathscr{K}(g)$ are contiguous as simplicial maps.

Now that we have presented the notion contiguity as it pertains to simplicial and $A$-space maps, we offer the following relationships between contiguity and homotopy that link equivalence in the colimit to equivalence after realization.

**Proposition 22.4.2.** *Suppose $\mathscr{K}(f) : \mathscr{K}(X) \to \mathscr{K}(Y)$ and $\mathscr{K}(g) : \mathscr{K}(\mathrm{Sd}^n X) \to \mathscr{K}(Y)$ are contiguous. Then $|\mathscr{K}(f)| \simeq |\mathscr{K}(g)|$.*

Since $\mathscr{K}(f)$ and $\mathscr{K}(g)$ are contiguous, they satisfy the definition of simplicial closeness given in [**48**]. Thus, there is a linear homotopy connecting their realizations.

**Theorem 22.4.3.** [**29**] *For a finite $A$-space $X$ and arbitrary $A$-space $Y$, if $f : |\mathscr{K}(X)| \to |\mathscr{K}(Y)|$ is continuous, then there exists an $n \geq 0$ and a map $g : \mathrm{Sd}^n X \to Y$ with $|\mathscr{K}(g)| \sim f$.*

Proof. By the classical Simplicial Approximation Theorem, there exists a simplicial map $u : \mathrm{Sd}^{n-1}\mathscr{K}(X) \to \mathscr{K}(Y)$ such that $|u| \simeq f$. Noting $\mathrm{Sd}^{n-1}\mathscr{K}(X) = \mathscr{K}(\mathrm{Sd}^{n-1}X)$, define $g = \sup \circ \mathscr{X}(u)$. Then $g$ is a map from $\mathrm{Sd}^n X$ to $Y$, and given $\sigma = \{S_1, S_2, ..., S_k\}$, an arbitrary simplex of $\mathscr{K}(\mathrm{Sd}^n(X))$, we have

$$(\mathscr{K}(\sup \circ \mathscr{X})(u))(\sigma) = \sup\{u(S_1), u(S_2), ..., u(S_k)\} = u(\sup(\sigma)),$$

so by definition $u$ and $\mathscr{K}(g)$ are contiguous. Following Proposition 22.4.2, this implies $|\mathscr{K}(g)| \simeq |u|$. Thus, $|\mathscr{K}(g)| \simeq f$. □

We say any map $g$ satisfying this condition is an *$A$-space approximation* of $f$.

**Proposition 22.4.4.** [**48**] *Let $X$ and $Y$ be $A$-spaces. If $g : \mathrm{Sd}^i X \to Y$ and $g' : \mathrm{Sd}^j X \to Y$ are both simplicial approximations of $f : |\mathscr{K}(X)| \to |\mathscr{K}(Y)|$, then $g$ and $g'$ are contiguous.*

**Proposition 22.4.5.** *Suppose $f$ and $g$ are maps from $X$ to $Y$, where $X$ is a finite $A$-space and $Y$ is an arbitrary $A$-space. If $f \simeq g$, then $|\mathscr{K}(f)| \simeq |\mathscr{K}(g)|$.*

Proof. Since $f \simeq g$ implies that there is a sequence of comparable maps connecting the two, and since both $\mathrm{Im}(f)$ and $\mathrm{Im}(g)$ are finite, it suffices to show the desired result for $f$ and $g$ such that $f(x) = g(x)$ for all but one $x'$, where $f(x') \leq g(x')$. For a simplex $\sigma$ of $X$ that does not contain $x'$, we have $f(\sigma) = g(\sigma)$, which is clearly contained in a simplex of $Y$. If $x' \in \sigma$, then $x' = x_i$ for some $i$ and both $f(\sigma)$ and $g(\sigma)$ are contained in the simplex given by

$$\{f(x_0), f(x_1), ..., f(x'), g(x'), g(x_{i+1}), ..., g(x_k)\}$$

after removing repetitions. By Proposition 22.4.2, $|\mathscr{K}(f)| \simeq |\mathscr{K}(g)|$. □

## 22.5. Modeling Homotopy Classes of Maps

We are ready to consider a method to model the homotopy classes of maps between topological spaces based on the framework we have laid.

**Theorem 22.5.1.** *Let $X$ be a finite $A$-space and let $Y$ be an arbitrary $A$-space. Then there is a natural bijection between $[|\mathscr{K}(X)|, |\mathscr{K}(Y)|]$ and the colimit of the system*

$$[X, Y] \xrightarrow{\sup^*} [\mathrm{Sd}X, Y] \xrightarrow{\sup^*} [\mathrm{Sd}^2 X, Y] \xrightarrow{\sup^*} \cdots$$

*This bijection, which we denote $K : \operatorname{colim}_n [\mathrm{Sd}^n X, Y] \to [|\mathscr{K}(X)|, |\mathscr{K}(Y)|]$, maps $[f] \in [\mathrm{Sd}^i X, Y]$ to $[|\mathscr{K}(f)|] \in [|\mathscr{K}(X)|, |\mathscr{K}(Y)|]$.*

Proof. From Proposition 22.4.5, homotopies are preserved by each component map from $[\mathrm{Sd}^n X, Y]$ to $[|\mathscr{K}(X)|, |\mathscr{K}(Y)|]$ so $K$ is well-defined. Suppose $[f] \in [\mathrm{Sd}^i X, Y]$ and $[g] \in [\mathrm{Sd}^j X, Y]$. If $[f]$ and $[g]$ are identified in the colimit, then there exists an $N$ such that $f \circ \sup^{(N-i)} \simeq g \circ \sup^{(N-j)}$. Again by Proposition 22.4.5, $|\mathscr{K}(f \circ \sup^{(N-i)})| \simeq |\mathscr{K}(g \circ \sup^{(N-j)})|$. Thus, we need only check that $|\mathscr{K}(f)| \simeq |\mathscr{K}(f \circ \sup^{(N-i)})|$ to ensure $|\mathscr{K}(f)| \simeq |\mathscr{K}(g)|$. Noting that $f$ and $f \circ \sup^{(N-i)}$ are tautologically contiguous, we have by Proposition 22.4.2 that $|\mathscr{K}(f)| \simeq |\mathscr{K}(f \circ \sup^{(N-i)})|$. Thus, $|\mathscr{K}(f)| \simeq |\mathscr{K}(g)|$.

By Theorem 22.4.3, each homotopy class of maps has associated to it at least one $A$-space approximation, so $K$ is surjective.

To prove injectivity, suppose $|\mathscr{K}(f)| \simeq |\mathscr{K}(g)|$. Then $f$ and $g$ are by definition $A$-space approximations for, say, $|\mathscr{K}(f)|$. Suppose $[f] \in [\mathrm{Sd}^i X, Y]$ and $[g] \in [\mathrm{Sd}^j X, Y]$. Then by Proposition 22.4.4, $f$ and $g$ are contiguous. This is sufficient co conclude $[f] = [g]$ in $\operatorname{colim}_n [\mathrm{Sd}^n X, Y]$. Thus, we have shown all necessary criteria for the existence of this canonical bijection. □

While a bijection certainly creates a beautiful relationship here, the surjectivity of this map should be regarded as the most interesting component of this result; It means that given an arbitrary continuous map between $|\mathscr{K}(X)|$ and $|\mathscr{K}(Y)|$, only a finite number of subdivisions on the level of $A$-spaces are necessary to model this map up to homotopy. The fact that we can transfer the study of continuous maps between topological spaces to that of order preserving maps between posets should be startling. The rest of the framework and proof simply collapses down maps redundant up to homotopy.

## 22.6. Suspensions and the Group Structure on $\langle X, Y\rangle$

In this section, we narrow our focus to a particular class of topological spaces that induce additional structure on homotopy classes of maps. For arbitrary topological spaces $X$ and $Y$, $[X, Y]$ is in general simply a set, so a bijection is the strongest relationship we can hope for. However, the fact that under certain conditions based homotopy classes admit a group structure may raise the questions as to whether our bijection can also be extended to a group isomorphism in these cases. Keeping the motivating example of homotopy groups in mind, we widen our attention to suspensions, of which spheres are an example, to demonstrate a way to define a group structure on our colimit using subdivisions.

First, we show the bijection outlined in Section 7 can be modified to a correspondence, to borrow Hatcher's notation for pointed homotopy classes in [31], $\langle|\mathscr{K}(X)|, |\mathscr{K}(Y)|\rangle$ and the colimit of

$$\langle X, Y\rangle \xrightarrow{\sup^*} \langle \mathrm{Sd}X, Y\rangle \xrightarrow{\sup^*} \langle \mathrm{Sd}^2X, Y\rangle \xrightarrow{\sup^*} \cdots$$

by noticing that fixing basepoints $x_0 \in X$ and $y_0 \in Y$ in partially ordered set allows for a notion of based maps where we require $f(x_0) = y_0$. The corresponded totally ordered subsets $\{x_0\}$ and $\{y_0\}$ indeed realize to points in $|\mathscr{K}(X)|$ and $|\mathscr{K}(Y)|$ respectively, since they denote 0-simplices and thus realize to 0-subcomplexes. Our choice of 0-cell for the basepoint is preserved as expected under subdivisions of $X$, since there is a canonical inclusion map $i : X \to \mathrm{Sd}X$ that takes an element $x$ to the corresponding single element subset of $\{x\}$ of $X$, which denotes an element of $\mathrm{Sd}X$. Since under the sup map, $\{x_0\} \mapsto x_0$, taking pointed spaces is compatible with our colimit and our definition of contiguity, which importantly extends the Simplicial Approximation Theorem to this new result.

However, to make this variation play well with the sup map and contiguity, we must require that $x_0$ is a maximal point of $X$, which is no trouble since we have already assumed $X$ is finite. This will agree with our methods in the following section.

Now that we have sufficiently restricted our homotopy class bijection, we may proceed with our group structure construction.

Recall that given a topological space $X$, we define the *suspension of* $X$, denoted $SX$, to be the quotient space $(X \times I)/(\sim)$, where $(x, t) \sim (y, s)$ if and only if $(x, t) = (y, s)$ or $s = t = 0$ or $s = t = 1$. We say the suspension is *reduced* if we additionally identify all points of the form $(x_0, t)$ for $t \in [0, 1]$. For this, we write $\Sigma X$. Clearly, this collapsing map gives a homotopy equivalence from $SX$ to $\Sigma X$.

**Definition 22.6.1.** We define the finite analog of these constructions to be the *non-Hausdorff suspension* $\mathbb{S}X$, which is the resulting space after adding two points $+$ and $-$ to $X$ such that the only open sets containing them are $\mathbb{S}X$ itself, $\{X \cup +\}$, and $\{X \cup -\}$.

In terms of posets, where we will set our following constructions, this amounts to adding two new maximal points to the poset of $X$.

**Theorem 22.6.2.** [48] *For any space $X$, the map $\gamma : SX \to \mathbb{S}X$ is a weak homotopy equivalence. For any weak homotopy equivalence $f : X \to Y$, the maps $Sf : SX \to SY$ and $\mathbb{S}f : \mathbb{S}X \to \mathbb{S}Y$ are weak homotopy equivalences. Therefore $\gamma^n : S^nX \to \mathbb{S}^nX$ is a weak homotopy equivalence for any space $X$.*

Further, this implies that there is a basepoint-preserving weak homotopy equivalence $\gamma^n : (\Sigma^nX, x_0) \to (\mathbb{S}^nX, x_0)$.

Unfortunately, the non-Hausdorff suspension of a space will not give quite the structure we need to make our desired map continuous. Thus, we define an alternative configuration:

**Definition 22.6.3.** For a topological space $X$, the *non-Hausdorff opposite suspension* of $X$ is given formally by $(\mathbb{S}(X^{op}))^{op}$, which we interpret as adding two new minimal points to $X$, and denote $\mathbb{S}^{op}X$.

From this will arise a nice space intimately related to the classical suspension that will allow us to model a defining property of suspensions of topological spaces.

There is a natural map from $\mathscr{K}(\mathbb{S}X)$ to $\mathscr{K}(\mathbb{S}^{op}X)$ which, for example, takes a simplex $\sigma = \{x_0, x_1, ..., x_k, +\}$ to the simplex $\{+, x_0, x_1, ..., x_k\}$ in $\mathscr{K}(\mathbb{S}^{op}X)$ and preserves ordering. From this, it follows that $|\mathscr{K}(\mathbb{S}X)| = |\mathscr{K}(\mathbb{S}^{op}X)|$, as the geometric realization functor acts independently of the order on the underlying poset.

The map $\gamma$ in Theorem 22.6.2 and the map $p$ defined by McCord together generate a weak equivalence $\tilde{p} : \Sigma|\mathscr{K}(X)| \to \mathbb{S}X$. This, coupled with the above equivalence, gives that $\Sigma|\mathscr{K}(X)| \simeq |\mathscr{K}(\mathbb{S}^{op}X)|$. Keeping this relationship in mind, we digress to lay some mathematical framework that will help us make use of this homotopy equivalence.

**Definition 22.6.4.** [5] A pointed topological space $(X, x_0)$ is a *co-H-space* if there exists a map $\psi : X \to X \vee X$ such that $p_1\psi \simeq id_X$ rel $x_0$ and $p_2\psi \simeq id_X$ rel $x_0$, where $p_1, p_2 : X \vee X \to X$ are the two projection maps. $(X, x_0)$ is a *co-group* if it is co-associative and there is a co-inverse map $\xi : X \to X$ up to based homotopy such that the co-group axioms are satisfied.

Above, we require that $X \vee X$ be wedged at $x_0$ and call $x_0$ the identity of the co-H-space.

**Example 22.6.5.** $(\Sigma X, x_0)$ is a co-group (and thus a co-H-space) for all pointed topological spaces $(X, x_0)$.

We are particularly interested in suspensions and their properties because of their presence in many important constructions in algebraic topology. For example, the $n$-sphere is homeomorphic to $n$ iterative suspensions of the 0-sphere, generating a class of topological spaces central to algebraic topology. Additionally, the co-group structure of a suspension plays an important role in the Eckmann-Hilton duality, as understanding the loop spaces of certain topological spaces often gives a wealth of information.

Given that our motivation for looking at $A$-spaces is to encode information about topological spaces more generally, one may ask if we are able to model this co-H-space structure using finite $A$-spaces. Unfortunately, following a dual result for H-spaces from Stong, we have the following:

**Theorem 22.6.6.** [33] *Suppose $X$ is a finite connected space. Then $X$ admits a co-H-space structure if and only if it is contractible.*

Thus, any finite model that is also a co-H-space is a model of a space with only trivial homotopy groups, which is both terribly narrow and uninteresting. This seems to imply that we have no way to model such a structure using finite spaces. For this reason, we turn to the colimit we have constructed for inspiration on how to model the co-H-space structure of a suspension that makes use of the subdivision functor in a way that strengthens the main result of Section 8.

**Theorem 22.6.7.** *Suppose that $X$ is a finite $A$-space. Then there exists an order-preserving map $\phi : \mathrm{Sd}(\mathbb{S}^{op}X) \to \mathbb{S}^{op}X \vee \mathbb{S}^{op}X$ such that $p_1|\mathscr{K}(\phi)| \simeq id_{\Sigma|\mathscr{K}(X)|}$ and $p_2|\mathscr{K}(\phi)| \simeq id_{\Sigma|\mathscr{K}(X)|}$, where $p_1$ and $p_2$ are the respective projection maps.*

**Construction 22.6.8.** We assume that the wedge of two copies of $\mathbb{S}X$ identifies different minimal points (i.e. $+ \sim -$) in each copy of $X$ to make the description of our map cleaner, but the reader should check that a similar map exists regardless of choice of minimal basepoint on each copy.

Let $\sigma = \{x_0, x_1, ..., x_n\}$ be an element of $\mathrm{Sd}(\mathbb{S}^{op}X)$. Define an equivalence relation on $\mathrm{Sd}(\mathbb{S}^{op}X)$ by the following two generating conditions:

(i) Identify all such $\sigma$ where $x_0$ was an original element of $X$ (in other words, $\inf(\sigma) \neq +$ or $-$).
(ii) $\sigma \sim \sigma'$ if $\sup(\sigma) = \sup(\sigma')$ and $\inf(\sigma) = \inf(\sigma')$.

Call the identification space generated by this partition $X'$, equipped with the quotient topology (which corresponds to inheriting an order from $\mathbb{S}^{op}X$). Note that if $X$ is a finite $A$-space, $X'$ is a finite $A$-space. Let $\rho$ denote the identification map from $\mathrm{Sd}(\mathbb{S}^{op}X)$ to $X'$.

**Proposition 22.6.9.** *$X' \cong \mathbb{S}^{op}X \vee \mathbb{S}^{op}X$.*

Proof. Define a map $f : X' \to \mathbb{S}^{op}X \vee \mathbb{S}^{op}X$ as follows. Given an element $\sigma$ of $\mathrm{Sd}(\mathbb{S}^{op}X)$ such that $\inf(\sigma)$ is not minimal in $\mathbb{S}^{op}X$, let $f(\sigma) = *$, the basepoint of the wedge. Otherwise, $\inf(\sigma)$ is one of two minimal points of $\mathbb{S}^{op}X$. Let which minimal point $\sigma$ contains be regarded as a coordinate to denote one of the copies of $\mathbb{S}^{op}X$. Then, restricted to that copy, let $f(\sigma) = \sup(\sigma)$.

That this map is surjective follows from the fact that sup is surjective. To see that it is injective, consider $\sigma$ and $\sigma'$ such that $f(\sigma) = f(\sigma')$. Then either $f(\sigma) = f(\sigma') = *$, in which case neither contained a minimal element of $\mathbb{S}^{op}X$ so they were identified together in $X'$, or both the inf and sup map agree, which implies $\sigma \sim \sigma'$ in $\mathrm{Sd}\ \mathbb{S}^n$, so $\sigma = \sigma'$ in $X$.

To prove $f$ is continuous, suppose $\sigma \leq \sigma'$. Then $\rho(\sigma) \leq \rho(\sigma')$ by the quotient topology, and there are four cases to consider: first, that neither $\sigma$ nor $\sigma'$ have a minimal element. Then $f\rho(\sigma) = * = f\rho(\sigma')$. Suppose $\sigma$ does not have a minimal element but $\sigma'$ does. Then $f\rho(\sigma) = *$ which is tautologically less than or equal to $f\rho(\sigma')$. Then, suppose both contain a minimal element. If they do not agree on the minimal element, they were not comparable subsets to begin with. Then supposing they do, this means that $\rho$ is determined by sup, which is continuous. So $f$ preserves order and is therefore continuous.

Let $g : \mathbb{S}^{\mathrm{op}} \vee \mathbb{S}^{\mathrm{op}} \to X'$ be the map that sends the wedge point to the equivalence class of all chains not containing a minimal element. Suppose $x \in \mathbb{S}^{\mathrm{op}} \vee \mathbb{S}^{\mathrm{op}}$ is not the wedge point. Then define where $x$ is sent by first noting which copy of $\mathbb{S}^{\mathrm{op}}$ $x$ belongs to (the copies correspond to the minimal points $+$ and $-$ respectively) and which point in $\mathbb{S}^{\mathrm{op}}$ maps to it under the inclusion $i : \mathbb{S}^{\mathrm{op}} \to \mathbb{S}^{\mathrm{op}} \vee \mathbb{S}^{\mathrm{op}}$ into that copy. Let the resulting $+$ or $-$ association determine the minimal point of the chain and the preimage under inclusion the maximal point, completely determining an element of $X'$. This construction makes it a clear inverse, and checking continuity is simply another process of checking assignment cases while recognizing the ordering of $X'$ is fully determined by that of $\mathrm{Sd}(\mathbb{S}^{op}X)$. □

We end by renaming the composition $f \circ \rho$ as $\phi$ for simplicity. For the next checks, we will use the construction above with instead the assumption that the points "+" were identified for the basepoint of the wedge. Again, one should check that this choice is arbitrary and that the inconsistency in choice in this paper is instead to give the reader the simplest and most elegant constructions, as should be the point.

**Proposition 22.6.10.** *Let*

$$\phi_* : (|\mathscr{K}(\mathrm{Sd}(\mathbb{S}^{op}X)|, |\mathscr{K}(+)|) \to (|\mathscr{K}(\mathbb{S}^{op}X)| \vee |\mathscr{K}(\mathbb{S}^{op}X)|, |\mathscr{K}(*)|)$$

*be the map induced by $\phi$. Then $p_1\phi_* \simeq id_{\Sigma|\mathscr{K}(X)|}$ rel $|\mathscr{K}(+)|$ and $p_2\phi_* \simeq id_{\Sigma|\mathscr{K}(X)|}$ rel $|\mathscr{K}(+)|$, i.e. $\phi$ induces a co-H-space structure on $(\Sigma|\mathscr{K}(X)|, |\mathscr{K}(+)|)$.*

Proof. To show that $\phi$ induces a co-H-space structure on $\Sigma|\mathscr{K}(X)|$, we must show that the induced map $\phi_* = |\mathscr{K}(\phi)|$ satisfies the co-H-structure properties of $\psi$ as above. Basepoint preservation comes from keeping track of its point-wise assignment, so we omit that tracking. Since our map is symmetric, we demonstrate the proof explicitly for only $p_1$.

Since there is a canonical homeomorphism from $|\mathscr{K}(\mathrm{Sd}(\mathbb{S}^{op}X))|$ to $|\mathscr{K}(\mathbb{S}^{op}X)|$ [26], the homotopy equivalence $|\mathscr{K}(\sup)| : |\mathscr{K}(\mathrm{Sd}(\mathbb{S}^{op}X))| \to |\mathscr{K}(\mathbb{S}^{op}X)|$ is in fact a map homotopic to the identity. This simplifies our problem to showing that the following diagram commutes up to homotopy:

$$\mathrm{Sd}^{k+1}(\mathbb{S}^{op}X) \xrightarrow{\phi} \mathrm{Sd}^k(\mathbb{S}^{op}X) \vee \mathrm{Sd}^k(\mathbb{S}^{op}X) \xrightarrow{p_1} \mathrm{Sd}^k(\mathbb{S}^{op}X), \qquad \sup : \mathrm{Sd}^{k+1}(\mathbb{S}^{op}X) \to \mathrm{Sd}^k(\mathbb{S}^{op}X)$$

We begin by demonstrating that $p_1\phi \simeq \sup$. Let $\sigma = \{x_0, x_2, ..., x_k\}$ be an element of $\mathrm{Sd}(\mathbb{S}^{op}X)$. We consider three cases. First, suppose $x_0$ is not a minimal element of $\mathbb{S}^{op}X$. Then $\phi(\sigma)$ is minimal in $\mathbb{S}^{op}X \vee \mathbb{S}^{op}X$, which is preserved under projection. If $x_0$ is a minimal element, but is the minimal element associated with the second copy of $\mathbb{S}^{op}X$, the projection map will ultimately crush $\sigma$ to $*$, which is minimal. In both of these cases, it trivially follows that $p_1\phi(\sigma) \leq \sup(\sigma)$. If $\sigma$ contains the minimal point sent to "first" sphere, then $p_1\phi$ sends $\sigma$ to $\sup(\sigma)$, so we tautologically have that $p_1\phi(\sigma) \leq \sup(\sigma)$.

It follows from Proposition 22.3.1 that if $p_1\phi(\sigma) \leq \sup(\sigma)$ is true for all $\sigma$, then $p_1\phi \simeq \sup$. By Proposition 22.4.5 we have that $p_1\phi \simeq \sup$ implies $|\mathscr{K}(p_1\phi)| \simeq |\mathscr{K}(\sup)|$. Thus, $p_1\phi_* \simeq id$ rel $|\mathscr{K}(+)|$. □

**Proposition 22.6.11.** [5] *Let $(X, x_0)$ and $(Y, y_0)$ be arbitrary topological spaces. Then*

*(i) There is a unital binary operation on $\langle X, Y\rangle$ if and only if $X$ is a co-H-space.*
*(ii) $\langle X, Y\rangle$ is a group if and only if $X$ is a co-group.*

Given this result, we should expect $\langle|\mathscr{K}(\mathbb{S}^{op}X)|, |\mathscr{K}(Y)|\rangle$ to be a group since $|\mathscr{K}(\mathbb{S}^{op}X)|$ is homotopic to the reduced suspension of the geometric realization of $X$, a CW complex, and thus has a co-group structure. Below we will demonstrate how our finite model of the co-H-space structure of $\mathbb{S}^{op}X$ captures enough information to model the group structure of these homotopy classes of maps.

Suppose $[f] \in \langle \mathrm{Sd}^k(\mathbb{S}^{op}X), Y\rangle$ and $[g] \in \langle \mathrm{Sd}^m(\mathbb{S}^{op}X), Y\rangle$ with $k \leq m$. Then we may construct the following chain of maps:

$$\mathrm{Sd}^{m+1}(\mathbb{S}^{op}X) \xrightarrow{\phi} \mathrm{Sd}^m(\mathbb{S}^{op}X) \vee \mathrm{Sd}^m(\mathbb{S}^{op}X) \xrightarrow{\sup\,\vee\,id} \mathrm{Sd}^k(\mathbb{S}^{op}X) \vee \mathrm{Sd}^m(\mathbb{S}^{op}X) \xrightarrow{f \vee g} Y \vee Y \xrightarrow{\nabla} Y\ ,$$

where $\nabla : Y \vee Y \to Y$ denotes the codiagonal map. Define the homotopy class of the composition to be $[f] + [g]$. Thus, this co-H-space model structure allows us to define a unital binary operation in the colimit.

We would like our induced binary operation to be associative and have inverses, as to model the full group structure on these homotopy classes. For this, we use the construction of the bijection in the previous section to induce this additional structure from the modeled homotopy class of maps.

Given $[[f] + [g]] + [h]$ in the colimit, we have $[[f] + [g]] + [h] \mapsto K([[f] + [g]]) + K([h])$, which we know equals $K([f]) + K([[g] + [h]])$ by the associativity of the group structure on $[S^n, X]$. Then via the existence of $K^{-1}$, we get $[[f] + [g]] + [h] = [f] + [[g] + [h]]$, showing the operation is associative. Likewise, suppose we have $[f]$ in the colimit. Define $[f]^{-1}$ as the unique element $K^{-1}((K([f])^{-1})$, whose existence and uniqueness is guaranteed by the existence and uniqueness of inverses in $\langle \Sigma|\mathscr{K}(X)|, |\mathscr{K}(Y)|\rangle$ and our one-to-one correspondence. Therefore we have constructed a model for the full group structure of $\langle \Sigma|\mathscr{K}(X)|, |\mathscr{K}(Y)|\rangle$ using the result of Theorem 22.5.1.

## 22.7. Applications to Homotopy Groups

One very important place we can use the group structure induced from suspensions is in the homotopy groups of a topological space $X$. Homotopy groups are an essential element of algebraic topology, and the ability to model them using a system built from $A$-spaces should be incredibly motivating. Since for $n > 1$, the $n$-sphere is nothing more than the suspension of $S^{n-1}$, it should be unsurprising that we will conclude this paper by discussing applications to homotopy groups.

For this section, we will restrict our attention to connected (and hence path-connected) CW complexes $X$, and must pay some mind of based homotopy classes, since we have the relation $\pi_k(X) \cong \langle S^k, X\rangle$. Thus, applying the above sections to conclude homotopy-theoretic results is a natural extension.

Here, we omit the opposite suspension construction for the sphere, as $\mathbb{S}^{op}\mathbb{S}^n \cong \mathbb{S}\mathbb{S}^n$ because of the symmetric nature of the minimal finite model for $S^n$. Thus, we simplify our notation by dropping the opposites, and leave the reader to check that the same map $\phi$ suffices to give a map from Sd $\mathbb{S}^n$ to $\mathbb{S}^n \vee \mathbb{S}^n$ which induces a co-H-space structure on $S^n$. Given that we demonstrated both a bijection and group homomorphism in the previous sections, and under the assumption that $X$ is a connected $A$-space, we may conclude $\operatorname*{colim}_n \langle (\mathrm{Sd}^k\ \mathbb{S}^n, *), (X, x_0)\rangle \cong \pi_n(|\mathscr{K}(X)|, |\mathscr{K}(x_0)|)$ as groups for $n \geq 2$.

We can say more about spheres, as we know that for $n \geq 2$, we have a cocommutative co-group that induces an abelian group structure on $\pi_n(X)$. We demonstrate this below:

**Proposition 22.7.1.** [5] *Let $(X, x_0)$ be a co-H-space. Then $\psi : X \to X \vee X$ is cocommutative if and only if $\langle X, Y\rangle$ is a cocommutative cogroup for all spaces $Y$.*

Given $[f] + [g]$ in the colimit, we have $[f] + [g] \mapsto K([f]) + K([g])$, which we know is equal to $K([g]) + K([f])$ in $\langle S^n, X\rangle$. Then $[f] + [g] = [g] + [f]$.

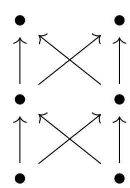
Applying the results of this paper to homotopy-theoretic results, we begin by showing that if a connected CW complex $X$ admits an $A$-space model $A$ with a sole maximal (or minimal) point, then $X$ is contractible.

Let $*$ denote this maximal (or minimal) point. Then any based map $f : \mathrm{Sd}^k(\mathbb{S}^n) \to A$ for some values $k$ and $n$, the image consists of finitely many points, and the constant map given by $g(x) \equiv *$ is always greater than or equal to $f$ since $* \geq \mathrm{Im}(f)$ (or less than or equal to $f$ since $* \leq \mathrm{Im}(f)$). Thus, $f \simeq g$ for all maps $f$ based at $*$. This gives that $\langle S^n, X\rangle$ is trivial in every grading, giving trivial homotopy groups by inclusion of $\pi_n$. Since trivial homotopy groups imply contractibility for CW complexes by Whitehead's theorem, we conclude $X$ is contractible.

Diverging to poset-theoretic results, this also implies that the geometric realization of a lattice is contractible. For a similar but perhaps more intriguing application, we set up the following definition:

**Definition 22.7.2.** The *infinite sphere* $S^\infty$ is defined to be the directed colimit of $S^n$, where the map from $S^{n-1}$ to $S^n$ is given by inclusion.

For our purposes, we would like to generalize the $A$-space model we have for $S^n$ to an $A$-space model for $S^\infty$. As we understand $\mathbb{S}^n$ as being $n$ successive suspensions of $S^0$, the same construction applies here. In particular, we typically represent the finite model of $S^2$ by the Hasse diagram below:



For $S^\infty$, we get much the same picture, except the tower stretches infinitely upwards. The $A$-space corresponding to this Hasse diagram we will denote $\mathbb{S}^\infty$.

**Proposition 22.7.3.** *$S^\infty$ is contractible.*

Proof. Given our $A$-space model for $S^\infty$, we can consider $\operatorname{colim}_n \langle \mathrm{Sd}^n\mathbb{S}^k, \mathbb{S}^\infty\rangle$ to model the $k$th homotopy group of $S^\infty$. While working with a colimit does not seem necessarily advantageous compared to working with a topological space itself, it does allow us to conclude some important results here.

It is clear to see that for any $n$ the image of a map $f : \mathrm{Sd}^n\mathbb{S}^k \to \mathbb{S}^\infty$ will be finite since $\mathrm{Sd}^n\mathbb{S}^k$ has only finitely many points. Since $\mathbb{S}^\infty$ is infinite, and in particular never attains maximal points in its poset representation, there will always exist an $x_K \in \mathbb{S}^\infty$ such that $f(x) < x_K$ for all $x \in \mathrm{Sd}^n\mathbb{S}^k$. Then the constant map that sends every element of $\mathrm{Sd}^n\mathbb{S}^k$ to $x_K$ is homotopic to $f$ by Proposition 22.3.1. Further, given maps $f$ and $g$, we are ensured by the nonexistence of a maximal element that we can always find some $x_K$ s.t. $f$ and $g$ are both dominated by the constant map to $x_K$. Thus, any two choices of $f$ and $g$ will be homotopic regardless of the number of subdivisions we take, and therefore by the same argument involving basepoints as above, $\pi_n(S^\infty)$ is trivial for all $n$. $S^\infty$ is also a CW complex, so this is sufficient to conclude contractibility. □

While the applications of this section involve little calculation, and do not require the results of Section 8, the potential applications of framing some homotopic information within the scope of combinatorial computation could be used more broadly. While the computation of these homotopy classes of maps between subdivided $A$-spaces remain unreasonably computable at present, there is no *theoretical* barrier to this method of finding the the homotopy classes of maps between spaces, particularly for finite homotopy groups. Following this paper, I hope to determine if some sort of upper bound on subdivisions for the bijection to hold can be obtained in the case where $[X, Y]$ is a finite set, and explore whether the Freudenthal Suspension Theorem can be combinatorially concluded.

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