L-FUNCTIONS OVER A NUMBER FIELD

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ABSTRACT. This paper discusses the formulation of L-functions over a number field, as formulated in Tate's thesis (1950). We begin with a classical discussion of the Riemann zeta function. Then we provide a short introduction to the adèles and idèles before discussing and comparing the meromorphic continuation and functional equation of global zeta integrals and Hecke L-functions with the classical case.

CONTENTS

1. INTRODUCTION

As setup for the main sections of this paper, we recall some preliminary definitions and facts necessary for the coming sections. Details and proofs of the following statements can be found in [1] and [5].

1.1. Gamma function.

Definition 1.1. Let $\mathrm{Re}(s) > 0$. The *gamma function* is defined as

$$\Gamma(s) := \int_0^\infty t^s e^{-t} \frac{dt}{t}.$$

Some important properties of the gamma function are:

Proposition 1.2 (Properties of $\Gamma(s)$)**.** *Let* $Re(s) > 0$.

- *The gamma function has functional equation:*
$$\Gamma(s+1) = s\Gamma(s).$$
- $\Gamma(s)$ *extends to a meromorphic function on* $\mathbb{C}$*, except for simple poles at* $0, -1, -2, \ldots$*.*
- $\Gamma(s)$ *has no zeroes.*
- $\Gamma(n) = (n-1)!$ *for all* $n \in \mathbb{Z}_+$*.*

1.2. **Schwartz functions and Fourier transforms.**

Definition 1.3. A function $f : \mathbb{R} \to \mathbb{C}$ is a *Schwartz function* if for all $k \geq 0$ the k^{th} derivative $f^{(k)}$ exists and
$$\lim_{|x| \to \infty} x^n f^{(k)}(x) = 0.$$
We denote the space of all such functions $\mathcal{S}(\mathbb{R})$, the Schwartz space.

For a Schwartz function $f \in \mathcal{S}(\mathbb{R})$, we are able to define its Fourier transform $\hat{f}$, which also belongs to $\mathcal{S}(\mathbb{R})$.

Definition 1.4. Let $f \in \mathcal{S}(\mathbb{R})$. The Fourier transform of f is
$$\hat{f}(y) := \int_{\mathbb{R}} f(x) e^{-2\pi i x y} \, dx.$$

By taking the derivative of $\hat{f}$, the Fourier transform can be seen to swap multiplication by a monomial and differentiation.

For $f \in \mathcal{S}(\mathbb{R})$, an important formula holds that relates sums of a Schwartz function and sums of its Fourier transform.

Theorem 1.5 (Poisson summation formula)**.** *Let* $f \in \mathcal{S}(\mathbb{R})$*. Then*
$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n).$$

1.3. **Theta function.** The theta function plays a crucial role in the proof of the analytic continuation of the Riemann zeta function.

Definition 1.6. Let $t \in \mathbb{R}_+$. Define the *theta function*
$$\Theta(t) := \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t}.$$

The functional equation of $\Theta(t)$ follows immediately as an application of the Poisson summation formula (Theorem 1.5) with $f = e^{-\pi t x^2}$ and gives the following result.

Proposition 1.7 (Functional equation for $\Theta(t)$)**.** *Let* $t > 0$*. Then*
$$\Theta(t) = t^{-\frac{1}{2}} \Theta\left(\frac{1}{t}\right).$$

Noting that $e^{-\pi n^2 t}$ is an even function with respect to n so that
$$\Theta(t) = 1 + 2 \sum_{n \geq 1} e^{-\pi n^2 t},$$
we can rearrange the functional equation of $\Theta(t)$ into a form that can be easily applied in the next section.

Corollary 1.8. *For $t > 0$. The equation*

$$\sum_{n\geq 1} e^{-\pi n^2 t} = \frac{\Theta(t) - 1}{2}$$

holds.

2. Riemann Zeta Function

We can now discuss the analytic continuation of the Riemann zeta function as the prototypical example of an L-function. We begin by recalling the definition of the Riemann zeta function.

Definition 2.1. Let $s \in \mathbb{C}$ and $\mathrm{Re}(s) > 1$. The *Riemann zeta function* is

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}},$$

where the latter expression is an Euler product.

The main theorem of the Riemann zeta function has two parts: the meromorphic continuation and the functional equation. The theorem is most concisely stated after introducing the follow definition.

Definition 2.2. Let $s \in \mathbb{C}$. The *completed zeta function* is defined as

$$\xi(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Then our main theorem can be stated:

Theorem 2.3. *Let $s \in \mathbb{C}$. Then the function $\xi(s)$*

(1) extends to a meromorphic function on $\mathbb{C}$, holomorphic except for simple poles at $s = 0$ and 1; and
(2) has functional equation

$$\xi(s) = \xi(1 - s).$$

We include Riemann's proof as described in [1] and [5].

Proof. We begin by inspecting the contribution of the nth term of $\zeta(s)$ to the completed zeta function $\xi(s)$:

$$\begin{aligned}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)n^{-s} &= \pi^{-s/2}n^{-s}\int_0^\infty e^{-x}x^{s/2}\,\frac{dx}{x} \\ &= \int_0^\infty e^{-\pi n^2 t}t^{s/2}\,\frac{dt}{t}\end{aligned}$$

by making the substitution $x = \pi n^2 t$. This now makes an expression that looks like $\Theta(t)$.

Summing over $n \geq 1$, we arrive back with $\xi(s)$ on LHS:

$$\xi(s) = \sum_{n\geq 1}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)n^{-s} = \sum_{n\geq 1}\int_0^\infty e^{-\pi n^2 t}\,t^{s/2}\,\frac{dt}{t}.$$

Since $\left|t^{s/2}\right|$ does not depend on the imaginary part of s, the RHS is absolutely convergent for $\mathrm{Re}(s) > 1$, and thus the integral and sum can be interchanged for

$\mathrm{Re}(s) > 1$. This, however, does not hold for $\mathrm{Re}(s) < 0$ as the integral blows up for $t \in (0,1)$, resulting in a "bad part" of the integral.

Interchanging the integral and summation and applying Corollary 1.8 for $\Theta(t)$, we continue from before:

$$\xi(s) = \int_0^\infty \sum_{n\geq 1} e^{-\pi n^2 t}\, t^{s/2}\, \frac{dt}{t} = \int_0^\infty \left(\frac{\Theta(t)-1}{2}\right) t^{s/2}\, \frac{dt}{t} \tag{2.4}$$

Remark 2.5. Recall that the Mellin transform is defined as

$$\{\mathcal{M}f\}(s) = \int_0^\infty x^{s-1} f(x)\, dx.$$

Thus we can see that

$$\xi(-2s) = \{\mathcal{M}f\}\left(\frac{\Theta(t)-1}{2}\right).$$

To fix the problem of the integral blowing up for $t \in (0,1)$, we split the integral into the good part $t \in (1,\infty)$ and the bad part $t \in (0,1)$.

On the good part, we have that

$$I(s) = \int_1^\infty \left(\frac{\Theta(t)-1}{2}\right) t^{s/2}\, \frac{dt}{t}$$

is a holomorphic entire function since $\lim_{t\to\infty} e^{\pi t}(\Theta(t)-1) = 0$.

On the bad part, we begin with the substitution $t \mapsto \frac{1}{t}$:

$$\begin{aligned}
\int_0^1 \left(\frac{\Theta(t)-1}{2}\right) t^{s/2}\, \frac{dt}{t} &= \int_1^\infty \left(\frac{\Theta\left(\frac{1}{t}\right)-1}{2}\right) t^{-s/2}\, \frac{dt}{t} \\
&= \int_1^\infty \left(\frac{t^{1/2}\Theta(t)-1}{2}\right) t^{-s/2}\, \frac{dt}{t} \qquad \text{by Proposition 1.2} \\
&= \int_1^\infty \left(t^{1/2}\left(\frac{\Theta(t)-1}{2}\right) t^{-s/2} + \frac{t^{1/2}}{2}\, t^{-s/2} - \frac{t^{-s/2}}{2}\right) \frac{dt}{t} \\
&= \int_1^\infty \left(\frac{\Theta(t)-1}{2}\right) t^{(1-s)/2}\, \frac{dt}{t} + \int_1^\infty \frac{t^{(1-s)/2}}{2} - \frac{t^{-s/2}}{2}\, \frac{dt}{t} \\
&= I(1-s) - \frac{1}{1-s} - \frac{1}{s}.
\end{aligned}$$

So for $\mathrm{Re}(s) > 1$,

$$\xi(s) = I(s) + I(1-s) + \frac{1}{1-s} - \frac{1}{s}.$$

We can now see that the RHS is meromorphic on $\mathbb{C}$, holomorphic except simple poles at 0 and 1; and symmetric with respect to $s \mapsto 1-s$, which gives us our desired meromorphic continuation of $\xi(s)$. □

Recalling the definition of $\xi(s)$ (Definition 2.2), we can recover the meromorphic continuation of $\zeta(s)$ by dividing both sides by $\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)$. Since the gamma function has simple poles at $0, -2, -4, ...$ and no zeroes by Proposition 1.2, it follows that $\zeta(s)$ has meromorphic continuation on all of $\mathbb{C}$ except a simple pole at $s = 1$.

3. Adèles and Idèles

Before we can discuss L-functions for number fields (i.e. finite extensions of $\mathbb{Q}$), it will be necessary to introduce the ring of adèles $\mathbb{A}$ as a means to "do" analysis on a global field, e.g. the rational numbers $\mathbb{Q}$, by considering all of the absolute values simultaneously. For example, the ring of adèles for $\mathbb{Q}$ allows us to do analysis on $\mathbb{Q}$ by considering classical analysis on $\mathbb{R}$ and ultrametric analysis on $\mathbb{Q}_p$ (for primes p) all at once.

This section will contain few proofs and will be primarily intended to give an intuitive understanding of the adèles and idèles while simultaneously providing the necessary facts for the final section. We frequently recall the case of $K = \mathbb{Q}$ for use in examples. Proofs and additional details can be found on [4].

3.1. **Adèles and Idèles for K.**

Definition 3.1. We say that a *place* v of number field K is an equivalence class of absolute values on K.

For example, if $K = \mathbb{Q}$, the set of places is classified by Ostrowski's Theorem: The places of $\mathbb{Q}$ are exactly the prime numbers and ∞, corresponding to the p-adic absolute values and the usual absolute value.

For this section, we will use the following notation:

Notation 3.2.

- K: a number field, i.e. a finite extension of $\mathbb{Q}$
- $\mathcal{P}_K$: set of non-archimedean places of K
- $\mathcal{P}_\infty$: set of archimedean places of K
- $\overline{\mathcal{P}}_K$: set of places of K
- $\mathcal{O}_K$: ring of integers of K

Additionally, for $v \in \mathcal{P}_K$, denote K_v the completion of K with respect to v and $\mathcal{O}_v$ the ring of integers of K_v.

Definition 3.3. An *adèle* is an infinite tuple $(x_v : v \in \overline{\mathcal{P}}_K)$ such that $x_v \in L_v$ for all $v \in \overline{\mathcal{P}}_K$ and $x_v \in \mathcal{O}_v$ for almost every $v \in \mathcal{P}_K$.

The ring of adèles of K is denoted $\mathbb{A}_K = \mathbb{A}$.

In other words, $\mathbb{A}$ is the restricted product

$$\mathbb{A} = \prod_{v \in \overline{\mathcal{P}}_K}{}' (K_v, \mathcal{O}_v),$$

where all but finitely many $x_v \in K_v$ belong to $\mathcal{O}_v$. In the case of $K = \mathbb{Q}$, we see that

$$\mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \prod_{p \text{ prime}}{}' \mathbb{Q}_p,$$

i.e. an adèle $\alpha \in \mathbb{A}_{\mathbb{Q}}$ is an infinite tuple such that all but finitely many $\alpha_p \in \mathbb{Q}_p$ belong to $\mathbb{Z}_p$.

Moreover, we can diagonally embed $\mathbb{Q}$ into the adèle ring $\mathbb{A}_{\mathbb{Q}}$ with the map $\alpha \mapsto (\alpha_\infty, \alpha_p : p \text{ prime})$, where $\alpha_p = \alpha$ for every prime p and $\alpha_\infty = \alpha$. In general, we have the following proposition for $\mathbb{A}$.

Proposition 3.4.

(1) The ring of adèles $\mathbb{A}$ is locally compact.

(2) K embeds diagonally in $\mathbb{A}$ as a discrete cocompact subgroup.

The proof of this proposition can be found in [3].

Definition 3.5. An *idèle* is an infinite tuple $(x_v : v \in \overline{\mathcal{P}}_K)$ such that $x_v \in K_v^\times$ for all $v \in \overline{\mathcal{P}}_K$ and $x_v \in \mathcal{O}_v^\times$ for almost every $v \in \mathcal{P}_K$.
The group of idèles is denoted $\mathbb{A}^\times$.

In other words, $\mathbb{A}^\times$ is the restricted product

$$\mathbb{A}^\times = \prod_{v \in \overline{\mathcal{P}}_K}{}' (K_v^\times, \mathcal{O}_v^\times),$$

where all but finitely many $x_v \in K_v^\times$ belong to $\mathcal{O}_v^\times$. The group of idèles is indeed the group of invertible elements of $\mathbb{A}$. Moreover, the group of idèles $\mathbb{A}^\times$ is also locally compact.

In particular, in the case of the idèles of $\mathbb{Q}$, we also have the following decomposition:

Proposition 3.6. *The homomorphism $\mathbb{Q}^\times \times \widehat{Z}^\times \times \mathbb{R}_+ \to \mathbb{A}_\mathbb{Q}^\times$ given by $(\alpha, u, t) \mapsto \alpha u t$ is an isomorphism of topological groups.*

Here $\widehat{Z}$ is the profinite completion of $\mathbb{Z}$. We can write

$$\widehat{Z} = \varprojlim_{n} \mathbb{Z}/n\mathbb{Z} \cong \prod_{p \text{ prime}} \mathbb{Z}_p,$$

where the inverse limit is taken over the set of positive integers ordered by divisibility. This decomposition, however, becomes much more complicated when generalizing to number fields due to the interference of global units and the non-triviality of the ideal class group.

3.2. **Idèlic norm.**

Definition 3.7. The idèlic norm $\|\cdot\|_K : \mathbb{A}_K^\times \to \mathbb{R}_+$ is defined by

$$\|x\|_K := \prod_{v \in \mathcal{P}_K \cup \mathcal{P}_\infty} \|x_v\|_v ,$$

for idèles $x = (x_v : v \in \overline{\mathcal{P}}_K) \in \mathbb{A}_K^\times$.

Since for almost every $v \in \mathcal{P}_K$, the v-adic norm $\|x_v\|_v = 1$, it follows that the infinite product $\|x\|_K$ is well-defined.

We now use the idèlic norm to provide a proof of a useful formula regarding the restriction of the idèlic norm to the diagonally embedded subgroup $K^\times$ in $\mathbb{A}^\times$.

Proposition 3.8 (Product Formula)**.** *If $a \in K^\times$, then $\|a\|_K = 1$.*

Proof. Let $a \in K^\times \subset \mathbb{A}^\times$. Note that multiplication by a on $\mathbb{A}$ is an isomorphism:

$$\mathbb{A} \xrightarrow{\sim} \mathbb{A}$$
$$\|a\|_K \, dx \leftarrow dx,$$

where dx and $\|a\|_K \, dx$ are Haar measures. In particular, multiplication by a "stretches" $\mathbb{A}$ by a factor of $\|a\|_K$.

Applying the same reasoning, we have that multiplication by a is an isomorphism $\mathbb{A}/K \xrightarrow{\sim} \mathbb{A}/K$ and in particular, $\mathrm{Vol}(\mathbb{A}/K) = \|a\|_K \mathrm{Vol}(\mathbb{A}/K)$. But since $\mathrm{Vol}(\mathbb{A}/K)$ is finite but non-zero, it follows that $\|a\|_K = 1$. $\square$

Notation 3.9. Denote the group of norms of idèles

$$\left|\mathbb{A}^\times\right| := \{\|x\|_K : x \in \mathbb{A}^\times\}.$$

For $t \in |\mathbb{A}^\times|$, denote the preimage set of norm t

$$\mathbb{A}_t^\times := \{x \in \mathbb{A}^\times : \|x\|_K = t\}.$$

Thus it follows that we have the disjoint union

$$\mathbb{A}^\times = \bigsqcup_{t \in |\mathbb{A}^\times|} \mathbb{A}_t^\times. \tag{3.10}$$

In particular, note that the kernel of the idèlic norm $\|\cdot\|_K$ is $\mathbb{A}_1^\times$. Equipping $\mathbb{A}_1^\times$ with induced topology as the closed subset of $\mathbb{A}^\times$, it follows that $\mathbb{A}_1^\times$ is also locally compact.

We end our discussion of adèles and idèles with an important theorem about the kernel of the idèlic norm. The proof is fairly lengthy and will be omitted, but can be found in [4].

Theorem 3.11. *$K^\times$ is a discrete cocompact subgroup of $\mathbb{A}_1^\times$.*

The compactness of $\mathbb{A}_1^\times/K^\times$ will prove useful in our discussion of the global zeta function in Section 4.3.

3.3. **Hecke characters.**

Definition 3.12. A *quasicharacter* or *Hecke character* or *Idèle-class character* is a continuous homomorphism $\chi : \mathbb{A}^\times \to \mathbb{C}^\times$ such that $\chi|_{K^\times} = 1$.

Denote $\mathcal{X}$ the group of Hecke characters.

Hecke characters (as we will refer to them in this paper), can thus be identified with characters of the *idèle class group* $\mathbb{A}^\times/K^\times$.

Example 3.13. Let $K = \mathbb{Q}$ and let $\chi : (\mathbb{Z}/n\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character. Then the composition

$$\mathbb{A}_\mathbb{Q}^\times \cong \mathbb{Q}^\times \times \widehat{\mathbb{Z}}^\times \times \mathbb{R}_+ \twoheadrightarrow \widehat{\mathbb{Z}}^\times \twoheadrightarrow (\mathbb{Z}/n\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times$$

defines a Hecke character on $\mathbb{A}_\mathbb{Q}^\times$.

By Theorem 3.11 and the following short exact sequence

$$1 \longrightarrow \mathbb{A}_1^\times/K^\times \longrightarrow \mathbb{A}^\times/K^\times \xrightarrow{\|\cdot\|_K} \left|\mathbb{A}^\times\right| \longrightarrow 1,$$

it follows that every Hecke character can be written as $\chi = \eta\,\|\cdot\|^s$ for some unitary character η and $s \in \mathbb{C}$.

Lastly, we define a list of objects related to the Hecke character.

Definition 3.14. Let χ be a Hecke character.

- If $|\chi| = |\cdot|^\sigma$, then σ is the *exponent* of χ.
- The *twisted dual* of χ is $\chi^\vee := \chi^{-1}\,|\cdot|$.
- For each $v \in \mathcal{P}_K$, define $\chi_v := \chi|_{K_v^\times}$.

3.4. **Multiplicative Haar measure.** We define a multiplicative Haar measure $d^\times x_v$ on $K_v^\times$.

Definition 3.15.

$$d^\times x_v := \begin{cases} \frac{dx_v}{\|x_v\|_v} & v \text{ archimedean} \\ (1-q_v^{-1})^{-1}\frac{dx_v}{\|x_v\|_v} & v \text{ non-archimedean,} \end{cases}$$

where $q_v = |\mathcal{O}_v/\mathcal{P}_v|$, where $\mathcal{P}_v$ is the maximal ideal of $\mathcal{O}_v$.

It is an easy exercise to check that this defines a Haar measure so that

$$\int_{\mathcal{O}_v^\times} d^\times x_v = 1$$

for almost every $v \in \overline{\mathcal{P}}_K$. (In fact, this is the reason for the $(1-q_v^{-1})^{-1}$ factor in the non-archimedean case. Otherwise, the infinite product in Definition 3.16 would diverge to 0.) And thus the following definition of multiplicative Haar measure on $\mathbb{A}^\times$ is well-defined:

Definition 3.16.

$$d^\times x := \prod_{v\in\overline{\mathcal{P}}_K} d^\times x_v$$

Since we are restricting ourselves to the case where K is a number field and $|\mathbb{A}^\times| = \mathbb{R}_+^\times$, we have the multiplicative Haar measure $\frac{dt}{t}$ on $|\mathbb{A}^\times|$.

Recalling the short exact sequence from before, we can equip $\mathbb{A}_1^\times$ with Haar measure d^*x to be compatible with measures $d^\times x$ and $\frac{dt}{t}$ on $\mathbb{A}^\times$ and $|\mathbb{A}^\times|$, respectively:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{A}_1^\times & \longrightarrow & \mathbb{A}^\times & \xrightarrow{\|\cdot\|_K} & |\mathbb{A}^\times| & \longrightarrow & 1 \\ & & d^*x & \longrightarrow & d^\times x & \longrightarrow & \dfrac{dt}{t} & & \end{array}$$

Moreover, the measure d^*x on $\mathbb{A}_1^\times$ induces a measure on $\mathbb{A}_t^\times$ (by homeomorphic map of multiplication by $a \in \mathbb{A}_t^\times$), and thus further induces a measure on $\mathbb{A}_t^\times/K^\times$. We abuse notation and still denote each of these measures as d^*x.

4. L-functions for Number Fields

4.1. **Notation & Review of Relevant Local and Global Properties.**

As a review/extension to Notation 3.2, we declare many of the frequently used notations for this section here.

Notation 4.1.

- K: a number field
- K_v: Completion of K at v, a local field
- $\mathcal{O}_v$: ring of integers of K_v, which contains ring of integers of K, $\mathcal{O}_K$.
- $\mathcal{P}_v$: maximal ideal of $\mathcal{O}_v$
- $\mathbb{F}_v = \mathcal{O}_v/\mathcal{P}_v$: residue field
- $q_v = |\mathcal{O}_v/\mathcal{P}_v|$: order of the residue field
- $\|\cdot\|_v$: normalized absolute value on K_v
- $V := \mathrm{Vol}(\mathbb{A}_1^\times/K^\times)$

Proposition 4.2. *The ring of adèles $\mathbb{A}$ is isomorphic to its Pontryagin dual $\widehat{\mathbb{A}}$ as locally compact abelian (LCA) groups.*

Here the Pontryagin dual of $\mathbb{A}$, $\widehat{\mathbb{A}} = \mathrm{Hom}(\mathbb{A}, \mathbb{T})$ is the group of continuous group homomorphisms from $\mathbb{A}$ to the circle group $\mathbb{T}$.

4.2. **Adèlic Fourier Transform and Poisson Summation Formula.** First we recall the definition of Schwartz-Bruhat functions for a local field F:

Definition 4.3. A function $f : F \to \mathbb{C}$ is called a *local Schwartz-Bruhat function* if

$$f = \begin{cases} \text{a Schwartz function as defined in Definition 1.3} & \text{if } F = \mathbb{R} \text{ or } \mathbb{C} \\ \text{a locally constant function with compact support} & \text{if F non-archimedean.} \end{cases}$$

Denote $\mathcal{S}(F)$ the $\mathbb{C}$-vector space of local Schwartz-Bruhat functions.

Now we wish to find a suitable definition of Schwartz-Bruhat functions on $\mathbb{A}$. Suppose that for all $v \in \overline{\mathcal{P}}_K$, we have $f_v \in \mathcal{S}(K_v)$ such that $f_v|_{\mathcal{O}_v} = 1$ for almost every v. Then the infinite product $\prod_{v \in \overline{\mathcal{P}}_K} f_v : \mathbb{A} \to \mathbb{C}$ is well-defined. We can use such functions as an infinite basis for the space of Schwartz-Bruhat functions on $\mathbb{A}$.

Definition 4.4. A *Schwartz-Bruhat function* $f : \mathbb{A} \to \mathbb{C}$ is a finite $\mathbb{C}$-linear combination of functions $\prod_{v \in \overline{\mathcal{P}}_K} f_v$ (as above).

Denote $\mathcal{S} = \mathcal{S}(\mathbb{A})$ the space of all Schwartz-Bruhat functions.

Fix a standard ψ and self-dual measure dx. Then we can define the adèlic Fourier transform similarly to the real case.

Definition 4.5. Let $f \in \mathcal{S}(\mathbb{A})$. The *adèlic Fourier transform* is defined as

$$\widehat{f}(y) := \int_{\mathbb{A}} f(x)\, \psi(xy)\, dx.$$

Moreover, as in the real case, $\widehat{f}(y) \in \mathcal{S}(\mathbb{A})$.

Since there is a correspondence between functions on $\mathbb{A}/K$ and K-periodic functions on $\mathbb{A}$, i.e. $\{f : \mathbb{A} \to \mathbb{C} \,\big|\, f(x+\kappa) = f(x), \forall \kappa \in K\}$, we can also define a Fourier transform for $\mathbb{A}/K$ and its Pontryagin dual $\widehat{\mathbb{A}/K} = K$.

Remark 4.6. Recall that the space of L^p functions on X, $L^p(X)$, is the set of measurable functions $f : X \to \mathbb{C}$ such that $|f|^p$ is integrable, quotiented by the null functions.

Definition 4.7. Let $f \in L^1(\mathbb{A}/K)$. The Fourier transform $\widehat{f} : K \to \mathbb{C}$ is defined as

$$\widehat{f}(\kappa) := \int_D f(x)\, \psi(\kappa x)\, dx,$$

where D is the fundamental domain of the quotient $\mathbb{A}/K$.

If $f \in L^1(\mathbb{A}/K)$ is continuous and $\widehat{f} \in L^1(K)$, then the *adèlic Fourier inversion formula* also holds:

$$f(x) = \sum_{\kappa \in K} \widehat{f}(\kappa)\overline{\psi(\kappa x)}.$$

Note that here we have a summation rather than the usual integral since we are integrating with respect to the counting measure.

We can now state the adèlic Poisson summation formula.

Theorem 4.8. *Let $f \in \mathcal{S}(\mathbb{A})$. Then*

$$\sum_{\kappa \in K} f(\kappa) = \sum_{\kappa \in K} \widehat{f}(\kappa).$$

The proof follows similarly to the real case, i.e. apply the Fourier inversion formula to the Fourier transform of $F(x) := \sum_{\kappa \in K} f(x+\kappa)$.

4.3. **Global Zeta Integrals and Hecke L-functions.**

We are now able to prove the meromorphic continuation of a more generalized zeta function, the global zeta function. Then from this, we can generalize a bit further to Hecke L-functions.

Definition 4.9. Let $f \in \mathcal{S}(\mathbb{A})$ and $\chi : \mathbb{A}^\times \to \mathbb{C}^\times$, a Hecke character. The *global zeta integral* is

$$\mathcal{Z}(f, x) := \int_{\mathbb{A}^\times} f(x)\chi(x)\, d^\times x.$$

Now for the main theorem for global zeta functions:

Theorem 4.10. *Let $f \in \mathcal{S}(\mathbb{A})$. Then the following hold:*

(1) The integral $\mathcal{Z}(f, \chi)$ converges for Hecke characters χ with exponent $\sigma > 1$.
(2) The function $\mathcal{Z}(f, \chi)$ extends to a meromorphic function on $\mathcal{X}$, holomorphic except for a simple pole at $|\cdot|^0$ with residue $-Vf(0)$ and a simple pole at $|\cdot|^1$ with residue $V\hat{f}(0)$.
(3) As meromorphic functions of $\chi \in \mathcal{X}$,

$$\mathcal{Z}(f, \chi) = \mathcal{Z}(\hat{f}, \chi^\vee).$$

Details regarding part (1) of the above theorem can be found in [1]. We will focus our discussion on parts (2) and (3), and their apparent similarity with the proof of meromorphic continuation of the Riemann zeta function in Theorem 2.3.

We begin by defining a function (and proving its functional equation) that will play an analogous role to that of $\Theta(t)$ (and its functional equation) in the proof of Theorem 2.3.

Integrating over each norm t-slice $\mathbb{A}_t^\times$ as per Equation 3.10 gives

$$\mathcal{Z}(f, \chi) = \int_{\mathbb{A}_t^\times} f(x)\, \chi(x)\, d^* x, \tag{4.11}$$

and if the exponent of χ, $\sigma > 0$, then

$$\mathcal{Z}(f, \chi) = \int_{|\mathbb{A}^\times|} \mathcal{Z}_t(f, \chi)\, \frac{dt}{t}.$$

Lemma 4.12. *Let χ be a Hecke character. Then*

$$\mathcal{Z}_t(f, \chi) + f(0) \cdot \int_{\mathbb{A}_t^\times / K^\times} \chi(x)\, d^* x = \mathcal{Z}_{1/t}(\widehat{f}, \chi^\vee) + \widehat{f}(0) \cdot \int_{\mathbb{A}_{1/t}^\times / K^\times} \chi^\vee(x)\, d^* x.$$

Through this lemma we can now see that Equation 4.11 with $\chi = |\cdot|^s$ is analogous to Equation 2.4 in the proof of the Riemann zeta function:

$$\xi(s) = \int_0^\infty \left(\frac{\Theta(t) - 1}{2}\right) t^{s/2}\, \frac{dt}{t}.$$

In particular, Lemma 4.12 tells us that $\mathcal{Z}_t(f, \chi)$ plays an analogous role to that of $\left(\frac{\Theta(t)-1}{2}\right) t^{s/2}$ in the Riemann proof.

Proof of Lemma 4.12. We begin by rewriting $\mathcal{Z}_t(f,\chi)$ by considering $\mathbb{A}_t^\times$ as translations of a fundamental domain for the action of multiplication by elements of $K^\times$:

$$\begin{aligned}\mathcal{Z}_t(f,\chi) &= \int_{\mathbb{A}_t^\times/K^\times} \sum_{a\in K^\times} f(ax)\,\chi(ax)\,d^*x \\ &= \int_{\mathbb{A}_t^\times/K^\times} \left(\sum_{a\in K^\times} f(ax)\right)\chi(x)\,d^*x \qquad \text{since } \chi\big|_{K^\times} = 1.\end{aligned}$$

Now to apply Poisson summation formula, we require our summation to go over all of K. Thus we add $f(0)$, the summand corresponding to $a = 0$, to both sides of the equation:

$$\begin{aligned}\mathcal{Z}_t(f,\chi) + f(0)\cdot\int_{\mathbb{A}_t^\times/K^\times}\chi(x)\,d^*x &= \int_{\mathbb{A}_t^\times/K^\times}\left(\sum_{a\in K} f(ax)\right)\chi(x)\,d^*x \\ &= \int_{\mathbb{A}_t^\times/K^\times}\left(\frac{1}{\|x\|_K}\sum_{a\in K}\widehat{f}\left(\frac{a}{x}\right)\right)\chi(x)\,d^*x \\ &= \int_{\mathbb{A}_{1/t}^\times/K^\times}\left(\|x\|_K\sum_{a\in K}\widehat{f}(ay)\right)\chi(y^{-1})\,d^*y \\ &\qquad \text{by substituting } x = y^{-1} \\ &= \int_{\mathbb{A}_{1/t}^\times/K^\times}\left(\sum_{a\in K}\widehat{f}(ay)\right)\chi^\vee(y)\,d^*y.\end{aligned}$$

By reversing the above argument on $\mathbb{A}_{1/t}^\times/K^\times$, the result follows. □

We require one additional lemma to prove the main theorem. Recall that $V = \mathrm{Vol}(\mathbb{A}_1^\times/K^\times)$ as per Notation 4.1.

Lemma 4.13. *Let $t \in |\mathbb{A}^\times|$ and $\chi \in \mathcal{X}$ a Hecke character. Then*

$$\int_{\mathbb{A}_t^\times/K^\times}\chi(x)\,d^*x = \begin{cases} Vt^s & \text{if } \chi = |\cdot|^s \text{ for some } s\in\mathbb{C} \\ 0 & \text{otherwise.}\end{cases}$$

Proof. Suppose $\chi = |\cdot|^s$ for some $s \in \mathbb{C}$. Then $\chi(x) = t^s$ a constant and $\mathrm{Vol}(\mathbb{A}_t^\times/K^\times) = \mathrm{Vol}(\mathbb{A}_1^\times/K^\times) = V$ since $\mathbb{A}_t^\times/K^\times$ is a coset of $\mathbb{A}_1^\times/K^\times$. So

$$\int_{\mathbb{A}_t^\times/K^\times}\chi(x)\,d^*x = Vt^s.$$

If $\chi \neq |\cdot|^s$ for any $s \in \mathbb{C}$, then χ does not factor through $\mathbb{A}^\times/\mathbb{A}_1^\times$ and it follows that χ is non-constant on compact $\mathbb{A}_1^\times/K^\times$. So the integral over any coset of $\mathbb{A}_1^\times/K^\times$ is 0. □

With these two lemmas in mind, we can now complete the proof of parts (2) and (3) of Theorem 4.10.

Proof of parts (2) and (3) of Theorem 4.10. As with Riemann's proof of the zeta function, we begin by splitting $\mathcal{Z}(f,\chi)$ into its good and bad parts, the integral of

$\mathcal{Z}_t$ over $(1,\infty)$ and $(0,1)$, respectively:

$$\mathcal{Z}(f,\chi) = \underbrace{\int_0^1 \mathcal{Z}_t(f,\chi)\,\frac{dt}{t}}_{J(f,\chi)} + \underbrace{\int_1^\infty \mathcal{Z}_t(f,\chi)\,\frac{dt}{t}}_{I(f,\chi)}.$$

Note that if the exponent of χ, $\sigma > 1$, then both $I(f,\chi)$ and $J(f,\chi)$ converge to holomorphic functions. In particular, $I(f,\chi)$ converges to a holomorphic function everywhere. To "fix" the bad part $J(f,\chi)$, we wish to rewrite it in terms of $I(f,\chi)$, as we did in Riemann's proof. We do this by using the functional equation for $\mathcal{Z}_t(f,\chi)$ (Lemma 4.12) and substituting $t \mapsto \frac{1}{t}$.

Suppose that $\sigma > 1$ and $\chi = |\cdot|^s$:

$$\begin{aligned} J(f,\chi) &= \int_0^1 \mathcal{Z}_t(f,\chi)\,\frac{dt}{t} \\ &= \int_0^1 \mathcal{Z}_{1/t}(\widehat{f},\chi^\vee)\,\frac{dt}{t} + \int_0^1 \left(V\widehat{f}(0)\left(\frac{1}{t}\right)^{1-s} - Vf(0)t^s \right)\frac{dt}{t} \\ &= \underbrace{\int_1^\infty \mathcal{Z}_y(\widehat{f},\chi^\vee)\,\frac{dy}{y}}_{=I(\widehat{f},\chi^\vee)} + V\widehat{f}(0)\cdot \int_0^1 t^{s-1}\,\frac{dt}{t} - Vf(0)\cdot\int_0^1 t^s\,\frac{dt}{t}. \end{aligned}$$

Then, adding $I(f,\chi)$ back in, we arrive again (in a similar fashion to the Riemann zeta proof) to

$$\mathcal{Z}(f,\chi) = I(f,\chi), + I(\widehat{f},\chi^\vee) + \frac{V\widehat{f}(0)}{s-1} - \frac{Vf(0)}{s},$$

which is meromorphic on $\mathbb{C}$ and symmetric with respect to $(f,\chi) \longleftrightarrow (\widehat{f},\chi^\vee)$, with the required residues.

If we instead had $\chi \neq |\cdot|^s$, the functional equation would follow similarly except without the latter two terms on the right hand side, by Lemma 4.13. □

From here, after recalling some additional local properties of ϵ-factors and L-factors, we can further generalize and find the functional equation for Hecke L-functions $L(s,\chi)$ with $s \in \mathbb{C}$ and $\chi \in \mathcal{X}$:

$$L(s,\chi) = \epsilon(s,\chi)L(1-s,\chi^{-1}),$$

where $\epsilon(s,\chi)$ is a global ϵ-factor. Although the proof of this follows fairly directly from the work done in Theorem 4.10, there is much setup required to adequately present the local case. A proof of both the local and global case can be found in Sections 4.9 and 5.11 of [1], respectively.

Acknowledgments

I would like to thank my mentor, Micah Gay, for her patience in helping me clear any confusing concepts I ran into while reading material for our weekly meetings. I also greatly appreciate for her advice on organizing the paper. I would also like to thank Professor Hyde for his additional help on understanding the reading material; in particular, for relating concepts in the readings to material we covered in the algebraic number theory course. Finally, I would like to thank Professor Emerton for giving several engaging talks on L-functions and Professor May for organizing the REU, which has allowed me to enjoy learning a lot more interesting math.

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