SIMPLICIAL SETS AND RELATED FUNCTORS

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Abstract. In this paper, we will first introduce simplicial sets combinatorially and geometrically. Then we will go over a few important properties of simplicial sets like regularity and nonsingularity. Along the way we will demonstrate how subdivision can give nice properties of simplicial sets while preserving its geometric structures. Then we turn our attention to the functors that link simplicial sets with other categories. In the end we provide a characterization of what simplicial sets are nerves of posets.

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1. Introduction

In algebraic topology, abstract simplicial complexes, ordered simplicial complexes, and CW complexes are used to decompose and analyze topological spaces. They form the foundation for many theories in algebraic topology. In this section, we will introduce these important structures, and then present a broader concept called simplicial sets that extend simplicial complexes and possess good geometric structures.

Definition 1.1. An abstract simplicial complex $K$ is a set of vertices $V$, together with a set $K$ of nonempty finite subsets of $V$ which satisfies the following properties:

- for any element $v$ in $V$, \{v\} is a set in $K$
- for each set $k \in K$, any subsets of $k$ is also in $K$

The elements in $V$ are called vertices, those in $K$ are simplicies, subsets of simplicies are faces, and facets are maximal simplicies which are not faces of any other
simplicies. Let $V(-)$ denote the vertex set of a simplex. For a simplex $\alpha \in K$, its dimension is $|V(\alpha)| - 1$.

**Definition 1.2.** The category of simplicial complexes $\mathcal{SC}$ has objects as abstract simplicial complexes and morphisms as simplicial maps which send simplicies to simplicies by specifying the vertices. That is, a simplicial map $f : A \to B$ can be determined by $V(f) : V(A) \to V(B)$.

**Definition 1.3.** An ordered simplicial complex is an abstract simplicial complex $K$ with a partial ordering on the vertices such that it becomes a total ordering when restricted to each simplex.

**Definition 1.4.** The category of ordered simplicial complexes $\mathcal{OSC}$ has objects as ordered simplicial complexes and morphisms as order-preserving simplicial maps.

**Definition 1.5.** An $n$-dim disk is the subspace $D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ of $\mathbb{R}^n$.

**Definition 1.6.** An $n$-dim open cell $e^n$ is a space homeomorphic to $\text{int}(D^n)$, the interior of an $n$-dim disk. We denote the closure of $e^n$ as $\bar{e}^n$, which is homeomorphic to $D^n$.

**Definition 1.7.** A CW complex $X$ is a space that is built inductively in the following way. First of all we start with the 0-skeleton $X^0$ which is a discrete set of points. Inductively, we can form the $n$-skeleton from $(n-1)$-skeleton by taking the pushout of the following diagram:

$$
\begin{array}{ccc}
\coprod \partial D^n & \xrightarrow{j} & X^{n-1} \\
\cap & \downarrow & \cap \\
\coprod D^n & \xrightarrow{j} & X^n \\
\end{array}
$$

where $j$ can be regarded as the disjoint union of maps defined on each piece of $D^n$ such that over each $\text{int}(D^n)$, $j$ is a homeomorphism. CW complex $X$ has weak topology, that is a set $A \subset X$ is open(closed) if and only if $A \cap X^n$ is open(closed) in $X^n$ for each $n$.

**Definition 1.8.** A regular CW complex has the closure of each open $n$-cell homeomorphic to $D^n$, which means the boundary of each $n$-cell is homeomorphic to $S^{n-1}$.

Now we turn our discussion to simplicial sets.

**Definition 1.9.** The category of finite ordered sets, denoted by $\Delta$, has objects as totally ordered sets $[n] = \{0, \ldots, n\}$ for any $n \in \mathbb{N}$ and morphisms as order-preserving maps $f : [m] \to [n]$ such that $i \leq j$ implies $f(i) \leq f(j)$. All of the order-preserving maps are generated by the following two types of maps, face maps $\delta_i : [n-1] \to [n]$ and degeneracy maps $\sigma_i : [n+1] \to [n]$ which are defined as:

$$
\delta_i(j) = \begin{cases} 
  j & \text{if } j < i \\
  j+1 & \text{if } j \geq i
\end{cases}
$$

$$
\sigma_i(j) = \begin{cases} 
  j & \text{if } j \leq i \\
  j-1 & \text{if } j > i
\end{cases}
$$
for $0 \leq i \leq n$. That is, $\delta_i$ jump the $i^{th}$ index in the codomain, and $\sigma_i$ repeat the $i^{th}$ index in the codomain.

The category of finite ordered sets is extremely important in understanding simplicial sets. In the following we introduce some special maps in $\Delta$ that will be useful in the later discussion of simplicial sets.

**Definition 1.10.** Define general face maps as order-preserving injective maps $\iota : [n] \to [m]$ in $\Delta$ for any $n \leq m \in \mathbb{N}$. If we take the descending sequence of elements $i_1 \geq \cdots \geq i_p$ which are not in $\iota([n])$, then $\iota$ can be decomposed into compositions of face maps $\iota = \delta_{i_1} \cdots \delta_{i_p}$.

**Definition 1.11.** Define general degeneracy maps as order-preserving surjective maps $\rho : [n] \to [k]$ in $\Delta$ for any $n \geq k \in \mathbb{N}$. If we take the ascending sequence of elements $j_1 \leq \cdots \leq j_q$ such that $\rho(j_k) = \rho(j_k + 1)$ for $0 \leq k \leq q$, then $\rho$ can be decomposed into compositions of degeneracy maps $\rho = \sigma_{j_1} \cdots \sigma_{j_q}$.

**Definition 1.12.** Define vertex maps as $v_i : [0] \to [n]$ with $0 \mapsto i$ in $\Delta$.

**Proposition 1.13.** Any map $f$ between finite ordered sets can be written as a composition of general degeneracy maps and general face maps, i.e. $f = \iota \circ \rho$.

**Proof.** By definition any map $f : [m] \to [n]$ in $\Delta$ is generated by the composition of degeneracy and face maps ($\delta_i$ and $\sigma_j$). Canonically we can denote $0 \leq i_p \leq \cdots \leq i_1 \leq n$ to be the descending sequence of indices that are not in $f([m])$ and the ascending chain of indices $0 \leq j_1 \leq \cdots \leq j_q < m$ such that $f(ik) = f(ik + 1)$. Then $f$ can be decomposed into:

$$f = \delta_{i_1} \cdots \delta_{i_p} \circ \sigma_{j_1} \cdots \sigma_{j_q}$$

Take $\iota = \delta_{i_1} \cdots \delta_{i_p}$ and $\rho = \sigma_{j_1} \cdots \sigma_{j_q}$. Then $f = \iota \circ \rho$. \hfill $\Box$

**Definition 1.14.** A simplicial set $K$ is a collection of sets $K_n$ for $n \geq 0$ with functions $d_i : K_n \to K_{n-1}$, $s_i : K_{n-1} \to K_n$ for $0 \leq i \leq n$ that satisfy the following simplicial relations:

$$d_i \circ d_j = d_{j-1} \circ d_i \text{ if } i < j$$

$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & \text{if } i < j \\ \text{id} & \text{if } i = j \text{ or } i + j + 1 \\ s_j \circ d_{i-1} & \text{if } i > j + 1 \end{cases}$$

$$s_i \circ s_j = s_{j+1} \circ s_i \text{ if } i \leq j$$

where $d_i$ are face maps, $s_i$ are degeneracy maps. Elements of the set $K_n$ are called $n$-simplices. A map $f : K \to L$ of simplicial sets is a sequence of functions $f_n : K_n \to L_n$ such that $f_{n-1} \circ d_i = d_{i} \circ f_n$ and $f_{n+1} \circ s_i = s_i \circ f_n$, i.e. the following diagrams commute:

$$\begin{array}{ccc}
K_n & \xrightarrow{f_n} & L_n \\
\downarrow d_i & & \downarrow d_i \\
K_{n-1} & \xrightarrow{f_{n-1}} & L_{n-1}
\end{array} \quad \begin{array}{ccc}
K_n & \xrightarrow{f_n} & L_n \\
\uparrow s_i & & \uparrow s_i \\
K_{n-1} & \xrightarrow{f_{n-1}} & L_{n-1}
\end{array}$$

This gives the category of simplicial sets, denoted as $sSet$. 

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The face and degeneracy maps in the category of simplicial sets is linked to the category of finite ordered set by the following functor.

**Definition 1.15.** Alternatively, we can regard simplicial sets as a contravariant functor from \( \Delta \) to \( \text{Set} \). The morphisms in simplicial sets become natural transformation between functors. Specifically, the face and degeneracy maps in \( \Delta \) are mapped to the face and degeneracy maps in Definition 1.14.

**Definition 1.16.** In the category of simplicial sets, we define *general degeneracy maps* \( \rho^* \) as those induced by general degeneracy maps \( \rho \) in \( \Delta \) using Definition 1.15. Similarly, define *general face maps* \( \iota^* \) and *vertex maps* \( v_i^* \) as induced by \( \iota \) and \( v_i \).

**Remark 1.17.** By Proposition 1.13, we can decompose any simplicial map \( f^* \) into a composition of general degeneracy and face maps \( \rho^* \circ \iota^* \).

**Definition 1.18.** For a simplicial set \( X \), a simplex \( x \in X \) is called *degenerate* if it can be written as \( \rho^* y \) for some simplex \( y \) and a nontrivial general degeneracy map \( \rho^* \). It is called *nondegenerate* otherwise.

**Lemma 1.19.** (Eilenberg-Zilber Lemma) For a simplicial set \( X \), any simplex \( x \) can be written in a unique decomposition of the form

\[
x = \rho_\# x
\]

where \( \rho_\# \) is a general degeneracy map and \( x_\# \) is a nondegenerate simplex.

**Proof.** Since \( x \in X_n \), we can check whether \( x \) is degenerate by enumerating through different general degeneracy maps. Since there are finite number of them this can be done. If \( x \) is nondegenerate, then the case is trivial; if \( x \) is degenerate, then by induction on the degeneracy we know \( x \) can be written in the form \( x = \rho_\# x_\# \). Suppose there are two different ways to decompose \( x \),

\[
x = \rho_1^* x_1 \quad \text{and} \quad x = \rho_2^* x_2
\]

since \( \rho_1 : [m] \to [n] \) is a surjective map, it has a right inverse \( \bar{\rho}_1 : [n] \to [m] \). Hence

\[
\rho_1^* x_1 = \rho_2^* x_2 \Rightarrow \bar{\rho}_1 \circ \rho_1^* x_1 = \bar{\rho}_1 \circ \rho_2^* x_2 \Leftrightarrow x_1 = \bar{\rho}_1 \circ \rho_2^* x_2 = (\rho_2 \circ \bar{\rho}_1)^* x_2
\]

since \( x_1 \) is nondegenerate we know \( \rho_2 \circ \bar{\rho}_1 \) cannot contain any degeneracy maps, so it must be a general face map, which implies \( \dim(x_1) \leq \dim(x_2) \). By symmetry of the previous step we also know \( \dim(x_2) \leq \dim(x_1) \). Together they give \( \dim(x_1) = \dim(x_2) \). Since \( \rho_2 \circ \bar{\rho}_1 \) is an injective order-preserving map with domain and codomain the same dimension, it must be the identity map. This gives \( x_1 = x_2 \) and the symmetry of the previous construction gives \( \rho_1 = \rho_2 \), which implies uniqueness. \( \square \)

Simplicial sets are a generalization for simplicial complexes. For simplicial complexes, each \( n \)-simplex is uniquely determined by its \( (n+1) \) distinct vertices, but for simplicial sets, a nondegenerate \( n \)-simplex might not have \( (n+1) \) distinct vertices. The following two propositions tell us more about simplicial sets.

**Proposition 1.20.** Any simplicial set \( K \) has an intrinsic ordering on its vertices( though not necessarily well-ordered).

**Proof.** Regard \( K \) as a contravariant functor from \( \Delta \) to \( \text{Set} \) as in Lemma 1.15. For any \( x \in K_n \) and vertex map \( v_i \), we have the \( i \)th vertex of \( x \) as \( K(v_i)(x) \). Therefore each simplex \( x \) has its vertices ordered, which gives an ordering on \( K \). \( \square \)
Next proposition shows a nice property of simplicial sets that we have not seen in the literature but will be important in the later part of the paper.

**Proposition 1.21.** Given a simplicial set $K$ and any arbitrary $n$-simplex $x$ with vertex set $V$, for any subset $S \subset V$ such that $|S| = m+1$, there exists an $m$-simplex in $X$ with $S$ as the set of vertices.

**Proof.** Regard the simplicial set $K$ as a contravariant functor $K : \Delta \to \text{Set}$. Define $g : [m] \to [n]$ which maps each element $i$ to the position of the $i^{th}$ element of $S$ in $V$. Consider the map $f_i : [0] \to [n]$ defined by $0 \mapsto g(i)$. Denote the $g(i)^{th}$ vertex of $x$ as $v_i$, then this means:

$$K(f_i)(x) = v_i$$

Define morphism $\delta_i : [0] \to [k]$ by $0 \mapsto i$, then $f_i = g \circ \delta_i$. Therefore any function in the collection $\{f_i\}_{i \in [m]}$ factors through $g$, as shown in the following diagram for $f_i$:

$$
\begin{array}{ccc}
[0] & \xrightarrow{f_i} & [n] \\
\downarrow{\delta_i} & & \downarrow{g} \\
[m] & & \\
\end{array}
$$

This corresponds to the following commutative diagram in the category of simplicial sets:

$$
\begin{array}{ccc}
K_0 & \xrightarrow{X(f_i)} & K_n \\
\downarrow{X(\delta_i)} & & \downarrow{X(g)} \\
K_m & & \\
\end{array}
$$

Therefore for given $x \in K_n$, $X(g)(x)$ gives a $m$-simplex with vertex set $S$. □

**Definition 1.22.** Define the standard simplicial $n$-simplex $\Delta[n]^s$ as the contravariant functor represented by $[n]$, i.e. $\text{Hom}_\Delta(-,[n])$. This means the $i$-simplicies $\Delta[n]^s_i$ of it are morphisms $\varphi : [i] \to [n]$ in $\Delta$.

**Lemma 1.23.** (Contravariant Version of Yoneda's Lemma). For any category $\mathcal{C}$, $\text{Hom}_\mathcal{C}(-,A)$ maps each object $X \in \text{Ob}(\mathcal{C})$ to the set $\text{Hom}_\mathcal{C}(X,A)$. For any contravariant functor $F : \mathcal{C} \to \text{Set}$, there is a natural bijection

$$y : \text{Nat}(\text{Hom}_\mathcal{C}(-,A), F) \cong F(A)$$

mapped by sending $\tau \mapsto \tau_A(id_A)$.

**Proof.** The proof exploits categorical facts about functors and natural transformations.

**Injectivity:** For any $B \in \text{Ob}(\mathcal{C})$ and $\varphi \in \text{Hom}_\mathcal{C}(B,A)$, we have the following commutative diagram since $\tau$ is a natural transformation:

$$
\begin{array}{ccc}
\text{Hom}_\mathcal{C}(A,A) & \xrightarrow{\tau_A} & F(A) \\
\varphi^* \downarrow & & \downarrow{F(\varphi)} \\
\text{Hom}_\mathcal{C}(B,A) & \xrightarrow{\tau_B} & F(B)
\end{array}
$$
This implies
\[ F(\varphi)\tau_A(id_A) = \tau_B\varphi^*(id_A) = \tau_B(id_A \circ \varphi) = \tau_B(\varphi) \]
Suppose for natural transformations \(\tau, \sigma\) such that \(y(\tau) = y(\sigma)\), this means \(\tau_A(id_A) = \sigma_A(id_A)\). Since \(\sigma\) is a natural transformation, we also have
\[ F(\varphi)\sigma_A(id_A) = \sigma_B(\varphi) \]
and therefore
\[ \tau_B(\varphi) = F(\varphi)\tau_A(id_A) = \tau_B(id_A \circ \varphi) = \tau_B(\varphi) \]
Since the choice of \(B\) and \(\varphi\) are arbitrary, this implies \(\tau = \sigma\). Hence injectivity is proved.

\textbf{Surjectivity:} Consider any object \(\alpha \in F(A)\), \(B \in \text{Ob}(C)\) and \(\varphi \in \text{Hom}_C(B, A)\), we construct the map \(\tau\) as
\[ \tau_B(\varphi) = F(\varphi)(\alpha) \]
Now we show \(\tau\) defined in this way is a natural transformation. For any other object \(C \in \text{Ob}(C)\) and \(\delta : C \to B\), we consider the following diagram:
\[
\begin{array}{ccc}
\text{Hom}_C(B, A) & \xrightarrow{\tau_B} & F(B) \\
\downarrow{\delta^*} & & \downarrow{F(\delta)} \\
\text{Hom}_C(C, A) & \xrightarrow{\tau_C} & F(C)
\end{array}
\]
Then we have
\[
F(\delta)\tau_B(\varphi) = F(\delta)F(\varphi)(\alpha) = F(\varphi \circ \delta)(\alpha) = \tau_C(\varphi \circ \delta) = \tau_C\delta^*(\varphi)
\]
Hence the diagram commutes, which shows \(\tau\) is a natural transformation, which gives surjectivity.

\textbf{Corollary 1.24.} For any simplicial set \(K\) and an \(n\)-simplex \(x\), there exists an associated map \(\bar{x} : \Delta[n] \to K\) with \(\bar{x}(id) = x\).

\textbf{Proof.} When regarding the simplicial set as a contravariant functor \(K : \Delta \to sSet\), by Yoneda’s Lemma we know for each simplex \(x \in K_n\), there exists a corresponding natural transformation \(\bar{x} : \text{Hom}_\Delta(-, [n]) \to K_n\). Equivalently, it can interpreted as a map \(\bar{x} : \Delta[n]^s \to K_n\) since \(\text{Hom}_\Delta(-, [n])\) is the standard \(n\)-simplex. \(\bar{x}(id) = x\) is given by the explicit map of bijection in Lemma 1.23.

\textbf{Remark 1.25.} As mentioned in Definition 1.15, simplicial sets can be regarded as a contravariant functor. As a dual construction, we can similarly define the category of \textit{cosimplicial sets} \(sSet^{op}\) as the category of covariant functor \(\Delta \to \text{Set}\). The characterization of (co)simplicial sets given in Definition 1.15 provides a way to generalize the construction to other categories. This means now we can take the category of contravariant functors \(\Delta \to \mathcal{C}\) as the category of simplicial objects in \(\mathcal{C}\), and the category of covariant functors \(\Delta \to \mathcal{C}\) as the category of cosimplicial objects in \(\mathcal{C}\). This generalization is very useful in defining homology in other categories.
2. **Geometric Realization and Subdivision**

In this section we will show how to turn simplicial sets into spaces. In addition, we will discuss subdivisions, which gives nice properties to simplicial sets while preserving the space associated to it up to homeomorphism.

**Definition 2.1.** A set of points \( \{v_0, ..., v_n\} \) in \( \mathbb{R}^m \) is geometrically independent if the vectors \( \{v_1 - v_0, ..., v_n - v_0\} \) are linearly independent.

**Definition 2.2.** Define the standard topological n-simplex \( \Delta[n]^t \) as the subspace \( \{(t_0, ..., t_n) : 0 \leq t_i \leq 1 \text{ and } \sum_{i=0}^n t_i = 1\} \) of \( \mathbb{R}^{n+1} \). Note that this is the simplex spanned by the standard basis of \( \mathbb{R}^{n+1} \). It has face maps and the degeneracy maps defined in the following sense:

- \( \delta_i : \Delta[n-1]^t \to \Delta[n]^t \) by \( \delta_i(t_0, ..., t_{n-1}) = (t_0, ..., t_{i-1}, 0, t_i, ..., t_{n-1}) \)
- \( \sigma_i : \Delta[n+1]^t \to \Delta[n]^t \) by \( \sigma_i(t_0, ..., t_{n+1}) = (t_0, ..., t_{i-1}, t_i + t_{i+1}, t_{i+2}, ..., t_{n+1}) \)

**Definition 2.3.** In general, a topological n-simplex is the simplex spanned by a set of \( n+1 \) geometrically independent points in some \( \mathbb{R}^m \) with face and degeneracy maps defined as in the case of Definition 2.2.

**Definition 2.4.** A geometric simplicial complex \( X \) is a collection of topological simplicies in \( \mathbb{R}^n \) such that

1. every face of simplicies in \( X \) is also in \( X \)
2. the intersection of any two simplicies is in \( X \)

**Definition 2.5.** A general geometric simplicial complex \( X \) can be defined based on a collection \( \{X_n\}_{n \in \mathbb{N}} \) where each \( X_n \) is a set of n-dim geometric simplicies defined in \( \mathbb{R}^n \). Now take the discrete topology for \( X_n \) and product topology for \( X_n \times \Delta[n]^t \). Consider the disjoint union \( \bigsqcup_{n \in \mathbb{N}} X_n \times \Delta[n]^t \). This disjoint union is too 'large' because many simplicies are repeated. Hence we need \( (fx, u) \sim (x, fu) \) as the quotient relation, where \( x \in X_n \), \( u \in \Delta[m]^t \). \( f \) is the map that takes the simplex \( x \) to a face in \( X_m \), and \( \bar{f} \) is the corresponding map that takes \( \Delta[m]^t \) as the corresponding face of \( \Delta[n]^t \). The resulting space \( \bigsqcup_{n \in \mathbb{N}} X_n \times \Delta[n]^t / \sim \) is a geometric simplicial complex.

**Remark 2.6.** For each abstract simplicial complex \( X \), we can associate it with a geometric simplicial complex by breaking its simplicies into a collection \( \{X_n\}_{n \in \mathbb{N}} \) and apply Definition 2.5. We call this the geometric realization of \( X \) and denote it as \( |X| \). For each general geometric simplicial complex \( X \), we can obtain an associated simplicial complex \( X' \) by taking the \( V(X') \) as the union of vertices of simplicies in \( X \), and for each simplex in \( X \) spanned by some vertex set \( B \), we take \( B \) as a simplex in \( X' \). Hence there is a correspondence between abstract simplicial complex and geometric simplicial complex, but this is not a bijection since for each abstract simplicial complex there are many homeomorphic geometric simplicial complex we
can have! Also given homeomorphic geometric simplicial complexes, there might be different abstract simplicial complexes give rise to them.

Following the logic of geometric realization for abstract simplicial complex, we obtain a analogous construction for simplicial sets.

**Definition 2.7.** Define the geometric realization functor $| \cdot | : sSet \to \text{Top}$ as the following: for a simplicial set $K$, regard each set $K_n$ as a space with discrete topology and then take the product topology for $K_n \times \Delta[n]^t$. Construct the space $\coprod_{n \geq 0} K_n \times \Delta[n]^t$ and take the topology of disjoint union. Now this disjoint union of spaces gives each $n$-simplex, degenerate or nondegenerate, a topological $n$-simplex, which is not what we want since we want to identify the degenerate simplicies with the nondegenerate part of it. Therefore we define an equivalence relation $\sim$ as:  

$$(f^* x, u) \sim (x, f^* u)$$

where $x \in K_n$, $u \in \Delta[n]^t$, and $f : [m] \to [n]$. By Remark 1.13, we know $f$ can be decomposed into a series of face and degeneracy maps, and $f^*$, $f_*$ are the corresponding composition of these maps in the sense of simplicial sets and topological simplicies. The resulting quotient space $|K| = \coprod_{n \geq 0} K_n \times \Delta[n]^t / \sim$ gives each nondegenerate $n$-simplex a corresponding topological simplex $\Delta[n]^t$ and also correctly identifies the faces of each simplex. Given a simplicial map $g : K \to L$, the map $|g| : |K| \to |L|$ is defined by $(x, t) \mapsto (g(x), g_* (t))$, where $g_*$ takes the unique linear combination of $t = c_0 x_0 + \cdots + c_n x_n$ to $c_0 g(x_0) + \cdots + c_n g(x_n)$. Note here $x_i$ are vertices in $\Delta[n]^t$ but we identify them with elements in $K_0$. Readers can check the map is continuous and therefore this induces a map $|g| : |K| \to |L|$.

Now we switch to the discussion of subdivision on simplicial sets, which again comes from the construction of subdivision of abstract simplicial complex, and before defining what this is we need to set up some background.

**Definition 2.8.** For any abstract simplicial complex $X \in \text{SC}$, define the cone $C(X, \ast)$ as the join of $X$ and $\{ \ast \}$, i.e.,  

$$C(X, \ast) = \{ y : y \in X \text{ or } y \in X \cup \{ \ast \} \}$$

where $y$ is any simplex represented by vertices in the cone. The cone is also an abstract simplicial complex.

**Definition 2.9.** A geometric simplicial complex $X'$ is the subdivision of another geometric simplicial complex $X$ in $\mathbb{R}^N$ if

1. any simplex of $X'$ is contained in a simplex of $X$
2. any simplex of $X$ is a finite union of simplices in $X'$

**Definition 2.10.** For a topological $n$-simplex with vertices $v_0, \ldots, v_n$ in $\mathbb{R}^N$, the barycenter is the point

$$b = \frac{1}{n+1} \sum_{i=0}^{n} v_i$$

**Definition 2.11.** For any geometric simplicial complex $X$, we abuse the notation by treating $X$ also as an abstract simplicial complex as mentioned in Remark 2.6 so that we can use our definition for the cone space. Define the barycentric subdivision $\text{Sd}(X)$ inductively as the following: take $\text{Sd}(X)_0 = X_0$ and suppose simplicies up to dimension $(n - 1)$ have been subdivided. For an $n$-simplex $x \in X_n$ with
barycenter $b_\alpha$ and the boundary $\partial x \in Sd(X)_{n-1}$, take the cone space $|C(\partial x, b_\alpha)|$ as the subdivision for $x$ and add this to $Sd(X)_n$. This indeed is a subdivision because each simplex of $|C(\partial x, b_\alpha)|$ is in $x$ and $x$ is the finite union of all simplicies in $|C(\partial x, b_\alpha)|$ as in Definition 2.11. Perform this operation over all $n$-simplicies in $X$ and we get the resulting space as $Sd(X)_n$. By induction $Sd(X) = \bigcup_{n \geq 0} Sd(X)_n$ is defined. By construction $Sd(X) \cong X$.

To define similar subdivision for simplicial sets, we need to transform the language from topology to combinatorics. The idea is for a geometric $n$-simplex $x$ we can denote its vertices using elements in $[n]$, and for the barycenter of each $q$-simplex of $x$ with vertices $v_{i_0} < \cdots < v_{i_q}$ we denote it using the $q$-tuple $(v_{i_0}, \ldots, v_{i_q})$. The following graph is an illustration of how this works for a 2-simplex:

```
0
/|
/ |\n(0, 1)  (0, 2)
|   |
|   |
|   |
(0, 1, 2)
```

Generalizing this observation, we derive the following purely combinatorial definition of barycentric subdivision for simplicial sets.

**Definition 2.12.** The barycentric subdivision for a simplicial set is a functor $Sd : sSet \to sSet$ defined as the following: for a simplicial set $K$, $Sd(K)$ has $q$-simplicies as equivalent classes of tuples $(x; S_0, \ldots, S_q)$ where $x \in K_n$ for $n \geq 0$, $S_i \subset [n]$ with $S_i \subseteq S_{i+1}$, with equivalence relation as

$$(f^*y, S_0, \ldots, S_q) \sim (y, f(S_0), \ldots, f(S_q))$$

for any $f : [n] \to [m]$, $y \in K_m$, $S_i \subset [n]$. Face and degeneracy maps are defined as:

$$d_i(x; S_0, \ldots, S_q) = (x; S_{\delta_i(0)}, \ldots, S_{\delta_i(q-1)})$$

$$s_i(x; S_0, \ldots, S_q) = (x; S_{\sigma_i(0)}, \ldots, S_{\sigma_i(q+1)})$$

where $\sigma_i$ and $\delta_i$ are defined as in Definition 1.9. It is not hard to check that $d_i, s_i$ satisfy simplicial relations. In addition $Sd$ is functorial in the sense that for any simplicial map $h : K \to L$, $Sd(h) : Sd(K) \to Sd(L)$ is obtained by

$$Sd(h)(x; S_0, \ldots, S_q) = (h(x); S_0, \ldots, S_q)$$

**Remark 2.13.** By an analysis on the definition of barycentric subdivision, we can show that barycenters of higher dimensional simplicies have higher orders. For example, consider the example at the top of this page, we have the following ordering on the vertices:

$$0 \leq (0, 1) \leq (0, 1, 2)$$

**Proposition 2.14.** The geometric realization of a simplicial set $K$ is homeomorphic to the geometric realization of $Sd(K)$.
Proof. Observe that the subdivision in Definition 2.12 gives each nondegenerate \( n \)-simplicial set a subdivision as in geometric simplicial complexes. Therefore for the simplicial set \( Sd(K) \), we can identify the geometric realization as

\[
|Sd(K)| = \coprod_{n \geq 0} Sd(K)_n \times \Delta[n]^t / \sim
\]

\[
\cong \coprod_{n \geq 0} K_n \times Sd(\Delta[n]^t) / \sim
\]

\[
\cong \coprod_{n \geq 0} K_n \times [n]^t / \sim
\]

That is, the homeomorphism between \( \Delta[n]^t \) and \( Sd(\Delta[n]^t) \) induces the homeomorphism between \( |Sd(K)| \) and \( |K| \). \( \Box \)

**Lemma 2.15.** For any simplicial set \( X \), \( Sd(X) \) can eliminate cycles in the ordering on the vertices of \( X \).

Proof. There is a simple proof using Proposition 1.21. For any cycle \( x_1 \leq \cdots \leq x_n \), \( x_n \leq x_1 \) in a simplicial set \( X \), there exists a nondegenerate 1-simplex \( \alpha_i \) with vertices \( x_i, x_{i+1} \) for all \( i \), where indices are taken mod \( n \). In \( Sd(X) \), denote the barycenter of \( \alpha_i \) as \((x_i, x_{i+1})\), then we have the order \( x_i, x_{i+1} \leq (x_i, x_{i+1}) \) replacing the original order \( x_i \leq x_{i+1} \). Hence the original cycle disappears in \( Sd(X) \). In addition, subdivision does not create cycles because if there is a cycle \( y_1 \leq \cdots \leq y_n \), \( y_n \leq y_1 \) in \( Sd(X) \), then this implies \( y_n \) must be a barycenter of some \( n \)-simplex with \( n \geq 1 \). However, as barycenters must have higher-order, it is impossible that \( y_n \leq y_1 \). \( \Box \)

**Proposition 2.16.** For any simplicial set \( X \), \( Sd(X) \) has a ordering on the vertices that satisfies reflexivity and antisymmetry but not transitivity.

Proof. This can be checked by the definition of \( Sd(X) \). \( \Box \)

### 3. Properties of Simplicial Sets

#### 3.1. Regularity.

**Definition 3.1.** Recall that the morphism between simplicial set \( f : K \to L \) consists of maps \( f_n : K_n \to L_n \) which commutes with face and degeneracy maps. \( f \) is **degreewise injective** if each \( f_n \) is injective.

**Definition 3.2.** Let \( \Delta[n]^s \) denote the standard \( n \)-simplicial set, \( \Delta[n]^s \) be the map defined based on Yoneda’s Lemma as in Lemma 1.23 with \( [d_n^s x] \) the subsimplicial set generated by \( d_n^s x \). Then a nondegenerate simplex \( x \in K_n \) in \( X \) is **regular** if the following pushout diagram has the canonical map from the pushout to \( X \) degreewise
in injective.

\[
\Delta[n-1]^* \xrightarrow{d_n x} [d_n x] \\
\downarrow \delta_n \\
\Delta[n]^* \xrightarrow{\subset} \Delta[n]^* \cup \Delta[n-1]^* [d_n x]
\]

A simplicial set is **regular** if every nondegenerate simplex is regular. Put in another way, a simplicial set is regular if each nondegenerate n-simplex \(x\) can be obtained by attaching \(\Delta[n]\) to \([d_n x]\) through the simplicial map \(d_n x : \Delta[n-1] \to d_n x\) where \(\Delta[n-1]\) is regarded as the \(n\)th face of \(\Delta[n]\).

**Definition 3.3.** For a simplicial set \(K\), an element in \(\text{Sd}(K)\) is written in the **minimal form** if in \((x; S_0, \cdots, S_q), x \in K_n\) is nondegenerate and \(S_q = [n]\).

**Lemma 3.4.** Every element in \(\text{Sd}(K)\) can be written in a unique minimal form.

**Proof.** Consider element \((x; S_0, \cdots, S_q)\) with \(x \in K_n\). If \(S_q = [n]\) then we are done. If not, take \(m_i = |S_i| - 1\) for all \(i \in [q]\) and write \(m = m_q\), and we can define a function \(f : [m] \to [n]\) based on the values in \(S_0, \cdots, S_q\) such that \(S_i = f([m_i])\).

Then

\[
(x; S_0, \cdots, S_q) = (x; f([m_0]), \cdots, f([m_q])) = (f^* x; [m_0], \cdots, [m_q])
\]

where \(f^* x \in K_m\) and \(|m_q| = [m]\). Here \(f\) is unique since the image of \(f\) is determined by elements in \(S_q\) and \(f\) is an order-preserving map. If \(f^* x\) is degenerate, then there exists a unique surjective map \(\mu : [m] \to [w]\) and a unique nondegenerate simplex \(y \in K_w\) such that \(\mu^* y = f^* x\) by Lemma 1.19. Therefore

\[
(x; S_0, \cdots, S_q) = (f^* x; [m_0], \cdots, [m_q]) = (\mu^* y; [m_0], \cdots, [m_q]) = (y; \mu([m_0]), \cdots, \mu([m_q]))
\]

where \(\mu([m_q]) = \mu([m]) = [w]\) because \(\mu\) is surjective. Hence \((y; \mu([m_0]), \cdots, \mu([m_q]))\) is in its minimal form. Since \(f, \mu, y\) are unique, we know the minimal form for each element is unique.

**Proposition 3.5.** The barycentric subdivision of any simplicial set is a regular simplicial set.

**Proof.** Based on Lemma 3.4, we know each element in \(\text{Sd}(X)\) can be written in a unique minimal form \(x' = (x; S_0, \cdots, S_q)\) with nondegenerate \(x \in K_n\) and \(S_q = [n]\).

Here \(S_q\) can be interpreted as the barycenter of \(x\). Then the \(q\)th face \(\partial_q x'\) of this simplex becomes

\[
\partial_q x' = (x; S_{\sigma_q(0)}, \cdots, S_{\sigma_q(q)}) = (x; S_0, \cdots, S_{q-1}, S_{q-1})
\]

We can identify the simplex \(x'\) as the standard simplex \(\Delta[q]\) attaching to the \(q\)-th face \(\partial_q x'\) since all other faces of \(x\) that are contained in \(\Delta[q] \setminus \Delta[q-1]\) are in the interior of \(x\) and these faces can be regarded as standard simplices embedded in \(x\). This gives regularity.
Lemma 3.6. For a geometric $n$-simplex $x$ and a proper face $x_1$ of $x$, for any proper face $x_2$ of $x_1$ and a simplicial retraction $\psi: x_1 \to x_2$, define $y$ as the geometric simplex obtained by attaching $x$ to $x_2$ via $\psi$, then there exists a homeomorphism between $y$ and $x$ extending the inclusion $x_2 \subset x$.

Proof. Conceptually, the homeomorphism comes from the observation that shrinking one face of a simplex does not change its topological property. However, for an explicit homeomorphism that extends the inclusion map, more work needs to be done. For details readers can consult Lemma 3.1.1 in [6]. $\square$

Proposition 3.7. The geometric realization of a regular simplicial set is a regular CW complex.

Proof. First of all we show that a regular simplicial complex can be defined by iteratively shrinking the $n$-th face of a standard $n$-simplex. Then we show that shrinking the $n$-th face preserves the shape homeomorphically and therefore induces a homomorphism to $\bar{e}^n$. For a regular simplicial set $X$, take $x$ as the nondegenerate $n$-simplex.

Case 1. If no face of $x$ is degenerate, then $x$ can be identified directly with $\Delta^n$, so $\Delta^n \cong [x]$. This is a regular CW complex.

Case 2. If there exists degenerate faces of $x$, then by regularity there exists a maximal degenerate face $x_1$, i.e., the first degenerate face we reach when tracing back to along the right column of the following diagram:

$$
\begin{array}{cccc}
\Delta[k] & \rightarrow & [x_1] \\
\downarrow \delta_k & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\Delta[n-2] & \rightarrow & [\delta^*_n \delta^*_{n-1} x] \\
\downarrow \delta_{n-1} & & \downarrow \\
\Delta[n-1] & \rightarrow & [\delta^*_n x] \\
\downarrow \delta_n & & \downarrow \\
\Delta[n] & \rightarrow & [x]
\end{array}
$$

By regularity we know $x_1 = \mu^*_1 x$, where $\mu^*_1 = \delta^*_k \cdots \delta^*_{n-1} \delta^*_n$. By the Eilenberg-Ziler Lemma, we know $x_1 = \gamma^*_{x_1}(x^*_1)$, where $\gamma^*_{x_1}$ is the degeneracy operator, which has a maximal left inverse $v^*_1$ such that

$$v^*_1 \circ \gamma^*_{x_1} = id$$

Define

$$y_1 = x^*_1 = v^*_1 \circ \gamma^*_{x_1}(x^*_1) = v^*_1 x_1$$

$y_1$ is the maximal nondegenerate simplex in the degenerate face $x_1$. This process can be repeated by regarding $y_1$ as the new nondegenerate simplex and find the next maximal degenerate face. Iteratively we have $x_j$ as the maximal degenerate face of $x_{j-1}$, and denote $y_j = x^*_j = v^*_j x_j$. Suppose the sequence ends at $p$, where
every face of $y_p$ is nondegenerate. Then we get two sequences of simplices with $x_0 = x$, $y_0 = x$, such that $y_i$ is the maximal nondegenerate part of $x_i$ for $i \geq 1$.

$$x_0 \ x_1 \ x_2 \ \ldots \ x_p$$

$$y_0 \ y_1 \ y_2 \ \ldots \ y_p$$

Denote $\dim(x_j) = m_j$, $\dim(y_j) = n_j$. Let $Z_1$ be the space obtained by attaching $\Delta_n$ to $\Delta_{n_1}$ based on the map $v_i^*$ which is a retraction. Based on Lemma 3.6, $Z_1$ is homeomorphic to $\Delta_n$. Similarly, define $Z_j$ as the space obtained by attaching $Z_{j-1}$ to $\Delta_{n_j}$ based on map $v_j^*$. $Z_p$ is the geometric realization of $[x]$. Since each iteration preserves homeomorphism, we know $Z_p$ is homeomorphic to $\Delta_n$, which means the closed space $[x]$ is homeomorphic to the closure of an $n$-cell. Since this is true for any nondegenerate simplex, we know the geometric realization of regular simplicial set is a regular CW complex. □

**Definition 3.8.** A topological space $X$ is called **triangulable** if there exists an abstract simplicial complex $K$ such that $\mu : |K| \to X$ is a homeomorphism. We call $(K, \mu)$ the **triangulation of $X$**.

**Proposition 3.9.** Any regular CW complex is triangulable.

*Proof.* This can be proved using induction. For a regular simplicial complex $X$, take triangulation on the vertices $X_0$ as vertices themselves. Then suppose the triangulation up to skeleton $X_{n-1}$ has been defined, we construct that of $X_n$. For the closure of each $n$-cell $\bar{e}^n$, its boundary is $\partial \bar{e}^n$ which lies in $X_{n-1}$ so it has been triangulated. Now take the triangulation of $\bar{e}^n$ as $|C(\partial \bar{e}^n, b)|$ where $b$ is the barycenter of $\bar{e}^n$, and $|C(\partial \bar{e}^n, b)| \cong \bar{e}^n$. Regularity is needed here for the barycenter to not lie inside $X_{n-1}$. Enumerate over all $n$-cells in $X_n$ and then we obtain the triangulation of $X$. □

**Corollary 3.10.** The geometric realization of any regular simplicial set is triangulable.

*Proof.* This is a direct application of Proposition 3.7 and 3.9. □

3.2. **Nonsingularity.** Now we turn to a special case of regular simplicial sets: nonsingular simplicial sets.

**Definition 3.11.** A simplicial set is **nonsingular** if for each nondegenerate simplex $x$, the map $\bar{x}$ is degreewise injective, i.e. the induced map on the geometric realization $|\Delta[n]| \to |[x]|$ is an embedding including the boundary.

**Proposition 3.12.** Nonsingular implies regular.

*Proof.* For any nonsingular simplicial set $K$ and a simplex $x \in K_n$, each of its faces is a nondegenerate $(n - 1)$-simplex so it can be identified as $\Delta[n]$ attaching along the $n^{th}$ face. This gives regularity. □

**Definition 3.13.** Define the **injection functor** $J : OSC \to sSet$ as the following: for each ordered simplicial complex $S \in OSC$, we take all its $n$-simplices as the set of nondegenerate $n$-simplicies in the simplicial set $X^s$. Define the degeneracy and face maps as the following

$$d_i(v_0, \ldots, v_n) = (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$$

$$s_i(v_0, \ldots, v_n) = (v_0, \ldots, v_i, v_{i+1}, \ldots, v_n)$$
Take the image of these maps as the degenerate faces of $X$. Any morphism $f : X \to Y$ in $OSC$ is entirely determined by the map of vertices, and therefore the induced map $J(f) : X^s \to Y^s$ can be determined by the vertices. We denote the image of functor $J$ as $OSC^*$, and this forms a subcategory inside $sSet$.

**Remark 3.14.** If $K$ is nonsingular, then the induced map on the geometric realization $|\Delta[n]|$ together with its boundary is an embedding to $|[x]|$. Nonsingular simplicial set is a structure in between ordered simplicial complex and simplicial set. The main difference between nonsingular simplicial sets and those induced by ordered simplicial complexes is that nondegenerate simplices in nonsingular simplicial sets are not uniquely determined by the vertices. Hence we have the following inclusion relations between the three types of simplicial sets:

\[ OSC^* \subset \text{nonsingular } sSet \subset sSet \]

### 3.3. Property A, B, C

Now we discuss Property A, B, C as introduced in [1]. We will also show that the definition of Property B in [1] is equivalent to nonsingular.

**Definition 3.15.** A simplicial set satisfies Property A if the faces of its nondegenerate simplices are also nondegenerate; it satisfies Property B if each nondegenerate simplex $x$ has distinct vertices; and it satisfies Property C if for a set of $n + 1$ distinct vertices, there exists at most one nondegenerate $n$-simplex whose vertices are in the set.

**Example 3.16.** Consider the simplicial set $X$ whose geometric realization has one vertex and one 1-simplex. $X$ satisfies Property A, C but not Property B.

**Example 3.17.** Consider the simplicial set whose geometric realization has three vertices \{0, 1, 2\}, three 1-simplex \{(0, 1), (0, 2), (1, 2)\}, and two 2-simplex with 0, 1, 2 as vertices. This simplicial set satisfies Property A, B, but not C.

**Definition 3.18.** The standard simplicial set $\Delta[n]^*$ satisfies Property A, B, and C.

**Proposition 3.19.** Property B implies A and there are no other implications among the properties.

**Proof.** For the nonimplications readers can try to find counterexamples. Now we show Property B implies Property A by proving that simplicies with distinct vertices are nondegenerate. Based on Eilenberg-Ziler Lemma, we know each simplex $x$ can be written as $\gamma^x_x x^\#$, where $\gamma^x_x : [n] \to [m]$ is a surjection. For every vertex map $v_i : [0] \to [n]$, we have the induced map $v^*_i : K_n \to K_0$, and

\[ v^*_i(x) = v^*_i \gamma^x_x x^\# \]

This shows vertices of $x$ come from vertices of $x^\#$. If $x$ is degenerate, that is $x$ is nontrivial and $m < n$, $x$ must have repetitive vertices because $x^\#$ has less vertices than $x$. By contrapositive, we know simplicies with distinct vertices must be nondegenerate. \[ \square \]

While nonsingular and Property B are defined separately in different sources, the following proposition will show these two definitions are actually the same.

**Proposition 3.20.** Property B is equivalent to nonsingular
Proof. By Yoneda’s Lemma,
\[ \bar{x}(\text{id}) = x \]
The maps between simplical sets are simplical maps which preserves the grading and commutes with operators, i.e. for \( \alpha : [m] \to [n] \),
\[ \bar{x}(\alpha^* \text{id}) = \alpha^* \bar{x}(\text{id}) = \alpha^* x \]
⇒ If \( K \) satisfies Property B, then \( x \) has distinct vertices so for each \( v_i : [0] \to [n] \), \( v_i^* \text{id} \) and \( v_i^* x \) give the \( i^{th} \) vertex of the standard simplex and \( x \) respectively. Since both of them have distinct vertices, we know \( \bar{x} \) is injective on the vertices, and since all other simplicies in both of them can be uniquely determined by the vertices, we know this \( \bar{x} \) is degreewise injective.
⇐ Assume \( x \) has its \( i_1^{th}, i_2^{th} \) vertices repeat. There exists \( v_{i_1} : [0] \to [n], v_{i_2} : [0] \to [n] \) which maps to \( i_1, i_2 \) respectively such that
\[ \bar{x}(v_{i_1}^* \text{id}) = v_{i_1}^* x = v_{i_2}^* x = \bar{x}(v_{i_2}^* \text{id}) \]
Therefore the map is not degreewise injective since it is not injective on the vertices. By contrapositive, degreewise injective implies Property B. Hence these two terms are equivalent. \( \square \)

4. Functors between sSets and other categories

In this section we introduce several functors that link simplicial sets with other categories.

Definition 4.1. A nerve functor \( N \) is a functor which takes the category of small categories to simplicial categories. For a small category \( C \), the simplicial set \( NC \) has \( NC_0 = \{\text{Ob}(C)\} \), and for \( n \geq 1 \), \( NC_n \) is defined as the set of \( n \) composable morphisms:
\[
c_0 \overset{f_1}{\to} c_1 \to \cdots \to c_{n-1} \overset{f_n}{\to} c_n
\]
Each element in \( NC_n \) can be denoted as an \( n \) tuple \( (f_1, \cdots, f_n) \). The degeneracy and face maps are defined as:
\[
d_i(f_1, \cdots, f_n) = \begin{cases} (f_2, \cdots, f_n), & i = 0 \\ (f_1, \cdots, f_{i+1} \circ f_{i}, \cdots, f_n), & 1 \leq i \leq n-1 \\ (f_1, \cdots, f_{n-1}), & i = n \\ \end{cases}
\]
\[
s_i(f_1, \cdots, f_n) = (f_1, \cdots, f_{i-1}, \text{id}, f_i, \cdots, f_n)
\]

For a functor between small categories \( F : C \to D \), it induces a simplicial map \( NF \) where \( NF_n : NC_n \to ND_n \) is defined as \( NF_n(\varphi) = F(\varphi) \).

Definition 4.2. The fundamental category functor \( \Pi : sSet \to \text{Cat} \) takes each simplicial set \( K \) to \( \Pi K \), where \( \Pi K \) has objects as vertices of \( K \), and morphisms are defined based on 1-simplicies. For any 1-simplex \( y \), there is a morphism \( \Pi(y) \) sending \( d_1 y \) to \( d_0 y \). To make morphisms compatible with compositions, the following condition need to be imposed on elements in \( K \):
\[
s_0(x) = \text{id}_x \text{ for } x \in K_0 \text{ and } d_1(z) = d_0(z) \circ d_2(z) \text{ for } z \in K_2
\]

Fact 4.3. The nerve functor and the fundamental category functor are adjoint. For the proof readers can consult Chapter 9 of [1].
**Definition 4.4.** Define the maximal realization functor $K : \text{Poset} \to \text{OSC}$ by taking a poset $P$ and turn all the $n$-chains into $n$-simplicies with the corresponding vertices and orderings. For any morphism $f : P_1 \to P_2$ of posets, there is an induced map $K(f) : K(P_1) \to K(P_2)$ by mapping the corresponding vertices.

**Definition 4.5.** For a simplicial set $X$, the poset $X^\#$ induced by $X$ has objects as nondegenerate simplicies in $X$ and $x \leq y$ if $y$ is a face of $x$. For each morphism $f : X \to Y$ between simplicial sets, we have the induced map $f^\# : X^\# \to Y^\#$ defined by taking $x \in X^\#$ to the nondegenerate part of $f(x)$. Therefore $-^\#$ is a functor and we call it the reordering functor.

**Definition 4.6.** Define the Barratt nerve functor $B : \text{sSet} \to \text{sSet}$ of a simplicial set $X$ as $N(X^\#)$. This is a functor because $B(X_n)$ is the set of $n$-chains in $X^\#$, and for a morphism $f : X \to Y$ and for an $n$-simplex in $B(X)$ written as

$$\alpha_0 \subseteq \alpha_1 \subseteq \cdots \subseteq \alpha_{n-1} \subseteq \alpha_n$$

$B(f) : B(X) \to B(Y)$ is defined by

$$B(f)(\alpha_0 \subseteq \alpha_1 \subseteq \cdots \subseteq \alpha_{n-1} \subseteq \alpha_n) = (f(\alpha_0) \subseteq f(\alpha_1) \subseteq \cdots \subseteq f(\alpha_{n-1}) \subseteq f(\alpha_n))$$

**Proposition 4.7.** Take $N, J, K$ as defined in Definition 4.1, 3.13, 4.4 and $I$ as the inclusion functor that sends $\text{Poset}$ as a subcategory to $\text{Cat}$, then the following diagram commutes.

![Diagram](unnamed.png)

**Proof.** To prove commutativity it is enough to show that $N \circ I$ and $J \circ K$ commutes. Given an arbitrary poset $P$, there exists a bijection between

$$\left\{ \text{nondegenerate } n \text{-simplicies in } N \circ I(P) \right\} \sim \sim \left\{ \text{chains of length } n \text{ in } P \right\}$$

because the nerve functor maps each $n$ composable morphism in $I(P)$, which is equivalent to a chain of length $n$ in $P$, to a nondegenerate $n$ simplex in the simplicial set. Also, there exists a bijection between

$$\left\{ \text{nondegenerate } n \text{-simplicies in } J \circ K(P) \right\} \sim \sim \left\{ \text{chains of length } n \text{ in } P \right\}$$

because each chain of length $n$ in $P$ is mapped to a nondegenerate $n$-simplex in $\text{OSC}$. Hence there exists a bijection between nondegenerate $n$-simplicies in $N \circ I(P)$ and in $I \circ K(P)$, which has the same vertex set. This implies $N \circ I(P) = J \circ K(P)$. Hence the diagram commutes.

**Corollary 4.8.** For any simplicial set $X$, $B(X) = N(X^\#, \subseteq)$ is an ordered simplicial complex.

**Proof.** This follows from the commutivity of the diagram proved in Proposition 4.7.

**Lemma 4.9.** If $X$ is regular, then $B(X)$ is isomorphic to the triangulation of $X$. 

□
Proof. Refer to [8]. □

Now we state a more refined version of Corollary 3.10.

**Theorem 4.10.** Given a simplicial set $X$, $BSd(X)$ gives an explicit triangulation.

**Proof.** For any simplicial set $X$, $Sd(X)$ is a regular simplicial set by Proposition 3.5. If $Sd(X)$ is regular, then $B(Sd(X))$ is isomorphic to the triangulation of $Sd(X)$ by Lemma 4.9. As $Sd(X)$ preserves homeomorphism and $B$ preserves isomorphism for regular simplicial set, we know $BSd(X)$ preserves homeomorphism for $X$ and also gives rise to a triangulation. □

5. New Results

5.1. **Characterizing Nerves of Posets.** In this section, we want to find conditions that characterize which simplicial sets are nerves of posets, i.e. those are in the image of $N \circ I$. We will also show that the intersection of the images of the functors $N$ and $J$ is exactly the subcategory the nerves of posets.

To be the nerve of a poset, the left side of the commutative diagram above shows that the following three conditions are necessary:

1. its vertices must have a partial ordering
2. it must be the nerve of a category
3. there is at most one nondegenerate $n$-simplex for every set of distinct $(n+1)$ vertices

The right side of the commutative diagram indicates the following necessary conditions:

1. its vertices must have a partial ordering and
2. it must be the maximal OSC realization based on the ordering of the vertices

These observations give us some intuition about what properties the nerves of a poset need to satisfy. First of all we characterize a property that gives partial ordering on the vertices.

**Definition 5.1.** Let $X$ be a simplicial set, then

1. Define $v < w$ for vertices $v$ and $w$ in $X$ if there is a nondegenerate $n$-simplex $x \in X$ such that the $i^{th}$ vertex is $v$ and the $j^{th}$ vertex is $w$, with $i < j$.
2. Define $v \leq w$ for vertices $v$ and $w$ in $X$ if there is a 1-simplex $x$ such that $d_0x = w$ and $d_1x = v$.

$X$ satisfies **Property D** if the ordering based on (1) is a strict partial ordering and that of (2) is a partial ordering.

**Remark 5.2.** By definition the vertices of a simplicial set $X$ that satisfies Property $D$ form a poset.

**Lemma 5.3.** **Property D** implies **Property B**.
Proof. Given a simplicial set $X$, if there exists a nondegenerate $n$-simplex that has a repeated vertex $v$, then by Property $D$ we have $v < v$, which is a contradiction to strict partial ordering. □

Now we ask how near Property $D$ is to being a characterization of the nerves of posets? Are there simplicial sets satisfying Property $D$ that are not nerves of posets? It turns out there exists a simplicial set $K$ that satisfies Property $D$ but is not the nerve of a poset because simplicial sets that satisfy Property $D$ might not be the nerve of a category. For example, consider the following simplicial set that has nondegenerate 0-simplicies and 1-simplicies as shown below, and no nondegenerate 2-dim simplicies.

\[
\begin{array}{c}
a \\
\downarrow \\
\downarrow \\
b & \searrow & c
\end{array}
\]

This satisfies Property $D$ but is not the nerve of a category. Now we consider the following theorem which characterizes those simplicial sets which are the nerves of categories [1, Chp13].

**Theorem 5.4.** Let $K$ be a simplicial set. Then the following conditions are equivalent:

- $K$ is isomorphic to the nerve of a category
- Every inner horn of $K$ has a unique filler

**Remark 5.5.** The second condition in Theorem 5.4 can be explicitly characterized as the following: for any $n \geq 2$ and any $n$-tuple of simplicies $\{x_i \in K_1|1 \leq i \leq n\}$, such that $d_0x_{i-1} = d_1x_i$ for $2 \leq i \leq n$, there is a unique $y \in K_n$ such that $v_i^*y = x_i$, where $v_i : [1] \to [n]$ with the image as $\{i - 1, i\}$.

In the following we denote the condition in Theorem 5.4 as the **unique inner horn filling condition**.

Now we ask the question whether Property $D$ and the unique inner horn filling condition imply Property $C$, which is another condition necessary to be nerves of poset. The answer is no with the following counterexample.

**Example 5.6.** Consider the simplicial set $K$ with $K_0 = \{0, 1\}$ and two nondegenerate 1-simplicies with 0, 1 as vertices and the ordering as $0 < 1$. It has the geometric realization in the form:

\[
\begin{array}{c}
0 \\
\leftrightarrow \\
\leftrightarrow \\
1
\end{array}
\]

$K$ satisfies Property $D$ and the unique inner horn filling condition, but it does not satisfy Property $C$ because there are two 1-simplicies with the same vertices. The key to this contradiction lies in the fact that the unique inner horn filling is defined only for simplicies with dimension $\geq 2$, so there are counterexamples in 1-simplicies.

**Theorem 5.7.** $K$ is the nerve of a poset if and only if it satisfies the following conditions:

1. Property $C$
2. Property $D$
3. Unique inner horn filling condition
Proof. ⇒ If \( K \) is the nerve of a poset \( P \), then the partial ordering on the vertices implies Property \( D \), and the uniqueness of morphisms between objects implies Property \( C \). Being the nerve of a category implies the unique inner horn filling condition.

⇐ If for a given simplicial set \( K \) which satisfies Property \( C \), \( D \) and the unique inner horn filling condition, we construct a poset \( P \) using \( K_0 \) along with their partial orderings in \( K \). By Property \( C \) and the unique inner horn filling condition, there exists a bijection between

\[
\left\{ \text{nondegenerate } n\text{-simplex in } K \right\} \sim \sim \left\{ \text{chains of length } n \text{ in } P \right\}
\]

By the construction of nerves, there’s also a bijection between

\[
\left\{ \text{chains of length } n \text{ in } P \right\} \sim \sim \left\{ \text{nondegenerate } n\text{-simplex in } N(P) \right\}
\]

Hence there is a bijective simplicial map from \( K \) to \( N(P) \) by mapping the corresponding vertices, which shows \( K \) is the nerve of a poset. □

Remark 5.9. Picking any two of the previous three conditions cannot give Property \( D \). This shows that Property \( D \) is indeed a very delicate condition.

Corollary 5.10. \( K \) is the nerve of a poset if and only if it satisfies the following conditions:

(i) Property \( B \)

(ii) Property \( C \)

(iii) Unique inner horn filling condition

Proof. ⇐ This is given by Theorem 5.7 and Lemma 5.3. ⇒ This is given by Theorem 5.7 and Lemma 5.8. □

Remark 5.11. These three conditions are easier to check than the one using Property \( D \). Also, careful readers should also realize that we can characterize the nerves of posets as the simplicial sets coming from some maximal realization of posets based
on the right side of the commutative diagram at the beginning of the section. However, this characterization does not give much insight to the nerves of poset and is hard to check so we did not choose this approach.

With the characterization in Corollary 5.10, we can derive the following interesting result.

**Corollary 5.12.** The intersection of the image of the nerve functor $N$ and the inclusion functor $J$ is exactly the nerves of posets.

**Proof.** By Theorem 5.4 being the nerve of a category is equivalent to the unique inner horn filling condition. It’s proven in [1] that $(OSC)^*$ satisfies Property $B$ and $C$. Hence by Corollary 5.10 the intersection of the image of $N$ and $J$ is the nerves of posets. □

**Proposition 5.13.** Any simplicial sets in $J(OSC) = OSC^*$ satisfy Property $A, B, C, D$.

**Proof.** Theorem 12.1.5 in [1] shows that Property $B$ implies Property $A$, and $J(OSC)$ satisfies Property $B$ and $C$. Any element in $(OSC)^*$ satisfies Property $D$ because the vertices of ordered simplicial complex always satisfy the two partial ordering conditions. □

**Proposition 5.14.** The nerves of posets satisfy Property $A, B, C, D$.

**Proof.** This follows from the commutivity of the diagram, i.e., $N \circ I = J \circ K$ and Proposition 5.13. □

**Remark 5.15.** Note that the main difference between the nerves of posets and $OSC^*$ is that the nerves of posets must satisfies the unique inner horn filling condition, while $OSC^*$ doesn’t need to. In the following we will show that this is the only condition that makes a difference between the two.

**Proposition 5.16.** A simplicial set $X$ belongs to $OSC^*$ if and only if $X$ satisfies Property $C$ and $D$.

**Proof.** $\Rightarrow$ follows from Proposition 5.13.

$\Leftarrow$ By Lemma 5.3, we know $X$ also satisfies Property $B$. For any simplicial set $X$ that satisfies Property $B$ and $C$, all of its nondegenerate simplicies are uniquely determined by its vertices. For any nondegenerate $n$-simplex $x$ with vertex set $V$, for any subset $S \subset V$, there exists a simplex with vertex set $S$ based on Proposition 1.21. Hence the collection of nondegenerate simplicies in $X$ form an abstract simplicial complex. Since Property $D$ ensures a partial ordering on the vertices, we know the collection of nondegenerate simplicies in $X$ form an $OSC$, which means $X$ is in $(OSC)^*$. □

After answering the question of when is a simplicial set the nerve of a poset, we ask the question when does a simplicial set have its barycentric subdivision the nerve of a poset. In the following we will see a neat characterization.
Theorem 5.17. A simplicial set $X$ has its subdivision $Sd(X)$ as the nerve of a poset if and only if $X$ satisfies Property B.

Proof. We need the following facts as proved in Chapter 12 of [1]:

1. $Sd(X)$ satisfies Property C if and only if $X$ satisfies Property B
2. $Sd(X)$ satisfies Property B if and only if $X$ satisfies Property A
3. $Sd(X)$ is the nerve of a category if and only if $X$ satisfies Property A

Since Property B implies Property A, we can deduce that $X$ satisfies Property B if and only if $Sd(X)$ satisfies Property B, C, and the unique inner horn filling condition.

Corollary 5.18. A simplicial set $X$ has its double subdivision $Sd^2(X)$ as the nerve of a poset if and only if $X$ satisfies Property A.

Proof. A simplicial set $X$ satisfies Property A if and only if $Sd(X)$ satisfies Property B. Applying Theorem 5.17, we know $Sd(X)$ satisfies Property B if and only if $Sd^2(X)$ is the nerve of a poset.

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You should thank anyone who deserves thanks, and for sure you should thank your mentor. “It is a pleasure to thank my mentor, his/her name, for ...”. Or add anyone else, for example “I thank [another participant] for helping me understand [something or other]”

6. Bibliography

The bibliography should list all sources that you have used and referenced. And you should reference anything you use. Especially if you quote any result without proof, you MUST give a reference. And never ever should you copy material directly or more or less directly, from a source.

References