TATE COHOMOLOGY

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Abstract. Tate cohomology is a useful synthesis of group homology and cohomology. It has applications in class field theory—in particular in proving the Artin reciprocity—as well as in the study of Tate spectra. In this paper we provide three equivalent definitions of Tate cohomology and define the cup product on it.

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1. Introduction

Singular homology and cohomology are powerful tools in algebraic topology and constructing analogous invariants for other situations can prove useful. Group homology and cohomology are one example. For a finite group G and G-module M we can construct homology and cohomology groups $H_\bullet(G, M)$ and $H^\bullet(G, M)$. These group invariants can be linked together to form the Tate cohomology $\hat{H}^\bullet(G, M)$ indexed over all of $\mathbb{Z}$. The elegance of this linking construction is itself interesting and it allows for the packaging of the homology and cohomology into one. Moreover, somewhat surprisingly, Tate cohomology takes on some of the usual properties we would expect from cohomology: it inherits the long exact sequence property from group homology and cohomology, as well as a cup product from group cohomology.

In certain cases—when the first Tate cohomology group is trivial and the second is cyclic for all subgroups—the cup product allows us to define an isomorphism $\hat{H}^n(G, \mathbb{Z}) \to \hat{H}^{n+2}(G, M)$ given by $\beta \mapsto \alpha \smile \beta$ for $\langle \alpha \rangle = \hat{H}^1(G, M)$. This is "Tate's theorem" and is a powerful tool in proving Artin reciprocity.

We will discuss three equivalent definitions for Tate cohomology. The first is the linking of group homology and cohomology already mentioned. The second uses complete resolutions. Though perhaps less computationally explicit, it directly expresses Tate cohomology as the cohomology of a complex. We will mention one final definition which, like the second, computes Tate cohomology directly from a

complex. The final definition is of interest because of how it generalizes to *Tate spectra*. If $P_\bullet \xrightarrow{\varepsilon} \mathbb{Z}$ is a projective resolution of G-modules and $\widetilde{P}_\bullet$ is the mapping cone of ε then the third definition for Tate cohomology takes the cohomology of the complex $(\widetilde{P}_\bullet \otimes \mathrm{Hom}(P_\bullet, M))^G$.

This is generalized as follows: For G a compact Lie group, let EG_+ be the union of EG with a disjoint G-fixed basepoint, $\tilde{EG}$ the mapping cone of the map $EG_+ \to S^0$ with EG sent to the non-basepoint, and $F(EG_+, X)$ the function spectrum $EG_+ \to X$. The Tate spectrum—or occasionally the "G-fixed" Tate spectrum—of some G-spectrum X is then defined to be $(\tilde{EG} \wedge F(EG_+, X))^G$. The connection between Tate cohomology and spectra is deeper than the analogy between their definitions. This is seen in the conditional convergence of the *Tate spectral sequence* constructed from Tate spectra.[1,2]

In the following three subsections of this paper we introduce the three aforementioned definitions for Tate cohomology of a finite group G. We will also discuss the cup product defined on Tate cohomology.

2. Three Definitions of Tate Cohomology

In each of the following subsections we will present a definition for the Tate cohomology of a group G and prove its equivalence to the others. In all cases G will denote a finite group.

2.1. Group Homology and Cohomology.

As with singular homology and cohomology, the group-theoretic equivalents are dependent on the "coefficients" we choose. More precisely, we will define the homology and cohomology of a group G with coefficients in a G-module:

Definition 2.1. A *(left) G-module* is an Abelian group $(M, +)$ with a (left) G-action which distributes over addition: $g \cdot (m + n) = g \cdot m + g \cdot n$.

Note that this definition is equivalent to that of a $\mathbb{Z}[G]$-module. We will think of G-modules as generalizations of G-representations. In other words, we define $\otimes$ and Hom as follows:

Definition 2.2. Given G-modules M and N we define G-actions on $M \otimes N = M \otimes_{\mathbb{Z}} N$ and $\mathrm{Hom}(M, N) = \mathrm{Hom}_{\mathbb{Z}}(M, N)$ by

$$g \cdot (m \otimes n) = (g \cdot m) \otimes (g \cdot n) \qquad (g \cdot \varphi)(m) = g \cdot (\varphi(g^{-1} \cdot m))$$

for any $m \in M$, $n \in N$, and $\varphi \in \mathrm{Hom}(M, N)$.

It is also useful to define:

Definition 2.3. The *augmentation ideal* $A_G \subseteq \mathbb{Z}[G]$ is the kernel of the *augmentation map*

$$\sum_{g \in G} n_g g \mapsto \sum_{g \in G} n_g.$$

Equivalently, $A_G = \langle \{1 - g : g \in G\} \rangle$.

[1] Greenlees and May[3] theorem 10.5, page 64.
[2] Blumberg[2] page 100.

Definition 2.4. The *orbits* and *fixed points* of a G-module M are denoted M_G and M^G respectively. Each inherits the structure of an Abelian group from M. On M_G this is given by $M_G = M/A_G M$, and the linearity of the G-action makes M^G a subgroup of M.

The above two definitions let us define $\otimes_G$ and Hom_G:

Definition 2.5. For G-modules M and N

$$M \otimes_G N = (M \otimes N)_G \qquad \mathrm{Hom}_G(M, N) = \mathrm{Hom}(M, N)^G.$$

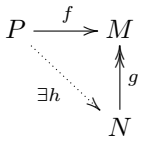
Equivalently, we could simply define them as

$$M \otimes_G N = M \otimes_{\mathbb{Z}[G]} N \qquad \mathrm{Hom}_G(M, N) = \mathrm{Hom}_{\mathbb{Z}[G]}(M, N).$$

The only technicality is that for $M \otimes_{\mathbb{Z}[G]} N$, we must assume that M is taken to be a right G-module in the natural manner $mg = g^{-1}m$. We then have $mg \otimes_G n = g^{-1}m \otimes_G n = gg^{-1}m \otimes_G gn = m \otimes_G gn$.

In order to define group homology and cohomology, we also need some homological algebra:

Definition 2.6. A *projective R-module* P is one such that any homomorphism $f : P \to M$ out of P factors through every surjection $g : N \twoheadrightarrow M$.



All free modules are projective and for our purposes free modules will be sufficient. However, projective modules lead more naturally to *projective resolutions*:

Definition 2.7. A *projective resolution* of a module M is a chain complex

$$\ldots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0$$

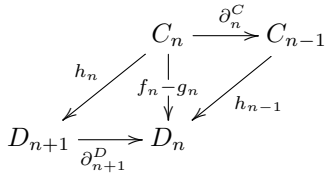
of projective modules along with an *augmentation map* $\varepsilon : P_0 \to M$ such that

$$\ldots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \to 0$$

is exact.

A key property of resolutions is their uniqueness up to *chain homotopy*.

Definition 2.8. A *homotopy* h between maps $f, g : C_\bullet \to D_\bullet$ of chain complexes is a sequence of maps $h_n : C_n \to D_{n+1}$ such that $f_n - g_n = \partial^D_{n+1} \circ h_n + h_{n-1} \circ \partial^C_n$ which can be visualized via the following (non-commutative) diagram:



Chain homotopy generalizes topological homotopy—Homotopic maps of spaces induce chain homotopic maps of the spaces' chain complexes.[3] As one might expect,

[3]Weibel[10] page 17.

it is true in general that chain homotopic maps induce the same homomorphism on homology.[4] It follows that homotopy equivalent chain complexes have isomorphic homology groups.

Moreover, the definition of chain homotopy relies only on composition and addition so any *additive functor* F which preserves addition of homomorphisms $F(\phi + \psi) = F(\phi) + F(\psi)$ will preserve homotopy equivalence of chain complexes. With this in mind, the following result becomes very useful:

Lemma 2.9. *Any two projective resolutions of the same module M are homotopy equivalent.*[5]

Of particular concern to us are projective resolutions of $\mathbb{Z}$ with the trivial G-action (going forward, we assume that any instance of $\mathbb{Z}$ is equipped with this trivial action). With the above lemma in mind it suffices to work with the following resolution of group rings:

$$\dots \xrightarrow{d_2} \mathbb{Z}[G^2] \xrightarrow{d_1} \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

with boundary maps $d_n : \mathbb{Z}[G^{n+1}] \to \mathbb{Z}[G^n]$ given by

$$d_n(g_0, \dots, g_n) = \sum_{i=0}^{n} (-1)^i (g_0, \dots, \hat{g_i}, \dots, g_n)$$

where the $\hat{g_i}$ indicates the omission of g_i. The degenerate case of the augmentation map $\varepsilon = d_0$ sends all $g \in G$ to 1. Note that $\mathbb{Z}[G^n]$ is a free $\mathbb{Z}[G]$-module generated by the elements $(g_1, g_2, \dots, g_{n-1}, e)$ for $(g_1, \dots, g_{n-1}) \in G^{n-1}$. This resolution provides a computational reference point for our later definitions. Now, as promised, we can now construct the homology and cohomology of a group G with coefficients in a G-module M:

Definition 2.10. Consider some projective resolution $P_\bullet$ of $\mathbb{Z}$. To provide the "coefficients in M" we tensor with M

$$\dots \to M \otimes_G P_2 \to M \otimes_G P_1 \to M \otimes_G P_0 \to 0$$

and apply $\mathrm{Hom}_G(-, M)$

$$0 \to \mathrm{Hom}_G(P_0, M) \to \mathrm{Hom}_G(P_1, M) \to \mathrm{Hom}_G(P_2, M) \to \dots.$$

The *group homology* $H_\bullet(G, M)$ is the homology of the chain complex $M \otimes_G P_\bullet$ and *group cohomology* $H^\bullet(G, M)$ is the cohomology of the cochain complex $\mathrm{Hom}_G(P_\bullet, M)$. These are well defined because $M \otimes_G -$ and $\mathrm{Hom}_G(-, M)$ are additive functors and thus preserve chain homotopy.

The groups $H_0(G, M)$ and $H^0(G, M)$ are of particular importance because they are isomorphic to the orbits and fixed points of M, respectively. To demonstrate this, note that $\mathbb{Z} \otimes_G M = (\mathbb{Z} \otimes M)_G = M_G$ and $\mathrm{Hom}_G(\mathbb{Z}, M) = \mathrm{Hom}(\mathbb{Z}, M)^G = M^G$. We can then think of $M_G = \mathbb{Z} \otimes_G M$ and $M^G = \mathrm{Hom}_G(\mathbb{Z}, M)$ as continuing the chain complexes used to construct group homology and cohomology. More precisely since $M \otimes_G -$ and $\mathrm{Hom}_G(-, M)$ are respectively right and left exact, when we apply them to the first few terms of the exact sequence

$$\dots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

[4] Weibel[10] page 17.

[5] Follows from the comparison theorem. See Weibel[10] page 35.

we get that

$$M \otimes_G P_1 \to M \otimes_G P_0 \to M \otimes_G \mathbb{Z} = M_G \to 0$$
$$0 \to M^G = \mathrm{Hom}_G(\mathbb{Z}, M) \to \mathrm{Hom}_G(P_0, M) \to \mathrm{Hom}_G(P_1, M)$$

are exact. Thus removing the M_G and M^G terms to compute homology and cohomology gives us $H_0(G, M) \cong M_G$ and $H^0(G, M) \cong M^G$.

In order to define Tate cohomology, we will string together group homology and cohomology at their zeroth groups:

Definition 2.11. The *norm map* $N : M \to M$ is defined $N(x) = \sum_{g \in G} gx$.

Note that for any $h \in G$

$$\sum_{g \in G} ghx = \sum_{gh^{-1} \in G} gx = \sum_{g \in G} gx$$

and, similarly,

$$h \cdot \sum_{g \in G} gx = \sum_{g \in G} hgx = \sum_{g \in G} gx.$$

Therefore N induces a map $\hat{N} : M_G \to M^G$.

Definition 2.12. The *Tate cohomology* of G with coefficients in M is

$$\hat{H}^n(G, M) = \begin{cases} H_{-n-1}(G, M) & n \leq -2 \\ \ker \hat{N} & n = -1 \\ \mathrm{coker}\, \hat{N} & n = 0 \\ H^n(G, M) & n \geq 1 \end{cases}.$$

In order to justify this definition for Tate "cohomology" we need to show that short exact sequences induce long exact sequences in cohomology. We will first show that this property holds for group homology and cohomology. We require the following lemma:

Lemma 2.13. *If P is a projective R-module, then $- \otimes_R P$ and $\mathrm{Hom}_R(P, -)$ are exact functors.*[6]

Long exact sequences for group homology and cohomology follow immediately

Theorem 2.14. *If $0 \to L \to M \to N \to 0$ is a short exact sequence of G-modules then we have long exact sequences*

$$\cdots \to H_n(G, L) \to H_n(G, M) \to H_n(G, N) \to H_{n-1}(G, L) \to \cdots \to H_0(G, N) \to 0$$

and

$$0 \to H^0(G, L) \to \cdots \to H^{n-1}(G, N) \to H^n(G, L) \to H^n(G, M) \to H^n(G, N) \to \cdots$$

Proof. Consider some projective resolution $P_\bullet \xrightarrow{\varepsilon} \mathbb{Z}$. By lemma 2.13 we have exact sequences

$$0 \to L \otimes_G P_\bullet \to M \otimes_G P_\bullet \to N \otimes_G P_\bullet \to 0$$

and

$$0 \to \mathrm{Hom}_G(P_\bullet, L) \to \mathrm{Hom}_G(P_\bullet, M) \to \mathrm{Hom}_G(P_\bullet, N) \to 0$$

of chain complexes. The theorem follows. □

[6] Dummit and Foote[5] proposition 30, page 389 for $Hom_R(P, -)$ and corollary 42, page 400 for $- \otimes_R P$.

We can now construct the long exact sequence for Tate cohomology:

Theorem 2.15. *If $0 \to L \to M \to N \to 0$ is a short exact sequence of G-modules then we have a long exact sequence*

$$\cdots \to \hat{H}^n(G, L) \to \hat{H}^n(G, M) \to \hat{H}^n(G, N) \to \hat{H}^{n+1}(G, L) \to \cdots .$$

Proof. Suppose we have

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

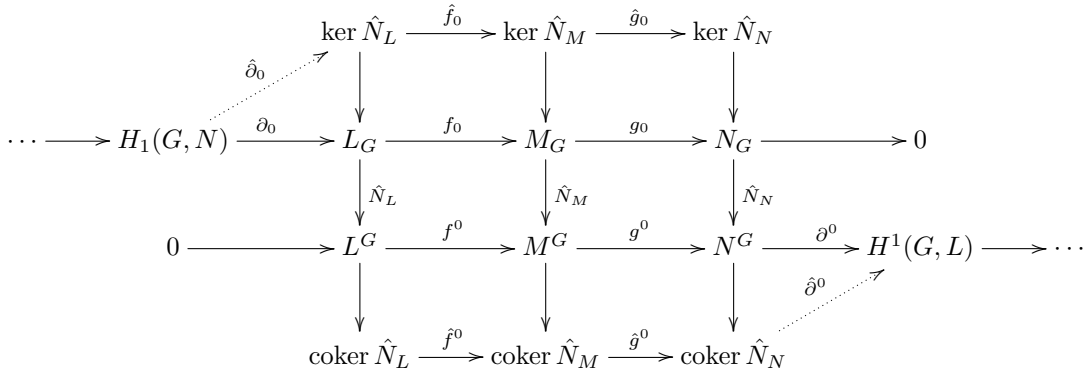
so that we get long exact sequences

$$\cdots \to H_1(G, N) \xrightarrow{\partial_0} H_0(G, L) \xrightarrow{f_0} H_0(G, M) \xrightarrow{g_0} H_0(G, N) \to 0$$

and

$$0 \to H^0(G, L) \xrightarrow{f^0} H^0(G, M) \xrightarrow{g^0} H^0(G, N) \xrightarrow{\partial^0} H^1(G, L) \to \cdots .$$

Note that the zeroth homology and cohomology groups are the orbits and fixed points, so we get the following commutative diagram:



The snake lemma gives us the exact sequence

$$\ker \hat{N}_L \xrightarrow{\hat{f}_0} \ker \hat{N}_M \xrightarrow{\hat{g}_0} \ker \hat{N}_N \xrightarrow{\partial} \operatorname{coker} \hat{N}_L \xrightarrow{\hat{f}^0} \operatorname{coker} \hat{N}_M \xrightarrow{\hat{g}^0} \operatorname{coker} \hat{N}_N$$

but we still need to show that this exact sequence fits in between the two long exact sequences from homology and cohomology.

First we show that ∂_0 maps into $\ker \hat{N}_L$, and that $\operatorname{im} \hat{N}_N$ is contained in $\ker \partial^0$ which will give us well defined maps $\hat{\partial}_0$ and $\hat{\partial}^0$. Note that $\operatorname{im} \partial_0 = \ker f_0$. If $f_0(x) = 0$ then $f^0 \hat{N}_L(x) = \hat{N}_M f_0(x) = 0$ and since f^0 is injective it follows that $x \in \ker \hat{N}_L$. Similarly, if $x \in \operatorname{im} \hat{N}_N$ then since g_0 is surjective $x \in \operatorname{im} \hat{N}_N g_0 = \operatorname{im} g^0 \hat{N}_M \subseteq \operatorname{im} g^0 = \ker \partial^0$.

We now have a well defined sequence as in the theorem, but we still need to check exactness at $\ker \hat{N}_L$ and $\operatorname{coker} \hat{N}_N$. Fortunately exactness at each follows from the work we just did. Since $\hat{f}_0$ is the restriction of f_0 to $\ker \hat{N}_L$ and, as we just proved, $\ker f_0 \subseteq \ker \hat{N}_L$, exactness is immediate. Similarly, $\ker \hat{\partial}^0$ is the quotient of $\ker \partial^0 = \operatorname{im} g^0$ by $\operatorname{im} \hat{N}_N$. Therefore we just need to show that $\operatorname{im} \hat{g}^0 = \operatorname{im} g^0 / \operatorname{im} \hat{N}_N$. But by definition $\hat{g}^0$ is the map induced by g^0 between $M^G / \operatorname{im} \hat{N}_M$ and $N^G / \operatorname{im} \hat{N}_N$ so this must hold.

□

2.2. **Complete Resolutions.** [7]

Definition 2.16. A *complete resolution* $X_\bullet$ for a finite group G is an exact sequence

$$\cdots \to X_1 \to X_0 \xrightarrow{d_0} X_{-1} \to \dots$$

of finitely-generated free $\mathbb{Z}[G]$-modules along with an element $e \in X_{-1}$ which is G-fixed and generates $\operatorname{im} d_0$.

We would like some examples of complete resolutions to show they exist. It turns out that we can "splice" together any two finitely-generated free resolutions to get a complete resolution. In fact, all complete resolutions can be obtained in this manner. In order to prove this characterization, we require the following lemma:

Lemma 2.17. *Suppose G is a finite group and $F_\bullet$ is an exact sequence of finitely-generated free $\mathbb{Z}[G]$-modules. Then the "dual" $F^\bullet = \operatorname{Hom}(F_\bullet, \mathbb{Z})$ is also an exact sequence of finitely-generated free $\mathbb{Z}[G]$-modules. The G-action on any F^n is $(g\varphi)(x) = g\varphi(g^{-1}x) = \varphi(g^{-1}x)$ since G acts trivially on $\mathbb{Z}$.*

Proof. We will first prove that the dual of a $\mathbb{Z}[G]$-free module F is $\mathbb{Z}[G]$-free. It suffices to consider the case $F = \mathbb{Z}[G]$. Consider the $\mathbb{Z}$-basis G. Define a dual basis $\{f_g\}_{g\in G}$ such that $f_g(h) = \delta_{g,h}$ is 1 if $g = h$ and 0 otherwise. The map $g \mapsto f_g$ is a $\mathbb{Z}$-isomorphism. We have

$$f_{gg'}(h) = \delta_{gg',h} = \delta_{g',g^{-1}h} = f_{g'}(g^{-1}h) = (gf_{g'})(h).$$

Since the isomorphism is G-linear, $\operatorname{Hom}(\mathbb{Z}[G], \mathbb{Z}) \cong \mathbb{Z}[G]$ as $\mathbb{Z}[G]$-modules.

It remains to prove that the dual of an exact sequence of $\mathbb{Z}$-free modules is exact. To this end, it suffices to show that for an exact sequence of $\mathbb{Z}$-free modules $F_\bullet$, there is a chain homotopy $\mathrm{id} \sim 0 : F_\bullet \to F_\bullet$. This chain homotopy is preserved by the $\operatorname{Hom}(-, \mathbb{Z})$ functor and so the dual sequence will be exact too (the identity and zero maps will induce the same homomorphism on homology).

We first break up

$$\dots \xrightarrow{\partial_{n+2}} F_{n+1} \xrightarrow{\partial_{n+1}} F_n \xrightarrow{\partial_n} F_{n-1} \xrightarrow{\partial_{n-1}} \dots$$

into short exact sequences

$$0 \to \ker \partial_n \xrightarrow{\subseteq} F_n \xrightarrow{\partial_n} \ker \partial_{n-1} \to 0.$$

Since everything is free, this short exact sequence splits. Let $h_n : F_n \to F_{n+1}$ be the composite map which projects F_n onto $\ker \partial_n$ and then includes that into F_{n+1}. Given the decomposition of $F_n \cong \ker \partial_n \oplus \ker \partial_{n-1}$ we see that $\partial_{n+1}h_n$ is projection onto $\ker \partial_n$ and $h_{n-1}\partial_n$ is projection onto $\ker \partial_{n-1}$. Therefore

$$\partial_{n+1}h_n + h_{n-1}\partial_n = \mathrm{id}$$

and so $h_\bullet$ is a chain homotopy $\mathrm{id} \sim 0$.[8]

□

[7] the following presentation of complete resolutions is based on Cartan and Eilenberg[4] Chapter XII, 1-3, pages 232-241.

[8] proof from Cartan and Eilenberg[4] proposition 3.3, page 239

Given two resolutions $P_\bullet \xrightarrow{\varepsilon} \mathbb{Z}$ and $Q_\bullet \xrightarrow{\phi} \mathbb{Z}$ consisting of finitely-generated free $\mathbb{Z}[G]$-modules, we can construct a complete resolution. First dualize $Q_\bullet$ to get $Q^\bullet$ with $Q^n = \mathrm{Hom}(Q_n, \mathbb{Z})$. By the lemma above,

$$0 \to \mathbb{Z} \xrightarrow{\phi^*} Q^0 \to Q^1 \to \dots$$

is an exact sequence and all Q^n are finitely-generated free $\mathbb{Z}[G]$-modules. We want to show that

$$\cdots \to P_1 \to P_0 \xrightarrow{\phi^*\varepsilon} Q^0 \to Q^1 \to \dots$$

is a complete resolution with $X_0 = P_0$. We can take $e = \phi^*(1)$ which is G-fixed because $\mathbb{Z}$ is. It also must generate the image of $\phi^*\varepsilon$ because $\phi^*\varepsilon$ factors through a copy of $\mathbb{Z}$ which is generated by 1. Since ε is surjective, $\operatorname{im}\phi^*\varepsilon = \operatorname{im}\phi^*$ so we have exactness at Q^0. Analogously, since ϕ^* is injective, $\ker\phi^*\varepsilon = \ker\varepsilon$ and so our spliced complete resolution is exact at P_0 as well.

Now consider some complete resolution $X_\bullet$. Since e is G-fixed and X_{-1} is $\mathbb{Z}[G]$-free, e generates a copy of $\mathbb{Z}$ inside X_{-1}. It follows that d_0 can be factored into

$$X_0 \xrightarrow{\varepsilon} \mathbb{Z} \xrightarrow{\mu} X_{-1}$$

with ε a surjection and μ an injection mapping $1 \mapsto e$. We can then reverse the steps of the construction above by dualizing $\mathbb{Z} \xrightarrow{\mu} X_{-1} \to \dots$ to get

$$\dots X^{-1} \xrightarrow{\phi=\mu^*} \mathbb{Z}.$$

Thus all complete resolutions are the result of splicing together one finitely-generated free resolution with the dual of another one.

Complete resolutions give us an alternate definition of Tate cohomology:

Definition 2.18. The *Tate cohomology* of G with coefficients in M is

$$\hat{H}^n(G, M) = H^n(\mathrm{Hom}_G(X_\bullet, M))$$

where $X_\bullet$ is a complete resolution and H^n takes the cohomology of the cochain complex $\mathrm{Hom}_G(X_\bullet, M)$.

We need to show that this definition agrees with our original definition of Tate cohomology. An important preliminary is necessary before we proceed: It seems reasonable that $\hat{H}^n(G, M) \cong H^n(G, M)$ for $n \geq 1$ considering $\hat{H}^n$ is defined by the complex $\mathrm{Hom}_G(X_\bullet, M)$ and H^n by $\mathrm{Hom}_G(P_\bullet, M)$. To see how group homology fits into the picture we must prove that $\mathrm{Hom}_G(X_\bullet, M) \cong X^\bullet \otimes_G M$.

We want to show that for G-modules X and M, with X $\mathbb{Z}[G]$-free and finitely generated

$$\mathrm{Hom}(X, \mathbb{Z}) \otimes_G M \cong \mathrm{Hom}_G(X, M).$$

The map which sends $f \otimes m$ to $x \mapsto f(x)m$ defines an isomorphism $\mathrm{Hom}(X, \mathbb{Z}) \otimes M \cong \mathrm{Hom}(X, M)$. Therefore

$$(\mathrm{Hom}(X, \mathbb{Z}) \otimes M)^G \cong \mathrm{Hom}(X, M)^G = \mathrm{Hom}_G(X, M)$$

and since

$$\mathrm{Hom}(X, \mathbb{Z}) \otimes_G M = (\mathrm{Hom}(X, \mathbb{Z}) \otimes M)_G$$

it suffices to prove that

$$(\mathrm{Hom}(X, \mathbb{Z}) \otimes M)_G \cong (\mathrm{Hom}(X, \mathbb{Z}) \otimes M)^G.$$

Since $\mathrm{Hom}(X, \mathbb{Z})$ is free and finitely generated as a $\mathbb{Z}[G]$-module, the following lemma suffices:

Lemma 2.19. *If F is a finitely generated free $\mathbb{Z}[G]$-module and M is any $\mathbb{Z}[G]$-module then the norm map*

$$\hat{N} : (F \otimes M)_G \to (F \otimes M)^G$$

is an isomorphism.

Proof. First note that $\hat{N}$, $(- \otimes M)_G$, and $(- \otimes M)^G$ all commute with direct sum. Therefore it suffices to consider the case $F = \mathbb{Z}[G]$.

Equip the underlying Abelian group of M with the trivial G-action. Call this new G-module M'. For clarity we will temporarily use ρ, ρ', $\rho_\otimes$, and $\rho'_\otimes$ to denote the actions of G on M, M', $\mathbb{Z}[G] \otimes M$, and $\mathbb{Z}[G] \otimes M'$ respectively. There is an isomorphism

$$\Phi : \mathbb{Z}[G] \otimes M' \to \mathbb{Z}[G] \otimes M$$

given by $\Phi(g \otimes m) = g \otimes \rho(g)m = \rho_\otimes(g)(1 \otimes m)$. The inverse can be described $\Phi^{-1}(g \otimes m) = g \otimes \rho(g^{-1})m \in \mathbb{Z}[G] \otimes M'$. To see equivariance note that

$$\Phi(\rho'_\otimes(h)(g \otimes m)) = \Phi(hg \otimes \rho'(h)m) = \Phi(hg \otimes m) = \rho_\otimes(h)\rho_\otimes(g)(1 \otimes m).$$

Since $\mathbb{Z}[G] = \bigoplus_{g \in G} \langle g \rangle$ as a $\mathbb{Z}$-module, and tensors distribute over direct sum, $\mathbb{Z}[G] \otimes M' = \bigoplus_{g \in G} \langle g \rangle \otimes M'$. In other words, $\mathbb{Z}[G] \otimes M'$ is $|G|$ copies of M' with a G-action given by permuting those copies.[9]

We now want to show that $\hat{N} : (\mathbb{Z}[G] \otimes M')_G \to (\mathbb{Z}[G] \otimes M')^G$ is an isomorphism. We will discontinue the use of the tedious $\rho, \rho', \rho_\otimes, \rho'_\otimes$ notation since $\mathbb{Z}[G] \otimes M$ will not be directly discussed for the remainder of the proof. Consider some element $x = \sum_{h \in G} h \otimes m_h \in \mathbb{Z}[G] \otimes M'$. If $x = 0$, then by the characterization of $\mathbb{Z}[G] \otimes M'$ above as a direct sum of $\langle g \rangle \otimes M'$, it follows that $m_h = 0$ for each $h \in G$.

Take $x = \sum_{h \in G} h \otimes m_h$ as above, but this time we do not require $x = 0$. Suppose $Nx = \sum_{g \in G} gx = 0$ and let $N = N1 = \sum_{g \in G} g$. Since N is constant over orbits, $Nh = N1 = N$ and so

$$\begin{aligned} 0 = Nx &= \sum_{g \in G} g \cdot \sum_{h \in G} h \otimes m_h = \sum_{h \in G} \sum_{g \in G} gh \otimes m_h = \sum_{h \in G} Nh \otimes m_h \\ &= \sum_{h \in G} N \otimes m_h = N \otimes \sum_{h \in G} m_h = \sum_{g \in G} g \otimes \sum_{h \in G} m_h. \end{aligned}$$

By the argument in the previous paragraph, it follows that $\sum_{h \in G} m_h = 0$. We then have that $m_1 = -\sum_{h \in G \setminus \{1\}} m_h$ and so

$$\begin{aligned} x &= \sum_{h \in G} h \otimes m_h = \sum_{h \in G \setminus \{1\}} h \otimes m_h + 1 \otimes m_1 \\ &= \sum_{h \in G \setminus \{1\}} h \otimes m_h - \sum_{h \in G \setminus \{1\}} 1 \otimes m_h = \sum_{h \in G \setminus \{1\}} (h - 1) \otimes m_h. \end{aligned}$$

Since $h - 1$ is in the augmentation ideal A_G for any $h \in G$, it follows that x is in $A_G(\mathbb{Z}[G] \otimes M')$. Therefore $\ker N = A_G(\mathbb{Z}[G] \otimes M')$ and $\hat{N}$ is injective.

Now suppose $gx = x$ for all $g \in G$. Since

$$gx = \sum_{h \in G} gh \otimes m_h = \sum_{h \in G} h \otimes m_{g^{-1}h}$$

[9] this trick for decomposing $\mathbb{Z}[G] \otimes M$ comes from Hesselholt and Madsen[6] page 60.

it follows that

$$0 = x - gx = \sum_{h \in G} h \otimes (m_h - m_{g^{-1}h})$$

which implies that $(m_h - m_{g^{-1}h}) = 0$ for all $g, h \in G$. It follows that $m_g = m$ is constant over $g \in G$. We can therefore write $x = \sum_{h \in G} h \otimes m = N \otimes m = N(1 \otimes m)$. Thus $\hat{N}$ is surjective and we are done.

□

We are now ready to prove the equivalence of definitions 2.12 and 2.18. Consider the complex $\mathrm{Hom}_G(X_\bullet, M)$ for some complete resolution $X_\bullet$. For all $n < 0$ replace $\mathrm{Hom}_G(X_n, M)$ with the isomorphic group $X^n \otimes_G M$. By our characterization of complete resolutions, we can write $X_\bullet$ as

$$\cdots \to X_1 \to X_0 \xrightarrow{\varepsilon} \mathbb{Z} \xrightarrow{\mu} X_{-1} \to X_{-2} \to \dots$$

so that both $\cdots \to X_1 \to X_0 \xrightarrow{\varepsilon} \mathbb{Z}$ and $\cdots \to X^{-2} \to X^{-1} \xrightarrow{\mu^*} \mathbb{Z}$ are free (and thus projective) $\mathbb{Z}[G]$-resolutions of $\mathbb{Z}$. It is therefore immediate that

$$H^n(\mathrm{Hom}_G(X_\bullet, M)) \cong \begin{cases} H_{-n-1}(G, M) & n \leq -2 \\ H^n(G, M) & n \geq 1 \end{cases}.$$

We then have the following diagram

$$\begin{array}{ccccc} X^{-1} \otimes_G M & \xrightarrow{\hat{N}} & (X^{-1} \otimes M)^G & \longrightarrow & \mathrm{Hom}_G(X_{-1}, M) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z} \otimes_G M & \xrightarrow{\hat{N}} & (\mathbb{Z} \otimes M)^G & \longrightarrow & \mathrm{Hom}_G(\mathbb{Z}, M) \end{array}$$

where the vertical arrows are induced by precomposition with μ and horizontal arrows are norm maps and isomorphisms induced by $\mathrm{Hom}(X, \mathbb{Z}) \otimes M \cong \mathrm{Hom}(X, M)$. Note that every horizontal arrow is an isomorphism except for the norm map $\mathbb{Z} \otimes_G M \to (\mathbb{Z} \otimes M)^G$ since $\mathbb{Z}$ is not $\mathbb{Z}[G]$-free. The left hand square commutes because

$$(\mu^* \otimes \mathrm{id}) \left(\sum_{g \in G} gf \otimes gm \right) = \sum_{g \in G} g(\mu^* f) \otimes gm$$

and the right hand square commutes because

$$f(\mu x)m = (\mu^* f)(x)m.$$

The corners of the diagram are therefore a commutative square. Moreover, if we isolate the right side of this square, we get a second diagram

$$\begin{array}{ccc} \mathrm{Hom}_G(X_{-1}, M) & \longrightarrow & \mathrm{Hom}_G(X_0, M) \\ \downarrow & \nearrow & \\ \mathrm{Hom}_G(\mathbb{Z}, M) & & \end{array}.$$

The added diagonal arrow is induced by the augmentation map ε and it commutes with everything else since the map $X_0 \to X_{-1}$ is by definition $\mu\varepsilon$. Swapping out

the right side of the commutative square with the right side of this commutative triangle we get a new commutative diagram

$$\begin{array}{ccc} X^{-1} \otimes_G M & \longrightarrow & \mathrm{Hom}_G(X_0, M) \\ \downarrow & & \uparrow \\ \mathbb{Z} \otimes_G M & \xrightarrow{\hat{N}} & \mathrm{Hom}_G(\mathbb{Z}, M) \end{array}$$

proving that the map $X^{-1} \otimes_G M \to \mathrm{Hom}_G(X_0, M)$ in $\mathrm{Hom}_G(X_\bullet, M)$ factors through the norm map $\mathbb{Z} \otimes_G M \to \mathrm{Hom}_G(\mathbb{Z}, M)$. Since $\mathbb{Z} \otimes_G M = M_G$ and $\mathrm{Hom}_G(\mathbb{Z}, M) = M^G$ this norm map is the same one we used to define Tate cohomology in definition 2.12.

Because

$$X_1 \to X_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0 \qquad X^{-2} \to X^{-1} \xrightarrow{\mu} \mathbb{Z} \to 0$$

are exact,

$$0 \to \mathrm{Hom}_G(\mathbb{Z}, M) \xrightarrow{\psi = \varepsilon^*} \mathrm{Hom}_G(X_0, M) \xrightarrow{\partial^0} \mathrm{Hom}_G(X_1, M)$$

and

$$X^{-2} \otimes_G M \xrightarrow{\partial^{-2}} X^{-1} \otimes_G M \xrightarrow{\varphi = \mu \otimes_G \mathrm{id}} \mathbb{Z} \otimes_G M \to 0$$

will be too. Therefore

$$\begin{aligned} H^{-1}(\mathrm{Hom}_G(X_\bullet, M)) &= \ker(\psi \hat{N} \varphi) / \operatorname{im}(\partial^{-2}) = \ker(\psi \hat{N} \varphi) / \ker(\varphi) \\ &= \ker(\hat{N} \varphi) / \ker(\varphi) \cong \ker \hat{N} \end{aligned}$$

and

$$\begin{aligned} H^0(\mathrm{Hom}_G(X_\bullet, M)) &= \ker(\partial^0) / \operatorname{im}(\psi \hat{N} \varphi) = \operatorname{im}(\psi) / \operatorname{im}(\psi \hat{N} \varphi) \\ &= \operatorname{im}(\psi) / \operatorname{im}(\psi \hat{N}) \cong \operatorname{coker} \hat{N}. \end{aligned}$$

We have found isomorphisms between the Tate cohomologies given by definitions 2.12 and 2.18. It is also true that any homomorphism $\phi : M \to N$ induces the same maps on both definitions of cohomology (up to the already given isomorphisms).[10] Though this definition for Tate cohomology via complete resolutions might seem more obscure than our original one, it can be convenient to express Tate cohomology directly as the cohomology of a cochain complex. For example, the long exact sequence in $H^n(\mathrm{Hom}_G(X_\bullet, M))$ follows immediately from lemma 2.13 and requires none of the additional verification of theorem 2.15.

2.3. **A Third Definition.**

Definition 2.20. For cochain complexes $X^\bullet$ and $Y^\bullet$, their *tensor product* is

$$(X^\bullet \otimes Y^\bullet)^n = \bigoplus_{r+s=n} X^r \otimes Y^s$$

with maps

$$d_{X \otimes Y}(x \otimes y) = d_X(x) \otimes y + (-1)^{\deg(x)} x \otimes d_Y y$$

Note that if we take M to denote the complex $\cdots \to 0 \to M \to 0 \to \ldots$ then this definition aligns with the "piecewise" definition we assumed for $M \otimes P_\bullet$.

[10] Cartan and Eilenberg[4] theorem 3.2, page 241.

Definition 2.21. For a map of cochain complexes $f : X^\bullet \to Y^\bullet$ we define the *mapping cone*

$$Cf^n = X^{n+1} \oplus Y^n$$

with maps

$$d_{C(f)}(x, y) = (-d_X(x), f(x) + d_Y(y)).$$

The projective resolution $P_\bullet$ of $\mathbb{Z}$ can be treated as a map of cochain complexes $P_\bullet \to \mathbb{Z}$ where $(P_\bullet)^n = P_{-n}$ and $\mathbb{Z}$ is $\cdots \to 0 \to \mathbb{Z} \to 0 \to \ldots$. Denote this mapping cone $\widetilde{P}_\bullet$. More explicitly, $\widetilde{P}_\bullet$ is

$$\ldots \xrightarrow{-d_3} P_2 \xrightarrow{-d_2} P_1 \xrightarrow{-d_1} P_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0.$$

Switching the signs of the differentials makes no difference up to homotopy equivalence. Taking $\widetilde{P}_\bullet$ to be a cochain complex in the manner already described we can provide our final definition of Tate cohomology:

Definition 2.22. Let $P_\bullet$ be a projective resolution of $\mathbb{Z}$ as previously. The *Tate cohomology* of G with coefficients in M is the cohomology of the cochain complex

$$(\widetilde{P}_\bullet \otimes \mathrm{Hom}(P_\bullet, M))^G$$

This definition of Tate cohomology is naturally isomorphic to the two previously given.[11]

3. The Cup Product

In analogy to singular cohomology, we will define a "cup product" on Tate cohomology:

$$\hat{H}^p(G, M) \otimes \hat{H}^q(G, M') \to \hat{H}^{p+q}(G, M \otimes M').$$

For $p = q = 0$ we already have a candidate for this map: If $m \in M^G$ and $m' \in M'$ then

$$m \otimes Nm' = m \otimes \sum_{g \in G} gm' = \sum_{g \in G} m \otimes gm' = \sum_{g \in G} g(m \otimes m') = N(m \otimes m')$$

and similarly for $Nm \otimes m'$ with $m \in M$, $m' \in M'^G$. Since $\hat{H}^0$ is the cokernel of $\hat{N}$, it follows that the natural map $M^G \otimes M'^G \to (M \otimes M')^G$ induces a map

$$\hat{H}^0(G, M) \otimes \hat{H}^0(G, M') \to \hat{H}^0(G, M \otimes M').$$

In order to extend this to higher cohomology groups, we require the following theorem:

Theorem 3.1. *Suppose $X_\bullet$ is a complete resolution with differential d and distinguished element $e \in X_{-1}$. There exist G-module homomorphisms*

$$\phi_{p,q} : X_{p+q} \to X_p \otimes X_q$$

such that

(1) $\phi_{p,q} \circ d = (d \otimes \mathrm{id}) \circ \phi_{p+1,q} + (-1)^p(\mathrm{id} \otimes d) \circ \phi_{p,q+1}$
(2) if $x \in X_0$ such that $dx = e$ then $(d \otimes d)\phi_{0,0}x = e \otimes e$.

[11] Hesselholt and Madsen[6] page 61.

Proof. We leave the proof of the general case to Cartan and Eilenberg[12] but we can use the resolution of group rings $\cdots \to \mathbb{Z}[G^2] \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$ to construct a complete resolution. Defining the $\phi_{p,q}$ for this complex is sufficient for our purposes.

Via the splicing procedure described in subsection 2.2, we can construct a complete resolution out of two copies of the standard resolution

$$\cdots \to \mathbb{Z}[G^2] \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$$

to get

$$\cdots \to \mathbb{Z}[G^2] \to \mathbb{Z}[G] \to \mathrm{Hom}(\mathbb{Z}[G], \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}[G^2], \mathbb{Z}) \to \cdots .$$

We will use the notation

$$(g_1, \ldots, g_n)^* \in \mathrm{Hom}(\mathbb{Z}[G^n], \mathbb{Z})$$

to denote the map which sends $(g_1, \ldots, g_n)$ to 1 and all other $(h_1, \ldots, h_n)$ to zero. Our definition for $\phi_{p,q}$ is "piece-wise".

For $p, q \geq 0$ let

$$\phi_{p,q}(g_0, \ldots, g_{p+q}) = (g_0, \ldots, g_p) \otimes (g_p, \ldots, g_{p+q}).$$

For $p, q \geq 1$ let

$$\phi_{-p,-q}(g_1, \ldots, g_{p+q})^* = (g_1, \ldots, g_p)^* \otimes (g_{p+1}, \ldots, g_{p+q})^*.$$

For $p \geq 0$, if $p \geq q \geq 1$ let

$$\phi_{p,-q}(g_0, \ldots, g_{p-q}) = \sum_{s_1, \ldots, s_q \in G} (g_0, \ldots, g_{p-q}, s_1, \ldots, s_q) \otimes (s_q, \ldots, s_1)^*$$

and if $p < q \geq 1$ let

$$\phi_{p,-q} = (g_1, \ldots, g_{q-p})^* = \sum_{s_1, \ldots, s_p \in G} (g_1, s_1, \ldots, s_p) \otimes (s_p, \ldots, s_1, g_1, \ldots, g_{q-p})^*.$$

For $q \geq 0$, if $q \geq p \geq 1$ let

$$\phi_{-p,q}(g_0, \ldots, g_{q-p}) = \sum_{s_1, \ldots, s_p \in G} (s_1, \ldots, s_p)^* \otimes (s_p, \ldots, s_1, g_0, \ldots, g_{q-p})$$

and if $q < p \geq 1$ let

$$\phi_{-p,q}(g_1, \ldots g_{p-q})^* = \sum_{s_1, \ldots, s_q \in G} (g_1, \ldots, g_{p-q}, s_1, \ldots, s_q)^* \otimes (s_q, \ldots, s_1, g_{p-q}).$$

We will not explicitly verify properties (1) and (2) due to the length of the computation, but they do hold.[13]

□

Using the homomorphisms $\phi_{p,q}$ we can now define the cup product:

Definition 3.2. For any $\varphi \in \mathrm{Hom}_G(X_p, M)$ and $\psi \in \mathrm{Hom}_G(X_q, M')$ define

$$\varphi \smile \psi = (\varphi \otimes \psi) \circ \phi_{p,q} \in \mathrm{Hom}_G(X_{p+q}, M \otimes M').$$

It follows from part (1) of Theorem 3.1 that this descends to a well defined cup product on Tate cohomology.

[12] Cartan and Eilenberg[4] Chapter XII, 4, pages 242-244.

[13] Atiyah and Wall[1] page 107.

As we would expect, this product is associative and skew-commutative (through the associations $(L \otimes M) \otimes N \cong L \otimes (M \otimes N)$ and $M \otimes M' \cong M' \otimes M$).[14]

There is still one issue remaining. Though we now have a proper definition, it seems to rely on a choice of $X_\bullet$ and $\phi_{p,q}$. The following theorem verifies that the cup product is in fact canonical:

Theorem 3.3. *The cup product is the unique family of homomorphisms*

$$\xi_{p,q} : \hat{H}^p(G, M) \otimes \hat{H}^q(G, M') \to \hat{H}^{p+q}(G, M \otimes M')$$

functorial in M and M' such that[15]

1. *$\xi_{0,0}$ is induced by the map $M^G \otimes M'^G \to (M \otimes M')^G$ (as described at the beginning of this section)*
2. *If $0 \to L \to M \to N \to 0$ and $0 \to L \otimes A \to M \otimes A \to N \otimes A \to 0$ are exact then for any $n \in \hat{H}^p(G, N)$ and $a \in \hat{H}^q(G, A)$,*

$$\partial n \smile a = \partial(n \smile a) \in \hat{H}^{p+q+1}(G, N \otimes A)$$

where ∂ denotes the boundary homomorphisms of the long exact sequences.

3. *If $0 \to L \to M \to N \to 0$ and $0 \to A \otimes L \to A \otimes M \to A \otimes N \to 0$ are exact then for any $a \in \hat{H}^p(G, A)$ and $n \in \hat{H}^q(G, N)$,*

$$a \smile \partial n = (-1)^p \partial(a \smile n) \in \hat{H}^{p+q+1}(G, A \otimes N).$$

4. Tate's Theorem

As mentioned in the introduction, Tate cohomology is particularly important in relation to Tate's theorem. In particular, the case $n = -2$ is useful since $\hat{H}^0(G, M) = \operatorname{coker} \hat{N} \subseteq M^G$ and it can be proven that $\hat{H}^{-2}(G, \mathbb{Z}) = H_1(G, \mathbb{Z})$ is the Abelianization of G.[16] Now having defined the cup product, we can provide a proper statement of the theorem:

Theorem 4.1. ***Tate's Theorem.***[17] *Suppose G is a finite group, M a G-module, and $\alpha \in \hat{H}^2(G, M)$ such that for any subgroup $H \subseteq G$*

1. *$\hat{H}^1(H, M) = 0$*
2. *$H^2(H, M)$ is cyclic of order $|H|$ and is generated by the restriction of α.*

Then for any n the map

$$\begin{array}{ccc} \hat{H}^n(G, \mathbb{Z}) & \to & \hat{H}^{n+2}(G, M) \\ \beta & \mapsto & \alpha \smile \beta \end{array}$$

is an isomorphism.

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[14] Atiyah and Wall[1] proposition 9, page 107.
[15] Atiyah and Wall[1] theorem 4, page 105.
[16] Tan[9] example 2.6, page 6.
[17] Sharifi[8] page 51.

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