THEORY OF LAPLACE TRANSFORMS AND THEIR APPLICATIONS

SAMANTHAK THIAGARAJAN

Abstract. The Laplace Transform is a critical tool used in the theory of differential equations with important applications to fields such as electrical engineering. Despite its many applications, the transform is mathematically rich, leading to several important theorems considering its behavior on different functions and its own structure. In this paper, we consider the above applications and develop the theory behind the Bromwich integral, an integration technique used for computing the inverse Laplace Transform. We assume an understanding of differential and integral calculus, along with some elementary complex analysis.

Contents

1. Introduction

The Laplace Transform is a mathematical object that is a critical tool in several fields. At its heart, the Laplace Transform is an integral transform, which itself has its set of unique properties, such as its actions on different functions. In addition, the transform, moving between the time and frequency domains and thus between real and complex numbers, has an inverse transform due to its uniqueness, which itself can be studied.

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We will begin by introducing the Laplace Transform in section two, in its two primary versions, along with some of its important properties, including its linearity and uniqueness, the latter of which is critical to the study of the inverse Laplace Transform. We will also do some example calculations of the Laplace Transform of common functions.

From here, we will discuss some important applications of the transform in section three, especially to solving problems that arise in electrical engineering. Within this field, this technique is critical for solving differential equations. In essence, the Laplace Transform transforms differential equations into algebraic equations, which are far easier to solve. We discuss another application, which is to evaluating integrals, a more mathematically-oriented application. In particular, Laplace Transform identities can be manipulated to evaluate certain integrals. We will close this section with the use of Laplace Transforms in the field of control theory and see how it can be used to simplify certain problems, in particular those involving linear time-invariant systems.

Section four will introduce the main focus of this paper: the inverse Laplace Transform. We will explore the relationship between the Fourier Transform and the Laplace Transform, and then investigate the inverse Fourier Transform and how it can be used to find the Inverse Laplace Transform, for both the unilateral and bilateral cases. We will conclude this section by directly applying the inverse Laplace Transform to a common function's Laplace Transform to recreate the original function.

2. Laplace Transforms

2.1. **Definition of the Laplace Transform.** The Laplace Transform has two primary versions: The Laplace Transform is defined by an improper integral, and the two versions, the unilateral and bilateral Laplace Transforms, differ in their bounds on the improper integral. Here, we introduce the two versions. We adopt the definitions in [2].

Definition 2.1. The *unilateral Laplace Transform* of a function $f : \mathbb{R}^+ \to \mathbb{R}$, denoted by $\mathfrak{L}_1\{f\}$, is defined as

$$\mathfrak{L}_1\{f\}(s) = \int_0^\infty f(t)e^{-st}\,dt$$

for any values of $s \in \mathbb{C}$ where the above integral converges. We note that $\mathfrak{L}_1\{f\}$ is complex valued.

Definition 2.2. The *bilateral Laplace Transform* of a function $f : \mathbb{R} \to \mathbb{R}$, denoted by $\mathfrak{L}_2\{f\}$, is defined as

$$\mathfrak{L}_2\{f\}(s) = \int_{-\infty}^\infty f(t)e^{-st}\,dt$$

for any values of $s \in \mathbb{C}$ where the above integral converges. We note that $\mathfrak{L}_2\{f\}$ is complex valued.

For this paper, when referencing the Laplace Transform, we will use Definition 2.1 as our definition. This is because the conditions for convergence of the bilateral Laplace Transform are far more restrictive than in the unilateral case. Also, we will

denote the Laplace Transform of a function f by $\mathfrak{L}\{f\}$, and will use the numbered subscripts if we need to use Definition 2.2.

2.2. **Properties of the Laplace Transform.** We will begin by considering some important properties of Laplace Transform.

Theorem 2.3 (Linearity)**.** *The Laplace Transform is linear. In other words, for two functions f and g,*

$$\mathfrak{L}\{mf + ng\} = m\mathfrak{L}\{f\} + n\mathfrak{L}\{g\}.$$

Proof. This can be shown using the linearity of integrals and the definition of the Laplace Transform. This also holds for both the unilateral and bilateral cases. □

Another important property of the Laplace Transform is its uniqueness.

Theorem 2.4 (Uniqueness)**.** *For two functions $f(t)$ and $g(t)$, if $F(s) = G(s)$ for all $\operatorname{Re} s \geq \operatorname{Re} s_0$ for some $s_0 \in \mathbb{C}$, then $f(t) - g(t)$ is a null function.*

Proof. See Theorem 5.1 in [2]. □

2.3. **Example Calculations with the Laplace Transform.** We will now consider some examples in which we calculate the Laplace Transformations of two common functions: e^{kt} and $\cos(kt)$.

Example 2.5. The Laplace transform of $f(t) = e^{kt}$ is given by

$$\mathfrak{L}\{f\}(s) = \frac{1}{s-k}.$$

Proof. We have that

$$\begin{aligned}\mathfrak{L}\{f\}(s) &= \int_0^\infty e^{kt}e^{-st}\,dt \\ &= \int_0^\infty e^{t(k-s)}\,dt \\ &= \frac{1}{k-s}\left[e^{t(k-s)}\right]_0^\infty = \frac{1}{s-k}.\end{aligned}$$

Note that, in this case, the Laplace Transform of f is defined for $s > k$. In the following examples, for convenience, we will just say "for an appropriate s", instead of giving exact values of s that are sufficient.

□

Example 2.6. The Laplace Transform of $f(t) = \cos(kt)$ is given by

$$\mathfrak{L}\{f\}(s) = \frac{s}{s^2+k^2}.$$

Proof. For an appropriate s,

$$\begin{aligned}\mathfrak{L}\{f\}(s) &= \int_0^\infty \cos(kt)e^{-st}\,dt \\ &= \left[\frac{-1}{s}\cos(kt)e^{-st}\right]_0^\infty - \frac{k}{s}\int_0^\infty \sin(kt)e^{-st}\,dt,\end{aligned}$$

where the above step follows from integration by parts.

Applying this technique again, we see that

$$\int_0^\infty \sin(kt)e^{-st}\,dt = \left[\frac{-1}{s}\sin(kt)e^{-st}\right]_0^\infty + \frac{k}{s}\int_0^\infty \cos(kt)e^{-st}\,dt.$$

Noting that the last integral above is the integral we want to find, we can combine the two equalities above to see that

$$\mathfrak{L}\{f\}(s) = \frac{1}{s} - \frac{k^2}{s^2}\mathfrak{L}\{f\}(s),$$

which gives the desired result.

□

The following proposition is useful for problems involving the derivative of a function. Our goal is to find the Laplace Transform of a function's derivative in terms of the Laplace Transform of the function.

Proposition 2.7. *The Laplace Transform of $f'(t)$ is given by*

$$\mathfrak{L}\{f'\}(s) = sF(s) - f(0),$$

where $F(s) = \mathfrak{L}\{f\}$.

Proof. For an appropriate s,

$$\mathfrak{L}\{f'\} = \int_0^\infty f'(t)e^{-st}\,dt.$$

Applying integration by parts,

$$\mathfrak{L}\{f'\} = \left[f(t)e^{-st}\right]_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt = sF(s) - f(0),$$

again for an appropriate s.

□

To end this section, we will include another proposition whose importance will become clear in the next section. This proposition is concerned with the convolution between two functions. As a reminder, for two functions f and g, the convolution of these two functions, denoted by $f * g$, is defined by

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau.$$

Proposition 2.8. *For two functions f and g,*

$$\mathfrak{L}\{f * g\}(s) = \mathfrak{L}\{f\}(s) \cdot \mathfrak{L}\{g\}(s),$$

provided that the Laplace Transforms of f and g are absolutely convergent for some $s_0 \in \mathbb{C}$. The above result holds for $Re(s) \geq Re(s_0)$.

Proof. See Theorem 10.1 in [2]. □

3. Applications of Laplace Transforms

3.1. **Circuit Analysis.** An important area of application for Laplace Transforms is circuit analysis. We shall see how applying the Laplace Transform to a differential equation results in a simpler problem. Our reference is [1].

To begin, consider a closed circuit with three elements: a battery, supplying a voltage $u(t)$, an inductor with inductance L, and a resistor with resistance R. Assume these elements are all present in the same loop. Using Kirchoff's Voltage Law, we have that the sum of the voltages of each component is 0. Applying Lenz's Law and Ohm's Law, the resulting voltage drops across the inductor and resistor are $Li'(t)$ and $Ri(t)$, respectively. Noting the correct signs on each term, we have the following differential equation:

$$u(t) - Li'(t) - Ri(t) = 0. \tag{3.1}$$

This is a tough differential equation to solve. However, the problem becomes more manageable by using the Laplace Transform. Applying the Laplace Transform to each side and using linearity and Proposition 2.7 gives

$$U(s) - L\left[sI(s) - i(0)\right] - RI(s) = 0,$$

where $U(s)$ and $I(s)$ are the Laplace transforms of $u(t)$ and $i(t)$, respectively. At this point, it is useful to revisit what we actually are trying to find. For example, we may be interested in the voltage across the resistor. In this case, let us denote the voltage across the resistor by $v_R(t) = Ri(t)$. So, $\mathfrak{L}\{v_R(t)\} = RI(s)$, which we will redefine as $V_R(s)$. Let's not substitute this directly into the differential equation above as there is an extra term with a factor of I. Instead, we can solve for I first. So,

$$I(s) = \frac{Li(0) + U(s)}{Ls + R}.$$

Now, making the substitution $V_R(s) = RI(s)$, we have that

$$V_R(s) = \frac{Li(0) + U(s)}{Ls/R + 1} = \frac{Li(0)}{Ls/R + 1} + \frac{U(s)}{Ls/R + 1}. \tag{3.2}$$

We can now apply the "Inverse Laplace Transform" in the sense that we can match the transforms of certain functions back to those functions. I use quotation marks because we shall rigorously define the Inverse Laplace Transform in the next section.

After referencing a standard table of Laplace Transforms, one finds that the first term of (3.2) has inverse Laplace Transform

$$\mathfrak{L}\{Ri(0)e^{-Rt/L}\} = \frac{Li(0)}{sL/R + 1}.$$

The second term is a bit more difficult. In this case, we are assuming that the voltage emitted by the battery is not constant. We therefore have that the second term is a product of Laplace Transforms. We can therefore use Proposition 2.8, which gives

$$\mathfrak{L}\left\{\frac{R}{L}e^{-Rt/L} * u(t)\right\} = \frac{R}{L}\mathfrak{L}\{e^{-Rt/L}\} \cdot \mathfrak{L}\{u(t)\} = \frac{U(s)}{Ls + R}.$$

Thus, we can write the solution to the differential equation in (3.1) as

$$v_R(t) = \frac{R}{L}\int_0^t e^{-R\tau/L}\, u(t-\tau)\, d\tau + Ri(0)e^{-Rt/L}.$$

What does this mean in the context of our physical system? Let's think of this solution in terms of its two components. The second term above has a factor of $Ri(0)$, which is the initial voltage supplied to the system. We can thus think of this term as only dependent on the initial voltage, which decays as time progresses. However, we see that the first term depends on the voltage emitted from the battery, which we can perceive as the contribution to the voltage across the resistor from accounting for the dynamically changing voltage of the battery.

This technique becomes especially interesting when considering circuits with several branches and components. After applying Kirchoff's Laws and Ohm's Law, we have multiple equations involving the resistance, current, and other electrical qualities of different circuit components. Some of these may be dependent only on a quality's actual value, such as the voltage across a resistor, and others may involve the derivative of a quality, as seen above with the voltage across the inductor. Therefore, we may have several differential equations. The Laplace Transform becomes an important tool as it allows us to transform every differential equation into an algebraic equation, while keeping algebraic equations as algebraic equations. This simplifies the problem of understanding the circuit.

3.2. **Improper Integrals.** Another interesting and unexpected application of Laplace Transforms is to evaluating integrals. Our reference is [7]. Consider the following integral, known as the Dirichlet Integral (not to be confused with the Dirichlet Integral in Section 4):

$$\int_0^\infty \frac{\sin x}{x}\, dx.$$

The value of this integral is $\pi/2$. To obtain this one can use Feynman's technique of differentiating under the integral sign, for instance. However, the Laplace Transform provides a completely new way to compute this integral. To do so, we will rely on the following property:

$$\mathfrak{L}\left\{\frac{f(t)}{t}\right\} = \int_0^\infty \frac{f(t)}{t} e^{-st}\, dt = \int_s^\infty \mathfrak{L}\{f\}(\tau)\, d\tau. \tag{3.3}$$

Using this, we can write that

$$\int_0^\infty \frac{\sin x}{x}\, dx = \mathfrak{L}\left\{\frac{f(t)}{t}\right\}\Big|_{s=0}.$$

by letting s to go to 0.

Now, in the same way that we derived the Laplace Transform of $\cos(kt)$ in Example 2.6, we can find that

$$\mathfrak{L}\{\sin(t)\} = \frac{1}{s^2+1}.$$

Applying this to (3.3) gives

$$\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^\infty \frac{1}{\tau^2+1}\,d\tau = \left[\tan^{-1}\tau\right]_0^\infty = \frac{\pi}{2}.$$

Because Laplace Transforms are defined with an improper integral (involving infinity in the limits of the integrals), they become a fantastic tool for computing the values of such integrals that may not be computable using traditional integration techniques.

3.3. **Control Theory.** The Laplace Transform is an important tool used in control theory due to its use in analyzing linear time-invariant (LTI) systems. We will begin by introducing LTI systems, and will reference [8]. An LTI system has an input and output signal along with a map between the two that is linear and time-invariant. This map is simply a way to get from the input signal to the output signal that satisfies these two conditions. This map is linear in the sense that it preserves addition and scalar multiplication on any signals that it can act on. The time-invariance of the map means that an offset in the input signal and the resulting offset in the output signal are the same.

The importance of Laplace Transforms in this field comes from the way LTI systems are described. Denote our input and output signals as $x(t)$ and $y(t)$, respectively. For a continuous time system, there is a function $h(t)$, known as the *impulse response* of the system, that satisfies

$$y(t) = x(t) * h(t),$$

where the above operation is a convolution. Laplace Transforms become a key tool here because, as discussed in Proposition 2.8, the Laplace Transform of a convolution is simply a product of Laplace Transforms. Indeed, for the above convolution,

$$Y(s) = X(s)H(s),$$

where Y, X and H are Laplace transforms of y, x and h, respectively. In this case, the function $H(s)$ is known as the transfer function. Note that the above relation is only valid given that the functions x, y, and h have defined Laplace Transforms. In essence, our system is simpler to analyze with this relation because we can use the transfer function to understand the characteristics of our LTI system, and therefore can study $H(s)$ as a function to understand our system.

4. The Inverse Laplace Transform

The Laplace Transform takes, as an input, a real-valued function and returns a complex-valued function. As mentioned in Theorem 2.4, the Laplace Transform is unique. As such, we can define the Inverse Laplace Transform as taking, as an input, a complex valued function F which is the Laplace Transform of another function f and return the function f.

To derive the formula for the Inverse Laplace Transform, we can use a related result: the Fourier Inversion Theorem.

4.1. **Fourier Inversion Theorem.** First, we will introduce the Fourier Transform. Our reference is [2].

Definition 4.1. The *Fourier Transform* of a function $f(x)$, denoted by $\mathfrak{F}\{f\}$, is given by

$$F(y) = \int_{-\infty}^{\infty} f(x)e^{-ixy}\,dx,$$

for any values of $y \in \mathbb{R}$ where the above integral converges.

As a side note, many textbooks and other resources define the Fourier Transform with an additional factor of 2π in the exponent of Definition 4.1. However, in this paper, we shall not use this convention because, without it, we may more easily translate results on the Fourier Transform to results on the Laplace Transform.

The following theorem gives us a way to reconstruct a function from its Fourier Transform. Before reaching it, however, we shall need the following definition and lemma.

Definition 4.2. A function f is *absolutely integrable* if

$$\int_{-\infty}^{\infty} |f(x)|\,dx$$

is a convergent integral.

Definition 4.3. A function f, over an interval (a, b), is of *bounded variation* if for any points $x_0, x_1, ..., x_n$ satisfying $a = x_0 < x_1 < ... < x_n = b$, we have, for some number M, that

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leq M.$$

Lemma 4.4. *Assume that over* $(0, M)$, f *is absolutely integrable. Then,*

$$\lim_{T\to\infty} \int_0^M e^{-iTt} f(t)\,dt = \lim_{T\to-\infty} \int_0^M e^{-iTt} f(t)\,dt = 0.$$

Proof. See Theorem 23.3 in [2]. □

An important note to make is that if the absolute integrability condition is met by a function f, then its Fourier Transform is defined for all $y \in \mathbb{R}$. The integral in Definition 4.1 will also converge absolutely for each such y. We are now ready to move to the Fourier Inversion theorem, below. Again, we use [2] as our reference.

Theorem 4.5. *Let f be absolutely integrable and of bounded variation in an open interval containing x. Also, let $F(y) = \mathfrak{F}\{f\}$. Then,*

$$\frac{f(x^-) + f(x^+)}{2} = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} F(y)e^{ixy}\,dy,$$

where $f(x^-)$ and $f(x^+)$ denote the left and right limits at x, respectively.

Proof. First, note that the integral above is well defined as the Fourier Transform is defined for all $y \in \mathbb{R}$. We will complete this proof by showing the difference between the integral formula above and $f(x)$ is less, in absolute value, than an arbitrary $\epsilon > 0$.

By definition,

$$\frac{1}{2\pi}\int_{-T}^{T} F(y)e^{ixy}\,dy = \frac{1}{2\pi}\int_{-T}^{T}\left[\int_{-\infty}^{\infty} f(k)e^{-iky}dk\right]e^{ixy}\,dy.$$

Using the fact that f is absolutely integrable, we can proceed by noting that

$$\begin{aligned}\frac{1}{2\pi}\int_{-T}^{T}\left[\int_{-\infty}^{\infty} f(k)e^{-iky}dk\right]e^{ixy}\,dy &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k)\left[\int_{-T}^{T} e^{iy(x-k)}\,dy\right]dk\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k)\left[\frac{e^{iT(x-k)}-e^{-iT(x-k)}}{i(x-k)}\right]dk\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(k)\,\frac{\sin T(x-k)}{x-k}\,dk.\end{aligned}$$

Now we are going to choose an M such that $M > |x| + 1$ and a δ such that $0 < \delta < 1$ to meet our requirement that the absolute value of the integral is less than ϵ. Let us consider the following 5 intervals:

$$(-\infty, -M), (-M, x-\delta), (x-\delta, x+\delta), (x+\delta, M), (M, \infty).$$

On the first and last intervals, $|k| \geq M > |x| + 1$. In addition, it always holds that $\sin T(x-k) \leq 1$. Therefore,

$$\left|\int_{-\infty}^{-M} f(k)\frac{\sin T(x-k)}{x-k}\,dk\right| \leq \int_{-\infty}^{-M}\left|f(k)\frac{\sin T(x-k)}{x-k}\right|dk \leq \int_{-\infty}^{-M}|f(k)|\,dk,$$

and similarly,

$$\left|\int_{M}^{\infty} f(k)\frac{\sin T(x-k)}{x-k}\,dk\right| \leq \int_{M}^{\infty}|f(k)|\,dk.$$

Because f is absolutely integrable, the integral of $|f|$ over the real line converges. Therefore, we can choose a sufficiently large M such that the tail ends of this integral, which are over the intervals $(-\infty, -M)$ and (M, ∞), are sufficiently small. Therefore, for an appropriately large M,

$$\frac{1}{\pi}\left|\int_{-\infty}^{-M} f(k)\frac{\sin T(x-k)}{x-k}\,dk + \int_{M}^{\infty} f(k)\frac{\sin T(x-k)}{x-k}\,dk\right| < \epsilon/3. \tag{4.6}$$

Now, before we take a look at the second and fourth intervals, let's first note the following about Lemma 4.4 :

$$\int_{0}^{M} e^{-iTt}f(t)\,dt = \int_{0}^{M}\cos(Tt)f(t)\,dt - i\int_{0}^{M}\sin(Tt)f(t)\,dt.$$

Therefore, because the first integral goes to 0 when T goes to infinity, so do the other two integrals. Now, we want to analyze the integral over the second interval, which is

$$\int_{-M}^{x-\delta} f(k)\frac{\sin T(x-k)}{x-k}\,dk.$$

In order to use Lemma 4.4, we need to make a substitution $u = x - k$ so that the integral becomes

$$\int_{\delta}^{x+M} \frac{f(x-u)}{u}\sin Tu\ du$$

The function $\frac{f(x-u)}{u}$ is absolutely integrable on the interval in question, and so, by Lemma 4.4,

$$\lim_{T\to\infty}\int_{0}^{x+M} \frac{f(x-u)}{u}\sin Tu\ du = 0$$

So

$$\lim_{T\to\infty}\int_{-M}^{x-\delta} \sin T(x-k)\frac{f(k)}{x-k}\,dk$$

and

$$\lim_{T\to\infty}\int_{x+\delta}^{M} \sin T(x-k)\frac{f(k)}{x-k}\,dk$$

can be made arbitrarily small for δ small enough. So, for a sufficiently large T,

$$\frac{1}{\pi}\left|\int_{-M}^{x-\delta} f(k)\frac{\sin T(x-k)}{x-k}\,dk + \int_{x+\delta}^{M} f(k)\frac{\sin T(x-k)}{x-k}\,dk\right| < \epsilon/3. \tag{4.7}$$

when δ is made small enough, which we will do later.

Finally, we want to compute the following integral:

$$\int_{x-\delta}^{x+\delta} f(k)\frac{\sin T(x-k)}{x-k}\,dk = -\int_{\delta}^{-\delta} f(x-u)\frac{\sin Tu}{u}\,du = \int_{-\delta}^{\delta} f(x-u)\frac{\sin Tu}{u}\,du.$$

This is known as the *Dirichlet Integral* in Fourier analysis.

Because f is of bounded variation in an open interval I containing x, we may choose a δ such that, $(x-\delta, x+\delta) \subset I$ and so f is of bounded variation on $(x-\delta, x+\delta)$. Having this, from a conclusion of Dirichlet, we have that

$$\lim_{T\to\infty}\int_{-\delta}^{\delta} f(x-u)\frac{\sin Tu}{u}\,du = \frac{f(x^+)+f(x^-)}{2} := L.$$

Now, for a sufficiently small δ and large T,

$$\left|\frac{1}{\pi}\int_{-\delta}^{\delta} f(x-u)\frac{\sin Tu}{u}\,du - L\right| < \epsilon/3, \tag{4.8}$$

Therefore, for all $\epsilon > 0$, there exists a T_0 such that for all $T > T_0$,

$$\begin{aligned}\left|\frac{1}{2\pi}\int_{-T}^{T} F(y)e^{ixy}\,dy - L\right| &= \left|\frac{1}{\pi}\int_{-\infty}^{\infty} f(k)\frac{\sin T(x-k)}{x-k}\,dk - L\right| \\ &\leq \frac{1}{\pi}\left|\int_{-\infty}^{-M} f(k)\frac{\sin T(x-k)}{x-k}\,dk + \int_{M}^{\infty} f(k)\frac{\sin T(x-k)}{x-k}\,dk\right| \\ &+ \frac{1}{\pi}\left|\int_{-M}^{x-\delta} f(k)\frac{\sin T(x-k)}{x-k}\,dk + \int_{x+\delta}^{M} f(k)\frac{\sin T(x-k)}{x-k}\,dk\right| \\ &+ \left|\frac{1}{\pi}\int_{-\delta}^{\delta} f(x-u)\frac{\sin Tu}{u}\,du - L\right| < \epsilon.\end{aligned}$$

As a side note, in the case that f is continuous at x, the value of the limit above collapses to $f(x)$. This completes the proof.

□

4.2. **Inverse Laplace Transform.** Now, noting the similarities between the Fourier and Laplace Transforms, we have the motivation to prove the following theorem. We will follow proofs given in [2]. Before we do so, as a reminder, note the following definitions:

$$\mathfrak{L}_1\{f\}(s) = \int_0^{\infty} f(t)e^{-st}\,dt, \quad \mathfrak{L}_2\{f\}(s) = \int_{-\infty}^{\infty} f(t)e^{-st}\,dt.$$

Theorem 4.9. *Let f be a function of bounded variation in an open interval containing t, and let $F = \mathfrak{L}_2\{f\}$. Assume that for some $x \in \mathbb{R}$, $F(s)$ is absolutely convergent. Then,*

$$\frac{f(t^-) + f(t^+)}{2} = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{x-iT}^{x+iT} F(s)e^{st}\,ds.$$

Proof. In the integral above, $s \in \mathbb{C}$ and, based on the bounds of integration, has real part x. Therefore, we can write $s = x + iy$. Thus,

$$\mathfrak{L}_2\{f\} = \int_{-\infty}^{\infty} e^{-st}f(t)\,dt = \int_{-\infty}^{\infty} e^{-iyt}e^{-xt}f(t)\,dt = \mathfrak{F}\{e^{-xt}f(t)\}.$$

We can write the above integral as $F(s) = F(x + iy)$. Using Theorem 4.5, we see that

$$e^{-xt}\frac{f(t^-) + f(t^+)}{2} = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} F(x+iy)e^{ity}\,dy.$$

Moving the e^{-xt} term and making the substitution $s = x + iy$ gives

$$\frac{f(t^-) + f(t^+)}{2} = \lim_{T\to\infty} \frac{1}{2\pi i}\int_{x-iT}^{x+iT} F(s)e^{st}\,ds.$$

This completes the proof.

□

Theorem 4.9 deals with the inverse of the bilateral Laplace Transform. However, we can move from this to the inverse of the unilateral Laplace Transform using the following theorem.

Theorem 4.10. *Let f be a function of bounded variation in an open interval containing t, and let $\mathfrak{L}_1 = F(s)$. Assume that for $s = x_0 \in \mathbb{R}$, $F(s)$ is absolutely convergent. Then, for any $x \geq x_0$, we have the following: if $t > 0$, then*

$$\frac{f(t^-) + f(t^+)}{2} = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{x-iT}^{x+iT} F(s)e^{st}\, ds.$$

If $t = 0$, then

$$\frac{f(0^+)}{2} = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{x-iT}^{x+iT} F(s)\, ds.$$

Finally, if $t < 0$, then

$$0 = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{x-iT}^{x+iT} F(s)\, ds.$$

Proof. First, we will define a function g satisfying the following:

$$g(t) = \begin{cases} f(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Note that $\mathfrak{L}_1\{f\} = \mathfrak{L}_2\{g\}$. Now, if f satisfies the conditions above, then g will as well. Using Lemma 4.9 and defining $G(s) = \mathfrak{L}_2\{g\}$ gives

$$\frac{g(t^-) + g(t^+)}{2} = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{x-iT}^{x+iT} G(s)e^{st}\, ds = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{x-iT}^{x+iT} F(s)e^{st}\, ds.$$

There are three cases we must consider: $t < 0$, $t = 0$, and $t > 0$. For $t > 0$, $g(t) = f(t)$, so

$$\frac{g(t^-) + g(t^+)}{2} = \frac{f(t^-) + f(t^+)}{2}.$$

For $t = 0$, from the definition of g, $g(0^-) = 0$. So,

$$\frac{g(0^-) + g(0^+)}{2} = \frac{f(0^+)}{2}.$$

Finally, for $t < 0$, $g(t) = 0$, so

$$\frac{g(t^-) + g(t^+)}{2} = 0.$$

This completes the proof.

□

4.3. **Application of the Inverse Laplace Transform.** We will end this section with an example calculation using the Inverse Laplace Transform. This theorem has been proven, but is still a difficult integral to compute in practice. Our goal is to take a function, compute its Laplace Transform, then apply our inversion theorem to the Laplace Transform and recreate the original function. Consider the following example. We complete this proof using supporting ideas from [4], [5], and [3].

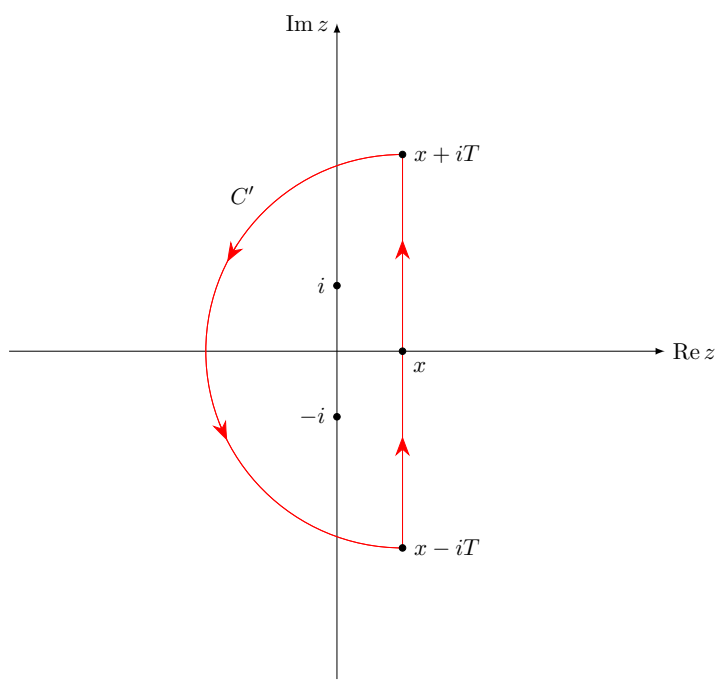
Example 4.11. Let $f(t) = \sin t$. In a similar fashion to the computation in Example 2.6, it can be shown that, in this case, $F(s) = \frac{1}{s^2+1}$. Now we wish to compute the following integral:

$$\lim_{T\to\infty} \frac{1}{2\pi i} \int_{x-iT}^{x+iT} F(s)e^{st}\, ds = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{x-iT}^{x+iT} \frac{e^{st}}{s^2+1}\, ds$$

for a suitable choice of x. In this case, note that the poles, each of order 1, of $F(s)$ are at $\pm i$. To compute the integral in question, we will apply the Residue Theorem and its associated techniques to the complex valued function

$$g(z) = \frac{e^{zt}}{z^2+1}$$

over the following curve C, which includes the vertical line from $x - iT$ to $x + iT$ and the semicircle from $x + iT$ to $x - iT$, the latter of which is denoted by C'. Note that this curve encloses the poles for sufficiently large T. In addition, we will take the limit as T tends to infinity, and so here have T as the radius of the semicircle and will analyze the behavior of our function g on this curve.



We now aim to compute the following integral:

$$\int_C \frac{e^{zt}}{z^2+1}\,dz.$$

The function g is holomorphic on $\mathbb{C}$ except for the points $\pm i$, each of which is a pole of order one. Therefore, we can compute the residue of g at each pole as follows:

$$\mathrm{Res}(g,i) = \lim_{z\to i}\left[(z-i)g(z)\right] = \lim_{z\to i}\frac{e^{zt}}{z+i} = \frac{e^{it}}{2i}.$$

Similarly,

$$\mathrm{Res}(g,-i) = \frac{e^{-it}}{-2i}.$$

So,

$$\int_C \frac{e^{zt}}{z^2+1}\,dz = 2\pi i\left[\mathrm{Res}(g,i)+\mathrm{Res}(g,-i)\right] = 2\pi i\left(\frac{e^{it}-e^{-it}}{2i}\right) = 2\pi i\sin t.$$

Now, considering the figure above, we see that

$$\int_C \frac{e^{zt}}{z^2+1}\,dz = \int_{x-iT}^{x+iT}\frac{e^{zt}}{z^2+1}\,dz + \int_{C'}\frac{e^{zt}}{z^2+1}\,dz.$$

We now seek to bound the integral over C'. First, note that we can parameterize the semicircle by $x+Te^{iy}$, with y ranging from $\pi/2$ to $3\pi/2$. We thus have, on the curve C', that

$$\left|\frac{e^{zt}}{z^2+1}\right| = \frac{e^{\mathrm{Re}(zt)}}{|z^2+1|} = \frac{e^{(x+R\cos y)t}}{|z^2+1|} \le \frac{Me^{xt}}{z^2}$$

for an appropriate constant $M\in\mathbb{R}$. We note that, in this case, our "starting" function is only defined for positive t, as written in Theorem 4.10. We therefore have assumed that $t\ge 0$. Finally, will bound the quantity $1/z^2$ for z lying on our curve C'. To do this, we need to find $\inf_{z\in C'} z$ to bound the quantity $1/z^2$ from above. To find this, consider the following reasoning: consider a circle in $\mathbb{R}^2$ shifted slightly to the right of the origin. Assume the formula for this circle is $(x-\epsilon)^2+y^2=T^2$, where T is the radius above. In this case, we have that

$$x^2+y^2 = T^2-\epsilon^2+2x\epsilon.$$

Therefore, the minimal distance from the origin to the circle occurs when $x=0$. Moving back to our curve C', this point is where C' intersects the imaginary axis. This distance is given by $\sqrt{T^2-x^2}$. For x sufficiently small, this is certainly greater than $T/2$. Therefore,

$$\left|\int_{C'}\frac{e^{zt}}{z^2+1}\,dz\right| \le \pi T\frac{4Me^{xt}}{T^2} = \frac{\pi 4Me^{xt}}{T}.$$

We thus have that, as T tends to infinity, the contribution to the integral over C from the integral over C' tends to 0. We thus have that all is left is the integral in question. Therefore,

$$\lim_{T\to\infty}\frac{1}{2\pi i}\int_{x-iT}^{x+iT}\frac{e^{st}}{s^2+1}\,ds = \frac{1}{2\pi i}\int_C\frac{e^{zt}}{z^2+1}\,dz = \sin t,$$

which was the given function.

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