ADVANTAGES AND APPLICATIONS OF QUANTUM GAME THEORY

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Abstract. This survey introduces the field of quantum game theory, assuming no previous knowledge of either game theory or quantum computing, and outlines the literature in the field. We present existing criticism of the field and discuss advantages of quantum game theory over classical game theory, focusing on the cases of CHSH Game and Prisoner's Dilemma. We list several real-live applications of quantum game theory in policy making for the sake of improving economy's efficiency. The survey finishes with discussion and suggestions for further research in the field.

Contents

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1. Introduction

1.1. **Background Information.** With the development of quantum computing, David A. Meyer was the first to merge them with game theory in 1999 [Mey99]. In order to find a better strategy that improves the expected payoff of individuals, Meyer applied general quantum algorithms and generated a quantum strategy that he proved to be always at least as good as the classical strategies. The result of this discovery thus serves as a foundation for a new chapter of game theory – *quantum game theory*. Since then, many mathematicians, physicists, and economists explored this area by constructing quantum versions of classical game theory models, such as the most famous game-theoretical model: Prisoner's Dilemma [EWL99].

The main difference between classical game theory and quantum game theory is the usage of *entanglement* mechanism in modeling. Instead of applying a pure or mixed strategy, as in the case of classical game theory, the players are assumed to share a *qubit* (or several, although such models are out of scoop of this survey), and their strategy is based on evaluating the *measurement* of this qubit in two different bases.

Since its introduction, quantum game theory has received both praise and criticism, and its exploration is not without challenges. Due to its complex nature that unites several fields of knowledge, quantum game theory requires background in physics, computer science, mathematics, and economics, which sets a high bar for researchers who want to enter this area. Another important criticism is that the study of quantum game theory, so far, has fallen within the existing knowledge in Economics and therefore fails to bring new input to it [Lev05]. However, there are reasons to suspect this will change [DJL05], therefore the applicability of quantum game theory to real-life scenarios is a topic of ongoing discussion.

1.2. **Organization of the Paper.** In this survey, we briefly introduce the basics, advantages, criticism, and real-world applications of quantum game theory.

The paper is organized as follows:

Section 2 sets up all the required background of the 'quantum' part of quantum game theory. In Subsection 2.1, we set up the model of quantum computations. We reveal advantages of quantum computations compared to classical computations in Subsection 2.2.

Section 3 introduces quantum game theory. We remind the reader some classical game theory definitions in Subsection 3.1, and then apply them to the so-called *CHSH Game* in Subsection 3.2. In Subsection 3.3, we analyze the same CHSH Game in quantum game-theoretical setting and prove that this approach improves the maximum winning probability compared to all possible classical strategies.

Section 4 discusses criticism of quantum game theory (Subsection 4.1) and introduces its advantages by comparing the classical (Subsection 4.2) and quantum (Subsection 4.3) game-theoretical analysis of the most well-known problem in game theory: the *Prisoner's Dilemma*. We go as far as to prove that the players can escape the dilemma entirely by applying a quantum strategy.

Section 5 presents real-life applications of quantum game theory to policy making, specifically in the case of oligopolistic markets (Subsection 5.1). In Subsection 5.2, we discuss ways in which governments can apply quantum game theory in order to improve social efficiency and prevent emergence of monopolies.

Section 6 discusses possible strategies for further exploration of quantum game theory, focusing on two particular topics: incomplete information games (Subsection 6.1) and random networks (Subsection 6.2).

2. Quantum Computing

2.1. **The Meaning of "Quantum".** In order to understand quantum game theory, we must first understand the word "quantum". In 1911, Danish physicist Niels Bohr and New Zealand physicist Ernest Rutherford proposed the Bohr model for presenting the structure of an atom [Rut1911]. This model suggests that electrons are orbiting around a dense nucleus. These electrons have different levels of discrete energy states.

Theorem 2.1 (**Superposition Principle** [Ill91])**.** *The net response caused by two or more stimuli is the sum of the responses that would have been caused by each stimulus individually. In other words, the net response can be represented as a linear combination of two responses.*

In quantum physics, the superposition principle gives rise to a surprising consequence: if a quantum system can be in one of two states, then any state that is a linear combination of the aforementioned is also realizable. The physical reason for this is the *uncertainty principle*: it is impossible to perfectly evaluate the speed and the position of a given electron at the same time. What we can say instead, intuitively, is that our electron has an inclination toward a certain state. We are going to use a coefficient a_i to indicate how inclined the electron is towards the state i (one may think that a_i is the probability that the electron is in the state i). In other words, instead of having discrete energy states as in the classical model, in quantum physics, one assumes the energy level to be a continuous range.

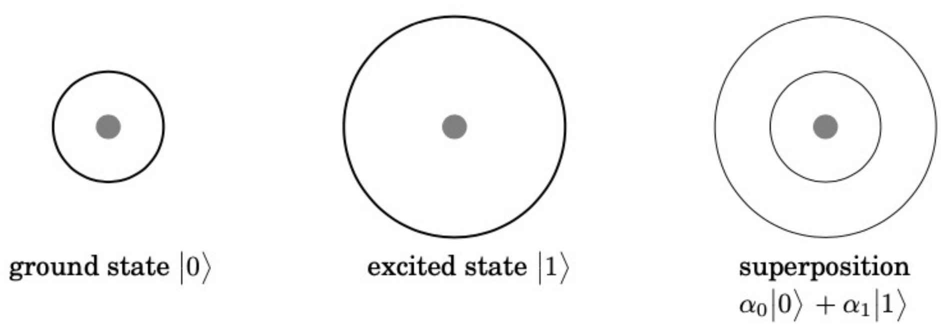

Figure 1. A model of an atom with one electron in a state that is a linear combination of two states. Adapted from Figure 10.1 of [DPV08].

Definition 2.2. Suppose we have an atom with one electron, and we know that this electron has two *energy states* (or *energy levels*): the *ground state* $|0\rangle$ and the *excited state* $|1\rangle$ (see Figure 1). The *quantum states* of this system are linear combinations of the ground state and the exited state: $a_0|0\rangle+a_1|1\rangle$. The coefficients

a_0, $a_1 \in \mathbb{C}$, called the *amplitudes* of the states $|0\rangle$ and $|1\rangle$, must satisfy the equation $|a_0|^2 + |a_1|^2 = 1$.

The system described above is called a *qubit*.

In general, the superposition principle holds for k*-level systems*.

Definition 2.3. Suppose we have an atom that has k energy states: a ground state $|0\rangle$ and $k-1$ excited energy states $|1\rangle, \ldots, |k-1\rangle$. Denote the amplitudes of the states by $a_0, a_1 \ldots a_{k-1} \in \mathbb{C}$, $|a_0|^2 + |a_1|^2 + \ldots + |a_{k-1}|^2 = 1$. Then we can write the quantum states of the system in the form

$$a_0|0\rangle + a_1|1\rangle + a_2|2\rangle + \ldots + a_{k-1}|k-1\rangle.$$

The system described above is called a k*-bit*.

One might ask: if I am given information in the classical way, with 0-1 bits, how do I make it quantum? There are several possible answers to this question, but the most common one is to take a $|0\rangle$ qubit for every 0 bit and a $|1\rangle$ qubit for every 1 bit. However, the question of how to get back to the classical world from the quantum one is much more interesting.

Definition 2.4. A *measurement projection* is a projection from the space of all qubits to $\{0, 1\}$. We say that we *measure* a qubit when we apply a measurement projection to it. The result is a *measurement* of the said qubit.

There exist many different measurement projections. Further on, we are not going to concern ourselves with choosing one of them and will just assume that it has been chosen and fixed.

Naturally, the idea of measurements can be extended to k-bits.

2.2. **Quantum Advantage.** *Quantum entanglement* is one of the most mysterious phenomenons of physics. Informally, one could say that two systems are *entangled* if they are strongly correlated to each other. Drawing from Chapter 10 of [DPV08], we will try our best to explain this enigmatic phenomenon.

Definition 2.5. Suppose we have two qubits $a_0|0\rangle + a_1|1\rangle$ and $b_0|0\rangle + b_1|1\rangle$. Their *joint* (or *entangled*) *state* is their tensor product

$$a_0b_0|00\rangle + a_0b_1|01\rangle + a_1b_0|10\rangle + a_1b_1|11\rangle.$$

In classical algorithms, we store information in 'bits', which have just two states: 0 and 1. Qubits allow to consider states of form $a_0|0\rangle + a_1|1\rangle$, where a_0, $a_1 \in \mathbb{C}$ and $|a_0|^2 + |a_1|^2 = 1$. With n classical bits, we can only store n possible classical states. However, if we have n qubits, then, using their joint states, we can store a linear superposition of 2^n possible classical states. In other words, with only n qubits of memory, a quantum algorithm can manipulate 2^n states.

Suppose that we have given a quantum computer n classical bits. By perceiving them as qubits, the quantum computer can "unpack" them into 2^n joint quantum states that it will manipulate simultaneously in the same time that a classical computer requires to manipulate n states, and then just *measure* the result, transforming it back to the form of classical bits, at the end. As a result, as n becomes large enough, quantum computers become exponentially more powerful than classical computers. As a result, quantum computers are able to solve in polynomial time many problems that are proven to require exponential time to be solved by classical computers.

Quantum computing has a significant advantage over classical computing, making it a very powerful tool.

3. Quantum Game Theory

The following section is mostly based on Chapters 13 and 14 of [Aar18].

3.1. **Defining "Best" in "Best Strategy".** How do define the optimal state, or strategy, in a classical game? There exist several options, but we are going to consider two most well-known.

Definition 3.1. [OR94, p. 14] *Nash Equilibrium* is a strategy profile (i.e. assignment of a strategy to each player) such that no player has a reason to change their strategy provided that all other strategies in the profile remain unchanged.

John Forbes Nash Jr. proved that in mixed strategies, every game with a finite number of players in which each player can choose from finitely many pure strategies has at least one Nash equilibrium. In general, the Nash Equilibrium position can be found through solving an algebraic expression for the payoff matrix. Note that a game can have more than one Nash equilibrium.

Economists also use a different definition of optimality of state.

Definition 3.2. [Var10, p. 15] An economical state is called *Pareto-efficient* if no individual can make themselves better off without making anybody else worse off.

Note that a Pareto-efficient state is not necessarily a Nash equilibrium, and vice versa.

3.2. **CHSH Game.** In 1969, John Clauser, Michael Horne, Abner Shimony, and Richard Holt introduced the idea of the *CHSH Game* to explain, in their own words, "how to think about what Bell did" [CHSH69]. This game model doesn't directly mention quantum mechanics but becomes evidence of the existence of a new form of game theory – quantum game theory.

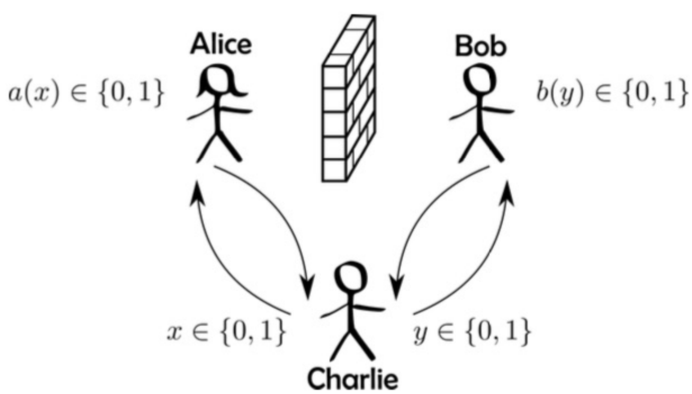

Figure 2. The setup of the CHSH game. Alice and Bob are the two players in the game, and Charlie is the referee. Adapted from Figure 13.1 of [Aar18].

The setup of the CHSH Game involves three people: two players A (Alice) and B (Bob) and a referee C (Charlie). Each player is sent into a separate room, and both are given a challenge bit, x and y respectively, by the referee. The challenge bits are independent random variables that take values in $\{0, 1\}$ with equal probability. The challenge bits (we will call them x and y from now on) are independent of each other. Each of the two players has to respond with an answer bit (a for Alice, b for Bob). The players are allowed to communicate in advance; however, once the players enter the rooms, they will not be able to communicate with each other.

The players win if and only if the sum of their answer bits has the same parity as the product of the challenge bits:

$$a + b \equiv xy \pmod 2.$$

Consider the following strategy: both Alice and Bob always answer 0. It is easy to prove that the probability of a win is 75%: the players win iff $xy \equiv 0 \pmod 2$. The only situation when this does not happen is when $x = y = 1$. Since the challenge bits x and y are independent random variables that take values in $\{0, 1\}$ with equal probability, the chance of the event $x = y = 1$ happening is 25%.

This strategy can be proven to be the most optimal strategy by the argument called the convexity argument [Aar18, pp. 101–102]. Intuitively, we can think about it as follows. Let $a(x)$ and $b(y)$ be Alice's and Bob's outputs respectively. The winning condition is $a(x) + b(y) \equiv xy \pmod 2$. Then the only possible pure strategies are

(1) Always send 0: $a(x) = 0$, $b(y) = 0$.
(2) Always send 1: $a(x) = 1$, $b(y) = 1$.
(3) Send input: $a(x) = x$, $b(y) = y$.
(4) Send the negation of the input: $a(x) = 1 - x$, $b(y) = 1 - y$.

Figure 3 shows the probability of a win for each of these strategies.

Since x and y are random, independent, and output 0 and 1 with equal probability, no mixed strategy can achieve better results than the best pure strategy. So, in the setup of classical game theory, the maximum possible probability of a win in the CHSH Game is 75%.

3.3. **Quantum Game Theory Analysis on CHSH Game.** We claim that we can find a quantum game strategy that yields the probability of a win higher than 75%. If it is, indeed, possible, this proves that quantum game theory has an advantage over classical game theory.

Let us use the setting of quantum game theory to analyze the CHSH Game.

Definition 3.3. The *Bell states* are four entangled states of two qubits:

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \ \ |\Phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}},$$

$$|\Psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \ \ |\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

One can think about two qubits in one of the Bell states as follows. The qubit $A = a_0|0\rangle + a_1|1\rangle$, held by Alice, is in a superposition of 0 and 1. If Alice measured her qubit in the standard basis, the outcome would be either 0 or 1, each with probability $\frac{1}{2}$; but if Bob also measured his qubit $B = b_0|0\rangle + b_1|1\rangle$, his outcome would necessarily be the same as Alice's in the cases $|\Phi^+\rangle$, $|\Phi^-\rangle$ and the opposite

Strategy	x	y	a	b	a+b	xy
Always Send 0	0	0	0	0	0	0
	0	1	0	0	0	0
	1	0	0	0	0	0
	1	1	0	0	0	1
Always Send 1	0	0	1	1	0	0
	0	1	1	1	0	0
	1	0	1	1	0	0
	1	1	1	1	0	1
Same as Input	0	0	0	0	0	0
	0	1	0	1	1	0
	1	0	1	0	1	0
	1	1	1	1	0	1
Opposite of Input	0	0	1	1	0	0
	0	1	1	0	1	0
	1	0	0	1	1	0
	1	1	0	0	0	1

FIGURE 3. Table of all wins (black) and losses (red) under each classical pure strategy for the CHSH Game. Adapted from Table 13.1 of [Aar18].

of it in the cases $|\Psi^+\rangle$, $|\Psi^-\rangle$. Thus, Alice and Bob each has random outcome, but these random outcomes are perfectly correlated.

Further on, we will call the Bell state $|\Phi^+\rangle$ the *Bell pair*.

Consider a quantum strategy in which, before the players are separated, they each get a qubit and entangle them, obtaining the Bell pair. When both players enter their room, they start measuring their respective qubits in different bases depending on the challenge bit they receive. The answer bit of each player is the result of their measurement.

Our standard basis is $\{|0\rangle, |1\rangle\}$. Let us introduce the notion of an *angle state* $|\psi\rangle = \cos(\psi)|0\rangle + \sin(\psi)|1\rangle$. One can perceive $|\psi\rangle$ as a rotation of a state $|0\rangle$ by angle ψ. For this strategy, we are going to require 6 angle states:

$$|+\rangle = |\frac{\pi}{4}\rangle = \cos(\frac{\pi}{4})|0\rangle + \sin(\frac{\pi}{4})|1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad |-\rangle = |-\frac{\pi}{4}\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}},$$

$$|\frac{\pi}{8}\rangle = \cos(\frac{\pi}{8})|0\rangle + \sin(\frac{\pi}{8})|1\rangle, \qquad |\frac{5\pi}{8}\rangle = \cos(\frac{5\pi}{8})|0\rangle + \sin(\frac{5\pi}{8})|1\rangle,$$

$$|-\frac{\pi}{8}\rangle = \cos(-\frac{\pi}{8})|0\rangle + \sin(-\frac{\pi}{8})|1\rangle, \qquad |\frac{3\pi}{8}\rangle = \cos(\frac{3\pi}{8})|0\rangle + \sin(\frac{3\pi}{8})|1\rangle.$$

The states $|\frac{\pi}{8}\rangle$ and $|-\frac{\pi}{8}\rangle$ are symmetrical with respect to $|0\rangle$-axis:

$$\cos(\frac{\pi}{8}) = \cos(-\frac{\pi}{8}), \qquad \sin(\frac{\pi}{8}) = -\sin(-\frac{\pi}{8}).$$

The states $|\frac{5\pi}{8}\rangle$ and $|\frac{3\pi}{8}\rangle$ are symmetrical with respect to $|1\rangle$-axis:

$$\cos(\frac{5\pi}{8}) = -\cos(\frac{3\pi}{8}), \quad \sin(\frac{5\pi}{8}) = \sin(\frac{3\pi}{8}).$$

Measuring a qubit in a basis $\{|\phi\rangle, |\theta\rangle\}$ is equivalent to applying a unitary transformation to the standard basis, followed by a measurement in the standard basis. For example, measuring a qubit in the basis $\{|\frac{\pi}{8}\rangle, |\frac{5\pi}{8}\rangle\}$ is equivalent to applying the unitary transformation $R_{\frac{\pi}{8}}$, followed by the measurement in the standard basis.

We propose the following strategy for Alice:

(1) If $x = 0$, Alice uses the standard basis $\{|0\rangle, |1\rangle\}$. If the measurement is $|0\rangle$, Alice's answer bit is 0. Otherwise, 1.
(2) If $x = 1$, Alice uses the basis $\{|+\rangle, |-\rangle\}$. If the measurement is $|+\rangle$, Alice's answer bit is 0. Otherwise, 1.

We propose the following strategy for Bob:

(1) If $y = 0$, Bob uses the basis $\{|\frac{\pi}{8}\rangle, |\frac{5\pi}{8}\rangle\}$. If the measurement is $|\frac{\pi}{8}\rangle$, Bob's answer bit is 0. Otherwise, 1.
(2) If $y = 1$, Bob uses the basis $\{|\frac{-\pi}{8}\rangle, |\frac{3\pi}{8}\rangle\}$. If the measurement is $|\frac{-\pi}{8}\rangle$, Bob's answer bit is 0. Otherwise, 1.

We exploit a very powerful result:

Theorem 3.4 (No-communication Theorem [PT04]). *It is impossible for one observer to communicate information to the other observer by measuring an entangled quantum state.*

Without loss of generality, we can assume that Alice measures first. We have 4 cases

(1) Suppose $x = y = 0$ (the probability of which is $\frac{1}{4}$). Then Alice has $\frac{1}{2}$ chance of obtaining the measurement $|0\rangle$, and $\frac{1}{2}$ chance of obtaining the measurement $|1\rangle$. Therefore Bob has $\cos^2(\frac{\pi}{8})$ chance of obtaining the measurement $|\frac{\pi}{8}\rangle$, and $\sin^2(\frac{5\pi}{8})$ chance of obtaining the measurement $|\frac{5\pi}{8}\rangle$. Since the winning condition for the players is $a + b \equiv xy(\mod 2) \equiv 0$, the win happens in two cases:
   (a) Alice measures $|0\rangle$ and Bob $|\frac{\pi}{8}\rangle$. This happens with probability
   $$\frac{1}{2} \times \cos^2(\frac{\pi}{8}).$$
   (b) Alice measures $|1\rangle$ and Bob $|\frac{5\pi}{8}\rangle$. This happens with probability
   $$\frac{1}{2} \times \sin^2(\frac{5\pi}{8}) = \frac{1}{2} \times \cos^2(-\frac{\pi}{8}) = \frac{1}{2} \times \cos^2(\frac{\pi}{8}).$$
   Therefore, the probability of winning is $\cos^2(\frac{\pi}{8})$.
(2) Suppose $x = 0$, $y = 1$ (the probability of which is $\frac{1}{4}$). Then Alice has $\frac{1}{2}$ chance of obtaining the measurement $|0\rangle$, and $\frac{1}{2}$ chance of obtaining the measurement $|1\rangle$. Therefore Bob has $\cos^2(-\frac{\pi}{8})$ chance of obtaining the measurement $|-\frac{\pi}{8}\rangle$, and $\sin^2(\frac{3\pi}{8})$ chance of obtaining the measurement $|\frac{3\pi}{8}\rangle$. Since the winning condition for the players is $a+b \equiv xy(\mod 2) \equiv 0$, the win happens in two cases:
   (a) Alice measures $|0\rangle$ and Bob $|-\frac{\pi}{8}\rangle$. This happens with probability
   $$\frac{1}{2} \times \cos^2(-\frac{\pi}{8}) = \frac{1}{2} \times \cos^2(\frac{\pi}{8}).$$

(b) Alice measures $|1\rangle$ and Bob $|\frac{3\pi}{8}\rangle$. This happens with probability

$$\frac{1}{2} \times \sin^2(\frac{3\pi}{8}) = \frac{1}{2} \times \cos^2(\frac{\pi}{8}).$$

Therefore, the probability of winning is $\cos^2(\frac{\pi}{8})$.

(3) Suppose $x = 1$, $y = 0$ (the probability of which is $\frac{1}{4}$). Then Alice has $\frac{1}{2}$ chance of obtaining the measurement $|+\rangle$, and $\frac{1}{2}$ chance of obtaining the measurement $|-\rangle$. Therefore Bob has $\cos^2(\frac{\pi}{8})$ chance of obtaining the measurement $|\frac{\pi}{8}\rangle$, and $\sin^2(\frac{5\pi}{8})$ chance of obtaining the measurement $|\frac{5\pi}{8}\rangle$. Since the winning condition for the players is $a + b \equiv xy(\mod 2) \equiv 0$, the win happens in two cases:

(a) Alice measures $|+\rangle$ and Bob $|\frac{\pi}{8}\rangle$. This happens with probability

$$\frac{1}{2} \times \cos^2(\frac{\pi}{8}).$$

(b) Alice measures $|-\rangle$ and Bob $|\frac{5\pi}{8}\rangle$. This happens with probability

$$\frac{1}{2} \times \sin^2(\frac{5\pi}{8}) = \frac{1}{2} \times \cos^2(-\frac{\pi}{8}) = \frac{1}{2} \times \cos^2(\frac{\pi}{8}).$$

Therefore, the probability of winning is $\cos^2(\frac{\pi}{8})$.

(4) Suppose $x = y = 1$ (the probability of which is $\frac{1}{4}$). Then Alice has $\frac{1}{2}$ chance of obtaining the measurement $|+\rangle$, and $\frac{1}{2}$ chance of obtaining the measurement $|-\rangle$. Therefore Bob has $\cos^2(-\frac{\pi}{8})$ chance of obtaining the measurement $|-\frac{\pi}{8}\rangle$, and $\sin^2(\frac{3\pi}{8})$ chance of obtaining the measurement $|\frac{3\pi}{8}\rangle$. Since the winning condition for the players is $a+b \equiv xy(\mod 2) \equiv 0$, the win happens in two cases:

(a) Alice measures $|+\rangle$ and Bob $|-\frac{\pi}{8}\rangle$. This happens with probability

$$\frac{1}{2} \times \cos^2(-\frac{\pi}{8}) = \frac{1}{2} \times \cos^2(\frac{\pi}{8}).$$

(b) Alice measures $|-\rangle$ and Bob $|\frac{3\pi}{8}\rangle$. This happens with probability

$$\frac{1}{2} \times \sin^2(\frac{3\pi}{8}) = \frac{1}{2} \times \cos^2(\frac{\pi}{8}).$$

Therefore, the probability of winning is $\cos^2(\frac{\pi}{8})$.

Therefore, regardless of what the challenge bit each player receives, this strategy will always yield a probability of a win equal to

$$\cos^2(\frac{\pi}{8}) = \frac{1}{4}(2 + \sqrt{2}) \approx 0.85 > 0.75.$$

This proves that quantum game theory allows to create strategies that outperform the ones available in the setting of the classical game theory.

4. How useful is Quantum Game Theory?

4.1. **Criticism of Quantum Game Theory.** Doubts in the usefulness of quantum game theory have been voiced. David K. Levine has stated that quantum games theory gives no new advantages to the field of economics since its results fall within the already existing framework [Lev05].

We disagree with this claim, and will further try to convince the reader in our position.

4.2. **Prisoner's Dilemma in Classical Game Theory.** The most famous model in game theory is Prisoner's Dilemma. It is a nonzero-sum game, which means that players can achieve extra benefit via cooperation.

The setup is as follows. The game has two players, A and B. Each of them independently chooses to either Defect (D) or Cooperate (C). Based on the choices of both player, they will receive a certain payoff. Both players' goal is to maximize their individual payoff. We will consider the following payoff matrix:

	B: Cooperate	B: Defect
A: Cooperate	$(3, 3)$	$(0, 5)$
A: Defect	$(5, 0)$	$(1, 1)$

Here, the payoffs are presented in the form $(a,\, b)$, where a is the payoff for player A, and b is the payoff for player B.

It is well-known that, in the classical game-theoretical setting, the only Nash equilibrium is when both players choose to Defect, meaning that they get the least possible total payoff.

4.3. **Prisoner's Dilemma in Quantum Game Theory.** Jens Eisert proved that the players in the Prisoner's Dilemma can improve their results if they are allowed quantum strategies [EWL99]. In this subsection, we relay his argument.

Let us give one qubit to each of the players. The classical pure strategies (Cooperate and Defect) correspond to two basis vectors $|C\rangle$ and $|D\rangle$ in the Hilbert space of a two-state system. The state of the game can be described by a vector in the tensor product of $|AB\rangle$, where A represents player A's strategy and B represents player B's strategy. Therefore, there are four possible states: $|CC\rangle$, $|CD\rangle$, $|DC\rangle$, and $|DD\rangle$.

Suppose that the initial state of the game is $|\psi_0\rangle = \hat{J}|CC\rangle$, where $\hat{J}$ is a unitary operator that is known to both players. Both players' moves are associated with unitary operators $\hat{U_A}$ and $\hat{U_B}$.

The strategy $\hat{U}$ can be written in the form of a 2-parameter set of unitary 2×2 matrices:

$$\hat{U}(\theta, \phi) = \begin{pmatrix} e^{i\phi}\cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \\ \cos(-\frac{\theta}{2}) & e^{-i\phi}\sin(\frac{\theta}{2}) \end{pmatrix},$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq \frac{\pi}{2}$.

Let all the possible strategies be taken from the subset $S_0 \equiv \{\hat{(U)}(\theta, 0) | \theta \in [0, \pi]\}$.

The Cooperate strategy $\hat{C}$ can be written as

$$\hat{C} \equiv \hat{U}(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and the Defect strategy $\hat{D}$ is a spin flip:

$$\hat{D} \equiv \hat{U}(\pi, 0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The expected payoff $\$_A$ for player A can be written as:

$$\$_A = rP_{CC} + pP_{DD} + tP_{DC} + sP_{CD},$$

where $P_{CC} = p_A^{(C)} \times p_B^{(C)}$. The probability of the choice being cooperation c is:

$$p^{(c)} = \cos^2(\frac{\theta}{2}).$$

Similarly, the probability of the choice being defect d is:

$$p^{(d)} = 1 - \cos^2(\frac{\theta}{2}).$$

Let us introduce a variable $\gamma \in [0, \frac{\pi}{2}]$, which we will call the entanglement of the model. If we consider this game as a separable game, we set $\gamma = 0$. Figure 4 shows the potential payoff of player A if the game is a separable game. From this figure, we can see that player A's payoff is maximized if he chooses to play the Defect strategy $\hat{D}$ regardless of player B's choices.

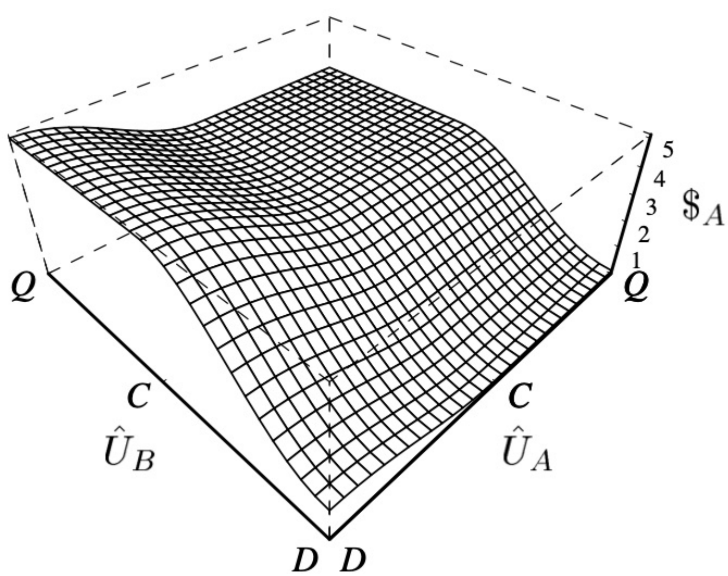

FIGURE 4. Player A's expected payoff under different strategies with entanglement $\gamma = 0$. Adapted from Figure 2 of [EWL99].

Since this game is symmetric for both players, the Strategy that maximizes player B's payoff is also $\hat{D}$. This is just the classical case presented in the terms of quantum game theory.

Let us now consider case of entanglement $\gamma = \frac{\pi}{2}$. Figure 5 shows the potential payoff of player A in this case.

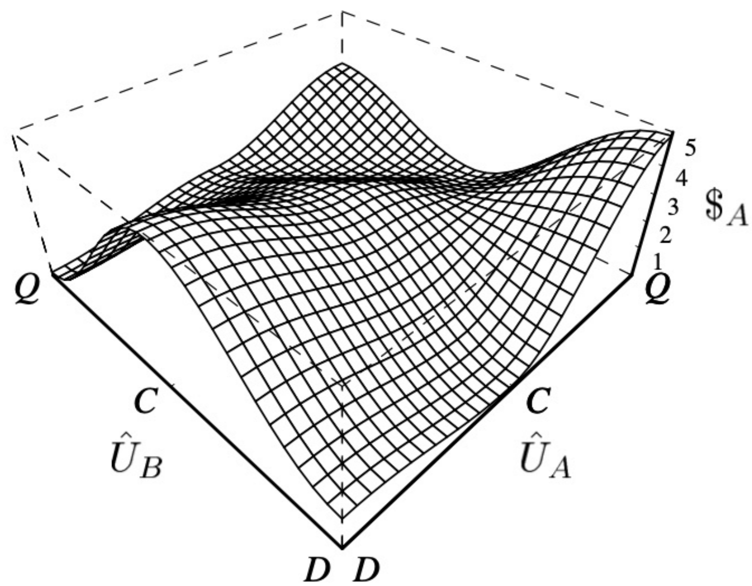

FIGURE 5. Player A's expected payoff under different strategies with entanglement $\gamma = 0$. Adapted from Figure 3 of [EWL99].

If player B chooses the Defect strategy $\hat{D}$, player A's best strategy is:

$$\hat{Q} \equiv \hat{U}(0, \frac{pi}{2}) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

If player B chooses the Cooperate strategy $\hat{C}$, player A's best strategy is the Defect strategy $\hat{D}$. As a result, there is no dominant strategy for player A. And since the game is symmetric for both players, player B also has no dominant strategy. Thus, mutual defection is no longer the Nash Equilibrium.

However, one can prove a much more powerful statement: the new Nash Equilibrium for both players is to choose $\hat{Q}$. This is because, if we assume that player B chooses $\hat{Q}$, the expected payoff of player A is

$$\$_A[\hat{U}(\theta, \phi), \hat{Q}] = \cos^2(\frac{\theta}{2})(3\sin^2(\phi) + \cos^2(\phi)) \leq 3$$

for all $\theta \in [0, \pi]$ and $\phi \in [0, \frac{\pi}{2}]$. And when we chose $\theta = 0$, $\phi = \frac{\pi}{2}$, meaning if we choose $\hat{Q}$, the expected payoff of player A becomes

$$\$_A[\hat{Q}, \hat{Q}] = \$_A[\hat{U}(0, \frac{\pi}{2}), \hat{Q}] = \cos^2(0)(3\sin^2(\frac{\pi}{2}) + \cos^2(\frac{\pi}{2})) = 3 \geq \$_A[\hat{U}(\theta, \phi), \hat{Q}]$$

for all $\theta \in [0, \pi]$ and $\phi \in [0, \frac{\pi}{2}]$.

Therefore, player A should always choose $\hat{Q}$. And since the game is symmetric for player B, they should also choose $\hat{Q}$. We have shown that a player's best strategy is to choose $\hat{Q}$ if the other player is choosing $\hat{Q}$. This means that the player will not have any more benefits from changing his strategy. Therefore, we have obtained a new Nash Equilibrium of both players choosing $\hat{Q}$. Under the new Nash Equilibrium, the expected payoff of both players has improved.

With the help of quantum game theory, the players can improve their payoffs in the Prisoner's Dilemma.

5. Application of Quantum Game Theory to Oligopoly Markets

Quantum game theory can provide many advantages in the field of Economics. For example, Quantum Game Theory can be applied by the government for policy making in order to improve the efficiency of the market. The following analysis is based on [DJL05].

Definition 5.1. *Oligopoly* is a market structure in which the market is dominated by a small number of firms. Each of the firms has significant market power to influence the market price.

Oligopoly has been a familiar problem within Economics. Due to imperfect competition, markets under this structure are oftentimes very inefficient. The firms usually produce an insufficient amount of goods, leading to increases in selling price and unnecessary strain on consumers. The situation becomes the most inefficient when there is only one firm left in the market, causing the market structure to become a *monopoly.* As a result, the governments strive to impose policies that prevent oligopolies from emerging.

In the following subsection, we are going to discuss how quantum game theory can help with this task.

5.1. Cournot's Duopoly Game.

The classical model for oligopoly is Cournot's duopoly game. In Cournot's duopoly game, there are two firms producing homogeneous products. Each of the firms simultaneously decides on the quantities q_i, $i \in \{0, 1\}$, of the goods they are going to produce. The total quantity is $Q = q_1 + q_2$. The market price for the unit of product is determined by a function of the total quantity

$$P(Q) = \{ \begin{array}{cc} a - Q & Q < a \\ 0 & Q \geq a \end{array} ,$$

where a is a constant.

For each firm, the production cost per unit is c. The profit of each firm is

$$u_i(q_1, q_2) = q_i[P(Q) - c] \; for \; i \in \{0, 1\}.$$

However, in reality, there is often a problem of asymmetric information in the market, which causes one of the firms to abuse another. Consider the following scenario. While Firm 2 knows that Firm 1 has the unit cost c, Firm 1 thinks that Firm 2 has the unit cost c'. Then Firm 1 thinks the opponent will decide on the quantity of production $q_2 = q_2^*$, where q_2^* is the quantity that maximizes Firm 2's profit

$$u_2(q_1^*, q_2) = q_2[a - q_1^* - q_2 - c']$$

subject to Firm 1 having the quantity of production equal to q_1^*. Firm 1 will then choose the quantity $q_1 = q_1^*$ to maximize the profit

$$u_1(q_1, q_2^*) = q_1[a - q_1 - q_2^* - c].$$

Let $k = a - c$, and let us denote by $\sigma = \frac{c-c'}{k}$ the informational incorrectness of Firm 1.

Solving these two optimization problems, we will find out that

$$q_1^* = \frac{1}{3}(1-\sigma)k \quad \text{and} \quad q_2^* = \frac{1}{6}(2+\sigma)k.$$

As a result, the product cost is

$$P(Q) - c = a - \frac{1}{3}(1-\sigma)k - \frac{1}{6}(2+\sigma)k - c = \frac{1}{6}(2+\sigma)k,$$

So the total profit for each firm under quantity q_1^* and q_2^* is:

$$u_1^* = q_1^*(P(Q)-c) = \frac{1}{3}(1-\sigma)k \times \frac{1}{6}(2+\sigma)k = \frac{1}{18}(2-\sigma-\sigma^2)k^2, \quad \text{and}$$

$$u_2^* = q_2^*(P(Q)-c) = \frac{1}{6}(2+\sigma)k \times \frac{1}{6}(2+\sigma)k = \frac{1}{36}(4+4\sigma+\sigma^2)k^2.$$

However, in reality, there are some restrictions on the model. Since q_1^* and q_2^* are actual quantities that the firms decide to produce, they have to be positive values. When q_1^* or q_2^* becomes negative or zero, a firm will choose to leave the market.

Since $q_2^* = \frac{1}{6}(2+\sigma)k$ is a monotone increasing function with respect to σ, increasing σ would not cause Firm 2 leaving the market. At the same time, since the quantity for Firm 1 is $q_1^* = \frac{1}{3}(1-\sigma)k$, when $\sigma \geq 1$, the quantity for Firm 1 will be non-positive, meaning that Firm 1 would leave the market.

Therefore, in order to maintain both firms in the market, we need to have $\sigma \in [0, 1)$. When $\sigma \geq 1$, Firm 1 will exit the market, leaving Firm 2 with the quantity $q_2^* = \frac{k}{2}$ and the profit $u_2^* = \frac{k}{2} \times \frac{1}{6}(2+1) = \frac{k^2}{4}$. Since the total quantity produced $Q = q_1^* + q_2^* = \frac{1}{6}(4-\sigma)$ is a decreasing monotone function respecting to σ, increase in σ will cause the decrease of the total quantity Q, putting he market in the most inefficient position possible with the minimum quantity produced and the highest price for the consumer.

5.2. **Policy using Quantum Game Theory.** Let us now apply quantum game theory. Let us again introduce entanglement γ. Instead of simply choosing the quantity q_i, each firm should now exploit a new strategy to determine the quantity through the use of strategy parameters x_i, where $x_i \in (-\infty, \infty)$ for $i \in \{0, 1\}$. Then, the quantity for Firm 1 can be written as

$$q_1(x_1, x_2) = x_1 \cosh(\gamma) + x_2 \sinh(\gamma).$$

Similarly, the quantity for firm 2 can be written as

$$q_2(x_1, x_2) = x_2 \cosh(\gamma) + x_1 \sinh(\gamma).$$

Therefore, Firm 1 would like to choose x_1^* that maximizes their profit under the assumption that Firm 2 chooses x_2^* that maximizes their profit too. In other words, Firm 1 chooses $x_1 = x_1^*$ that maximizes

$$u_1(x_1, x_2^*) = q_1(x_1, x_2^*)[P(x_1, x_2^*) - c]$$

while assuming Firm 2 to choose $x_2 = x_2^*$ that maximizes

$$u_2(x_1^*, x_2) = q_2(x_1^*, x_2)[P(x_1^*, x_2) - c'].$$

Solving an optimization problem, we will find out that:

$$x_1^* = \frac{(1-e^{2\gamma}\sigma)\cosh\gamma}{1+2e^{2\gamma}}k \text{ and}$$
$$x_2^* = \frac{(1+e^{2\gamma}+e^{4\gamma}\sigma)}{2e^{\gamma}(1+2e^{2\gamma})}k,$$

where $\sigma = \frac{c-c'}{k}$, where $k = a - c$, is the informational incorrectness of the firm. Therefore, we can find the quantity for Firm 1:

$$q_1^* = x_1^*\cosh(\gamma) + x_2^*\sinh(\gamma) = \frac{1+e^{2\gamma}}{4(1+2e^{2\gamma})}(2 - \frac{1+3e^{2\gamma}}{1+^{2\gamma}}\sigma)k,$$

and we can find the quantity for Firm 2:

$$q_2^* = x_2^*\cosh(\gamma) + x_1^*\sinh(\gamma) = \frac{1+e^{2\gamma}}{4(1+2e^{2\gamma})}(2+\sigma)k.$$

Then we can also find the profit of Firm 1:

$$u_1^* = q_1^*[k - q_1^* - q_2^*] = \frac{e^{2\gamma}(1+e^{2\gamma})}{8(1+2e^{2\gamma})^2}(4 - \frac{4e^{2\gamma}}{1+^{2\gamma}}\sigma - \frac{1+3e^{2\gamma}}{1+e^{2\gamma}}\sigma^2)k^2$$

and the profit for Firm 2

$$u_2^* = q_2^*[k - q_1^* - q_2^*] = \frac{e^{2\gamma}(1+e^{2\gamma})}{8(1+2e^{2\gamma})^2}(4+4\sigma+4\sigma^2)k^2.$$

Since q_2 is a monotone increasing function with respect to σ, no increase σ would lead to Firm 2 leaving the market. However, notice that when $\sigma \geq \frac{2(1+e^{2\gamma})}{1+3e^{2\gamma}}$, q_1^* becomes non-positive, meaning that Firm 1 would leave the market. If Firm 1 leaves the market, the market will become a monopoly, with Firm 2 producing the quantity $q_2^* = \frac{k}{2}$ and getting the profit $u_2^* = \frac{k^2}{4}$.

Let us introduce a function

$$\sigma'(\gamma) = \frac{2(1+e^{2\gamma})}{1+3e^{2\gamma}}.$$

Note that $\sigma'(\gamma)$ is bounded by

$$\frac{2}{3} \leq \sigma'(\gamma) \leq 2.$$

The government can adjust the entanglement γ to increase $\sigma'(\gamma)$ to its upper bound. As long as $\sigma < 2$, the government can adjust the entanglement to keep Firm 1 in the market, preventing monopoly. Notice that the total quantity is equal to

$$Q = q_1^* + q_2^* = \frac{1+e^{2\gamma}}{2(1+2e^{2\gamma})}(2 - \frac{e^{2\gamma}}{1+e^{2\gamma}}\sigma)k.$$

Q increases monotonically as the entanglement γ decreases. This means that the government can also control the total production by adjusting the entanglement. As long as $\sigma < \frac{2}{3}$, the government can even decrease γ to increase the total production and lower the price for the consumers.

Thus, by applying quantum game theory, the government can prevent the occurrence of monopoly. In addition, by reducing entanglement, the government can also increase the total production and decrease the price for the consumers, which improves the efficiency of the economy.

6. Further Research Directions

6.1. **Incomplete Information GHZ Game.** All the models we have considered in this survey only have two players. It would be very interesting to extend the results mentioned in the survey for the cases of three and more players.

Allen Deckelbaum [Dec11] proposed an incomplete information game called the *GHZ game*, with three players, which is similar to the CHSH game with three players. In the GHZ game, each of the players receives a challenge bit x, y, z such that $x + y + z \equiv 0(\mod 2)$, and they respond with answer bits a, b, and c. The players win if $a + b + c(\mod 2) \equiv x \vee y \vee z$.

Classical game theory offers no good strategy for this game. However, the players can always win if they play in the setting of quantum game theory and share an entangled state

$$|\psi_{GHZ}\rangle = \frac{1}{2}(|000\rangle - |011\rangle - |101\rangle - |110\rangle),$$

where the first bit belongs to player A, the second bit belongs to player B, and the third bit belongs to player C. Allen Deckelabum proposed a quantum strategy in which all players apply a Hadamard transformation to their qubit iff the challenge bit is 1 and then measure their qubit in the standard basis. Under this strategy, the players can always win the GHZ game.

6.2. **Quantum Game Theory on Random Networks.** Another very interesting potential direction to explore is applications of quantum game theory to evolving random networks, in which case even more players are involved.

Qiang Li has compared the classical and quantum game theory strategies for the Prisoner's Dilemma analogue with random networks [LICA12]. He proved that quantum strategy outperforms classical strategy under a static network. As the probability of structural update increases or the intensity of the punishment decreases, quantum strategy can also dominate in the cases involving network evolution.

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