BROWNIAN MOTION AND RANDOM TRIANGULATIONS OF THE CIRCLE

ZONGJIN WU

Abstract. This expository paper is an introduction to Brownian motion and its close relations with the random triangulation of the circle. We establish the correspondence between a simple random walk and several combinatorial objects, including the triangulation of a polygon. The limit of such a polygon triangulation, i.e. the random triangulation of the circle, has a correspondence with Brownian motion. As the limiting distribution of scaled random walks, Brownian motion has dense strict local minima pairs that point to chords of a random triangulation of the circle. Using this result, we prove that the fractal dimension of the random triangulation of the circle is almost surely $\frac{3}{2}$.

Contents

1. Introduction

A stochastic process, sometimes simply referred to as a "random process," is a collection of random variables X_t indexed by time. Our main subject Brownian motion models a stochastic process with continuous motions. To understand the path of Brownian motion, we first take a look at a (simple) random walk, a discrete stochastic process with independent increments.

Definition 1.1. A *simple random walk* S_n on the integer number line is defined as the sum of n random variables, $X_1, X_2, \cdots, X_n$, such that $\mathbb{P}\{X_j = 1\} = \mathbb{P}\{X_j = -1\} = \frac{1}{2}$ for any $1 \leq j \leq n$:

$$S_n = \sum_{j=1}^{n} X_j.$$

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The starting value S_0 is set to be 0.

Random walks are among a special class of stochastic processes that satisfy the *Markov property*, which states that the prediction of the future events of the system is only dependent on the current state of the system but not the past events. Consider a discrete-time stochastic process X_n with $n = 0, 1, 2, \cdots$, with X_n taking values in a finite set.

Definition 1.2. We say that X_n is a *Markov process* if

$$\mathbb{P}\{X_n = i_n : X_0 = i_0, \cdots, X_{n-1} = i_{n-1}\} = \mathbb{P}\{X_n = i_n : X_{n-1} = i_{n-1}\}.$$

The Markov property is sometimes referred to as the "memory-loss" property, indicating its independence of the past history of the event.

In addition to being a Markov process, the random walk S_n is also *time-homogeneous*, which means that the transition probabilities do not depend on time. On a integer line, S_n has equal probability of going either right or left. In this case, the transition probability $p(S_n, S_n + 1) = p(S_n, S_n - 1) = \frac{1}{2}$ for any n.

To visualize a simple random walk of n steps, we can construct a contour function $f : \{0, \cdots, n\} \to \mathbb{Z}$ such that $f(i) = S_i$ for all $i \in \{0, \cdots, n\}$. Figure 1 shows the path of the contour function of a simple random walk from 0 to 20.

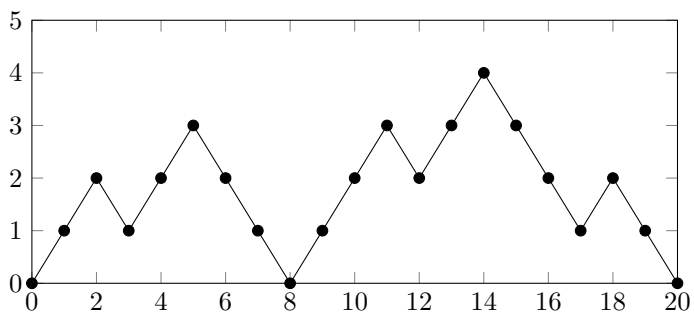

FIGURE 1. Contour function of a simple random walk

The following definitions are useful in the definition of Brownian motion.

Definition 1.3. Two stochastic processes $\{X(t) : t \geq 0\}$ and $\{Y(t) : t \geq 0\}$ are called *independent* if for any set of times $t_1, \cdots, t_n \geq 0$ and $s_1, \cdots, s_m \geq 0$, the vectors $(X(t_1), \cdots, X(t_n))$ and $(Y(s_1), \cdots, Y(s_m))$ are independent.

We will discuss an important property of the distribution of Brownian motion later, which requires the following definition of normal distribution:

Definition 1.4. A random variable X has a *normal distribution* with mean μ and variance σ^2 if it has the following density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty.$$

A stronger property following up the definition of normal distribution states that the drift of sample average from the mean should be normally distributed as well:

Theorem 1.5. *(Central limit theorem) Let $X_1, X_2, \cdots$ be independent, identically distributed random variables with mean μ and finite variance σ^2. If $-\infty < a < b < \infty$, then*

$$\lim_{n\to\infty} \mathbb{P}\{a \leq \frac{(\sum_1^n Xi) - n\mu}{\sigma\sqrt{n}} \leq b\} = \int_a^b \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx.$$

The theorem implies that as n gets larger, the difference between the sample average and the mean μ is normally distributed. A detailed establishment and proof of the theorem can be found in [6].

Now we introduce Brownian notion as a continuous stochastic process. Brownian motion models the random motion of a particle in a fluid with its collision with the surrounding environment. On a macroscopic level, a key property of such movements is that the particle moves randomly without making big jumps. Mathematically, Brownian motion is defined as follows:

Definition 1.6. A d-dimensional *Brownian motion* with variance parameter σ^2 is an $\mathbb{R}^d$ valued stochastic process $\{B_t : t \geq 0\}$ starting at $x \in \mathbb{R}^d$ with the following properties:

(a) Starting condition: $B_t(0) = x$.
(b) Independent increments: For any $s_1 \leq t_1 \leq s_2 \leq t_2 \leq \cdots \leq s_n \leq t_n$, the random variables $B_{t_1} - B_{s_1}, B_{t_2} - B_{s_2}, \cdots, B_{t_n} - B_{s_n}$ are independent.
(c) Normality: For any $s < t$, the random variable $B_t - B_s$ has normal distribution with mean 0 and variance $(t - s)\sigma^2$.
(d) Continuity: Almost surely, the function $t \to B_t$ is a continuous function.

We call a Brownian motion *standard* if $x = 0$ and $\sigma^2 = 1$. We call a Brownian motion *linear* if $d = 1$.

The independence of increments and the normality of Brownian motion follow from its discrete analogue random walks. Brownian motion can be constructed using the limit of a random walk. For a 1-dimensional standard Brownian motion, consider a simple random walk S_n on the integers with $S_0 = 0$. Instead of having time increments ± 1, we now need the increments of a re-scaled walk W to be $\Delta t = \frac{1}{n}$ where $n \in \mathbb{N}$. Set

$$W_{k\Delta t^{(n)}} = \frac{S_k}{\sqrt{n}}.$$

The definition of S_k follows from Definition 1.1. Note that $n = k$ in the case of $W_1^{(n)}$. Since each X_i of the simple random walk is an independent random variable, S_k is the sum of i.i.d. variables with mean 0 and variance σ^2. By the Central Limit Theorem, as $n = k \to \infty$, $W_1^{(n)}$ approaches a normal distribution. The normalizing constant $\frac{1}{\sqrt{n}}$ ensures the variance of $W_1^{(n)} = \frac{S_n}{\sqrt{n}}$ to be 1. This makes $W_1^{(n)}$ a standard linear Brownian motion almost surely.

An intuitive way of thinking about this limit is drawing an infinite random walk on a finite piece of paper. We need to scale the contour function of the random walk over an infinite time interval to $[0, 1]$. The initial condition of the random walk that the increments are ± 1 ensures that the path will have no big jumps when scaled down. The independence of increments also follow naturally from time-homogeneity of the random walk.

As a continuous stochastic process, Brownian motion has the Markov property similar to that of a discrete simple random walk. The Markov property and some

further properties of Brownian motion are often discussed using measure theory. However, we will not assume any measure theoretical background in this paper and will adopt alternative definitions inspired by [1].

Brownian motion has a close relationship with the random triangulation of the circle, which will appear later in this paper. We begin this discussion by showing that the contour function of a simple random walk corresponds to several combinatorial objects, including a polygon triangulation. Then we take the limit on both objects and show that a random triangulation of the circle can be defined using Brownian excursion, a scaled Brownian motion.

2. Properties of Brownian Motion

2.1. **Markov Properties of Brownian Motion.** Similar to a simple random walk, Brownian motion is a *time-homogeneous Markov process.* This means that the transition probabilities do not depend on time and the process satisfies the Markov property, which will be introduced below. For some time $s < t$, in order to predict B_t given all the information up to time s, it is enough to consider only B_s, the Brownian motion at time s. In other words, all the information we need to predict B_t is included in B_s alone. We say that Brownian motion has the Markov property:

Theorem 2.1. [3] *(Markov Property) Let $\{B_t : t \geq 0\}$ be a Brownian motion starting at $x \in \mathbb{R}^d$ and let $s > 0$. Then the process $\{B_{s+t} - B_s : t \geq 0\}$ is again a Brownian motion starting at the origin and it is independent of $\{B_t : 0 \leq t \leq s\}$.*

The Markov property follows from the definition of Brownian motion. The property of independent increments (b) implies that $B_{t+s} - B_s$ is independent of the original Brownian motion before time s.

We can interpret the Markov property by considering the transition density function of B_t. Let $p_t(x, y)$ be the distribution function of $\{B_t : B_0 = x, t \geq 0, \sigma^2 = 1\}$. Since $B_t - B_0$ is a standard Brownian motion, it has a normal distribution of mean 0 and variance t:

$$p_t(x, y) = \frac{1}{\sqrt{2\pi t}} e^{\frac{-(y-x)^2}{2t}}, \quad -\infty < y < \infty.$$

From the Markov property, we know that B_{s+t} is Brownian motion starting at B_s. We can represent $f_{s+t}(x, y)$ by considering all possible values of B_s. Since $B_{s+t} - B_s$ is independent of Brownian motion before s, we can divide the density function at s and average $f_s(x, z)$ for all possible values of B_s, denoted by z.

$$p_{s+t}(x, y) = \int_{-\infty}^{\infty} p_s(x, z) p_t(z, y) dz.$$

Brownian motion has further properties as a Markov process. In order to discuss them, we first make several definitions. Let B_t be a standard Brownian motion. We have seen that all the information needed to predict B_t is contained in B_s for some $0 < s \leq t$ alone. We now denote the information we can obtain by watching the Brownian motion up to time t by $\mathcal{F}_t$:

Definition 2.2. [1] $\mathcal{F}_t$ is all the information contained in B_s for some $s \leq t$.

Since B_t could be predicted by $\mathcal{F}_t$ alone, we say that B_t is $\mathcal{F}_t$ *measurable.* For a measure theoretical definition of filtration $\mathcal{F}_t$, see Chapter 2 of [3].

We now introduce the definition of a real-valued stopping time. A random time T is called a *stopping time* for Brownian motion if we can determine whether $T \leq t$ for each t by just looking at Brownian motion up to time t:

Definition 2.3. A random variable $T \in [0, \infty]$ is called a *stopping time* if $\{T \leq t\}$ is $\mathcal{F}_t$ measurable for every $t > 0$.

One example of a stopping time for Brownian motion is the first time B_t hits a certain value x:

$$T_x = \inf\{t : B_t = x\}. \tag{2.4}$$

For any t, event $\{T_x \leq t\}$ is the same as $B_s = x : 0 \leq s \leq t$. It suffices to watch the Brownian motion up to time t to decide whether this event happens, so T_x is a stopping time.

With the above definitions, a stronger version of the Markov property can be introduced. We say a stopping time T is almost surely finite if $\mathbb{P}\{T < \infty\} = 1$. The *strong Markov property* states that the Brownian motion $B_{T+t} - B_T$ is independent of the information contained in B_T for an almost surely finite stopping time T.

Theorem 2.5. [3] *(Strong Markov property) For every almost surely finite stopping time T, Brownian motion $\{B_{T+t} - B_T : t \geq 0\}$ is independent of $\mathcal{F}_T$.*

This property says that Brownian motion starts afresh at T. The only information we need to predict Brownian motion after time T is B_T itself. This indicates that Brownian motion is a time-homogeneous Markov process. With the strong Markov property, we can deduce the following *reflection principle*.

Theorem 2.6. *(Reflection principle) Let B_t be a Brownian motion starting at a with variance parameter σ^2. Let $b > a$. Then for any $t > 0$,*

$$\mathbb{P}\{B_s \geq b \text{ for some } 0 \leq s \leq t\} = 2\mathbb{P}\{B_t \geq b\}.$$

Proof. Let the stopping time $T_b = \inf\{t : B_t = b\}$ be defined as in (2.4), which is the first time B_t hits a given b. Then $\mathbb{P}\{B_s \geq b \text{ for some } 0 \leq s \leq t\} = \mathbb{P}\{T_b \leq t\}$. We can rewrite $\mathbb{P}\{B_t \geq b\}$ using the condition that B_t hits b before t:

$$\mathbb{P}\{B_t \geq b\} = \mathbb{P}\{T_b \leq t\}\mathbb{P}\{B_t \geq b : T_b \leq t\}.$$

By the strong Markov property, Brownian motion starts afresh at T_b, so $\{B_t - B_{T_b}\}$ is a Brownian motion with mean 0 and variance $(t - T_b)\sigma^2$. By normal distribution, we have $\mathbb{P}\{B_t - B_{T_b} \geq 0\} = \frac{1}{2}$, which implies that $\mathbb{P}\{B_t \geq b : T_b \leq t\} = \frac{1}{2}$. Therefore,

$$\mathbb{P}\{T_b \leq t\} = 2\mathbb{P}\{B_t \geq b\}.$$

□

A more generalized version of the reflection principle states that Brownian motion reflected at stopping time T is still a Brownian motion. This follows from the fact that $\{B_{T+t} - B_T\}$ is independent of $\{B_t : t \leq T\}$, as stated in the strong Markov property. Let us consider an example that can be solved using the reflection principle.

Example 2.7. [1] Let $t \geq 1$. We compute the probability that a standard Brownian motion B_t crosses the x-axis sometime between 1 and t:

$$\mathbb{P}\{B_s = 0 \text{ for some } 1 \leq s \leq t\}.$$

Suppose we have the condition that $B_1 = b \geq 0$. By the Markov property, B_t starting at $t = 1$ is again a Brownian motion that starts at b. Then the probability that $B_s = 0$ for some $1 \leq s \leq t$ in the new Brownian motion is the same as the probability that $B_s \leq -b$ for some $0 \leq s \leq t-1$ in the standard Brownian motion. By the reflection principle and normal distribution, we have

$$\mathbb{P}\{B_s \geq b \text{ for } 0 \leq s \leq t-1\} = 2\mathbb{P}\{B_t \geq b\} = 2\int_b^\infty \frac{1}{\sqrt{2\pi(t-1)}} e^{\frac{-x^2}{2(t-1)}}\, dx.$$

Suppose the condition is $B_1 = b' \leq 0$, then the probability is equal to the probability that $B_s \geq -b$ for some $0 \leq s \leq t-1$, which by symmetry of normal distribution is the same as the probability that $B_s \leq -b$ for some $0 \leq s \leq t-1$. Therefore, we can integrate the probability over all possible values of b:

$$\begin{aligned}
\mathbb{P}\{B_s = 0 \text{ for some } 1 \leq s \leq t\} &= \int_{-\infty}^\infty p_1(0,b)\mathbb{P}\{B_s \geq b : 0 \leq s \leq t-1\} db \\
&= 2\int_0^\infty p_1(0,b)\left[2\int_b^\infty \frac{1}{\sqrt{2\pi(t-1)}} e^{\frac{-x^2}{2(t-1)}}\, dx\right] db \\
&= 2\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{\frac{-b^2}{2}}\left[2\int_b^\infty \frac{1}{\sqrt{2\pi(t-1)}} e^{\frac{-x^2}{2(t-1)}}\, dx\right] db.
\end{aligned}$$

Substitute $y = \frac{x}{\sqrt{t-1}}$, the integral becomes

$$4\int_{b=0}^\infty \int_{y=\frac{b}{\sqrt{t-1}}}^\infty \frac{1}{2\pi} e^{\frac{-(y^2+b^2)}{2}}\, dy db.$$

Using polar coordinates, we have $b^2 + y^2 = r^2$ and $dydb = rdrd\theta$. Then $0 < b < \infty$ is converted to $0 < r < \infty$, and $\frac{b}{\sqrt{t-1}} < y < \infty$ is converted to $\arctan(t-1)^{-1} < \theta < \frac{\pi}{2}$. The integral becomes

$$\begin{aligned}
&4\int_{r=0}^\infty \int_{\theta=\arctan\frac{1}{\sqrt{t-1}}}^{\frac{\pi}{2}} \frac{1}{2\pi} e^{\frac{-r^2}{2}} r dr d\theta \\
&= 4\left(\frac{\pi}{2} - \arctan\frac{1}{\sqrt{t-1}}\right)\frac{1}{2\pi}\int_{r=0}^\infty e^{\frac{-r^2}{2}} r dr \qquad (2.8)\\
&= 1 - \frac{2}{\pi}\arctan\frac{1}{\sqrt{t-1}}. \quad \square
\end{aligned}$$

We will use this result later when we discuss the fractal dimension of the zero set of Brownian motion.

As the limiting process of a conditioned random walk, Brownian motion has interesting properties concerning its local behaviors. The path of Brownian motion is extremely "rough", a fact that follows from the independent increment property. If the path is smooth, we would be able to predict where the path is going based on local values. The randomness and independence of increments implies that the path has to be rough. In fact, Brownian motion has the following property:

Theorem 2.9. *The path of a Brownian motion B_t is nowhere differentiable.*

A sketch of the proof of this theorem is as follows [3]. To show that B_t is nowhere differentiable, we first show that it is not differentiable at a given time t. Taking the upper and lower right derivatives of a re-scaled Brownian motion at 0, one can show

that they diverge to $+\infty$ and $-\infty$, respectively. Applying the Markov property, we know that B_t is not differentiable at any $t \geq 0$. Then we can consider the global behavior of B_t and suppose that the upper (lower) right derivatives are less than infinity. Then the derivative must be less than or equal to some finite constant M. Some further computations show that the probability of such an event is 0.

Note that non-differentiability implies that for any time t and any positive $\delta > 0$, there always exists $s \in (t, t+\delta)$ such that $B_{t+s} - B_t > 0$. This follows from the fact that the upper right derivative of Brownian motion goes to infinity.

There exists a series of interesting properties concerning the local behavior of Brownian motion, one of which is non-monotonicity of Brownian motion.

Theorem 2.10. *Almost surely, for all $0 < a < b < \infty$, Brownian motion B_t is not monotonic on $[a, b]$.*

Proof. Let $\mathbb{P}$ be the probability that B_t is monotonic on a fixed interval $[a, b]$. Divide $[a, b]$ into n sub-intervals $[a_{i-1}, a_i]$ such that $a = a_0 \leq a_1 \leq \cdots \leq a_n = b$. Since B_t is monotonic, $B_i - B_{i-1}$ has the same sign for all $1 \leq i \leq n$. Since each increment $B_i - B_{i-1}$ is random and independent, $\mathbb{P}\{B_i - B_{i-1} > 0\} = \frac{1}{2}$ for any i. This gives us $\mathbb{P} = 2 \cdot \frac{1}{2^n}$. Take $n \to \infty$, we have $\mathbb{P} \to 0$. Since a, b can be arbitrary, we know that B_t is not monotonic on any interval with rational endpoints by taking a countable union of rational intervals. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, any interval in $\mathbb{R}$ has a rational sub-interval. Therefore, almost surely, B_t is nowhere monotonic. $\square$

As we have discussed, the standard deviation of simple random walks as well as Brownian motion is roughly $\sqrt{t}$ where t denotes the time. Over a long period of time, Brownian motion could go up or down to a large value. Those values are in fact gradually attained by building up little steps. Non-monotonicity suggests that the path of Brownian motion has "infinitesimal small jumps" everywhere but "big jumps" nowhere.

We are also interested in the local minima of Brownian motion for the sake of our discussion of random triangulation of the circle later. We now consider some properties concerning local extrema of Brownian motion. First, we define a strict local minimum:

Definition 2.11. A continuous function f attains a *strict local minimum* at $s \in I$ on the interval I if

$$f(s) < f(x) \quad \text{for any } x \in I, x \neq s.$$

In fact, strict local minima are dense for Brownian motion, a result that can be intuitively understood from non-differentiability of Brownian motion. To prove this useful result, we first prove the following lemma. Let $\{B_t : t \geq 0\}$ be a linear Brownian motion.

Lemma 2.12. *Let $[a_1, b_1]$, $[a_2, b_2]$ be two fixed time intervals such that $b_1 \leq a_2$. Then the local minima of B_t on $[a_1, b_1]$ and $[a_2, b_2]$ are almost surely distinct.*

Proof. Let m_1 be the local minimum of B_t on $[a_1, b_1]$ and m_2 be the local minimum of B_t on $[a_2, b_2]$. Let $T = \{t \geq a_2 : B_t = m_1\}$ be a stopping time. If $T \notin [a_2, b_2]$, then $m_2 \neq m_1$. Suppose that $T \in [a_2, b_2]$. From normal distribution, $\mathbb{P}\{B_{b_2} = m_1\} = 0$, so we can consider $[a_1, b_2)$. Let $T + s < b_2$ for some $s > 0$. From strong Markov property, $B_{T+s} - B_T$ is again a Brownian motion with mean 0 starting at B_T. From

non-differentiability of the new Brownian motion, almost surely there exists some $v \in (T, T+s)$ such that $B_v - B_T < 0$. Then $m_2 \leq B_v < B_T = m_1$. □

Remark 2.13. Note that the above lemma has the two intervals fixed as a condition. The lemma is not saying that no two non-overlapping intervals attain the same local minimum almost surely. In fact, as we will discuss in the section of triangulating the circle, a special case of a local minimum pairs will correspond to chords of the triangulation. This is another distinction between global behaviors and those of given local areas of Brownian motion.

With the above lemma, we can now prove that strict local minima are dense for Brownian motion.

Theorem 2.14. *For a linear Brownian motion $\{B_t : t \geq 0\}$, almost surely, every local minimum is strict, and the set of times where strict local minima are attained is dense in $[0, \infty]$.*

Proof. The fact that every local minimum is a strict local minimum follows from the lemma above. Suppose $B_t = m$ is a non-strict local minimum in $[a, b] \ni t$. Then there exists $B_s = m$ with $s \in I$ and $s \neq t$. Without loss of generality, let $t < s$, then there exists $t < v < s$ with $[a, v]$ and $[v, s]$ being two fixed non-overlapping intervals with the same minimum, which contradicts the lemma.

To prove that local minima are dense, consider a given interval $[a, b]$. By non-differentiability, local extrema are almost surely not attained at endpoints. Suppose for contradiction that a local minimum is attained at a, then there exists some $s \in (a, b)$ such that $B_t - B_a < 0$ for some $t \in (a, s)$, which means B_a is not a local minimum. Since a, b can be arbitrary, we take the countable union over all intervals with rational endpoints. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, any interval contains a sub-interval with rational endpoints. Thus local minima are almost surely dense in $[0, \infty]$ for Brownian motion. □

We will use this result later when we discuss the mapping from a linear Brownian motion to the random triangulation of the circle.

Another interesting property of Brownian motion involves the maximal process:

Theorem 2.15. *(Lévy's identity) Given a linear standard Brownian motion $\{B_t : t \geq 0\}$, define the maximal process $M_t = \sup\{B_s : 0 \leq s \leq t\}$. Then $\{M_t - B_t\}$ and $|B_t|$ have the same distribution.*

The main idea of the proof of this theorem is to show that $\{M_t - B_t\}$ is a Markov process and its Markov transition kernel has modulus normal distribution. A detailed proof can be found on page 50 of [3]. This identity of Brownian motion will be used later in our proof of the fractal dimension of the random triangulation of the circle.

2.2. **The Zero Set of Brownian Motion.**

Definition 2.16. (Zero Set) Let $\{B_t : t \geq 0\}$ be a linear Brownian motion. The random set

$$Z = \{t \geq 0 : B_t = 0\}$$

is called the *zero set*.

Remark 2.17. For a standard Brownian motion, the zero set is the set of times when B_t crosses the x-axis. From non-differentiability of Brownian motion, we have shown that B_t takes on both positive and negative values for any interval $(0, \epsilon)$ where $\epsilon > 0$. By continuity, Brownian motion also takes on the value 0 for arbitrarily small intervals about time 0.

To understand the path of Brownian motion, we are particularly interested in the zero set. How often does Brownian motion returns to zero? To calculate this, we need a notion of "dimension" of the zero set. We will now introduce the *Minkowski dimension*, also known as the "box-counting dimension."

Suppose E is a metric space with metric ρ such that $|E| = \sup\{\rho(x, y) : x, y \in E\}$ is finite. A *covering* of E is a finite or countable collection of sets $E_1, E_2, \cdots$ such that

$$E \subset \bigcup_{i=1}^{\infty} Ei.$$

Define, for $\epsilon > 0$,

$$\begin{aligned} M(E, \epsilon) = \min\{k \geq 1 :\ & \text{there exists a finite covering} \\ & E_1, E_2, \cdots \text{ of } E \text{ with } |E_i| \leq \epsilon \text{ for } i = 1, 2, \cdots, k\}. \end{aligned}$$

Definition 2.18. For a bounded metric space E, we define the *Minkowski dimension* as

$$\dim E = \lim_{\epsilon \to 0} \frac{\log M(E, \epsilon)}{\log(\frac{1}{\epsilon})}.$$

Intuitively, each E_i can be seen as a little box with a bounded diameter. If E has dimension d, then $M(E, \epsilon)$ should be of order ϵ^{-d}.

Example 2.19. The Minkowski dimension of the Cantor set can be calculated as follows:

$$\mathcal{C} = [0, 1] \setminus \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{3^n - 1} \left(\frac{3k+1}{3^{n+1}}, \frac{3k+2}{3^{n+1}} \right).$$

For each n, $[0, 1]$ is divided into 3^{n+1} small intervals, each has length $\frac{1}{3^{n+1}}$. Since all "middle thirds" of each interval are removed for the Cantor set, $M(\mathcal{C}, \epsilon) = 2^{n+1}$. Therefore,

$$\dim \mathcal{C} = \lim_{n \to \infty} \frac{\log 2^{n+1}}{\log 3^{n+1}} = \frac{\log 3}{\log 2}.$$

Minkowski dimension is one of the definitions of fractal objects. Another common definition is *Hausdorff dimension*. For objects discussed in this paper, Minkowski dimension and Hausdorff dimension are interchangeable. We use "fractal dimension" to refer to both in this paper.

We are now ready to discuss the fractal dimension of the zero set of Brownian motion. The following two scaling properties of Brownian motion will be used to calculate the dimension of the zero set.

Lemma 2.20. *(Scale invariance) Suppose B_t is a standard Brownian motion. If $a > 0$, then*

$$X_t = \frac{1}{\sqrt{a}} B_{at}$$

is a standard Brownian motion.

Proof. The starting condition, independent increments, and continuity of Brownian motion remain unchanged when scaled. We now show that for $0 \leq s < t$, $X_t - X_s$ has a normal distribution of mean 0 and variance $t - s$. We have the expectation $\mathbb{E}[X_t - X_s] = \frac{1}{\sqrt{a}}\mathbb{E}[B_{at} - B_{as}] = 0$ and the variance $Var(X_t - X_s) = (\frac{1}{\sqrt{a}})^2(at - as) = t - s$. □

Lemma 2.21. *(Time inversion) Suppose B_t is a standard Brownian motion. Then*

$$Y_t = \begin{cases} 0 & t = 0 \\ tB_{\frac{1}{t}} & t \neq 0 \end{cases}$$

is a standard Brownian motion.

Proof. The starting condition is satisfied. When $t > 0$, the continuity of Y_t follows directly from the continuity of B_t. We now show that Y_t is continuous at $t = 0$. Since $\mathbb{Q}$ is a countable set, the distribution of Y_t on $\mathbb{Q}$ is a Brownian motion. Since $\mathbb{Q} \cap (0, \infty)$ is dense in $(0, \infty)$ and Y_t is almost surely continuous on $(0, \infty)$, we have

$$\lim_{t \to 0} tB_{\frac{1}{t}} = \lim_{t \to 0, t \in \mathbb{Q}} tB_{\frac{1}{t}} = 0.$$

Thus, Y_t is almost surely continuous on $[0, \infty]$.

Now consider the variance of Y_t :

$$Var(Y_t) = \mathbb{E}(t^2 B_{\frac{1}{t}}^2) - \mathbb{E}(tB_{\frac{1}{t}})^2 = t^2(\frac{1}{t}) = t.$$

We show independent increments of Y_t by calculating the covariance between increments and Y_t:

$$\begin{aligned} Cov(Y_{t+s} - Y_t, Y_t) &= Cov(Y_{t+s}, Y_t) - Var(Y_t) = Cov((t+s)B_{\frac{1}{t+s}}, tB_{\frac{1}{t}}) - t \\ &= t(t+s)Cov(B_{\frac{1}{t+s}}, B_{\frac{1}{t}}) - t = t(t+s)\frac{1}{(t+s)} - t = 0. \end{aligned}$$

This shows the independence of increments. □

The above two lemmas imply that the *self-similarity* of the zero set Z, which means that Z looks similar when it is scaled. This is why we are able to calculate the fractal dimension of Z. How space-filling is Z exactly?

Theorem 2.22. *The zero set Z of Brownian motion has fractal dimension $\frac{1}{2}$ almost surely.*

Proof. We first bound Z on the interval $(0, 1]$. Let $Z_1 = Z \cap (0, 1]$. Then we divide $(0, 1]$ into n equal intervals, each with length $\frac{1}{n}$:

$$\left(\tfrac{k}{n}, \tfrac{k+1}{n}\right] \quad k = 0, 1, 2, \cdots, n-1.$$

Now we need to calculate the number of such intervals that cover Z. For a given k, the probability that such a interval is used to cover Z is

$$\mathbb{P}\{Z_1 \cap \left(\frac{k}{n}, \frac{k+1}{n}\right] \neq \emptyset\}.$$

By the scaling property of Brownian motion, $X_t = \sqrt{\frac{n}{k}} B_{\frac{k}{n}t}$ is a standard Brownian motion. Therefore,

$$p(n, k) = \mathbb{P}\{Z_1 \cap \left(\frac{k}{n}, \frac{k+1}{n}\right] \neq \emptyset\} = \mathbb{P}\{Z_1 \cap \left(1, 1 + \frac{1}{k}\right] \neq \emptyset\}.$$

We have calculated the latter previously in (2.8), which gives us

$$p(n,k) = 1 - \frac{2}{\pi}\arctan\sqrt{k}.$$

Then the expected number of intervals needed to cover Z is

$$\mathbb{E}(N) = \sum_{k=0}^{n-1} 1 - \frac{2}{\pi}\arctan\sqrt{k}.$$

To estimate the sum above, we consider the Taylor expansion of *arctan* :

$$\arctan\frac{1}{t} = \frac{\pi}{2} - t + O(t^2) + \cdots .$$

We can see that when $t \to 0$, the higher order terms tend to 0, which means when $x \to \infty$, $\arctan x \to \frac{\pi}{2} - \frac{1}{x}$. We can now approximate $\mathbb{E}(N)$:

$$\mathbb{E}(N) = \sum_{k=0}^{n-1} 1 - \frac{2}{\pi}\arctan\sqrt{k} \approx \frac{2}{\pi}\int_0^n (\frac{1}{\sqrt{x}} + \delta)dx \approx \frac{4}{\pi}\sqrt{n}.$$

The fractal dimension of Z is

$$\dim Z = \lim_{n\to\infty} \frac{\log\frac{4}{\pi}\sqrt{n}}{\log n} = \lim_{n\to\infty} \log_n \frac{4}{\pi} n^{\frac{1}{2}} \approx \frac{1}{2}.$$

□

2.3. **Brownian Excursion.** A Brownian excursion process is a Brownian motion process conditioned to be positive and attain the value 0 at endpoints. The following construction which can be found in [5] defines Brownian excursion. Let $B = \{B_t : t \geq 0\}$ be a standard Brownian motion. For each $t > 0$, We define

$$\gamma_t = \sup\{s : s \leq t, B_s = 0\}$$
$$\beta_t = \inf\{s : s \geq t, B_s = 0\}.$$

These two times indicate the last zero before t and the first zero after t, respectively. Since B_t is continuous and $\mathbb{P}\{B(t) = 0\} = 0$, we know that

$$\gamma(t) < t < \beta(t)$$

almost surely. This is true for each $t > 0$, and thus it is true for all rational $t > 0$. The interval (γ_t, β_t) is called the *interval of excursion straddling* t. Note that Brownian motion B on (γ_t, β_t) is either all positive or all negative. We define Brownian excursion by taking the absolute value of B on the interval of excursion.

Definition 2.23. For each $t > 0$, the process $E_t = \{B(s) : s \in (\gamma_t, \beta_t)\}$ is called the *Brownian excursion straddling* t.

More specifically, Brownian excursion can be conditioned to attain the value 0 at time 1 :

Definition 2.24. [4] The process defined on $s \in [0,1]$,

$$e(s) = \frac{|B(s\beta_1 + (1-s)\gamma_1)|}{\sqrt{\beta_1 - \gamma_1}},$$

is called the *unsigned scaled Brownian excursion.*

The above γ_1 and β_t follow from the definition of interval of excursion straddling 1. This definition takes a normalized Brownian excursion straddling 1 and scales it up to the interval of $[0, 1]$.

Although Brownian excursion is defined as a conditioned Brownian motion, the local properties of the former remain the same as the latter. Brownian excursion is closely related to the random triangulation of the circle, which we will discuss in the next section.

3. Random Triangulations of the Circle

Brownian motion is related to the random triangulation of the circle through finding local minima pairs. To show this correspondence, we will first show that simple random walks that return to zero can be mapped to triangulations of a polygon. Then we will take the limit of both objects to construct a algorithm that matches a random triangulation of the circle to Brownian motion.

3.1. Polygon Triangulations, Rooted trees, and Simple random walks.

In this section, we will show that there exist maps between simple random walks and several combinatorial objects.

Map 1: (Simple random walks and ordered trees) First, consider a simple random walk of $2n$ steps that returns to 0 at $2n$ and attains non-negative values from 0 to $2n$. A concrete example of a simple random walk of 10 steps is shown in figure 2. Then consider the correspondence between such a walk and a ordered tree with a root (example shown in figure 3). As defined in the previous section, a simple random walker take steps of ± 1 each time. Starting from time 0, when the walker takes step $+1$, an edge is drawn from the previous vertex in the order from left to right. When the walker takes step -1, we go back to the previous vertex. When the walker hits time $2n$, the corresponding position returns to the root. In this way, an ordered tree is created from the simple random walk.

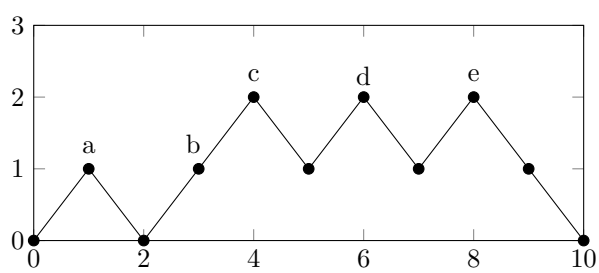

Figure 2. A simple random walk of $2n = 10$ steps

The reverse direction goes from a given rooted tree of n edges to a simple random walk of $2n$ steps that returns to 0 at $2n$ for the first time. The walk is thus obtained by traversing the tree in the order from left to right starting from and ending at the root. Note that each edge of the tree is traversed twice with the first one going away from the root and the second one coming back to the root. Thus a tree with n edges correspond to a walk of $2n$ steps.

Map 2: (Ordered trees and binary trees) Next, we will show that there exists a bijection between an order tree with n edges and a binary tree (every vertex has exactly two children) with n interior vertices (example shown in figure 4).

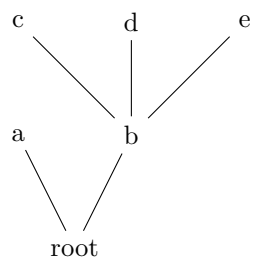

FIGURE 3. A rooted plane tree of $n = 5$ edges

To construct a binary tree from a given ordered tree, we start from drawing the first child of the root in the ordered tree as the root in the binary tree. For a given vertex v in the ordered tree, its first child, w, is drawn as the left child in the binary tree. Its first sibling is drawn as the right child in the binary tree. If the vertex has no children, a left leaf is drawn from it. If the vertex has no siblings, a right leaf is drawn from it.

The reverse algorithm starts by planting a root. If the left child of a given vertex of the binary tree is not a leaf (i.e. an interior vertex), then a child is drawn from the vertex. If the right child of a given vertex of the binary tree is not a leaf, then a sibling of the vertex is drawn to its right. Otherwise, nothing is drawn.

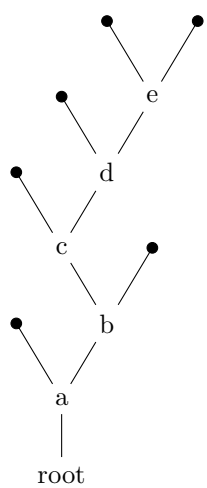

FIGURE 4. A binary tree of $n = 5$ interior vertices

Map 3: (Binary trees and triangulations) We will show that there exists a one-to-one correspondence between a binary tree with n interior vertices and a triangulation of a $(n + 2)$-gon. For simplicity, we consider the triangulation of a regular polygon. First, we define polygon triangulation:

Definition 3.1. (Polygon Triangulations) A triangulation of a regular $(n+2)$-gon P is a partition of P into triangles whose vertices lie on the vertices of P.

A triangulation of P consists of n non-intersecting triangles whose union is P. The number of triangulations for P is known to be the **Catalan Number** $C_n =$

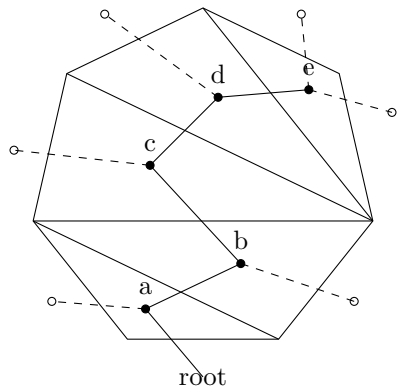

FIGURE 5. A triangulation of a polygon of $n+2=7$ sides

$\frac{1}{n+1}\binom{2n}{n}$, a formula found by Euler. In fact, the number of rooted plane trees with n edges is also equal to C_n, a fact following from the series of bijections we are proving.

There exists an one-to-one correspondence between a triangulation of P, S_p, and a binary tree with n interior vertices, BT_n (shown in figure 5). Each chord of the triangulation corresponds to an interior edge of the binary tree. Each interior vertex of the binary tree corresponds to the center of a triangle. The sides of P, except the chosen base side, can be identified with a leaf in the binary tree.

To construct the tree from the triangulation, first identify a base side and centers of each triangle. Then connect adjacent centers with an edge, which corresponds to the interior of the binary tree. Draw an edge between a center and the outside of the polygon that intersects with a side of the polygon to represent the leaves. To identify a triangulation from a binary tree, again start by choosing a base side and plant the root above the base side. If the right child of the root is a leaf, then draw a triangle to the right side of the base side. Otherwise, draw a triangle to the left. Then repeat the above process until the last triangle is reached.

3.2. **Random Triangulations of the Circle.** The three maps above link random walks to polygon triangulations. As shown earlier in this paper, rescaling random walks gives us Brownian motion. Brownian motion can be thought of as the "limit" of random walks as time $t \to \infty$ or step $\frac{1}{n} \to 0$. It is natural to think of the question whether the "limit" of triangulations of a n-gon (namely, as $n \to \infty$, the polygon eventually becomes the circle) gives us Brownian motion. In this section, we aim to investigate the random triangulation of the circle and link it to our previous discussion of Brownian motion.

The question leaves the scope of finite combinatorics as $n \to \infty$ for a polygon. The definition of a triangulation of the circle is first given by David Aldous:

Definition 3.2. [2] A triangulation of the circle is a closed subset of the closed disc whose complement is a disjoint union of open triangles with vertices on the circumference of the circle.

The length of the chords of triangulations of a circle C with radius R, unlike the one in the polygon case, can be anything within the boundary $(0, 2R]$. Therefore, it is useful only to consider the random triangulation of the circle. The defining

property of a triangulation of the circle is that the chords made up triangles that are non-intersecting. There is an analogue between this property and the local minimum of a continuous function shown below. Define a continuous function $f : [0,1] \to [0,\infty)$ such that all values on $(0,1)$ are positive and the start and end points are 0:

$$\begin{cases} f(0) = f(1) = 0 \\ f(t) > 0 \qquad\qquad t \in (0,1). \end{cases} \tag{3.3}$$

Let t_2 be a strict local minimum: for t in some neighborhood of t_2, $f(t) > f(t_2)$. Since f is continuous and fixed to be 0 at the boundary, there exists $t_1 = \sup\{t < t_2 : f(t) = f(t_2)\}$ and $t_3 = \inf\{t > t_2 : f(t) = f(t_2)\}$. Now we have $f(t) > f(t_2)$ for all $t \in (t_1, t_3)$. Let $[0,1]$ correspond with the circumference of the circle we are triangulating. For any such triple t_1, t_2, t_3, draw a triangle on the corresponding points on the circle. This formation has the property of non-intersecting triangles. Consider a local minimum t_2' and the corresponding two nearest points with the same value as $f(t_2')$. Without loss of generality, let $f(t_2') > f(t_2)$. Then t_1', t_2', t_3' necessarily lie on the same interval (t_1, t_2) or (t_2, t_3). When drawing on the circle, one can see that this condition guarantees that the triangles don't intersect. Such a function is almost successful at representing a triangulation of the circle.

One thing left to be considered is the number of local minima. Based on the algorithm above, the number of local minima on the function determines the number of chords on the circle. For a n-gon, the number of chords of a triangulation is $n-3$. As $n \to \infty$, the number of chords should go to infinity to be a triangulation of the circle. Therefore, we are not only looking for a continuous function with the above properties but one whose local minima are dense in $[0,1]$. Previously in this paper (Theorem 2.14), we have seen that the set of times that a Brownian motion attains a local minimum is dense in $[0,\infty)$. Since Brownian excursion $e(s)$ retains the local properties of a Brownian motion, the local minima of $e(s)$ are also dense. Thus, Brownian excursion matches the random triangulation of the circle. By taking all local minima of $e(s)$ and drawing vertical lines on the graph, we obtain the three points corresponding to the triangles in the circle.

3.3. **Fractal Dimension of A Random Triangulation of the Circle.** The random triangulation of the circle obtained by a Brownian excursion is a fractal object, like the zero set of Brownian motion. We can in fact compute the fractal dimension of this set.

Theorem 3.4. *Random triangulation of the circle has fractal dimension $\frac{3}{2}$ almost surely.*

Proof. Let S_ϵ be the the set of endpoints of some chord in the random triangulation of the circle for some given $\epsilon > 0$. First, we want to show that the fractal dimensional of S_ϵ is $\frac{1}{2}$. We approach this from the perspective of the Brownian excursion $e(t)$ generating the random triangulation of the circle. Consider intervals $[s, s']$ straddling the time 0.5 such that $e(s) = e(s') < e(t)$ for all $t \in (s, s')$. For $0 < y < e(0.5)$, define $s_y = \sup\{t < 0.5 : e(t) = y\}$ and $s_y' = \inf\{t > 0.5 : e(t) = y\}$. Combining Theorem 2.15 (Lévy's identity) with the dimension of the zero set Z, we know that the set

$$\{t_y : y > 0\} \text{ where } t_y = \inf\{t : B(t) = y\} \text{ has dimension } \frac{1}{2}.$$

From the Markov property of Brownian motion, $\widetilde{B}(t) = B(0.5-t)$ is also a Brownian motion. Combining the above two results we have the following

$$\{s_y : 0 < y < e(0.5)\} \text{ has dimension } \frac{1}{2}.$$

Since 0.5 can be any arbitrary rational number, the set of endpoints of some chord in the triangulation S_ϵ for some given $\epsilon > 0$ has dimension $\frac{1}{2}$ as well. Since three disjoint endpoints compose a triangle in the triangulation, the dimension of the random triangulation of the circle is $\frac{3}{2}$. □

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