SIMPLICIAL SETS AND RELATED FUNCTORS

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Abstract. In this paper, we characterize those simplicial sets which are nerves of posets. Toward this, we begin by reviewing the known theory. We first define simplicial sets combinatorially. In the process, we define geometric realization, subdivision, and discuss some properties of simplicial sets like regularity and nonsingularity. Finally, we turn our attention toward the functors relating $sSet$, $Poset$, OSC, and Cat.

Contents

1. Simplicial Complexes and CW Complexes

In this section, we define abstract simpicial complexes, ordered simplicial complexes, and CW complexes. These structures provide concrete models for topological spaces.

Definition 1.1. An *abstract simplicial complex* K is a set of vertices V together with a set K of nonempty finite subsets of V such that

- for any element v in V, $\{v\}$ is a set in K; and
- for each set $k \in K$, any subset of k is also in K.

The elements in V are called *vertices* and elements of K are called *simplicies*. Subsets of simplicies are called *faces*, and *facets* are maximal simplicies which are not faces of any other simplicies. Let $V(-)$ denote the vertex set of a simplex. For a simplex $\alpha \in K$, its *dimension* is $|V(\alpha)| - 1$. Let A, B be abstract simplicial complexes. A *morphism of simplicial complexes* is a map of simplicies that is determined by some function $V(A) \to V(B)$. We denote by $\mathcal{SC}$ the category of abstract simplicial complexes.

Definition 1.2. An *ordered simplicial complex* is an abstract simplicial complex K with a partial ordering on the vertices such that it becomes a total ordering when restricted to each simplex. Their morphisms are order-preserving simplicial maps. We denote by $\mathcal{OSC}$ the category of ordered simplicial complexes.

Definition 1.3. The *n-disk* is the subspace $D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ of $\mathbb{R}^n$.

Definition 1.4. An n-dimensional *open cell* e^n is a space homeomorphic to $int(D^n)$, the interior of an n-disk. We denote the closure of e^n as $\bar{e}^n$, which is homeomorphic to D^n.

Definition 1.5. A *CW complex* X is a space that is built inductively. We start with a discrete set of points X^0 which we call the 0-skeleton. Inductively, we form the n-skeleton X^n from the $(n-1)$-skeleton $X^{(n-1)}$ by taking the pushout in the following diagram:

$$\begin{tikzcd} \coprod \partial D^n \arrow[r, "\bar{j}"] \arrow[d, "\subset"'] & X^{n-1} \arrow[d, "\subset"] \\ \coprod D^n \arrow[r, "j"'] & X^n \end{tikzcd}$$

Here j can be regarded as the disjoint union of maps defined on each piece of D^n such that over each $int(D^n)$, j is a homeomorphism. $\bar{j}$ is j restricted to the boundaries of pieces of D^n. CW complexes have the weak topology. That is a set $A \subset X$ is open(closed) if and only if $A \cap X^n$ is open(closed) in X^n for each n.

Definition 1.6. A CW complex is called *regular* if the closure of each open n-cell is homeomorphic to D^n. In other words, if the boundary of each n-cell is homeomorphic to S^{n-1}.

2. Simplicial Sets

In this section we introduce simplicial sets, a structure that generalizes simplicial complexes and plays a very important role in modern algebraic topology.

Definition 2.1. The category of finite ordered sets, denoted by Δ, has objects as totally ordered sets $[n] = \{0, \cdots, n\}$ for any $n \in \mathbb{N}$ and morphisms as order-preserving maps $f : [m] \to [n]$ such that $i \leq j$ implies $f(i) \leq f(j)$. All order-preserving maps are generated by compositions of *face maps* $\delta_i : [n-1] \to [n]$ and *degeneracy maps* $\sigma_i : [n+1] \to [n]$. They are defined as

$$\delta_i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

$$\sigma_i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}$$

for all $0 \leq i \leq n$. In words, δ_i skips the ith index in the codomain, and σ_i repeats the ith index in the codomain.

Definition 2.2. Define *general face maps* as order-preserving injective maps $\iota : [n] \to [m]$ in Δ for any $n \leq m \in \mathbb{N}$. If we take the descending sequence of elements $i_1 \geq \cdots \geq i_p$ which are not in $\iota([n])$, then ι can be decomposed into compositions of face maps, i.e. $\iota = \delta_{i_1} \cdots \delta_{i_p}$.

Definition 2.3. Define *general degeneracy maps* as order-preserving surjective maps $\rho : [n] \to [k]$ in Δ for any $n \geq k \in \mathbb{N}$. If we take the ascending sequence of elements $j_1 \leq \cdots \leq j_q$ such that $\rho(j_k) = \rho(j_k + 1)$ for $0 \leq k \leq q$, then ρ can be decomposed into compositions of degeneracy maps, i.e. $\rho = \sigma_{j_1} \cdots \sigma_{j_q}$.

Definition 2.4. Define *vertex maps* as $v_i : [0] \to [n]$ with $0 \mapsto i$ in Δ for $0 \leq i \leq n$.

Proposition 2.5. *Any map f between finite ordered sets can be written as a composition of a general degeneracy map and a general face map, i.e. $f = \iota \circ \rho$.*

Proof. By definition any map $f : [m] \to [n]$ in Δ is generated by compositions of degeneracy and face maps (δ_i and σ_j). Canonically we can denote $n \geq i_1 \geq \cdots \geq i_q \geq 1$ to be the descending sequence of indicies that are not in $f([m])$ and $0 \leq j_1 \leq ... \leq j_q < m$ to be the descending sequence of indicies such that $f(i_k) = f(i_k + 1)$. Then

$$f = \delta_{i_1} \cdots \delta_{i_p} \circ \sigma_{j_1} \cdots \sigma_{j_q}$$

Take $\iota = \delta_{i_1} \cdots \delta_{i_p}$ and $\rho = \sigma_{j_1} \cdots \sigma_{j_q}$. The previous equation becomes $f = \iota \circ \rho$. $\square$

Definition 2.6. A *simplicial set* X is a collection of sets X_n for $n \geq 0$ with functions $d_i : X_n \to X_{n-1}$, $s_i : X_{n-1} \to X_n$ for $0 \leq i \leq n$ such that they satisfy the following *simplicial relations*:

$$d_i \circ d_j = d_{j-1} \circ d_i \text{ if } i < j$$

$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & \text{if } i < j \\ \text{id} & \text{if } i = j \text{ or } i + j + 1 \\ s_j \circ d_{i-1} & \text{if } i > j + 1 \end{cases}$$

$$s_i \circ s_j = s_{j+1} \circ s_i \text{ if } i \leq j$$

where d_i are *face maps* and s_i are *degeneracy maps*. Elements of the set X_n are called *n-simplicies*. A map $f : K \to L$ of simplicial sets is a sequence of functions $f_n : K_n \to L_n$ such that $f_{n-1} \circ d_i = d_i \circ f_n$ and $f_{n+1} \circ s_i = s_i \circ f_n$, i.e. the following diagrams commute:

$$\begin{tikzcd} K_n \arrow[r, "f_n"] \arrow[d, "d_i"'] & L_n \arrow[d, "d_i"] \\ K_{n-1} \arrow[r, "f_{n-1}"'] & L_{n-1} \end{tikzcd} \qquad \begin{tikzcd} K_n \arrow[r, "f_n"] & L_n \\ K_{n-1} \arrow[r, "f_{n-1}"'] \arrow[u, "s_i"] & L_{n-1} \arrow[u, "s_i"'] \end{tikzcd}$$

This gives the category of simplicial sets, denoted as $sSet$.

Definition 2.7. Alternatively, we can regard a simplicial set X as a contravariant functor $X : \Delta \to Set$ such that $X([n]) = X_n$, and any morphism $f : [n] \to [m]$ becomes $X(f) : X_m \to X_n$. Concretely, if

$$f = \delta_{i_1} \cdots \delta_{i_p} \circ \sigma_{j_1} \cdots \sigma_{j_q}$$

by the decomposition in Proposition 2.5, then

$$X(f) = s_{j_q} \cdots s_{j_1} \circ d_{i_p} \cdots d_{i_1}$$

where s_j, d_i are defined in Definition 2.6. In words, the functor takes degeneracy and face maps in Δ to those in simplicial sets. In addition, the morphisms between simplicial sets become natural transformations between functors.

Definition 2.8. In the category of simplicial sets, we define *general degeneracy maps* ρ^* as those induced by general degeneracy maps $\rho \in \Delta$ in the sense that we regard simplicial sets as contravariant functors. Similarly, we define *general face maps* ι^* and *vertex maps* v_i^* as those induced by ι and v_i respectively.

Remark 2.9. By Proposition 2.5, we can decompose any simplicial map f^* into a composition of a general degeneracy map and a face map $\rho^* \circ \iota^*$. For any map f in Δ, we use f^* to denote the map in $sSet$ induced by f.

Definition 2.10. For a simplicial set X, a simplex $x \in X$ is called *degenerate* if it can be written as $\rho^* y$ for some simplex y and a nontrivial general degeneracy map ρ^*. It is called *nondegenerate* otherwise.

Lemma 2.11. *(Eilenberg-Ziler Lemma) For a simplicial set X, any simplex x can be written in a unique decomposition of the form*

$$x = \rho_x^* x^{\#}$$

where ρ_x^* is a general degeneracy map and $x^{\#}$ is a nondegenerate simplex.

Proof. Since $x \in X_n$, we can check whether x is degenerate by enumerating through the images of different general degeneracy maps. This is doable because there are finitely many such maps to check. If x is nondegenerate, then the case is trivial; if x is degenerate, then by induction on the degeneracy we know x can be written in the form $x = \rho_x^* x^{\#}$. Suppose there are two different ways $x = \rho_1^* x_1$, $x = \rho_2^* x_2$ to decompose x. Because $\rho_1 : [m] \to [n]$ is a surjective map, it has a right inverse $\bar{\rho}_1 : [n] \to [m]$. Hence,

$$\begin{aligned} \rho_1^* x_1 &= \rho_2^* x_2 \\ \Rightarrow \bar{\rho}_1^* \circ \rho_1^* x_1 &= \bar{\rho}_1^* \circ \rho_2^* x_2 \\ \Leftrightarrow x_1 = \bar{\rho}_1^* \circ \rho_2^* x_2 &= (\rho_2 \circ \bar{\rho}_1)^* x_2 \end{aligned}$$

Since x_1 is nondegenerate, we know $\rho_2 \circ \bar{\rho}_1$ cannot contain any degeneracy maps. Hence it must be a general face map, which implies $dim(x_1) \leq dim(x_2)$. By the symmetry of the previous step we also know $dim(x_2) \leq dim(x_1)$. Together they give $dim(x_1) = dim(x_2)$. Since $\rho_2 \circ \bar{\rho}_1$ is an injective order-preserving map whose domain and codomain are in the same dimension, it must be the identity map. This gives $x_1 = x_2$ and $\rho_1 = \rho_2$. □

Simplicial sets are a generalization of simplicial complexes. For simplicial complexes, each n-simplex is uniquely determined by its $(n+1)$ distinct vertices, but for simplicial sets, a nondegenerate n-simplex might not have $(n+1)$ distinct vertices. Next proposition shows a nice property of simplicial sets that we have not seen in the literature. It will be useful in the later part of the paper.

Proposition 2.12. *Let X be a simplicial set and x an n-simplex with its vertex set V. Here V is can be a multiset. For each multisubset $S \subset V$, there exists an $(|S|-1)$-simplex in X with vertex set S.*

Proof. Note that because V is the vertex set of the n-simplex x, it has an ordering. Concretely, the i-th vertex of x is given by $v_i^*(x)$. Recall that $v_i : [0] \to [n]$ is the function $0 \mapsto i$.

Now, let $m = |S| - 1$. Define $g : [m] \to [n]$ to map i to the i-th element of $S \subset V$. Further, define $f_i : [0] \to [n]$ as the function $0 \mapsto g(i)$, i.e. $f_i := v_{g(i)}$. By definition, this factors as in the following commutative diagram:

$$\begin{tikzcd} {[0]} \arrow[rr, "f_i"] \arrow[dr, "0 \mapsto i"'] & & {[n]} \\ & {[m]} \arrow[ur, "g"'] & \end{tikzcd}$$

This induces the following commutative diagram in $sSet$:

$$\begin{tikzcd} X_0 & & X_n \arrow[ll, "X(f_i)"'] \arrow[dl, "X(g)"] \\ & X_m \arrow[ul, "X(0 \mapsto i)"] & \end{tikzcd}$$

Therefore for a given $x \in X_n$, $X(g)(x)$ gives an m-simplex with vertex set S. $\square$

Definition 2.13. Define the *standard simplicial n-simplex* $\Delta[n]^s$ as the contravariant functor represented by $[n]$, i.e. $\Delta[n]^s := Hom_\Delta(-, [n])$. The i-simplicies $\Delta[n]^s_i$ of $\Delta[n]^s$ are morphisms $\varphi : [i] \to [n]$ in Δ.

Lemma 2.14. *(Contravariant Version of Yoneda's Lemma). For any category $\mathcal{C}$, $Hom_\mathcal{C}(-, A) : \mathcal{C} \to Set$ is a contravariant functor. For any contravariant functor $F : \mathcal{C} \to Set$, there is a natural bijection*

$$y : Nat(Hom_\mathcal{C}(-, A), F) \cong F(A)$$

by sending $\tau \mapsto \tau_A(id_A)$.

Proof. The proof exploits categorical facts about functors and natural transformations.
Injectivity: For any $B \in Ob(\mathcal{C})$ and $\varphi \in Hom_\mathcal{C}(B, A)$, we have the following commutative diagram given by the natural transformation τ:

$$\begin{tikzcd} Hom_\mathcal{C}(A, A) \arrow[r, "\tau_A"] \arrow[d, "\varphi^*"'] & F(A) \arrow[d, "F(\varphi)"] \\ Hom_\mathcal{C}(B, A) \arrow[r, "\tau_B"'] & F(B) \end{tikzcd}$$

This implies

$$F(\varphi)\tau_A(id_A) = \tau_B\varphi^*(id_A) = \tau_B(id_A \circ \varphi) = \tau_B(\varphi)$$

Take natural transformations τ, σ such that $y(\tau) = y(\sigma)$, i.e. $\tau_A(id_A) = \sigma_A(id_A)$. Since σ is a natural transformation,

$$F(\varphi)\sigma_A(id_A) = \sigma_B\varphi^*(id_A) = \sigma_B(id_A \circ \varphi) = \sigma_B(\varphi)$$

and therefore

$$\tau_B(\varphi) = \tau_B\varphi^*(id_A) = F(\varphi)\tau_A(id_A) = F(\varphi)\sigma_A(id_A) = \sigma_B(\varphi)$$

Since the choice of B and φ are arbitrary, this implies $\tau = \sigma$. Hence injectivity is proved.

Surjectivity: Consider any object $\alpha \in F(A)$, $B \in Ob(\mathcal{C})$ and $\varphi \in Hom_{\mathcal{C}}(B, A)$. We construct the map τ as

$$\tau_B(\varphi) = F(\varphi)(\alpha)$$

Now we show τ defined in this way is a natural transformation. For any other object $C \in Ob(\mathcal{C})$ and $\delta : C \to B$,

$$\begin{aligned} F(\delta)\tau_B(\varphi) &= F(\delta)F(\varphi)(\alpha) \\ &= F(\varphi \circ \delta)(\alpha) \\ &= \tau_C(\varphi \circ \delta) \\ &= \tau_C \delta^*(\varphi) \end{aligned}$$

This shows the commutivity of the diagram:

$$\begin{tikzcd} Hom_{\mathcal{C}}(B, A) \arrow[r, "\tau_B"] \arrow[d, "\delta^*"'] & F(B) \arrow[d, "F(\delta)"] \\ Hom_{\mathcal{C}}(C, A) \arrow[r, "\tau_C"'] & F(C) \end{tikzcd}$$

Hence τ is a natural transformation. This gives surjectivity. □

Corollary 2.15. *For any simplicial set X and an n-simplex x, there exists an associated map $\bar{x} : \Delta[n]^s \to X$ with $\bar{x}(id) = x$.*

Proof. When regarding a simplicial set X as a contravariant functor $X : \Delta \to sSet$, by Yoneda's Lemma for each simplex $x \in X_n$, there exists a corresponding natural transformation $\bar{x} : Hom_{\Delta}(-, [n]) \to X_n$. Equivalently, it can interpreted as a map $\bar{x} : \Delta[n]^s \to X_n$ because $Hom_{\Delta}(-, [n]) := \Delta[n]^s$. Then $\bar{x}(id) = x$ is given by the explicit map of bijection in Yoneda's Lemma, i.e. Lemma 2.14. □

Remark 2.16. A simplicial set can be regarded as a contravariant functor as in Definition 2.7. As a dual construction, we can similarly define the category of *cosimplical sets* $sSet^{op}$ as the category of covariant functors $\Delta \to Set$. The characterization of (co)simplicial sets given in Definition 2.7 provides a way to generalize the construction to other categories. This means now we can take the category of contravariant functors $\Delta \to \mathcal{C}$ as the category of simplicial objects in $\mathcal{C}$, and the category of covariant functors $\Delta \to \mathcal{C}$ as the category of cosimplicial objects in $\mathcal{C}$. This generalization is very useful in defining homology in other categories.

3. Geometric Realization

Definition 3.1. A set of points $\{v_0, ..., v_n\}$ in $\mathbb{R}^m$ is *geometrically independent* if the vectors $\{v_1 - v_0, ..., v_n - v_0\}$ are linearly independent.

Definition 3.2. Define the *standard geometric n-simplex* $\Delta[n]^t$ as the subspace

$$\{(t_0, ..., t_n) : 0 \leq t_i \leq 1 \text{ and } \sum_{i=0}^{n} t_i = 1\}$$

of $\mathbb{R}^{n+1}$. Note that this is the simplex spanned by the standard basis of $\mathbb{R}^{n+1}$. It has *face maps* $\bar{\delta}_i : \Delta[n-1]^t \to \Delta[n]^t$ and *degeneracy maps* $\bar{\sigma}_i : \Delta[n+1]^t \to \Delta[n]^t$

defined as:

$$\bar{\delta}_i(t_0, ..., t_{n-1}) = (t_0, ..., t_{i-1}, 0, t_i, ..., t_{n-1})$$
$$\bar{\sigma}_i(t_0, ..., t_{n+1}) = (t_0, ..., t_{i-1}, t_i + t_{i+1}, t_{i+2}, ..., t_{n+1})$$

Alternatively, $\Delta[\cdot]^t$ can be interpreted as a covariant functor $\Delta \to Top$.

Definition 3.3. In general, a *geometric n-simplex* is a simplex spanned by a set of $n+1$ geometrically independent vectors V in some $\mathbb{R}^m$. The face and degeneracy maps are similar to those in Definition 3.2 if we write them with respect to a basis that contains V as the first $n+1$ elements.

Definition 3.4. For a simplicial set X, regard each set X_n as a space with discrete topology. Take the product topology for $X_n \times \Delta[n]^t$. The topological space associated to X is

$$|X| = \coprod_{n \geq 0} X_n \times \Delta[n]^t / \sim$$

where the equivalence relation is $(f^*x, u) \sim (x, \bar{f}u)$ for $x \in K_n$, $u \in \Delta[m]^t$, $f : [m] \to [n]$, and $\bar{f} : X_n \to X_m$. Concretely, if

$$f = \delta_{i_1} \cdots \delta_{i_p} \circ \sigma_{j_1} \cdots \sigma_{j_q}$$

by Proposition 2.5, then

$$\begin{aligned} f^* &= \sigma^*_{j_q} \cdots \sigma^*_{j_1} \circ \delta^*_{i_p} \cdots \delta^*_{i_1} \\ &= s_{j_q} \cdots s_{j_1} \circ d_{i_p} \cdots d_{i_1} \\ \bar{f} &= \bar{\delta}_{i_1} \cdots \bar{\delta}_{i_p} \circ \bar{\sigma}_{j_1} \cdots \bar{\sigma}_{j_q} \end{aligned}$$

In words, the face and degeneracy maps in Δ are mapped to those in $sSet$ and $\Delta[n]^t$. By construction, the resulting quotient space $|K|$ gives each nondegenerate n-simplex a corresponding geometric simplex $\Delta[n]^t$ and also correctly identifies the faces of each simplex. Given a simplicial map $g : K \to L$, the map $|g| : |K| \to |L|$ is defined by $(x, t) \mapsto (g(x), g_*(t))$. Here we take x_i as vertices in $\Delta[n]^t$ and identify them with elements in K_0 for convenience, and similarly for $g(x_i)$ and L_0. Then g_* maps the unique linear combination of $t = c_0x_0 + \cdots c_nx_n$ to $c_0g(x_0) + \cdots c_ng(x_n)$. Readers can check the map is continuous and therefore this induces a map $|g| : |K| \to |L|$. This shows $|\cdot| : sSet \to Top$ is a functor, named as the *geometric realization functor*.

4. Subdivision

In this section we will discuss subdivision, which gives nice properties to simplicial sets while preserving its topology up to homeomorphism.

Definition 4.1. For any abstract simplicial complex $X \in \mathcal{SC}$, define the *cone* $C(X, *) \in \mathcal{SC}$ as the join of X and $\{*\}$. Its simplicies are either simplicies in X or simplicies in X union $\{*\}$.

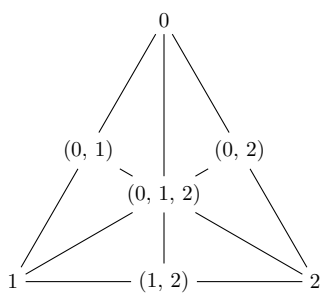
Definition 4.2. For a topological n-simplex with vertices $v_0, ..., v_n$ in $\mathbb{R}^N$, the *barycenter* is the point

$$b = \frac{1}{n+1} \sum_{i=0}^{n} v_i$$

Definition 4.3. For any abstract simplicial complex X, we define the *barycentric subdivision* $Sd(X)$ inductively as follows: take $Sd(X)_0 = X_0$ and suppose simplicies up to dimension $(n-1)$ have been subdivided. Consider an n-simplex $x \in X_n$ and denote the simplex corresponding to the boundary of x in $Sd(X)_{n-1}$ as ∂x. Add an extra point b_x and take the cone $C(\partial x, b_x)$ as the subdivision for x. Add this to $Sd(X)_n$. Perform this operation over all n-simplicies in X and we get the resulting abstract simplicial complex as $Sd(X)_n$. By induction $Sd(X) = \bigcup_{n\geq 0} Sd(X)_n$ is defined.

Definition 4.4. For each geometric n-simplex $\Delta[n]^t$, we can regard it as an abstract simplicial complex by representing each geometric simplex using its vertex set. Then we can reuse the construction in Definition 4.3 but replace $C(\partial x, b_x)$ with $|C(\partial x, b_x)|$ to obtain the *barycentric subdivision* $Sd(\Delta[n]^t)$ for geometric n-simplicies. By construction $Sd(\Delta[n]^t) \cong \Delta[n]^t$.

To define similar subdivision for simplicial sets, we need to transform the language from topology to combinatorics. The idea is for a geometric n-simplex x we can denote its vertices using elements in $[n]$. For the barycenter of each q-simplex of x with vertices $v_{i_0} < \cdots < v_{i_q}$, we denote it using the q-tuple $(v_{i_0}, \cdots, v_{iq})$. The following graph is an illustration of how this works for a 2-simplex:



Generalizing this observation, we derive the following purely combinatorial definition of barycentric subdivision for simplicial sets.

Definition 4.5. The *barycentric subdivision* for a simplicial set is a functor $Sd : sSet \to sSet$ defined as follows: for a simplicial set X, $Sd(X)$ has q-simplicies as equivalent classes of tuples

$$(x; S_0, \cdots, S_q)$$

for some $x \in X_n$, $S_i \subset [n]$ for all i and $S_i \subseteq S_{i+1}$ for $0 \leq i < q$. Given any $f : [n] \to [m]$, $y \in X_m$, and $S_i \subset [n]$, the equivalence relation is defined by

$$(f^*y, S_0, \cdots, S_q) \sim (y, f(S_0), \cdots, f(S_q))$$

In the simplicial set $Sd(X)$, the face and degeneracy maps are defined as:

$$d_i(x; S_0, \cdots, S_q) = (x; S_{\delta_i(0)}, \cdots, S_{\delta_i(q-1)})$$
$$s_i(x; S_0, \cdots, S_q) = (x; S_{\sigma_i(0)}, \cdots, S_{\sigma_i(q+1)})$$

where σ_i and δ_i are as in Definition 2.1. It is not hard to check that d_i, s_i satisfy simplicial relations. In addition Sd is functorial in the sense that for any simplicial map $h : K \to L$, $Sd(h) : Sd(K) \to Sd(L)$ is

$$Sd(h)(x; S_0, ..., S_q) = (h(x); S_0, ..., S_q)$$

Proposition 4.6. *The geometric realization of a simplicial set X is homeomorphic to the geometric realization of $Sd(X)$.*

Proof.

$$\begin{aligned}|Sd(X)| &= \coprod_{n\geq 0} Sd(X)_n \times \Delta[n]^t/\sim \\ &\cong \coprod_{n\geq 0} X_n \times Sd(\Delta[n]^t)/\sim \\ &\cong \coprod_{n\geq 0} X_n \times \Delta[n]^t/\sim \\ &= |X|\end{aligned}$$

That is, the homeomorphism between $\Delta[n]^t$ and $Sd(\Delta[n]^t)$ induces the homeomorphism between $|Sd(X)|$ and $|X|$. □

5. Properties of Simplicial Sets

5.1. Regularity.

Definition 5.1. Recall that a morphism between simplicial sets $f : K \to L$ consists of maps $f_n : K_n \to L_n$ which commute with face and degeneracy maps. We call f *degreewise injective* if each f_n is injective.

Definition 5.2. Given a simplicial set X and a nondegenerate simplex $x \in X_n$, let $[d_n x]$ be the subsimplicial set generated by $d_n x$ and $\overline{d_n x}$ be the corresponding map by Yoneda's Lemma. Then x is *regular* if the following pushout diagram has the canonical map from the pushout to X degreewise injective.

$$\begin{tikzcd}
\Delta[n-1]^s \arrow[r, "\overline{d_n x}"] \arrow[d, "\delta_n"] & {[d_n x]} \arrow[d, "\subset"'] \arrow[ddr, bend left] & \\
\Delta[n]^s \arrow[r, "\subset"] \arrow[drr, bend right] & \Delta[n]^s \cup_{\Delta[n-1]^s} {[d_n x]} \arrow[dr, dotted] & \\
& & X
\end{tikzcd}$$

A simplicial set is *regular* if every nondegenerate simplex is regular. Put in another way, a simplicial set is regular if each nondegenerate n-simplex x can be obtained by attaching $\Delta[n]$ to $[d_n x]$ through the simplicial map $\overline{d_n x} : \Delta[n-1] \to d_n x$ where $\Delta[n-1]$ is regarded as the nth face of $\Delta[n]$.

Definition 5.3. For a simplicial set X, an element $(x; S_0, \cdots, S_q)$ in $Sd(X)$ is in the *minimal form* if $x \in X_n$ is nondegenerate and $S_q = [n]$.

Lemma 5.4. *Every element in $Sd(X)$ can be written in a unique minimal form*

Proof. Consider any element $(x; S_0, \cdots, S_q)$ with $x \in X_n$. Take $m_i = |S_i| - 1$ for all $i \in [q]$ and write $m = m_q$. We define a function $f : [m] \to [n]$ based on the values in $S_0, \cdots, S_q$ such that $S_i = f([m_i])$. Then

$$(x; S_0, \cdots, S_q) = (x; f([m_0]), \cdots, f([m_q])) = (f^* x; [m_0], \cdots, [m_q])$$

where $f^*x \in X_m$ and $[m_q] = [m]$. Here f is unique since the image of f is determined by elements in S_q and f is an order-preserving map. If f^*x is degenerate, then there exists a unique surjective map $\mu : [m] \to [w]$ and a unique nondegenerate simplex $y \in X_w$ such that $\mu^* y = f^* x$ by Lemma 2.11. Therefore

$$(x; S_0, \cdots, S_q) = (f^*x; [m_0], \cdots, [m_q]) = (\mu^* y; [m_0], \cdots, [m_q]) = (y; \mu([m_0]), \cdots, \mu([m_q]))$$

where $\mu([m_q]) = \mu([m]) = [w]$ because μ is surjective. Hence $(y; \mu([m_0]), \cdots, \mu([m_q]))$ is in its minimal form. Since f, μ, y are unique, we know the minimal form for each element is unique. □

Proposition 5.5. *The barycentric subdivision of any simplicial set is a regular simplicial set.*

Proof. Based on Lemma 5.4, we know each element in $Sd(X)$ can be written in a unique minimal form $x' = (x; S_0, \cdots, S_q)$ with nondegenerate $x \in X_n$ and $S_q = [n]$. Here S_q can be interpreted as the barycenter of x. Then the qth facet of x', denoted as $\partial_q x'$, is

$$\partial_q x' = (x; S_{\delta_q(0)}, \cdots, S_{\delta_q(q-1)}) = (x; S_0, \cdots, S_{q-1})$$

where δ_q is the map that skips the qth index. We can identify the simplex x' as the standard simplex $\Delta[q]$ attaching to the q-th face $\partial_q x'$ since all other faces of x that are contained in $\Delta[q] \backslash \partial_q x'$ are in the interior of x and these faces can be regarded as standard simplicies embedded in x. This gives regularity. □

Lemma 5.6. *Take a geometric n-simplex x as in Definition 3.3, a proper face x_1 of x, a proper face x_2 of x_1, and a simplicial retraction $\psi : x_1 \to x_2$. Define y as the geometric simplex obtained by attaching x to x_2 via ψ. Then there exists a homeomorphism between y and x extending the inclusion $x_2 \subset x$.*

Proof. Conceptually, the homeomorphism comes from the observation that shrinking one face of an n-simplex does not change its topological property. However, for an explicit homeomorphism that extends the inclusion map, more work needs to be done. For details readers can consult Lemma 3.1.1 in [4]. □

Proposition 5.7. *The geometric realization of a regular simplicial set is a regular CW complex.*

Proof. This follows the proof of Proposition 4.6.11 in [4]. Given a regular simplicial set X and a nondegenerate simplex $x \in X_n$, without loss of generality we can assume that X is generated by x. We show that the geometric realization of this regular simplicial set can be defined by iteratively shrinking the n-th face of $\Delta[n]^t$. Then we show that shrinking the n-th face preserves the shape homeomorphically and therefore induces a homeomorphism to $\bar{e}^n$.
Case 1. If no face of x is degenerate, then x can be identified directly with $\Delta[n]$. Therefore $\Delta[n]^t \cong |[x]|$. This is a regular CW complex.
Case 2. If there exists degenerate faces of x, then by regularity there exists a maximal degenerate face x_1, i.e., the first degenerate face we reach when tracing

back along the right column of the following diagram from the bottom to top:

$$\begin{tikzcd}
\Delta[k] \arrow[r] \arrow[d, "\delta_k"'] & {[x_1]} \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
\Delta[n-2] \arrow[r] \arrow[d, "\delta_{n-1}"'] & {[\delta_{n-1}^*\delta_n^* x]} \arrow[d] \\
\Delta[n-1] \arrow[r] \arrow[d, "\delta_n"'] & {[\delta_n^* x]} \arrow[d] \\
\Delta[n] \arrow[r] & {[x]}
\end{tikzcd}$$

By regularity we know $x_1 = \mu_1^* x$, where $\mu_1^* = \delta_k^* \cdots \delta_{n-1}^* \delta_n^*$. Based on the Eilenberg-Ziler Lemma, we know $x_1 = \gamma_{x_1}^*(x_1^\#)$, where $\gamma_{x_1}^*$ is a degeneracy operator which has a maximal left inverse v_1^* such that

$$v_1^* \circ \gamma_{x_1}^* = id$$

Define

$$y_1 = x_1^\# = v_1^* \circ \gamma_{x_1}^*(x_1^\#) = v_1^* x_1$$

where y_1 is the maximal nondegenerate simplex in the degenerate face x_1. This process can be repeated by regarding y_1 as the new nondegenerate simplex and find the next maximal degenerate face. Iteratively we have x_j as the maximal degenerate face of x_{j-1}, and denote $y_j = x_j^\# = v_j^* x_j$. Suppose the sequence ends at p, where every face of y_p is nondegenerate. Now we have two sequences of simplices with $x_0 = x$, $y_0 = x$, such that y_i is the maximal nondegenerate part of x_i for $i \geq 1$.

$$\begin{matrix} x_0 & x_1 & x_2 & \dots & x_p \\ y_0 & y_1 & y_2 & \dots & y_p \end{matrix}$$

Denote $dim(x_j) = m_j$, $dim(y_j) = n_j$. Let Z_1 be the space obtained by attaching $\Delta[n]$ to $\Delta[n_1]$ based on the map v_1^* which is a retraction. Based on Lemma 5.6, Z_1 is homeomorphic to $\Delta[n]$. Similarly, define Z_j as the space obtained by attaching Z_{j-1} to $\Delta[n_j]$ through v_j^*. Z_p is the geometric realization of X. Since each iteration preserves homeomorphism, we know Z_p is homeomorphic to $\Delta[n]$, which means the closed space $|X|$ is homeomorphic to the closure of an n-cell. Since this is true for any nondegenerate simplex, we know geometric realizations of regular simplical sets are regular CW complexes. □

Definition 5.8. A topological space X is called *triangulable* if there exists an abstract simplicial complex K and a homeomorphism $\mu : |K| \to X$. We call (K, μ) a *triangulation* of X.

Proposition 5.9. *Any regular CW complex is triangulable.*

Proof. This can be proved using induction. For a regular CW complex X, take the triangulation on the vertices X_0 as themselves. Then suppose a triangulation up to skeleton X_{n-1} has been defined. We construct that of X_n. For each closed n-cell $\bar{e}^n$, its boundary has been triangulated. Now take the triangulation of $\bar{e}^n$ as $|C(\partial\bar{e}^n, b)|$ where b is the barycenter of $\bar{e}^n$. Regularity is needed here for the barycenter b to not lie inside X_{n-1} so that $|C(\partial\bar{e}^n, b)| \cong \bar{e}^n$. Enumerate over all n-cells in X_n. We obtain a triangulation of X. □

Corollary 5.10. *The geometric realization of any regular simplicial sets is triangulable.*

Proof. This is a direct application of Proposition 5.7 and 5.9. □

5.2. **Nonsingularity.** Now we turn to a special case of regular simplicial sets: nonsingular simplicial sets.

Definition 5.11. A simplicial set is *nonsingular* if for each nondegenerate simplex x, the map $\bar{x}$ induced by Yoneda's Lemma is degreewise injective, i.e. the induced map on the geometric realization $|\Delta[n]| \to |[x]|$ is an embedding including the boundary.

Proposition 5.12. *For simplicial sets, nonsingularity implies regularity.*

Proof. For any nonsingular simplicial set X and a simplex $x \in X_n$, each of its faces is a nondegenerate $(n-1)$-simplex and x can be identified as $\Delta[n]$ attaching along the nth face. This gives regularity. □

Definition 5.13. Define the *injection functor* $J : OSC \to sSet$ as follows: for each ordered simplicial complex $S \in OSC$, we take all n-simplicies as the set of nondegenerate n-simplicies in the simplicial set X^s. Define the degeneracy and face maps as the following:

$$d_i(v_0, \cdots, v_n) = (v_0, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n)$$
$$s_i(v_0, \cdots, v_n) = (v_0, \cdots, v_i, v_i, \cdots, v_n)$$

Take the image of these maps as degenerate faces of X^s. Because any morphism $f : X \to Y$ in OSC is entirely determined by the map of vertices, the induced map $J(f) : X^s \to Y^s$ is also determined by the map on vertices. We denote the image of functor J as OSC^s, and this forms a subcategory inside $sSet$.

Remark 5.14. If $X \in sSet$ is nonsingular, then the induced map on the geometric realization $|\Delta[n]|$ together with its boundary is an embedding to $|[x]|$. Nonsingular simplicial sets are in between ordered simplicial complexes and simplicial sets. The main difference between nonsingular simplicial sets and those induced by ordered simplicial complexes is that nondegenerate simplicies in nonsingular simplicial sets are not uniquely determined by the vertices. Hence we have the following inclusion relations between three types of simplical sets:

$$\mathcal{OSC}^s \subset \text{nonsingular } sSet \subset sSet$$

5.3. **Property A, B, C.** Now we discuss Property A, B, C as introduced in [1]. We will also show that the definition of Property B in [1] is equivalent to nonsingularity.

Definition 5.15. A simplicial set satisfies *Property A* if the faces of its nondegenerate simplicies are also nondegenerate; it satisfies *Property B* if each nondegenerate simplex x has distinct vertices; and it satisfies *Property C* if for a set of $n+1$ distinct vertices, there exists at most one nondegerate n-simplex whose vertices are in the set.

Example 5.16. Consider a simplicial set X generated by one 0-simplex and one nondegenerate 1-simplex. Then X satisfies Property A, C but not Property B.

Example 5.17. Consider the simplicial set generated by three vertices $\{0, 1, 2\}$, three nondegenerate 1-simplicies $\{(0, 1), (0, 2), (1, 2)\}$, and two nondegenerate 2-simplicies with $0, 1, 2$ as vertices. The order of this simplicial set respects the order of its vertices as natural numbers. This simplicial set satisfies Property A, B, but not C.

Definition 5.18. The standard simplicial set $\Delta[n]^s$ defined in Definition 3.2 satisfies Property A, B, and C.

Proposition 5.19. *Property B implies A and there are no other implications among the properties.*

Proof. For the nonimplications readers can try to find counterexamples. Now we show Property B implies Property A by proving that simplicies with distinct vertices are nondegenerate. Based on Eilenberg-Ziler Lemma, we know each simplex x can be written as $\gamma_x^* x^\#$, where $\gamma_x : [n] \to [m]$ is a surjection and $x^\#$ is nondegenerate. For every vertex map $v_i : [0] \to [n]$, we have the induced map $v_i^* : X_n \to X_0$, and

$$v_i^*(x) = v_i^* \gamma_x^* x^\#$$

This shows vertices of x come from vertices of $x^\#$. If x is degenerate, that is γ_x is nontrivial and $m < n$, x must have repetitive vertices because $x^\#$ has less vertices than x. By contrapositive, we know simplicies with distinct vertices must be nondegenerate. □

While nonsingularity and Property B are defined separately in different sources, the following proposition will show these two definitions are actually the same.

Proposition 5.20. *Property B is equivalent to nonsingularity.*

Proof. By Yoneda's Lemma,

$$\bar{x}(\mathrm{id}) = x$$

Note that the maps between simplical sets are simplicial maps which preserve the grading and commute with operators, i.e. for $\alpha : [m] \to [n]$,

$$\begin{aligned} \bar{x}(\alpha^* \mathrm{id}) &= \alpha^* \bar{x}(\mathrm{id}) \\ &= \alpha^* x \end{aligned}$$

$\Rightarrow$ If a simplicial set X satisfies Property B, then x has distinct vertices. For each $v_i : [0] \to [n]$, $v_i^* \mathrm{id}$ and $v_i^* x$ give the ith vertex of the standard simplex and x respectively. Since both of them have distinct vertices, we know $\bar{x}$ is injective on the vertices. Since all other simplicies in both of them can be uniquely determined by the map on vertices, $\bar{x}$ is degreewise injective.

$\Leftarrow$ Assume x has its i_1^{th}, i_2^{th} vertices repeat. There exists $v_{i_1} : [0] \to [n], v_{i_2} : [0] \to [n]$ which maps to i_1, i_2 repsectively such that

$$\bar{x}(v_{i_1}^* \mathrm{id}) = v_{i_1}^* x = v_{i_2}^* x = \bar{x}(v_{i_2}^* \mathrm{id})$$

Therefore the map is not degreewise injective since it is not injective on the vertices. By contrapositive, degreewise injectivity implies Property B. Hence these two terms are equivalent. $\square$

6. Functors between sSets and Other Categories

In this section we introduce several functors that link simplicial sets with other categories.

Definition 6.1. A *nerve functor* N is a functor which takes the category of small categories to $sSet$. For a small category $\mathcal{C}$, the simplicial set $N\mathcal{C}$ has $N\mathcal{C}_0 = \{Ob(C)\}$. For $n \geq 1$, $N\mathcal{C}_n$ is defined as the set of n-tuples of morphisms $(f_1, \cdots, f_n)$ such that they are composable. The degeneracy and face maps are defined as:

$$d_i(f_1, \cdots, f_n) = \begin{cases} (f_2, \cdots, f_n), & i = 0 \\ (f_1, \cdots, f_{i+1} \circ f_i, \cdots, f_n), & 1 \leq i \leq n-1 \\ (f_1, \cdots, f_{n-1}), & i = n \end{cases}$$

$$s_i(f_1, \cdots, f_n) = (f_1, \cdots, f_{i-1}, id, f_i, \cdots, f_n)$$

For a functor between two small categories $F : \mathcal{C} \to \mathcal{D}$, it induces a simplicial map $NF_n : N\mathcal{C}_n \to N\mathcal{D}_n$ defined as $NF_n((f_1, \cdots, f_n)) = (F(f_1), \cdots, F(f_n))$.

Definition 6.2. The *fundamental category functor* $\Pi : sSet \to Cat$ takes each simplicial set X to ΠX, where ΠX has objects as vertices of X and morphisms defined based on the 1-simplicies of X. Concretely, for any 1-simplex $y \in X$, there is a morphism $\Pi(y)$ sending $d_1 y$ to $d_0 y$. To make morphisms compatible with compositions, the following condition needs to be imposed on the morphisms.

$$\Pi(s_0(x)) = id_x \text{ for } x \in X_0 \text{ and } \Pi(d_1(z)) = \Pi(d_0(z)) \circ \Pi(d_2(z)) \text{ for } z \in X_2$$

Fact 6.3. The nerve functor and the fundamental category functor are adjoint. For the proof readers can consult Chapter 9 of [1].

Definition 6.4. Define the *maximal realization functor* $K : Poset \to OSC$ by taking a poset P and turn all the n-chains into n-simplicies with corresponding vertices and orderings. For any morphism $f : P_1 \to P_2$ of posets, there is an induced map $K(f) : K(P_1) \to K(P_2)$ defined by mapping the corresponding vertices.

Definition 6.5. For a simplicial set X, the poset $X^\#$ induced by X has objects as nondegenerate simplicies in X and relations $x \leq y$ if y is a face of x. For each morphism $f : X \to Y$ between simplicial sets, the induced map $f^\# : X^\# \to Y^\#$ is defined by taking $x \in X^\#$ to the nondegenerate part of $f(x)$. We call $-^\#$ the *reordering functor*.

Definition 6.6. Define the *Barratt nerve functor* $B : sSet \to sSet$ as the composition of functors $-^\#$ and N. That is for a simplicial set X, $B(X) = N(X^\#)$. Given a morphism $f : X \to Y$ and an n-simplex in $B(X)$ written as $(\alpha_0, \alpha_1, \cdots, \alpha_{n-1}, \alpha_n)$, $B(f) : B(X) \to B(Y)$ is defined by

$$B(f)(\alpha_0, \alpha_1, \cdots, \alpha_{n-1}, \alpha_n) = (f(\alpha_0), f(\alpha_1), \cdots, f(\alpha_{n-1}), f(\alpha_n))$$

Proposition 6.7. *Take N, J, K as in Definition 6.1, 5.13, 6.4 and I as the inclusion functor that embeds Poset to Cat. The following diagram commutes.*

$$\begin{tikzcd}
 & sSet \arrow[dd, "\#"] & \\
Cat \arrow[ur, "N"] & & OSC \arrow[ul, "J"'] \\
 & Poset \arrow[ul, "I"] \arrow[ur, "K"'] &
\end{tikzcd}$$

Proof. To prove commutivity it is enough to show that $N \circ I$ and $J \circ K$ commutes. Given an arbitrary poset P, there exists a bijection between

$$\left\{ \text{ nondegenerate } n\text{-simplicies in } N \circ I(P) \right\} \leftrightsquigarrow \left\{ \text{ chains of length } n \text{ in } P \right\}$$

because the nerve functor maps n composable morphisms in $I(P)$, which is equivalent to a chain of length n in P, to a nondegenerate n simplex in the simplicial set. Also, there exists a bijection between

$$\left\{ \text{nondegenerate } n\text{-simplicies in } J \circ K(P) \right\} \leftrightsquigarrow \left\{ \text{ chains of length } n \text{ in } P \right\}$$

because each chain of length n in P is mapped to a nondegenerate n-simplex in OSC. Hence there exists a bijection between nondegenerate n-simplicies in $N \circ I(P)$ and those in $J \circ K(P)$. The bijection maps simplicies to simplicies with the same vertex set. This implies $N \circ I(P) = J \circ K(P)$. Hence the diagram commutes. □

Corollary 6.8. *For any simplicial set X, $B(X) = N(X^{\#}, \subseteq)$ is an ordered simplicial complex.*

Proof. This follows from the commutivity of the diagram proved in Proposition 6.7. □

Lemma 6.9. *If a simplicial set X is regular, then $B(X)$ is isomorphic to the simplicial set generated by the triangulation of X.*

Proof. Refer to [8]. □

Now we state a more refined version of Corollary 5.10.

Theorem 6.10. *Given a simplicial set X, $BSd(X)$ gives an explicit triangulation on X.*

Proof. For any simplicial set X, $Sd(X)$ is a regular simplicial set by Proposition 5.5. Because $Sd(X)$ is regular, $B(Sd(X))$ is isomorphic to the simplicial set generated by the triangulation of $Sd(X)$ by Lemma 6.9. As $Sd(X)$ always preserves homeomorphism and B perserves homeomorphism for regular simplicial sets, we know $BSd(X)$ preserves homeomorphism for X and also gives rise to a triangulation. □

7. Characterizing Nerves of Posets

In this section, we want to find conditions that characterize which simplicial sets are nerves of posets, i.e. those in the image of $N \circ I$. We will also show that the intersection of the images of the functor N and J is exactly the subcategory the nerves of posets.

$$\begin{tikzcd}
 & sSet & \\
Cat \arrow[ur, "N"] & & OSC \arrow[ul, "J"'] \\
 & Poset \arrow[ul, "I"] \arrow[ur, "K"'] &
\end{tikzcd}$$

To be the nerve of a poset, the left side of the commutative diagram above shows that the following three conditions are necessary:

(1) its vertices must have a partial ordering
(2) it must be the nerve of a category
(3) there is at most one nondegenerate n-simplex for every set of distinct $(n+1)$ vertices

The right side of the commutative diagram indicates the following necessary conditions:

(1) its vertices must have a partial ordering and
(2) it must be the maximal OSC realization based on the ordering of the vertices

These observations give us some intuition about what properties the nerves of a poset need to satisfy. First of all we characterize a property that gives partial ordering on the vertices.

Definition 7.1. Given a set X and a relation $\leq$ on them, we call the relation a *partial ordering* if it satisfies reflexivity, antisymmetry, and transitivity.

Definition 7.2. Given a set X and a relation $<$ on them, we call the relation a *strict partial ordering* if it satisfies irreflexivity, antisymmetry, and transitivity.

Definition 7.3. Let X be a simplicial set, then

(1) define $v < w$ for vertices v and w in X if there is a *nondegenerate* n-simplex $x \in X$ such that the ith vertex is v and the jth vertex is w, with $i < j$.
(2) define $v \leq w$ for vertices v and w in X if there is a 1-simplex x such that $d_0 x = w$ and $d_1 x = v$.

X satisfies *Property D* if the ordering based on (1) is a strict partial ordering and that of (2) is a partial ordering.

Remark 7.4. By definition the vertices of a simplicial set X that satisfies Property D form a poset with relation $\leq$.

Lemma 7.5. *Property D implies Property B.*

Proof. Given a simplicial set X, if there exists a nondegenerate n-simplex that has a repeated vertex v, then by Property D we have $v < v$, which is a contradiction to strict partial ordering. □

Now we ask how near Property D is to being a characterization of the nerves of posets? Are there simplicial sets satisfying Property D that are not nerves of posets? It turns out there exists a simplicial set X that satisfies Property D but is not the nerve of a poset because simplicial sets that satisfy Property D might not even be the nerve of a category. For example, consider the following simplicial set that is generated by three nondegenerate 0-simplicies $\{a, b, c\}$, three 1-simplicies $\{(b, a), (a, c), (b, c)\}$ with ordering $b < a$, $a < c$, and $b < c$:

$$\begin{tikzcd} & a \arrow[dr] & \\ b \arrow[ur] \arrow[rr] & & c \end{tikzcd}$$

This satisfies Property D but is not the nerve of a category because there is no nondegerate 2-simplex with vertices a, b, c. Thus, we need to consider the following theorem which characterizes the nerves of categories from Chapter 13 of [1].

Theorem 7.6. *Let X be a simplicial set. Then the following conditions are equivalent:*

- *X is isomorphic to the nerve of a category*
- *Every inner horn of X has a unique filler*

This is called the unique inner horn filling condition*.*

Remark 7.7. The second condition in Theorem 7.6 can be explicitly characterized as follows: for any $n \geq 2$ and any n-tuple of 1-simplicies $\{x_i \in X_1 | 1 \leq i \leq n\}$ such that $d_0 x_{i-1} = d_1 x_i$ for $2 \leq i \leq n$, there is a unique $y \in X_n$ such that $a_i^* y = x_i$. Here $a_i : [1] \to [n]$ is defined as $0 \mapsto i - 1$, $1 \mapsto i \mod n$ for $0 \leq i \leq n$.

Now we ask the question whether Property D and the unique inner horn filling condition imply Property C, which is another condition necessary to be nerves of posets. The answer is no with the following counterexample.

Example 7.8. Consider a simplicial set X with $X_0 = \{0, 1\}$ and two nondegenerate 1-simplicies such that $0, 1$ are their verticies with ordering $0 < 1$. It has the geometric realization in the form:

$$\begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] & 1 \end{tikzcd}$$

X satisfies Property D and the unique inner horn filling condition, but it does not satisfy Property C because there are two 1-simplicies with the same vertices. The key to this contradiction lies in the fact that the unique inner horn filling is defined only for simplicies with dimension ≥ 2, so there are counterexamples in 1-simplicies.

Theorem 7.9. *A simplicial set X is the nerve of a poset if and only if it satisfies the following conditions:*
(i) Property C
(ii) Property D
(iii) Unique inner horn filling condition

Proof. $\Rightarrow$ If X is the nerve of a poset P, then the partial ordering on the vertices implies Property D, and the uniqueness of morphisms between objects in posets implies Property C. Being the nerve of a category implies the unique inner horn filling condition.
$\Leftarrow$ For a given simplicial set X which satisfies Property C, D and the unique inner

horn filling condition, we construct a poset P such that X_0 are its elements and the partial order is equivalent to $\leq$ in X as in Definition 7.3. By Property C and the unique inner horn filling condition, there exists a bijection between

$$\left\{\text{nondegenerate } n\text{-simplex in } X\right\} \leftrightsquigarrow \left\{\text{chains of length } n \text{ in } P\right\}$$

By the construction of nerves, there's also a bijection between

$$\left\{\text{chains of length } n \text{ in } P\right\} \leftrightsquigarrow \left\{\text{nondegenerate } n\text{-simplex in } N(P)\right\}$$

Hence there is a bijective simplicial map from X to $N(P)$ by mapping the corresponding vertices, i.e. $X \cong N(P)$. This shows X is the nerve of a poset. □

Since the definition of Property D is a bit complicated, it is hard to check whether a given simplicial complex satisfies property D. Therefore we develop another equivalent, easier-to-check condition that characterizes the nerves of posets.

Lemma 7.10. *Property B, Property C and the unique inner horn filling condition imply Property D.*

Proof. Suppose X is a simplicial set satisfying the three conditions, we first check the partial ordering property as in Definition 7.3. Given any vertex v, there always exists the degenerate 1-simplex $\sigma_0^* v$ such that it has both of its vertices as v. Hence reflexivity is satisfied; for any two vertices v, w, if there exists two 1-simplicies with v, w as its vertices but one gives $v \leq w$ and another gives $w \leq v$, then v, w must be the same because if they are distinct, then there exists two nondegenerate simplicies with the same vertex set, contrary to Property C. This gives antisymmetry. Transitivity is given by the unique inner horn filling condition. Now we show they also satisfies the condition for strict partial ordering. Property B ensures that irreflexivity is satisfied since any nondegenerate 1-simplex must have distinct vertices. Property C guarantees antisymmetry as in the previous case, and transitivity is given by the unique inner horn filling property. □

Corollary 7.11. *A simplicial set X is the nerve of a poset if and only if it satisfies the following conditions:*
(i) Property B
(ii) Property C
(iii) Unique inner horn filling condition

Proof. $\Leftarrow$ This is given by Theorem 7.9 and Lemma 7.5. $\Rightarrow$ This is given by Theorem 7.9 and Lemma 7.10. □

With the characterization in Corollary 7.11, we can derive the following interesting result.

Corollary 7.12. *The intersection of the image of the nerve functor N and the inclusion functor J is exactly the nerves of posets.*

$$\begin{tikzcd} & sSet & \\ Cat \arrow[ur, "N"'] & & OSC \arrow[ul, "J"] \end{tikzcd}$$

Proof. By Theorem 7.6 being the nerve of a category is equivalent to *the unique inner horn filling condition*. It's proven in [1] that $(OSC)^s$ satisfies Property B and C. Hence by Corollary 7.11 the intersection of the image of N and J is the nerves of posets. □

Proposition 7.13. *The nerves of posets satisfy Property A, B, C, D.*

Proof. This follows from Theorem 7.9, Corollary 7.11, and Proposition 5.19. □

After answering the question of when is a simplicial set the nerve of a poset, we ask the question when does a simplicial set have its barycentric subdivision the nerve of a poset. In the following we will see a neat characterization.

Theorem 7.14. *A simplicial set X has its subdivision $Sd(X)$ the nerve of a poset if and only if X satisfies Property B.*

Proof. We need the following facts from Chapter 12 of [1]:

(1) $Sd(X)$ satisfies Property C if and only if X satisfies Property B
(2) $Sd(X)$ satisfies Property B if and only if X satisfies Property A
(3) $Sd(X)$ is the nerve of a category if and only if X satisfies Property A

Since Property B implies Property A, we can deduce that X satisfies Property B if and only if $Sd(X)$ satisfies Property B, C, and the unique inner horn filling condition. Then Corollary 7.11 gives the result. □

Corollary 7.15. *A simplicial set X has its double subdivision $Sd^2(X)$ the nerve of a poset if and only if X satisfies Property A.*

Proof. A simplicial set X satisfies Property A if and only if $Sd(X)$ satisfies Property B. Applying Theorem 7.14, we know $Sd(X)$ satisfies Property B if and only if $Sd^2(X)$ is the nerve of a poset. □

Acknowledgments

I want to thank Peter May for being an extremely supportive, knowledgeable, and accessible mentor who makes this paper and REU possible. Your attitudes towards math and writing definitely make a long-lasting positive influence on me. Also I want to thank Matthew Niemiro for proofreading my paper and correcting my English. Last but not least, I am grateful for all the people I met in the REU. Thank you all for creating a friendly environment and supporting me mentally. This REU experience is definitely memorable.

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