AN INTRODUCTION TO THE THEORY OF OSCILLATORY INTEGRALS

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Abstract. This paper gives an introduction to the theory of oscillatory integrals of the first kind. Due to the complexity and multiplicity of critical points in higher dimensions, the decay of $|I(\lambda)|$ in $\mathbb{R}^d$ with $d > 1$ is far from clear-cut. However, using the method of stationary phase, one can establish an explicit decay property of the oscillatory integral in certain cases. We will first prove decay of $|I(\lambda)|$ in the more straightforward case in which the phase function does not have a critical point. Next, we will analyze the case of a nondegenerate critical point and demonstrate some applications of oscillatory integrals in the study of partial differential equations.

Contents

1. Introduction

This expository paper aims to discuss asymptotic behaviours and applications of oscillatory integrals of the first kind in the form

$$I_{a,\phi}(\lambda) = \int_{\mathbb{R}^d} e^{i\lambda\phi(x)} a(x) dx,$$

for real valued phase function $\phi \in \mathcal{C}^\infty(\mathbb{R}^d)$ and compactly supported $a \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. We are interested in the decay of this integral as $\lambda \to \infty$. Note that this is different from oscillatory integrals of the second kind, which is given in the form of the operator

$$T_\lambda(f)(x) = \int_{\mathbb{R}^d} e^{i\lambda\phi(x,y)} K(x,y) f(y)\ dy.$$

This characterization is given by Stein in [3].

There are many reasons to study such integrals as they have been an essential part of harmonic analysis since the beginning of the subject. A straightforward connection is that the Fourier transform itself is an oscillatory integral.

This paper mainly focuses on proposing some main ideas of the theory of oscillatory integrals in higher dimensions as well as some applications of them in deriving the decay properties of solutions to partial differential equations. The proofs for the nonstationary and stationary phases are based on those in [1]. The section about the Schrödinger equation is derived from an exercise in [2].

We will introduce some preliminary definitions in Section 2. In Section 3, we discuss separately the case in which $\phi(x)$ does not have a stationary point on $supp(a)$, and the case in which $\phi(x)$ has a nondegenerate stationary point on $supp(a)$, and prove some main results about the decay of $|I_{a,\phi}(\lambda)|$. In Section 4, we demonstrate applications of

Date: AUGUST 14.

these results to obtaining a universal rate of decay of the solution to the Schrödinger equation.

2. Preliminary Definitions

We first define the Fourier transform of complex valued Borel measures and the Schwartz space.

Definition 2.1. Let μ be a complex-valued Borel measure. We define the function

$$\widehat{\mu}(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\langle x,\xi\rangle} d\mu(x),$$

where $\langle x, \xi\rangle$ is the Euclidean inner product. Then $\widehat{\mu}(\xi)$ is called the **Fourier transform** of μ.

Definition 2.2. Let dx be the Lebesgue measure on $\mathbb{R}^d$. Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ be a measurable function. Then f is a **density** of a measure μ on $\mathbb{R}^d$ if for any $A \subseteq \mathbb{R}^d$,

$$\mu(A) = \int_A f(x)\ dx,$$

and we write $d\mu(x) = f(x)dx$.

Moreover, observe that if $d\mu(x) = f(x)dx$ for $f \in L^1(\mathbb{R}^d)$, then the Fourier transform of μ is well defined and we also write $\widehat{\mu}(\xi) = \widehat{f}(\xi)$.

Definition 2.3. The **Schwartz space** $\mathcal{S}(\mathbb{R}^d)$ is the collection of all functions in $\mathcal{C}^\infty(\mathbb{R}^d)$ that decay rapidly, together with all derivatives. i.e., $f \in \mathcal{S}(\mathbb{R}^d)$ if and only if

$$C_{\alpha,\beta} := \sup_{x\in\mathbb{R}^d} (1 + \|x\|)^\alpha \left|D^\beta f(x)\right| < \infty,$$

for all $\alpha \in \mathbb{N}_0$ and multi-index β.

Note that if f is a Schwartz function in $\mathbb{R}^d$, then for each multi-index β,

$$\int_{\mathbb{R}^d} \left|D^\beta f(x)\right| dx \le \int_{\mathbb{R}^d} \frac{C_{\alpha,\beta}}{(1 + \|x\|^\alpha)} dx, \text{ for any } \alpha \in \mathbb{N}_0. \tag{2.4}$$

Take α large enough so that the integral in (2.4) is finite. Then we have

$$\int_{\mathbb{R}^d} \left|D^\beta f(x)\right| dx < \infty.$$

Definition 2.5. A $\mathcal{C}^\infty$ function $f : \mathbb{R}^d \to \mathbb{C}$ is **compactly supported** if f vanishes identically outside some compact subset of $\mathbb{R}^d$. We write $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ for such f and define the **support** of f as

$$\text{supp}(f) := \overline{\{x \in \mathbb{R}^d : f(x) \neq 0\}}.$$

Observe that if $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, then $f(x) = 0$ for all $x \notin supp(f)$. Meanwhile, f is infinitely differentiable and all its derivatives are continuous. Thus,

$$\sup_{x\in\mathbb{R}^d} (1 + \|x\|)^\alpha \left|D^\beta f(x)\right| = \sup_{x\in supp(f)} (1 + \|x\|)^\alpha \left|D^\beta f(x)\right| < \infty,$$

for all $\alpha \in \mathbb{N}_0$ and multi-index β. It follows that $f \in \mathcal{S}(\mathbb{R}^d)$.

Finally, we define the adjoint of a linear operator in the sense of distributions.

Definition 2.6. Given a linear operator $L : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$, We say that L^* is the **ajoint of L in the sense of distributions** if for all $f, g \in \mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} L[f](x)g(x)\ dx = \int_{\mathbb{R}^d} f(x)L^*[g](x)\ dx.$$

3. Oscillatory Integrals

Recall that the oscillatory integral is defined as

$$I_{a,\phi}(\lambda) := \int_{\mathbb{R}^d} e^{i\lambda\phi(x)} a(x) dx,$$

for real valued phase function $\phi \in \mathcal{C}^\infty(\mathbb{R}^d)$ and compactly supported function $a \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. We are interested in the behaviour of the integral for large λ. To examine this, we will distinguish between the cases in which the function ϕ does and does not have a stationary point on $supp(a)$.

3.1. Nonstationary Phase.

This section addresses the case of nonstationary phase, namely the case where ϕ does not have a critical point on $supp(a)$. In Theorem 3.1, we show that for the nonstationary phase, $|I_{a,\phi}(\lambda)|$ decays as $O(\lambda^{-N})$ for arbitrary $N \in \mathbb{N}$.

Theorem 3.1. *Let $\phi \in \mathcal{C}^\infty(\mathbb{R}^d)$ and $a \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. Suppose there exists $c > 0$ such that $\|\nabla\phi(x)\| \geq c$ for all $x \in supp(a)$. Then for arbitrary $N \in \mathbb{N}$, there exists a constant $C(N, d, \phi, a)$ such that*

$$|I_{a,\phi}(\lambda)| \leq \frac{C(N, d, \phi, a)}{\lambda^N}$$

Proof. Consider the linear differential operators $L, L_0 : \mathcal{C}^1(\mathbb{R}^d) \to \mathcal{C}(\mathbb{R}^d)$ defined as

$$L_0[f](x) := \left\langle \frac{\nabla\phi(x)}{\|\phi(x)\|^2}, \nabla f(x) \right\rangle, \text{ and } L[f](x) := \frac{1}{i\lambda} L_0[f](x), \tag{3.2}$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product.

Let $L^N[f]$ denote L composed n times with itself. It is not hard to prove by induction that for each $N \in \mathbb{N}$,

$$L^N[f] = \frac{1}{(i\lambda)^N} \cdot L_0^N[f].$$

We further claim that the linear operator $L_0^* : \mathcal{C}^1(\mathbb{R}^d) \to \mathcal{C}(\mathbb{R}^d)$

$$L_0^*[f](x) := -\nabla \cdot \left(\frac{\nabla\phi(x)}{\|\phi(x)\|^2} \cdot f(x) \right)$$

is the adjoint of L_0 in the sense of distributions. Observe that for $f, g \in \mathcal{S}(\mathbb{R}^d)$, integration by parts yields

$$\begin{aligned} \int_{\mathbb{R}^d} L_0[f](x) g(x)\, dx &= \int_{\mathbb{R}^d} \left(\sum_{i=1}^d \left[\frac{\nabla\phi_i(x)}{\|\nabla\phi(x)\|^2} g(x) \right] \right) \cdot \frac{\partial f}{\partial x_i}(x)\, dx \\ &= -\int_{\mathbb{R}^d} \left(\sum_{i=1}^d \frac{\partial}{\partial x_i} \left[\frac{\nabla\phi_i(x)}{\|\nabla\phi(x)\|^2} g(x) \right] \right) \cdot f(x)\, dx \\ &= \int_{\mathbb{R}^d} -\nabla \cdot \left(\frac{\nabla\phi(x)}{\|\nabla\phi(x)\|^2} g(x) \right) f(x)\, dx. \end{aligned} \tag{3.3}$$

Moreover, the function $e^{i\lambda\phi(x)}$ is an eigenfunction of the linear operator L with eigenvalue one:

$$\begin{aligned} L[e^{i\lambda\phi}](x) &= \frac{1}{i\lambda} \left\langle \frac{\nabla\phi(x)}{\|\phi(x)\|^2}, \nabla \left(e^{i\lambda\phi(x)} \right) \right\rangle \\ &= \frac{i\lambda}{i\lambda} \frac{e^{i\lambda\phi(x)}}{\langle \nabla\phi(x), \nabla\phi(x) \rangle} \langle \nabla\phi(x), \nabla\phi(x) \rangle \\ &= e^{i\lambda\phi(x)}. \end{aligned}$$

Since a is a Schwartz function and $e^{i\lambda\phi(x)}$ is bounded for real ϕ, boundary term which arises in the computation of (3.3) decays to zero, and so we can apply the ajoint of L_0

to obtain the estimate

$$\begin{aligned}
|I_{a,\phi}(\lambda)| &= \left|\int_{\mathbb{R}^d} e^{i\lambda\phi(x)} a(x)\ dx\right| \\
&= \left|\int_{\mathbb{R}^d} L^N[e^{i\lambda\phi}](x) a(x)\ dx\right| \\
&= \frac{1}{\lambda^N}\left|\int_{\mathbb{R}^d} L_0^N[e^{i\lambda\phi}](x) a(x)\ dx\right| \\
&= \frac{1}{\lambda^N}\left|\int_{\mathbb{R}^d} e^{i\lambda\phi(x)} (L_0^*)^N[a](x)\ dx\right| \\
&\leq \frac{1}{\lambda^N}\int_{\mathbb{R}^d} \left|(L_0^*)^N[a](x)\right|\ dx. \qquad (3.4)
\end{aligned}$$

We proceed by showing that the integral $\int_{R^d} \left|(L_0^*)^N[a](x)\right|\ dx$ is bounded. Firstly, by Lemma 3.6 below, we know that

$$(L_0^*)^N[a] = \sum_{i_1=1}^{d} \cdots \sum_{i_{N+1}=1}^{d} \sum_{\substack{(\alpha_0,\alpha_1)\in \\ \{(e_{i_1},0),(0,e_{i_1})\}}} \cdots \sum_{\substack{(\alpha_0,\ldots\alpha_N)\in \\ M_{i_N}((\alpha_0,\ldots,\alpha_{N-1}))}} D^{\alpha_0}\{a\} \cdot \prod_{l=1}^{N} D^{\alpha_l}\left\{\frac{\partial_{i_l}\phi}{\|\nabla\phi\|^2}\right\}.$$

Since ϕ smooth, all of its partial derivatives are continuous and bounded on $supp(a)$. Also, recall that $\|\nabla\phi(x)\| \neq 0$ for all $x \in supp(a)$, so for each $1 \leq l \leq d$, and any multi-index α, the derivative

$$D^{\alpha}\left\{\frac{\partial_{i_l}\phi}{\|\nabla\phi\|^2}\right\}$$

is continuously differentiable and bounded on $supp(a)$. Hence,

$$\prod_{l=1}^{N}\left|D^{\alpha_l}\left\{\frac{\partial_{i_l}\phi}{\|\nabla\phi\|^2}\right\}\right| \leq \prod_{l=1}^{N}\sup\left\{\left|D^{a_l}\left\{\frac{\partial_{i_l}\phi(x)}{\|\nabla\phi(x)\|^2}\right\}\right| : x \in supp(a)\right\} =: K(\phi,\alpha_1,\ldots,\alpha_N) < \infty.$$

Therefore, we have the estimate

$$\left|(L_0^*)^N[a]\right| \leq \sum_{i_1=1}^{d} \cdots \sum_{i_N=1}^{d} \sum_{\substack{(\alpha_0,\alpha_1)\in \\ \{(e_{i_1},0),(0,e_{i_1})\}}} \cdots \sum_{\substack{(\alpha_0,\ldots\alpha_N)\in \\ M_{i_N}((\alpha_0,\ldots,\alpha_{N-1}))}} |K(\phi,\alpha_1,\ldots,\alpha_N)|\,|D^{\alpha_0}\{a\}|.$$

Since a is a Schwartz function, by (2.4) we know that the integral in $\int_{R^d} |D^{\alpha_0}\{a(x)\}|\ dx$ is finite for any multi-index α. Thus, the Sobolev norm of a defined below exists for each N:

$$\|a\|_{W^{N,1}} = \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|\leq N} \int_{\mathbb{R}^d} |D^{\alpha}a(x)|\ dx < \infty.$$

Note that this norm is translation invariant and monotone in N.

In each stage of the sum in $(L_0^*)^N[a]$, the operation $M_i(\alpha_0,\ldots\alpha_k)$ adds only one e_i to all matrices in the output set. Therefore, for each term of the sum, the multi-index for differentiation obeys $|a_0| + \ldots + |a_N| = N$. By monotonicity of the norm $\|.\|_{W^{N,1}(\mathbb{R}^d)}$, we obtain the final estimate

$$\begin{aligned}
&\int_{R^d} |(L_0^*)[a](x)|\ dx \\
&\leq \sum_{i_1=1}^{d} \cdots \sum_{i_N=1}^{d} \sum_{\substack{(\alpha_0,\alpha_1)\in \\ \{(e_{i_1},0),(0,e_{i_1})\}}} \cdots \sum_{\substack{(\alpha_0,\ldots\alpha_N)\in \\ M_{i_N}((\alpha_0,\ldots,\alpha_{N-1}))}} |K(\phi,\alpha_1,\ldots,\alpha_N)| \int_{R^d} |D^{\alpha_0}\{a(x)\}|\ dx \\
&\leq \|a\|_{W^{N,1}} \underbrace{\sum_{i_1=1}^{d} \cdots \sum_{i_N=1}^{d} \sum_{\substack{(\alpha_0,\alpha_1)\in \\ \{(e_{i_1},0),(0,e_{i_1})\}}} \cdots \sum_{\substack{(\alpha_0,\ldots\alpha_N)\in \\ M_{i_N}((\alpha_0,\ldots,\alpha_{N-1}))}} |K(\phi,\alpha_1,\ldots,\alpha_N)|}_{=:C(N,d,\phi)} < \infty.
\end{aligned}$$

In conclusion, line (3.4) yields

$$|I_{a,\phi}(\lambda)| \leq \frac{1}{\lambda^N} \int_{\mathbb{R}^d} \left|(L_0^*)^N[a](x)\right| \, dx \leq \|a\|_{W^{N,1}} \, C(N,d,\phi)\lambda^{-N}.$$

□

Remark 3.5. Notice that the argument for the nonstationary phase depends on the fact that $\|\nabla\phi(x)\| \neq 0$ for all $x \in supp(a)$, since one cannot divide by zero when we plug the function a into (3.2). Thus, the same argument does not apply to the case for which $\|\nabla\phi(x)\| = 0$ on $supp(a)$.

We conclude this section by proving the Lemma invoked in Theorem 3.1. We will also use this Lemma in the proof for the stationary phase (Theorem 3.7), in the next section.

Lemma 3.6. *Define the linear operator*

$$L_0^*[f] := \nabla \cdot \left(\frac{\nabla\phi}{\|\nabla\phi\|^2} f \right) = \sum_{j=1}^{d} \partial_j \left\{ \frac{\partial_j \phi}{\|\nabla\phi\|^2} f \right\}.$$

Let $S_n(k,x)$ denote the matrix

$$S_n(k,x) := \left(\overbrace{\underbrace{0,...,0,}_{k-1 \text{ zeros}} \; x, 0, ..., 0}^{n \text{ columns}} \right), \text{ for } x \in \mathbb{N}^d.$$

For multi-indices $a_0, ..., a_N \in \mathbb{N}^d$, let $M((a_0,...,a_N))$ denote the operation which takes the $d \times N$ matrix $(a_0,...,a_N)$ to a set of $d \times N+1$ matrices defined in the following way:

$$M_i((a_0,...,a_N)) := \{(a_0,...,a_N,0) + S_{N+1}(k,e_i) : 1 \leq k \leq N+1\}.$$

Let $f \in \mathcal{C}^\infty(\mathbb{R}^d)$. Then for each $N \in \mathbb{N}$, one has

$$(L_0^*)^N[f] = \sum_{i_1=1}^{d} \cdots \sum_{i_N=1}^{d} \sum_{\substack{(a_0,a_1)\in \\ \{(e_{i_1},0),(0,e_{i_1})\}}} \sum_{\substack{(a_0,a_1,a_2)\in \\ M_{i_2}((a_0,a_1))}} \cdots \sum_{\substack{(a_0,...a_N)\in \\ M_{i_N}((a_0,...,a_{N-1}))}} D^{a_0}\{f\} \cdot \prod_{l=1}^{N} D^{a_l} \left\{ \frac{\partial_{i_l}\phi}{\|\nabla\phi\|^2} \right\}.$$

Proof. We proceed by induction on N. For the ease of notation, we denote

$$a^n := (a_0,...,a_n).$$

For the base case ($N=1$), observe that

$$\begin{aligned} L_0^*[f] &= \sum_{i_1=1}^{d} \left(\frac{\partial_{i_1}\phi}{\|\nabla\phi\|^2} \partial_{i_1} f + f \partial_{i_1} \left\{ \frac{\partial_{i_1}\phi}{\|\nabla\phi\|^2} \right\} \right) \\ &= \sum_{i_1=1}^{d} \sum_{\substack{a^1\in \\ \{(e_{i_1},0),(0,e_{i_1})\}}} D^{a_0}\{f\} \cdot D^{a_1} \left\{ \frac{\partial_{i_1}\phi}{\|\nabla\phi\|^2} \right\}. \end{aligned}$$

For the inductive step, suppose that the statement about $(L_0^*)^N[f]$ holds. Then by the base case,

$$\begin{aligned} (L_0^*)^{N+1}[f] = &\underbrace{\sum_{i_{N+1}=1}^{d} \frac{\partial_{i_{N+1}}\phi}{\|\nabla\phi\|^2} \partial_{i_{N+1}} \left\{ (L_0^*)^N[f] \right\}}_{=:I_1} \\ &+ \underbrace{\sum_{i_{N+1}=1}^{d} (L_0^*)^N[f] \partial_{i_{N+1}} \left\{ \frac{\partial_{i_{N+1}}\phi}{\|\nabla\phi\|^2} \right\}}_{=:I_2}. \end{aligned}$$

Expanding $(L_0^*)^{N+1}[f]$ yields

$$I_1 = \sum_{i_1=1}^{d} \cdots \sum_{i_{N+1}=1}^{d} \sum_{\substack{a^1 \in \\ \{(e_{i_1},0),(0,e_{i_1})\}}} \cdots \sum_{\substack{a^N \in \\ M_{i_N}(a^{N-1})}} \left[D^{a_0+e_{i_{N+1}}}\{f\} \cdot D^0 \left\{ \frac{\partial_{i_{N+1}}\phi}{\|\nabla\phi\|^2} \right\} \cdot \prod_{l=1}^{N} D^{a_l} \left\{ \frac{\partial_{i_l}\phi}{\|\nabla\phi\|^2} \right\} \right.$$

$$\left. + D^{a_0}\{f\} \cdot D^0 \left\{ \frac{\partial_{i_{N+1}}\phi}{\|\nabla\phi\|^2} \right\} \cdot \sum_{l=1}^{N} D^{a_l+e_{i_{N+1}}} \left\{ \frac{\partial_{i_l}\phi}{\|\nabla\phi\|^2} \right\} \cdot \prod_{l\neq k} D^{a_k} \left\{ \frac{\partial_{i_k}\phi}{\|\nabla\phi\|^2} \right\} \right],$$

$$I_2 = \sum_{i_1=1}^{d} \cdots \sum_{i_{N+1}=1}^{d} \sum_{\substack{a^1 \in \\ \{(e_{i_1},0),(0,e_{i_1})\}}} \cdots \sum_{\substack{a^N \in \\ M_{i_N}(a^{N-1})}} D^{a_0}\{f\} \cdot D^{e_{i_{N+1}}} \left\{ \frac{\partial_{i_{N+1}}\phi}{\|\nabla\phi\|^2} \right\} \prod_{l=1}^{N} D^{a_l} \left\{ \frac{\partial_{i_l}\phi}{\|\nabla\phi\|^2} \right\}.$$

The multi-indices for differentiation in the terms of I_1+I_2 obey the following structure: for each term in the sum, the original matrix a^N(which has N columns) is taken and a new column of 0's is appended to the last entry to form a new matrix a^{N+1}. Then, for each column $a_k \in a^{N+1}$ with $1 \leq k \leq N+1$, a new vector $e_{i_{N+1}} \in \mathbb{N}^d$ is added to it to form a sum in the next stage. Explicit computation demonstrates that this operation is done for each $1 \leq k \leq N+1$ once and only once. Thus, this is equivalent to taking a sum over all the terms in the set $M_{i_{N+1}}(a^N)$. Therefore,

$$\begin{aligned}(L_0^*)^{N+1}[f] &= I_1 + I_2 \\ &= \sum_{i_1=1}^{d} \cdots \sum_{i_{N+1}=1}^{d} \sum_{\substack{a^1 \in \\ \{(e_{i_1},0),(0,e_{i_1})\}}} \cdots \sum_{\substack{a^N \in \\ M_{i_N}(a^{N-1})}} \sum_{\substack{a^{N+1} \in \\ M_{i_{N+1}}(a^N)}} D^{a_0}\{f\} \cdot \prod_{l=1}^{N+1} D^{a_l} \left\{ \frac{\partial_{i_l}\phi}{\|\nabla\phi\|^2} \right\},\end{aligned}$$

which proves the claim.

□

3.2. **Stationary Phase.** This section concerns the case in which ϕ does have a stationary point in $supp(a)$. In this case we do not have decay as $O(\lambda^{-N})$ for arbitrary N, and it can be very difficult to determine the rate of decay. However, in Theorem 3.7, we show that if the critical point of ϕ in $supp(a)$ is nondegenerate, then $|I_{a,\phi}(\lambda)|$ decays as $O(\lambda^{-d/2})$, where d is the dimension of the domain of a and ϕ.

Theorem 3.7. *Let $\phi \in \mathcal{C}^\infty(\mathbb{R}^d)$ and $a \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. Suppose that $\nabla\phi(x_0) = 0$ for some $x_0 \in supp(a)$, and the Hessian of ϕ at the stationary point x_0 is nondegenerate, that is, the $d \times d$ matrix*

$$D^2\phi(x_0) = \left[\frac{\partial\phi}{\partial x_i x_j} \right](x_0)$$

has non-zero determinant. Then for all $\lambda \geq 1$,

$$|I_{a,\phi}| \leq \frac{C(d,a,\phi)}{\lambda^{d/2}}.$$

Proof. By rotational and translational symmetry we center x_0 at the origin and assume that $\nabla\phi(0) = 0$. We know that if x_0 is a nondegenerate critical point, then it is an isolated critical point.

Because 0 is isolated, it is the only critical point of ϕ in a neighborhood U. This means that we can always separate the critical point at 0 from the non critical points by taking a neighborhood around it, no matter how small that neighborhood is. Thus, we may separate $I_{a,\phi}(\lambda)$ into two parts: the first part describes the behaviour of the integral in a neighborhood around the origin (the nondegenerate stationary point), while the second part describes the behaviour of the integral away from the stationary point. Without loss of generality, we assume that 0 is the only stationary point in $supp(a)$.

To carry out the separation, consider a smooth, compactly supported function $\chi_0 : \mathbb{R}^d \to \mathbb{R}$ such that

$$\chi_0(x) := \begin{cases} 1 & \text{if } \|x\| < 1 \\ 0 & \text{if } \|x\| \geq 2. \end{cases}$$

Using χ_0, we split up the integral in the following way:

$$\begin{aligned}|I_{a,\phi}(\lambda)| &= \left|\int_{\mathbb{R}^d} e^{i\lambda\phi(x)}a(x)\ dx\right| \\ &\leq \underbrace{\left|\int_{\mathbb{R}^d} e^{i\lambda\phi(x)}a(x)\chi_0(\sqrt{\lambda}x)\ dx\right|}_{=:I_1} \\ &\quad + \underbrace{\left|\int_{\mathbb{R}^d} e^{i\lambda\phi(x)}a(x)\left(1-\chi_0(\sqrt{\lambda}x)\right)\ dx\right|}_{=:I_2}.\end{aligned}$$

We will first bound I_1. Since a is a Schwartz function, the supremum norm $\|a\|_\infty := \sup_{x\in\mathbb{R}^d}|a(x)|$ is finite. Hence,

$$\begin{aligned}I_1 &\leq \int_{\mathbb{R}^d}\left|a(x)\chi_0(\sqrt{\lambda}x)\right|\ dx \\ &\leq \|a\|_\infty \int_{\mathbb{R}^d}\chi_0(\sqrt{\lambda}x)\ dx \\ &\leq \|a\|_\infty \int_{[-\frac{2}{\sqrt{\lambda}},\frac{2}{\sqrt{\lambda}}]^d} 1\ dx \\ &= \|a\|_\infty\, 4^d\lambda^{-d/2}.\end{aligned}$$

To estimate I_2, we define linear differential operators $L, L_0 : \mathcal{C}^1(\mathbb{R}^d) \to \mathcal{C}(\mathbb{R}^d)$ as in Theorem 3.1,

$$L_0[f](x) := \left\langle \frac{\nabla\phi(x)}{\|\phi(x)\|^2}, \nabla f(x)\right\rangle, \text{ and } L[f](x) := \frac{1}{i\lambda}L_0[f](x), \tag{3.8}$$

and obtain the estimate

$$\begin{aligned}I_2 &\leq \int_{\mathbb{R}^d}\left|e^{i\lambda\phi(x)}a(x)\left(1-\chi_0(\sqrt{\lambda}x)\right)\right|\ dx \\ &= \frac{1}{\lambda^N}\int_{\|x\|\geq\lambda^{-1/2}}\left|(L_0^*)^N\left[a(x)\left(1-\chi_0(\sqrt{\lambda}x)\right)\right]\right|\ dx.\end{aligned}$$

Notice that for I_2, $\phi(x)$ does not have a stationary point in the support of the function $a(x)\left(1-\chi_0(\sqrt{\lambda}x)\right)$ for any λ, since $0 \in supp\left(\chi_0(\sqrt{\lambda}x)\right)$ is the only stationary point and

$$supp\left(a(x)\left(1-\chi_0(\sqrt{\lambda}x)\right)\right) = supp\left(a(x)\right)\setminus supp\left(\chi_0(\sqrt{\lambda}x)\right).$$

Therefore, the use of the linear operators L, L_0 is justified.

Similar to the proof for Theorem 3.1, we denote $g_\lambda(x) := 1-\chi_0(\sqrt{\lambda}x)$ and proceed to obtain an estimate for the integral

$$\int_{\|x\|\geq\lambda^{-1/2}}\left|(L_0^*)^N\left[a(x)g_\lambda(x)\right]\right|\ dx.$$

First note that by Lemma 3.6 we have

$$\left|(L_0^*)^N\left[ag_\lambda\right]\right| \leq \sum_{i_1=1}^d\cdots\sum_{i_N=1}^d \sum_{\substack{(a_0,a_1)\in\\ \{(e_{i_1},0),(0,e_{i_1})\}}}\cdots\sum_{\substack{(a_0,\ldots a_N)\in\\ M_{i_N}((a_0,\ldots,a_{N-1}))}}\left|D^{a_0}\left\{ag_\lambda\right\}\right|\cdot\prod_{l=1}^N\left|D^{a_l}\left\{\frac{\partial_{i_l}\phi}{\|\nabla\phi\|^2}\right\}\right|.$$

We aim to find an estimate of $\left|(L_0^*)^N\left[ag_\lambda\right]\right|$ that is independent of the indices $\alpha_0, ..., \alpha_N$. Using Lemma 3.10 (see below), we may first obtain a bound for $|D^{a_0}\{ag_\lambda\}|$. Suppose

$\alpha_0 = (\beta_1, ..., \beta_d)$ for $\beta_1, ..., \beta_d \in \mathbb{N}_0$, then

$$\begin{aligned}|D^{a_0}\{ag_\lambda\}| &\leq \sum_{i_1=0}^{\beta_1} \cdots \sum_{i_d=0}^{\beta_d} \binom{\beta_1}{i_1} \cdots \binom{\beta_d}{i_d} \left| \frac{\partial^{\beta_1-i_1}}{\partial x_1^{\beta_1-i_1}} \cdots \frac{\partial^{\beta_d-i_d}}{\partial x_d^{\beta_d-i_d}} \{a\} \frac{\partial^{i_1}}{\partial x_1^{i_1}} \cdots \frac{\partial^{i_d}}{\partial x_d^{i_d}} \{g_\lambda\} \right| \\ &\leq \underbrace{\sum_{i_1=0}^{\beta_1} \cdots \sum_{i_d=0}^{\beta_d} \binom{\beta_1}{i_1} \cdots \binom{\beta_d}{i_d}}_{=:c(\alpha_0,d)} \|a\|_{C^{|\alpha_0|}(\mathbb{R}^d)} \|g_\lambda\|_{C^{|\alpha_0|}(\mathbb{R}^d)},\end{aligned}$$

where the norm $\|.\|_{C^M(\mathbb{R}^d)}$ is defined as

$$\|f\|_{C^M(\mathbb{R}^d)} := \sum_{\alpha \in \mathbb{N}_0^d : |\alpha| \leq M} \sup_{x \in \mathbb{R}^d} |D^\alpha f(x)|.$$

Note that this norm is translation invariant and monotone in M.

Recall that in each stage of the sum in $(L_0^*)^N [ag_\lambda]$, the operation $M_i(\alpha_0, ...\alpha_k)$ adds only one e_i to all matrices in the output set. Therefore, for each term of the sum, the multi-index for differentiation satisfies $|a_0| + ... + |a_N| = N$. By monotonicity of the norm $\|\cdot\|_{C^M(\mathbb{R}^d)}$, we have

$$\begin{aligned}\|a\|_{C^{|\alpha_0|}(\mathbb{R}^d)} &\leq \|a\|_{C^N(\mathbb{R}^d)}, \\ \|g_\lambda\|_{C^{|\alpha_0|}(\mathbb{R}^d)} &\leq \|g_\lambda\|_{C^N(\mathbb{R}^d)}.\end{aligned}$$

Moreover, note that for any $\alpha \in \mathbb{N}_0^d$

$$D^\alpha g_\lambda(x) = D^\alpha \left(1 - \chi_0(\lambda^{1/2} x)\right) = \lambda^{\frac{|\alpha|}{2}} D^\alpha g_1(\lambda^{\frac{1}{2}} x).$$

From the hypothesis we know that $\lambda \geq 1$, so by translation and scaling invariance of the norm $\|.\|_{C^M(\mathbb{R}^d)}$,

$$\begin{aligned}\|g_\lambda\|_{C^{\alpha_0}(\mathbb{R}^d)} &= \sum_{\alpha \in \mathbb{N}_0^d : |\alpha| \leq |\alpha_0|} \lambda^{\frac{|\alpha|}{2}} \sup_{x \in \mathbb{R}^d} \left| D^\alpha g_1(\lambda^{\frac{1}{2}} x) \right| \\ &\leq \lambda^{\frac{|\alpha_0|}{2}} \sum_{\alpha \in \mathbb{N}_0^d : |\alpha| \leq |\alpha_0|} \sup_{x \in \mathbb{R}^d} \left| D^\alpha g_1(\lambda^{\frac{1}{2}} x) \right| \\ &= \lambda^{\frac{|\alpha_0|}{2}} \|g_1\|_{C^{|\alpha_0|}(\mathbb{R}^d)}.\end{aligned}$$

It follows that

$$\begin{aligned}\left| D^{|a_0|}\{ag_\lambda\} \right| &\leq c(\alpha_0, d) \|a_0\|_{C^{|\alpha_0|}(\mathbb{R}^d)} \|g_\lambda\|_{C^{|\alpha_0|}(\mathbb{R}^d)} \\ &\leq c(\alpha_0, d) \|a\|_{C^N(\mathbb{R}^d)} \|g_1\|_{C^N(\mathbb{R}^d)} \lambda^{\frac{|\alpha_0|}{2}}.\end{aligned}$$

Now we proceed to bound the term $\prod_{l=1}^N \left| D^{a_l} \left\{ \frac{\partial_{i_l} \phi}{\|\nabla \phi\|^2} \right\} \right|$. We will take for granted the bound in [1, page 87], which states that for each $\alpha \in \mathbb{N}_0^d$, there exists a constant $K(\alpha) > 0$ such that for all $x \in \mathbb{R}^d$,

$$\left\| D^\alpha \left\{ \frac{\nabla \phi(x)}{\|\nabla \phi(x)\|^2} \right\} \right\| \leq K(\alpha) \|x\|^{-1-|\alpha|}.$$

Then,

$$\begin{aligned}\prod_{l=1}^N \left| D^{a_l} \left\{ \frac{\partial_{i_l} \phi}{\|\nabla \phi\|^2} \right\} \right| &\leq \prod_{l=1}^N \left\| D^{\alpha_l} \left\{ \frac{\nabla \phi(x)}{\|\nabla \phi(x)\|^2} \right\} \right\| \\ &\leq \|x\|^{-N-\sum_{l=1}^N |\alpha_l|} \prod_{l=1}^N K(\alpha_l) \\ &= S(\alpha_1, ..., \alpha_N) \|x\|^{-N-\sum_{l=1}^N |\alpha_l|},\end{aligned}$$

where $S(\alpha_1, ..., \alpha_N) := \prod_{l=1}^N K(\alpha_l)$.

Therefore, combining the above estimates yields

$$\begin{aligned}\left|\left(L_0^*\right)^N[ag_\lambda]\right| &\leq \sum_{i_1=1}^{d}\cdots\sum_{i_N=1}^{d}\sum_{\substack{(a_0,a_1)\in\\ \{(e_{i_1},0),(0,e_{i_1})\}}}\cdots\sum_{\substack{(a_0,\dots a_N)\in\\ M_{i_N}((a_0,\dots,a_{N-1}))}}\left|D^{a_0}\{ag_\lambda\}\right|\cdot\prod_{l=1}^{N}\left|D^{a_l}\left\{\frac{\partial_{i_l}\phi}{\|\nabla\phi\|^2}\right\}\right| \\ &\leq \|a\|_{C^N(\mathbb{R}^d)}\|g_1\|_{C^N(\mathbb{R}^d)}\sum_{i_1=1}^{d}\cdots\sum_{i_N=1}^{d}\sum_{\substack{(a_0,a_1)\in\\ \{(e_{i_1},0),(0,e_{i_1})\}}}\cdots\sum_{\substack{(a_0,\dots a_N)\in\\ M_{i_N}((a_0,\dots,a_{N-1}))}} \\ &\quad c(\alpha_0,d)S(\alpha_1,\dots,\alpha_N)\lambda^{\frac{|\alpha_0|}{2}}\|x\|^{-N-\sum_{l=1}^{N}|\alpha_l|}.\end{aligned}$$

We will further find an estimate for $\lambda^{\frac{|\alpha_0|}{2}}\|x\|^{-N-\sum_{l=1}^{k}|\alpha_l|}$ by removing dependence on $\alpha_0,\dots,\alpha_N$.

Observe that if $|\alpha_0|\neq 0$, since $\|x\|\geq\lambda^{-1/2}$ and $|\alpha_0|+\dots+|\alpha_N|=N$, we have

$$\begin{aligned}\lambda^{\frac{|\alpha_0|}{2}}\|x\|^{-N-\sum_{l=1}^{N}|\alpha_l|} &= \lambda^{\frac{N-\sum_{l=1}^{N}|\alpha_l|}{2}}\|x\|^{-N-\sum_{l=1}^{N}|\alpha_l|} \\ &= \lambda^{\frac{N}{2}}\|x\|^{-N}\left(\frac{\lambda^{-\frac{1}{2}}}{\|x\|}\right)^{\sum_{l=1}^{N}|\alpha_l|} \\ &\leq \lambda^{\frac{N}{2}}\|x\|^{-N}.\end{aligned}$$

If $|\alpha_0|=0$, then $|\alpha_1|+\dots+|\alpha_N|=N$ and we have

$$\lambda^{\frac{|\alpha_0|}{2}}\|x\|^{-N-\sum_{l=1}^{N}|\alpha_l|}=\|x\|^{-2N}.$$

Combining the two cases yields that for any $\|x\|\geq\lambda^{-1/2}$,

$$\begin{aligned}\lambda^{\frac{|\alpha_0|}{2}}\|x\|^{-N-\sum_{l=1}^{N}|\alpha_l|} &\leq \max\{\|x\|^{-2N},\lambda^{\frac{N}{2}}\|x\|^{-N}\} \\ &\leq \|x\|^{-2N}+\lambda^{\frac{N}{2}}\|x\|^{-N}.\end{aligned}$$

Hence, for all $\|x\|\geq\lambda^{-1/2}$,

$$\begin{aligned}\left|\left(L_0^*\right)^N[ag_\lambda]\right| &\leq \|a\|_{C^N(\mathbb{R}^d)}\|g_1\|_{C^N(\mathbb{R}^d)}\sum_{i_1=1}^{d}\cdots\sum_{i_N=1}^{d}\sum_{\substack{(a_0,a_1)\in\\ \{(e_{i_1},0),(0,e_{i_1})\}}}\cdots\sum_{\substack{(a_0,\dots a_N)\in\\ M_{i_N}((a_0,\dots,a_{N-1}))}} \\ &\quad c(\alpha_0,d)S(\alpha_1,\dots,\alpha_N)\left(\lambda^{\frac{|\alpha_0|}{2}}\|x\|^{-N-\sum_{l=1}^{k}|\alpha_l|}\right) \\ &\leq C(N,d)\|a\|_{C^N(\mathbb{R}^d)}\|g_1\|_{C^N(\mathbb{R}^d)}\left(\|x\|^{-2N}+\lambda^{\frac{N}{2}}\|x\|^{-N}\right),\end{aligned}$$

where the constant $C(N,d)$ is defined as:

$$C(N,d):=\sum_{i_1=1}^{d}\cdots\sum_{i_N=1}^{d}\sum_{\substack{(a_0,a_1)\in\\ \{(e_{i_1},0),(0,e_{i_1})\}}}\cdots\sum_{\substack{(a_0,\dots a_N)\in\\ M_{i_N}((a_0,\dots,a_{N-1}))}}c(\alpha_0,d)S(\alpha_1,\dots,\alpha_N).$$

Therefore, fixing any $N>d$ and switching to spherical coordinates (i.e., $x=rw$, for $r=\|x\|$ and $w\in S^{d-1}$) yields

$$\begin{aligned}&\int_{\|x\|\geq\lambda^{-1/2}}\left|\left(L_0^*\right)^N[ag_\lambda]\right|\,dx \\ &\quad\leq C(N,d)\|a\|_{C^N(\mathbb{R}^d)}\|g_1\|_{C^N(\mathbb{R}^d)}\int_{\|x\|\geq\lambda^{-1/2}}\|x\|^{-2N}+\lambda^{\frac{N}{2}}\|x\|^{-N}\,dx \\ &\quad= C(N,d)\|a\|_{C^N(\mathbb{R}^d)}\|g_1\|_{C^N(\mathbb{R}^d)}\int_{\lambda^{-1/2}}^{\infty}\int_{S^{d-1}}r^{d-1}(r^{-2N}+\lambda^{\frac{N}{2}}r^{-N})\,dwdr \\ &\quad= C(N,d)\|a\|_{C^N(\mathbb{R}^d)}\|g_1\|_{C^N(\mathbb{R}^d)}\frac{3N-2d}{(N-d)(2N-d)}\lambda^{\frac{2N-d}{2}}<\infty.\end{aligned}$$

Finally, we find a bound for $\int_{\|x\|\geq\lambda^{-1/2}} \left|(L_0^*)^N [ag_\lambda]\right| \, dx$ in terms of λ. Let $C^*(d,\phi)$ denote the constant

$$C^*(d,\phi) := \max\left\{C(N,d)\,\|g_1\|_{C^N(\mathbb{R}^d)}\,\frac{3N-2d}{(N-d)(2N-d)}, 4^d\right\},$$

for fixed integer $N > d \geq 1$. Then, combining all the earlier estimates yields

$$\begin{aligned}
I_1 &\leq 4^d\,\|a\|_\infty\,\lambda^{-d/2} \\
&\leq C^*(d,\phi)\,\|a\|_{C^N(\mathbb{R}^d)}\,\lambda^{-d/2}, \\
I_2 &\leq \lambda^{-N}\int_{\|x\|\geq\lambda^{-1/2}} \left|(L_0^*)^N [ag_\lambda]\right| \, dx \\
&\leq C(N,d)\,\|a\|_{C^N(\mathbb{R}^d)}\,\|g_1\|_{C^N(\mathbb{R}^d)}\,\lambda^{-d/2} \\
&= C^*(d,\phi)\,\|a\|_{C^N(\mathbb{R}^d)}\,\lambda^{-d/2}.
\end{aligned}$$

In summary, for fixed $N > d$, we have

$$|I_{\phi,a}(\lambda)| \leq I_1 + I_2 \leq 2C^*(d,\phi)\,\|a\|_{C^N(\mathbb{R}^d)}\,\lambda^{-d/2}.$$

□

Finally, we prove the following two Lemmas which justify the claims we made about the bound for $|D^{a_0}\{ag_\lambda\}|$ in the proof of Theorem 3.7.

Lemma 3.9. *Suppose $a, g : \mathbb{R}^d \to \mathbb{R}$ are smooth functions, then for any $i \in \{1, 2, ..., d\}$ and any $\alpha \in N_0$,*

$$\frac{\partial^\alpha}{\partial x_i^\alpha}\{ag\} = \sum_{k=0}^{\alpha}\binom{\alpha}{k}\frac{\partial^{\alpha-k}a}{\partial x_i^{\alpha-k}}\frac{\partial^k g}{\partial x_i^k}.$$

Proof. We proceed by induction on α. Fix $i \in \{1, 2, ..., d\}$. Then for $\alpha = 1$,

$$\frac{\partial}{\partial x_i}\{ag\} = a\frac{\partial g}{\partial x_i} + g\frac{\partial a}{\partial x_i} = \sum_{k=0}^{1}\binom{1}{k}\frac{\partial^{1-k}a}{\partial x_i^{1-k}}\frac{\partial^k g}{\partial x_i^k}.$$

Suppose that the statement holds for α. For $\alpha + 1$,

$$\begin{aligned}
\frac{\partial^{\alpha+1}}{\partial x_i^{\alpha+1}}\{ag\} &= \sum_{k=0}^{\alpha}\binom{\alpha}{k}\frac{\partial^{\alpha-k+1}a}{\partial x_i^{\alpha-k+1}}\frac{\partial^k g}{\partial x_i^k} + \sum_{k=0}^{\alpha}\binom{\alpha}{k}\frac{\partial^{\alpha-k}a}{\partial x_i^{\alpha-k}}\frac{\partial^{k+1} g}{\partial x_i^{k+1}} \\
&= \binom{\alpha+1}{0}\frac{\partial^{\alpha+1}a}{\partial x_i^{\alpha+1}}g + \sum_{k=1}^{\alpha}\left(\binom{\alpha}{k} + \binom{\alpha}{k-1}\right)\frac{\partial^{\alpha-k+1}a}{\partial x_i^{\alpha-k+1}}\frac{\partial^k g}{\partial x_i^k} + \binom{\alpha+1}{\alpha+1}a\frac{\partial^{\alpha+1}g}{\partial x_i^{\alpha+1}} \\
&= \sum_{k=0}^{\alpha+1}\binom{\alpha+1}{k}\frac{\partial^{\alpha+1-k}a}{\partial x_i^{\alpha+1-k}}\frac{\partial^k g}{\partial x_i^k}.
\end{aligned}$$

□

Lemma 3.10. *Suppose $a, g : \mathbb{R}^d \to \mathbb{R}$ are smooth functions, then for any muti-index $\alpha = (\alpha_1, ...\alpha_d) \in \mathbb{N}^d$, one has*

$$D^a(ag) = \sum_{i_1=0}^{\alpha_1}\cdots\sum_{i_d=0}^{\alpha_d}\binom{\alpha_1}{i_1}\cdots\binom{\alpha_d}{i_d}\frac{\partial^{\alpha_1-i_1}}{\partial x_1^{\alpha_1-i_1}}\cdots\frac{\partial^{\alpha_d-i_d}}{\partial x_d^{\alpha_d-i_d}}\{a\}\frac{\partial^{i_1}}{\partial x_1^{i_1}}\cdots\frac{\partial^{i_d}}{\partial x_d^{i_d}}\{g\}$$

Proof. We proceed by induction on d. The base case (when $d = 1$) follows from Lemma 3.9. Now suppose that the statement holds for all $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}_0^d$.

Consider the multi-index $\alpha^* = (\alpha_0, ..., \alpha_d, \alpha_{d+1}) \in \mathbb{N}_0^{d+1}$. By Lemma 3.9 we obtain

$$\begin{aligned}
D^{\alpha^*}(ag) &= \frac{\partial^{\alpha_{d+1}}}{\partial x_{d+1}^{\alpha_{d+1}}}\left(D^{\alpha}(ag)\right) \\
&= \sum_{i_1=0}^{\alpha_1} \cdots \sum_{i_d=0}^{\alpha_d} \binom{\alpha_1}{i_1} \cdots \binom{\alpha_d}{i_d} \frac{\partial^{\alpha_{d+1}}}{\partial x_{d+1}^{\alpha_{d+1}}} \left(\frac{\partial^{\alpha_1 - i_1}}{\partial x_1^{\alpha_1 - i_1}} \cdots \frac{\partial^{\alpha_d - i_d}}{\partial x_d^{\alpha_d - i_d}} \{a\} \frac{\partial^{i_1}}{\partial x_1^{i_1}} \cdots \frac{\partial^{i_d}}{\partial x_d^{i_d}} \{g\} \right) \\
&= \sum_{i_1=0}^{\alpha_1} \cdots \sum_{i_d=0}^{\alpha_d} \binom{\alpha_1}{i_1} \cdots \binom{\alpha_d}{i_d} \sum_{i_{d+1}}^{\alpha_{d+1}} \binom{\alpha_{d+1}}{i_{d+1}} \frac{\partial^{\alpha_1 - i_1}}{\partial x_1^{\alpha_1 - i_1}} \cdots \frac{\partial^{\alpha_{d+1} - i_{d+1}}}{\partial x_{d+1}^{\alpha_{d+1} - i_{d+1}}} \{a\} \frac{\partial^{i_1}}{\partial x_1^{i_1}} \cdots \frac{\partial^{i_{d+1}}}{\partial x_{d+1}^{i_{d+1}}} \{g\} \\
&= \sum_{i_1=0}^{\alpha_1} \cdots \sum_{i_{d+1}=0}^{\alpha_{d+1}} \binom{\alpha_1}{i_1} \cdots \binom{\alpha_{d+1}}{i_{d+1}} \frac{\partial^{\alpha_1 - i_1}}{\partial x_1^{\alpha_1 - i_1}} \cdots \frac{\partial^{\alpha_{d+1} - i_{d+1}}}{\partial x_{d+1}^{\alpha_{d+1} - i_{d+1}}} \{a\} \frac{\partial^{i_1}}{\partial x_1^{i_1}} \cdots \frac{\partial^{i_{d+1}}}{\partial x_{d+1}^{i_{d+1}}} \{g\}.
\end{aligned}$$

□

4. Uniform Rate of Decay in Schrödinger

One application of oscillatory integrals is their use in obtaining a uniform rate of decay for solutions to partial differential equations. We will provide a detailed example to how to find such rate of decay for the solution to the Schrödinger equation. For $(x,t) \in \mathbb{R}^d \times \mathbb{R}$, consider the Schrödinger equation

$$i\frac{\partial}{\partial t}\left(\psi(x,t)\right) - \Delta\psi(x,t) = 0, \quad \psi(x,0) = \psi_0(x), \tag{4.1}$$

where $\psi_0 : \mathbb{R}^d \to \mathbb{R}$ is a fixed Schwartz function such that $\widehat{\psi_0} \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and $\psi : \mathbb{R}^{d+1} \to \mathbb{R}$.

Observe that if $\psi(x,t)$ is also a Schwartz function in x for any t, then using the differentiation property of the Fourier transform (i.e., $\widehat{\partial_{x_j} f}(\xi) = i\xi^j \widehat{f}(\xi)$), the Fourier transform of $\Delta\psi$ with respect to the spatial variable x is given by

$$\begin{aligned}
\widehat{\Delta\psi}(\xi,t) = \widehat{\sum_{i=1}^{d} \frac{\partial^2 \psi}{\partial {x_i}^2}}(\xi,t) &= \left(\sum_{i=1}^{d} {\xi_i}^2\right) \cdot \left(-\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \psi(x,t) e^{-i\langle x,\xi\rangle}\, dx\right) \\
&= -\left\|\xi\right\|^2 \widehat{\psi}(\xi,t).
\end{aligned}$$

Therefore, applying the Fourier transform to both sides of (4.1) yields

$$i\frac{\partial}{\partial t}\left(\widehat{\psi}(\xi,t)\right) + \left\|\xi\right\|^2 \widehat{\psi}(\xi,t) = 0, \quad \widehat{\psi}(\xi,0) = \widehat{\psi_0}(\xi). \tag{4.2}$$

Notice that (4.2) is an ordinary differential equation with respect to the time variable t. The solution to this ordinary differential equation is thus given by

$$\widehat{\psi}(\xi,t) = e^{i\|\xi\|^2 t}\widehat{\psi_0}(\xi).$$

Applying the inverse Fourier transform on $\widehat{\psi}(\xi,t)$, we acquire the solution to the Schrödinger equation:

$$\psi(x,t) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{it\|\xi\|^2 + i\langle x,\xi\rangle} \widehat{\psi_0}(\xi)\, d\xi = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{it\left(\|\xi\|^2 + \left\langle \frac{x}{t},\xi\right\rangle\right)} \widehat{\psi_0}(\xi)\, d\xi.$$

For simplicity reasons, from now on we will drop the coefficient $\frac{1}{(2\pi)^{d/2}}$.

By completing the square, we have for $t > 0$,

$$\|\xi\|^2 + \left\langle \frac{x}{t}, \xi \right\rangle = \left\|\xi + \frac{x}{2t}\right\|^2 - \left\|\frac{x}{2t}\right\|^2.$$

Thus, treating the term $e^{-it\left\|\frac{x}{2t}\right\|^2}$ as a constant with absolute value one, and doing a change of variable, one has

$$\begin{aligned}|\psi(x,t)| &= \left|\int_{\mathbb{R}^d} e^{it\left(\left\|\xi+\frac{x}{2t}\right\|^2-\left\|\frac{x}{2t}\right\|^2\right)}\widehat{\psi_0}(\xi)\ d\xi\right| \\ &= \left|\int_{\mathbb{R}^d} e^{it\left\|\xi+\frac{x}{2t}\right\|^2}\widehat{\psi_0}(\xi)\ d\xi\right| \\ &= \left|\int_{\mathbb{R}^d} e^{it\|v\|^2}\widehat{\psi_0}\left(v-\frac{x}{2t}\right)\ dv\right|.\end{aligned}$$

If we further denote

$$\widehat{\psi_0^*}(v) := \widehat{\psi_0}\left(v-\frac{x}{2t}\right), \text{ and } \phi^*(v) := \|v\|^2,$$

then

$$|\psi(x,t)| = \left|\int_{\mathbb{R}^d} e^{it\phi^*(v)}\widehat{\psi_0^*}(v)\ dv\right| =: \left|I_{\phi^*,\widehat{\psi_0^*}}(t)\right|$$

defines an oscillatory integral.

Notice that

$$\|\nabla\phi^*(v)\| = \left\|\nabla\|v\|^2\right\| = \left(4v_1^2+...+4v_d^2\right)^{1/2} = 0$$

if and only if $v=0$. Moreover, the Hessian matrix of $\|v\|^2$ is the constant diagonal matrix $diag(2,...,2)$ at any $v\in\mathbb{R}^d$. Thus, we know that the determinant $Det(D^2\|v\|^2) = 2^d \neq 0$ at any $v\in\mathbb{R}^d$, and so the only stationary point of $\|v\|^2$ (which is the origin) is nondegenerate.

We first consider the case in which $0\notin supp(\widehat{\psi_0^*}(v))$. By the nonstationary phase argument in Theorem 3.1 we have for any $N\in\mathbb{N}$ and fixed $x\in\mathbb{R}^d$, $t>0$,

$$|\psi(x,t)| = \left|I_{\phi^*,\widehat{\psi_0^*}}(t)\right| \leq \left\|\widehat{\psi_0^*}\right\|_{W^{N,1}} C(N,d,\phi^*)t^{-N}.$$

Note that this bound is dependent on both x and t, and the $x-$dependence is exclusively given by the term $\widehat{\psi_0^*}$. We can remove the dependence by invoking the fact that the Sobolev norm is translation invariant: for any x and t,

$$\left\|\widehat{\psi_0^*}\right\|_{W^{N,1}} = \sum_{\alpha\in\mathbb{N}_0^d:|\alpha|\leq N}\int_{\mathbb{R}^d}\left|D^\alpha\widehat{\psi_0}\left(v-\frac{x}{2t}\right)\right|\ dx = \left\|\widehat{\psi_0}\right\|_{W^{N,1}},$$

and so for all $x\in\mathbb{R}^d$, we have the universal bound

$$|\psi(x,t)| \leq \left\|\widehat{\psi_0}\right\|_{W^{N,1}(\mathbb{R}^d)} C(N,d,\phi^*)t^{-N}, \text{ for } t>0.$$

If, on the other hand, $0\in supp(\widehat{\psi_0^*}(v))$, then Theorem 3.7 applies. In particular, there is a constant $C(\phi,d)$ for $N>d$ such that

$$|\psi(x,t)| = \left|I_{\phi^*,\widehat{\psi_0^*}}(t)\right| \leq \left\|\widehat{\psi_0^*}\right\|_{C^N(\mathbb{R}^d)} 2C(d,\phi^*)t^{-d/2}.$$

Since the norm $\left\|\widehat{\psi_0}\right\|_{C^N(\mathbb{R}^d)}$ is also translation invariant, we have

$$|\psi(x,t)| \leq \left\|\widehat{\psi_0}\right\|_{C^N(\mathbb{R}^d)} 2C(d,\phi^*)t^{-d/2}, \text{ for } t>0.$$

Acknowledgments

I would like to thank my mentor, Peter Morfe, for recommending me the readings and helping me to work through the details of the proofs. I am very grateful for his patience and support throughout our discussions as well as the editing process of this paper. I also want to thank Peter May for organizing this REU and being such a supportive and kind person to everyone. The REU was a very enjoyable experience.

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