TANGENT SPACE PROPERTIES OF MANIFOLDS AND RECTIFIABLE SETS

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ABSTRACT. Rectifiable sets are a measure-theoretic generalization of manifolds studied in geometric measure theory. In this paper, we will define rectifiable sets and study the properties of their tangent spaces by comparing them to manifolds. We will make rigorous the intuition that a manifold is "locally Euclidean" and that a rectifiable set is a manifold "almost everywhere".

CONTENTS

1. INTRODUCTION

To motivate the idea of rectifiable sets, I will briefly present the historical context of its creation.

One long-standing problem in calculus of variations is Plateau's Problem, where we aim to show the existence of an area minimizing surface with a given closed curve as its boundary. Notice that since we can always find an infinite sequence of surfaces whose area approaches the infimum, the difficulty is to prove that such a sequence converges and that the limit is a somewhat reasonable set. As a result, one main approach in solving Plateau's Problem can be described in the following three steps: first define an appropriate space of surfaces in which we work, then prove some compactness theorem in this space, and lastly prove regularity properties of our limit [2].

The specifics of solving Plateau's Problem depends on the class of surfaces and curves we consider, our definition of "boundary", and the dimensions of our ambient space, surface, and curve. However, one natural way to formalize our problem is to define our mathematical objects to mimic a realization of this problem in nature——dipping a wire-frame into a soap solution and obtaining a soap film that spans the wire-frame. From this perspective, one goal we'd have for the class of surfaces we consider is for it to imitate the set of possible soap films in nature.

Date: August 29, 2022.

One instinctive definition of our class of surfaces might be differential manifolds, since both manifolds and soap films look smooth. However, we'll quickly discover that working with manifolds is too restrictive, since there exist soap films with self-intersections and other singularities, which are not allowed for manifolds. Yet, even though soap films might have singularities, they are all contained on quite a small set, so we should still require some rigidity in our class of surfaces. From here stems the idea of rectifiable sets—–sets that are manifolds "almost everywhere".

There are many solutions to various versions of Plateau's Problem, and the one to which rectifiable sets are most intimately connected is Federer and Fleming's solution using integral currents. This solution can be broadly summarized as follows:

Given a rectifiable set, we define a corresponding function on the space of differential forms by integrating any given differential form over said rectifiable set. Following the approach described above, we first generalize this set of rectifiable set functions into a larger class of linear functions on the space of differential forms, which are termed "integral currents", then prove a compactness theorem on the space of integral currents, and finally show some regularity results that tell us an area-minimizing sequence of rectifiable sets will converge to some reasonable rectifiable set, which will be the minimal surface that we're looking for [4].

In other words, rectifiable set is a foundational concept in geometric measure theory crucial to solving the famous Plateau's Problem. We therefore hope to provide an intuitive yet rigorous introduction to this concept in the rest of this paper.

In the following sections, we will define Hausdorff measure, density, tangent cone, manifolds, and rectifiable sets. We will first prove a lemma about how the tangent cone of a set transforms under the derivative of a function. We will then prove that a manifold has tangent planes everywhere and a rectifiable set has tangent planes almost everywhere.

In the writing of this paper, I assumed that the reader has knowledge of undergraduate analysis and foundational general measure theory. In the rectifiable set section, we will also be assuming without proof a few theorems about Lipschitz functions and refer the readers to Federer's *Geometric Measure Theory* [1] for a more rigorous treatment of them.

2. Hausdorff measure and density

We begin by defining the Hausdorff measure. To find the area of an n-dimensional subset of $\mathbb{R}^n$, we use the Lebesgue measure. However, if our subset is of a lower dimension (i.e. a surface in 3D space or a line in 2D space), then the Lebesgue measure will always measure the set to be measure zero. To measure volume of lower dimensional subsets of $\mathbb{R}^n$, we define the m-dimensional Hausdorff measure (for $m \leq n$). Our goal is for this measure to agree with the classical area formula for submanifolds of $\mathbb{R}^n$ but be defined on a larger class of sets and coincide with the Lebesgue measure when $m = n$.

Definition 2.1. For any $S \subset \mathbb{R}^n$, let $\operatorname{diam}(S) = \sup\{|x-y| : x, y \in S\}$. Additionally, let $\alpha(m)$ be the m-dimensional volume of the unit m-ball. For $A \subset \mathbb{R}^n$, we define the *m-dimensional Hausdorff measure* to be:

$$\mathcal{H}^m(A) = \lim_{\delta \to 0} \inf_{\substack{A \subset \cup S_j \\ \mathrm{diam} S_j \leq \delta}} \sum_{j=1}^{\infty} \alpha(m) \left(\frac{\mathrm{diam}(S_j)}{2} \right)^m .$$

Intuitively, we obtain the Hausdorff measure of a set by covering it as efficiently as possible with sets of diameter less than δ, approximate the volume of each set with the volume of an m-ball of the same diameter, and then take the limit as δ approaches zero.

Since we are taking the infimum over all covers with sets of diameter less than δ, as we decrease δ, the infimum is non-increasing, which means that this limit always exists and $0 \leq \mathcal{H}^m(A) \leq \infty$.

Remark. Note that the limit above is well-defined for all subsets of $\mathbb{R}^n$. The Hausdorff measure is in fact defined using the Carathéodory extension method. It can be verified that the limit we've defined above is an outer measure, and for it to be a proper measure, we restrict ourselves to a sigma algebra of measurable sets. This includes all Borel sets, as will be proven later.

Remark. You might see the Hausdorff measure defined elsewhere where, instead of allowing the cover to be composed of arbitrary sets, we require the sets to all be n-balls. These two definitions are equivalent, though allowing covers using arbitrary sets have the advantage of simplifying certain proofs. For instance, in our definition, if $\{S_i\}$ is a cover of A, then $\{f(S_j)\}$ is a proper cover of $f(A)$. In contrast, if we require the covering sets to be n-balls, then $\{f(S_j)\}$ might not be a proper cover of $f(A)$, since we cannot guarantee that f maps balls to balls.

We will now present two nice properties of the Hausdorff measure.

Theorem 2.2. *$\mathcal{H}^m$ is Borel regular.*

Proof. We will first show that all Borel sets are measurable as they satisfy the Carathéodory criterion, then show that every set is contained in a Borel set of the same measure.

To prove our first claim, we will first prove a sufficient condition for all Borel sets passing the Carathéodory criterion, then show that the Hausdorff measure does fulfill this condition. Our condition proceeds as follows: Let μ be our outer measure defined on $\mathbb{R}^n$. We require

$$\mu(X \cup Y) = \mu(X) + \mu(Y) \tag{2.3}$$

for all $X, Y \subset \mathbb{R}^n$ such that

$$\mathrm{dist}(X, Y) := \inf\{|x - y| : x \in X, y \in Y\} > 0.$$

Since the collection of measurable sets is always a σ-algebra, and the Borel sets are the smallest σ-algebra containing the closed sets, it suffices to show that fulfilling (2.3) implies that $\mu(A) \geq \mu(A \cap C) + \mu(A \setminus C)$ for all closed sets C and arbitrary sets A in $\mathbb{R}^n$.

If $\mu(A) = \infty$, our inequality holds, so we may assume $\mu(A) < \infty$. Let B_n be

$$\{x \in \mathbb{R}^n : \inf(x, c) > 1/n \text{ for all } c \in C\}.$$

Then, since C is closed, we know that $C^c = \cup_{i=1}^{\infty} B_i$. By our condition, $\mu(A \cap C) + \mu(A \cap B_n) = \mu(A \cap (C \cup B_n)) \leq \mu(A)$, so we only need to show that $\mu(A \cap B_n)$ approaches $\mu(A \cap C^c)$ as n approaches infinity. Let $E_i = (B_{i+1} \setminus B_i) \cap A$. Notice

$$A \cap C^c = B_n \cup (\cup_{i=n+1}^{\infty} E_i),$$

hence

$$\mu(A \cap B_n) \leq \mu(A \cap C^c) \leq \mu(A \cap B_n) + \sum_{i=n+1}^{\infty} \mu(E_i).$$

It therefore suffices to show that $\sum_{i=n+1}^{\infty} \mu(E_i)$ approaches zero as n approaches infinity, or equivalently, $\sum_{i=1}^{\infty} \mu(E_i) < \infty$. This sum can be separated into

$$\sum_{i=1}^{\infty} \mu(E_{2i}) + \sum_{i=1}^{\infty} \mu(E_{2i-1}).$$

Notice $\operatorname{dist}(E_i, E_j) > 0$ if $|i - j| \geq 2$, hence by (2.3), the sum can be rewritten as $\mu(\cup_{i=1}^{\infty} E_{2i}) + \mu(\cup_{i=1}^{\infty} E_{2i-1}) \leq 2\mu(A) \leq \infty$, and we are done.

We will now show that the Hausdorff measure does fulfill (2.3). Observe that if we make δ sufficiently small, then for all sets X and Y positive distance apart, any efficient cover of $X \cup Y$ can be separated into two disjoint covers of X and Y respectively. This implies that the measure of our cover of $X \cup Y$ equals the sum of the measures of the two individual covers of X and Y, which is greater than or equal to $\mathcal{H}^m(X) + \mathcal{H}^m(Y)$. Taking a sequence of such covers of $X \cup Y$ whose measure approaches $\mathcal{H}^m(X \cup Y)$ then gives us our first claim.

To complete our proof that $\mathcal{H}^m$ is Borel regular, we will now show that all $A \subset \mathbb{R}^n$ are contained in a Borel set of the same measure.

Let us take a sequence of covers of A whose measure approaches $\mathcal{H}^m(A)$. Observe that replacing the covering sets with their closures does not change the diameter of the set hence volume of the cover. Hence, for each cover in the sequence, replace each set with its closure, take the union of all the covering sets, and we have obtained a sequence of Borel sets containing A whose measure approaches $\mathcal{H}^m(A)$. We now take the infinite intersection of all sets in this sequence and obtain a Borel set whose measure is $\mathcal{H}^m(A)$.

□

Theorem 2.4. *$\mathcal{H}^m = \mathcal{L}^m$ on $\mathbb{R}^m$.*

The proof of this theorem requires the isodiametric inequality and the Besicovitch covering theorem and is therefore unfortunately outside of the scope of this paper. We refer the reader to Federer section 2.10 for a rigorous treatment or Morgan 2.8 for a sketch of the proof [1] [2].

To finish off this section, we will now define density.

Let $A \subset \mathbb{R}^n$, $a \in \mathbb{R}^n$, and $m \leq n$. We define the m-dimensional upper density and lower density of A at a as follows:

$$\Theta^{*m}(A, a) = \limsup_{r \to 0} \frac{\mathcal{H}^m(A \cap B(a, r))}{\alpha(m) r^m},$$

$$\Theta^m_*(A, a) = \liminf_{r \to 0} \frac{\mathcal{H}^m(A \cap B(a, r))}{\alpha(m) r^m}.$$

When $\Theta^{*m}(A, a) = \Theta^m_*(A, a)$, we denote that as $\Theta^m(A, a)$ and call it the m-dimensional density of A at a.

Remark. Our main reference source Federer defines density more generally for measures, and density of sets is defined as the density of measure where we take our measure to be the Hausdorff measure restricted to the given set. Federer's definition is more versatile; however, only density of sets is relevant to our discussion, so we've simplified the definition here in hopes of aiding the reader's understanding. To see the definition of density in its full generality, refer to Federer 2.9.10 and 2.10.19 [1].

3. The tangent cone

Next, we define an idea representing the local behavior of a set around a point.

Definition 3.1. Let $A \subset \mathbb{R}^n$ and $a \in \mathbb{R}^n$. The *tangent cone* of A at a, denoted $\text{Tan}(A, a)$, is the collection of $v \in \mathbb{R}^n$ for which for all $\epsilon > 0$, there exists $x \in A$ and $0 < r \in R$ such that $|x - a| < \epsilon$ and $|r(x - a) - v| < \epsilon$.

Reading the definition intuitively, the tangent cone contains all the vectors for which given an arbitrarily small neighborhood around a, we can take a step from a to a point in the neighborhood in A and walk roughly in the direction of that vector. As illustrated by the three properties below, we can equivalently understand $\text{Tan}(A, a)$ to be the closure of the set of directions you can walk in starting from a and still stay on the set A. Here, we observe these properties of $\text{Tan}(A, a)$:

(1) If $a \notin \text{Clos } A$, then $\text{Tan}(A, a) = \emptyset$.
(2) $\text{Tan}(A, a)$ is a closed subset of $\mathbb{R}^n$.
(3) If $v \in \text{Tan}(A, a)$ and $0 < k \in R$, then $kv \in \text{Tan}(A, a)$.

Property (1) makes sense intuitively because if $a \notin \text{Clos } A$, then if we take a sufficiently small neighborhood around a, there will be no directions in which you can walk starting from a and still stay on A. Notice that (2) is true because we've required the existence of r and x for arbitrary $\epsilon > 0$ such that $|r(x - a) - v| < \epsilon$ instead of r and x such that $|r(x - a) - v|$ equals zero. If we, instead, defined $\text{Tan}(A, a)$ to be

$$\{v \in \mathbb{R}^n : \forall \epsilon > 0, \exists x \in A \cap B(a, \epsilon),\, 0 < r \in \mathbb{R} \text{ such that } r(x - a) = v\},$$

then $\text{Tan}(A, a)$ will still be the set of directions in which you can walk starting from a and still stay on A but just not the closure of it. (3) makes sense because $\text{Tan}(A, a)$ represents a set of directions, which is closed under scaling.

In addition to the tangent cone, we will also define the approximate tangent cone, which is a more stringent version of the tangent cone that ignores sets of density zero. Note that certain sets' tangent cones will not be a k-dimensional vector subspace but their approximate tangent cones will be, which makes this concept useful in dealing with less regular sets.

Remark. Federer also defines the approximate tangent cone for measures instead of sets, which is more general. However, the only measures for which we'll take the approximate tangent cone in this paper is the Hausdorff measure restricted to a subset of $\mathbb{R}^n$, so we will also only be defining it for sets to aid the readers' understanding. To see the definition of the approximate tangent cone in its full generality, refer to Federer 3.2.16 [1].

Definition 3.2. Let $W \subset \mathbb{R}^n$ and $a \in \mathbb{R}^n$. We define the *m-dimensional approximate tangent cone* of W at a as follows:

$$\text{Tan}^m(W, a) = \cap\{\text{Tan}(S, a) : S \subset \mathbb{R}^n, \Theta^m(W \cap S^c, a) = 0\}.$$

Notice that W itself always fulfills the two criteria that we require of S above, hence $\text{Tan}^m(W, a) \subset \text{Tan}(W, a)$, and the motivation for defining $\text{Tan}^m(W, a)$ is that it deletes sets of lower dimension out of the tangent cone.

Let $E(a, v, \epsilon) = \{x : |r(x - a) - v| < \epsilon \text{ for some } r > 0\}$, we can then prove an equivalent condition for a vector belonging to the approximate tangent cone of a set at a point.

Theorem 3.3. *Let $W \subset \mathbb{R}^n$ and $a \in \mathbb{R}^n$. Then $v \in Tan^m(W, a)$ if and only if $\Theta^{*m}[W \cap E(a, v, \epsilon), a] > 0$ for all $\epsilon > 0$.*

Proof. Given $\Theta^{*m}[W \cap E(a, v, \epsilon), a] > 0$ for all $\epsilon > 0$, we will show that given arbitrary S such that $\Theta^m(W \cap S^c, a) = 0$ and $\epsilon > 0$,

$$E(a, v, \epsilon) \cap B(a, \epsilon) \cap S \neq \emptyset,$$

which would then imply that $v \in \mathrm{Tan}^m(W, a)$. Suppose for contradiction that $E(a, v, \epsilon) \cap B(a, \epsilon) \cap S = \emptyset$, this means that within $B(a, \epsilon)$, $E(a, v, \epsilon) \subset S^c$. However, since $\Theta^m(W \cap S^c, a) = 0$, this would imply that $\Theta^{*m}[W \cap E(a, v, \epsilon), a]$ also equals zero, a contradiction.

Given $v \in \mathrm{Tan}^m(W, a)$, we will now also show that $\Theta^{*m}[W \cap E(a, v, \epsilon), a] > 0$ for all $\epsilon > 0$. Suppose otherwise, then for the ϵ for which the density of $E(a, v, \epsilon)$ at a is zero, take $S = E(a, v, \epsilon)^c$. $\Theta^m(W \cap S^c, a) = 0$, yet for all $\delta < \epsilon$

$$E(a, v, \delta) \cap B(a, \delta) \cap S \subset E(a, v, \epsilon) \cap B(a, \delta) \cap S = \emptyset,$$

which implies $v \notin \mathrm{Tan}(S, a)$ and hence $v \notin \mathrm{Tan}^m(W, a)$, a contradiction.

□

4. The manifold

We will now introduce the idea of a differential C^k manifold. We will then prove a lemma about how the tangent cone of a set at a point transforms under the derivative of a function. Finally, we'll prove that everywhere on a C^k m-dimensional manifold, the tangent cone is an m-dimensional vector subspace of $\mathbb{R}^n$.

Before we present the rigorous definition of a manifold, let's do a simple exercise to gain some intuition of this concept. I'll present six subsets of $\mathbb{R}^2$ or $\mathbb{R}^3$ below and tell you which ones are manifolds. From the examples, try to guess what a manifold is.

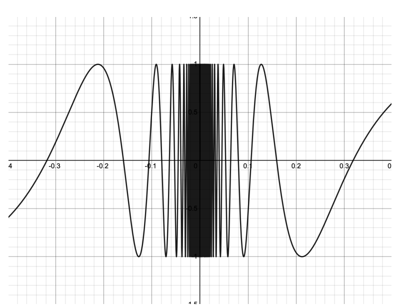

Figure 1. is not a manifold

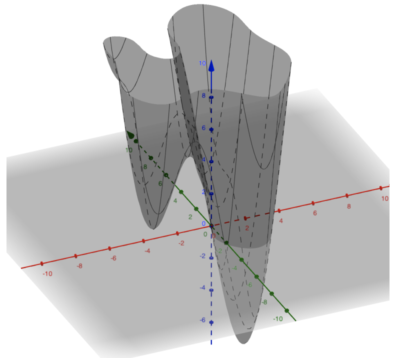

FIGURE 2. is a manifold

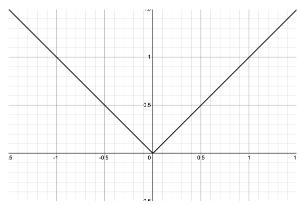

FIGURE 3. is not a manifold

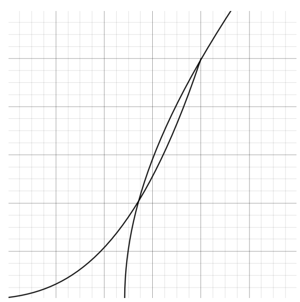

FIGURE 4. is not a manifold

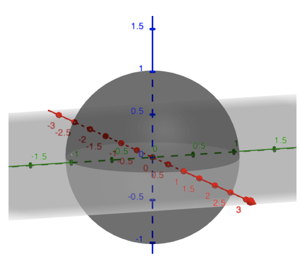

FIGURE 5. is a manifold

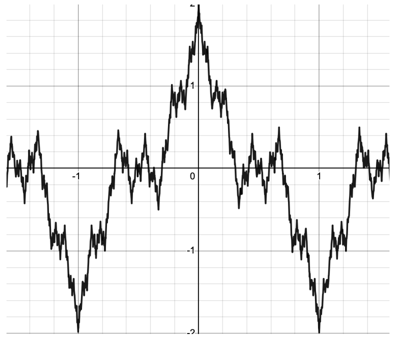

FIGURE 6. is not a manifold

From the examples above, you might have been able to gather that an m-dimensional manifold is a subset of $\mathbb{R}^n$ that, when you zoom in, is smooth and doesn't have any "edges" or "corners". This intuition is indeed what we'll formalize in our subsequent definition of a manifold and Theorem 4.3.

Definition 4.1. Given $m \le n$, $M \subset \mathbb{R}^n$ is a *C^k, m-dimensional manifold* if for all $x \in M$, there exists an open set $U \subset \mathbb{R}^m$ and a C^k injective function $\phi : U \to \mathbb{R}^n$ such that $D\phi(y)$ is injective for all $y \in U$, ϕ^{-1} is continuous, and $\phi(U)$ equals $V \cap M$ for some open set V around x.

Remark. There exist many equivalent definitions of a differential manifold. I presented this one as I find it to be the most intuitive and relevant to this paper. To see other definitions of a manifold, refer to Federer 3.1.19 [1].

For notational simplicity, we will assume that our manifold is C^k, m-dimensional, and embedded in $\mathbb{R}^n$ from now on unless specified otherwise.

Intuitively, this definition says that in order for M to qualify as a manifold, we require around every point the existence of a set open relative to M that can be described as the image of a "nice enough" function ϕ mapping from an open set in $\mathbb{R}^m$, with "nice enough" rigorously defined as C^k, injective, having injective derivative, and whose inverse is continuous. One key characteristic of well-behaved functions is that they cannot distort their domains too much, so we can roughly interpret Definition 4.1 to mean that a manifold around each point is a slightly distorted open set of $\mathbb{R}^m$.

We will later further formalize this explanation by proving in Theorem 4.3 that the tangent cone of a manifold around every point is an m-dimensional vector subspace of $\mathbb{R}^n$. Recall that the tangent cone of a set around a point is the closure of the directions in which one can walk and still stay on the set. Hence, Theorem 4.3 then tells us that on every point of the manifold, the manifold locally looks like $\mathbb{R}^m$.

Lemma 4.2. *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ and $A \subset \mathbb{R}^m$. If f is differentiable at a, then $Df(a)[Tan(A,a)] \subset Tan(f(A), f(a))$. Furthermore, if $f|A$ is injective, $Df(a)$ is injective, and $(f|A)^{-1}$ is continuous at $f(a)$, then equality holds.*

Proof. We will first prove that $Df(a)[\mathrm{Tan}(A,a)] \subset \mathrm{Tan}(f(A), f(a))$. Suppose $v \in \mathrm{Tan}(A,a)$, then there exists a sequence of points $\{x_i\} \to a$ and a sequence of real numbers $\{r_i\}$ such that $|r_i(x_i - a) - v| \to 0$ as $i \to \infty$. We will now prove that

$|r_i(f(x_i) - f(a)) - Df(a)(v)| \to 0$ as $i \to \infty$.

$$\begin{aligned}
&|r_i(f(x_i) - f(a)) - Df(a)(v)| \\
&\qquad = |r_i(f(x_i) - f(a)) - Df(a)(v - r_i(x_i - a)) - Df(a)(r_i(x_i - a))| \\
&\qquad = ||r_i(x_i - a)| \frac{f(x_i) - f(a) - Df(a)(x_i - a)}{|x_i - a|} - Df(a)(v - r_i(x_i - a))|.
\end{aligned}$$

Notice that $r_i(x_i - a)$ approaches v as $i \to \infty$. Therefore, by the definition of the derivative, the first term approaches zero. And since $v \in \mathrm{Tan}(A, a)$, the second term approoahces zero too, and we have our desired claim.

We will now prove that if $f|A$ is injective, $Df(a)$ is injective, and $(f|A)^{-1}$ is continuous at $f(a)$, then equality also holds. Suppose $v \in \mathrm{Tan}(A, a)$; then, we can find a sequence of points $\{y_i\} \to f(a)$ and a sequence of real numbers r_i such that $|r_i(y_i - f(a)) - w| \to 0$. Since $f|A$ is injective and $(f|A)^{-1}$ is continuous, we know that $\{y_i\}$ corresponds to a well defined sequence of points $\{x_i\}$ in A approaching a where $f(x_i) = y_i$.

By the definition of the derivative, we know that $r_i Df(a)(x_i - a) \to r_i(f(x_i) - f(a))$ as $i \to \infty$, but we also know that $r_i(f(x_i) - f(a)) \to w$ as $i \to \infty$. Hence, $Df(a)(r_i(x_i - a)) \to w$. Because $Df(a)$ is injective, it has a inverse when restricted to the range, which we'll call $Df(a)^{-1}$. Then we know $r_i(x_i - a) \to Df(a)^{-1}(w)$, which means $Df(a)^{-1}(w) \in \mathrm{Tan}(A, a)$ and $w \in Df(a)[\mathrm{Tan}(A, a)]$.

$\square$

A direct application of this lemma gives us our desired theorem.

Theorem 4.3. *Let $M \subset \mathbb{R}^n$ be a C^k m-dimensional manifold. For all $x \in M$, Tan(M, x) is an m-dimensional vector subspace of $\mathbb{R}^n$.*

Proof. By Definition 4.1, for all $x \in M$, we can find a C^k function ϕ that maps an open set $U \in \mathbb{R}^m$ onto an open set on the manifold around x. Notice that since this function fulfills the requirement for equality given in Lemma 4.2 and $\phi(U)$ is open in M, we get $\mathrm{Tan}(M, x) = \mathrm{Tan}(\phi(U), x) = D\phi(\phi^{-1}(x))[\mathrm{Tan}(U, \phi^{-1}(x))]$. Because U is an open set in $\mathbb{R}^m$ around $\phi^{-1}(x)$, $\mathrm{Tan}(U, \phi^{-1}(x))$ is just $\mathbb{R}^m$, hence $D\phi(\phi^{-1}(x))[\mathrm{Tan}(U, \phi^{-1}(x))]$ equals $\mathrm{Im}[D\phi(\phi^{-1}(x))]$, an m-dimensional vector subspace of $\mathbb{R}^n$.

$\square$

Remark. A great real-life example of a manifold is the surface of the Earth: In actuality, it is not $\mathbb{R}^2$ but, instead, a sphere. However, since a sphere is a 2-dimensional manifold and we're so small compared to the Earth that our perspective is basically local, the Earth looks like $\mathbb{R}^2$ to us, and we find flat-earthers in this world.

5. The rectifiable set

We finally define rectifiable sets and prove that except for a set of Hausdorff measure zero, the tangent cone of a rectifiable set around its points is a vector subspace of $\mathbb{R}^n$. This gives us the intuition that a rectifiable set is a manifold almost everywhere.

Definition 5.1. $W \subset \mathbb{R}^n$ is $(\mathcal{H}^m, m)$ rectifiable if W is $\mathcal{H}^m$ measurable, $\mathcal{H}^m(W) < \infty$, and $\mathcal{H}^m$ almost all of W can be written as a countable union of images of bounded subsets of $\mathbb{R}^m$ under Lipschitz functions $\phi_i : \mathbb{R}^m \to \mathbb{R}^n$.

Remark. Notice that since the only requirement we place on the domains of the Lipschitz functions is that they are bounded, all $(\mathcal{H}^m, m)$ rectifiable sets are also $(\mathcal{H}^n, n)$ rectifiable if $n > m$.

To prove that rectifiable sets have tangent planes almost everywhere, we will first need a few theorems about Lipschitz functions, which we'll present below.

Theorem 5.2. *Let $\phi : \mathbb{R}^m \to \mathbb{R}^n$ have Lipschitz constant λ, then for any set A, $\mathcal{H}^m(\phi(A)) \leq \lambda^m \mathcal{H}^m(\phi(A))$.*

Proof. Given any cover $\cup_{i=1}^{\infty} U_i$ of A, notice that $\cup_{i=1}^{\infty} \phi(U_i)$ is a cover of $\phi(A)$ whose diameter is be scaled up by at most λ. Hence,

$$\sum_{i=1}^{\infty} \alpha(m) \left(\frac{\operatorname{diam}(\phi(U_i))}{2} \right)^m \leq \sum_{i=1}^{\infty} \alpha(m) \lambda^m \left(\frac{\operatorname{diam}(U_i)}{2} \right)^m.$$

Taking a sequence of covers of A whose measure approaches $\mathcal{H}^m(A)$ then gives us our theorem.

□

Remark. One immediate consequence of this theorem is that Lipschitz functions map sets of m-Hausdorff measure zero to sets of m-Hausdorff measure zero.

Theorem 5.3. (Rademacher's Theorem) If $\phi : \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz, then it is differentiable almost everywhere.

Proof. The proof of this theorem is outside of the scope of this paper, but we will offer a sketch of the proof. The proof of Rademacher's Theorem can be summarized in five steps:

(1) If $\phi : \mathbb{R} \to \mathbb{R}$ is monotonic, then it is differentiable almost everywhere.
(2) If $\phi : \mathbb{R} \to \mathbb{R}$ locally has bounded variation (which includes all Lipschitz functions), then it is differentiable almost everywhere.
(3) If $\phi : \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz, then it has partial derivatives almost everywhere.
(4) If $\phi : \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz, then it is approximately differentiable (essentially means differentiable except for a set of density zero) almost everywhere.
(5) If $\phi : \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz and approximately differentiable almost everywhere, then it is differentiable almost everywhere.

The first step is a standard result from undergraduate analysis. The second step is proven by showing that all $\phi : \mathbb{R} \to \mathbb{R}$ of locally bounded variation can be written as the sum of two monotonic functions. The third step follows with marginal effort from the second step, and the last two steps are more advanced but technical results stemming from the definition of approximate differentiability.

We refer the interested reader to Morgan 3.2 for a more thorough partial proof and Federer 3.1.6 for a rigorous treatment of the theorem [2] [1].

□

Theorem 5.4. (Kirszbraun's Theorem) Let $U \subset \mathbb{R}^m$ and $\phi : U \to \mathbb{R}^n$ have Lipschitz constant λ, then there exists $\phi' : \mathbb{R}^m \to \mathbb{R}^n$ with Lipschitz constant λ that agrees with ϕ on U.

Theorem 5.5. *Let $m \leq n$ and $f : \mathbb{R}^m \to \mathbb{R}^n$ be Lipschitz. For all $\mathcal{L}^m$ measurable A, $\int_A J_m f(x) d\mathcal{L}^m \geq \mathcal{H}^m(f(A))$, where $J_m f(x)$ (the Jacobian) is defined as the square root of the sum of the squares of the determinants of the $m \times m$ submatrices of $Df(x)$.*

The proof of both theorems above are also outside of the scope of this paper, and we refer the interested reader to Federer 2.10.43 and 3.2.3 [1].

Now, we finally get to the proof that everywhere on an $(\mathcal{H}^m, m)$ rectifiable set (except for possibly a set of $\mathcal{H}^m$ measure zero), the m-approximate tangent cone is an m-dimensional vector subspace.

The proof of this theorem can be broken down into two main steps. First, we use Lemma 4.2 to prove that if a Lipschitz function from $\mathbb{R}^m \to \mathbb{R}^n$ is sufficiently well-behaved, then the tangent cone of the image around any given point in the image is an m-dimensional vector subspace (Lemma 5.6). Then, we prove that $\mathcal{H}^m$ almost all of an $(\mathcal{H}^m, m)$ rectifiable set can be written as a countable union of the images of such well-behaved Lipschitz functions (Lemma 5.7). Finally, we put it all together by using Lemma 5.7 to break down any $(\mathcal{H}^m, m)$ rectifiable set into the images of such Lipschitz functions and applying Lemma 5.6 to obtain our desired result.

Lemma 5.6. *Let $\phi : \mathbb{R}^m \to \mathbb{R}^n$, $a \in K \subset \mathbb{R}^m$, and K have m-dimensional density one at a. If some $1 < \lambda < \infty$ is a Lipschitz constant for $\phi|K$ and $(\phi|K)^{-1}$, $\phi|K$ is injective, and ϕ has injective derivative at a, then*

$$\lambda^{-2m} \leq \Theta^m_*[\phi(K), \phi(a)] \leq \Theta^{*m}[\phi(K), \phi(a)] \leq \lambda^{2m},$$

$$Tan^m[\phi(K), \phi(a)] = Tan[\phi(K), \phi(a)] = Im(D\phi(a)).$$

Proof. First, we observe that given $\delta \in R$, since both $\phi|K$ and $(\phi|K)^{-1}$ have λ as a Lipschitz constant,

$$K \cap B(a, \delta/\lambda) \subset \phi^{-1}[\phi(K) \cap B(a, \delta)]$$

and

$$\phi(K) \cap B(\phi(a), \delta) \subset \phi(K \cap B(a, \delta\lambda))$$

As a result, by Theorem 5.2, we see

$$\mathcal{H}^m[K \cap B(a, \delta/\lambda)] \leq \lambda^m \mathcal{H}^m[\phi(K) \cap B(\phi(a), \delta)]$$

and similarly

$$\mathcal{H}^m[\phi(K) \cap B(\phi(a), \delta)] \leq \lambda^m \mathcal{H}^m[K \cap B(a, \delta\lambda)].$$

Some simple algebraic manipulation then gives us

$$\lambda^{-2m} \frac{\mathcal{H}^m[K \cap B(a, \delta/\lambda)]}{\alpha(m)(\delta/\lambda)^m} \leq \frac{\mathcal{H}^m[\phi(K) \cap B(\phi(a), \delta)]}{\alpha(m)(\delta)^m} \leq \lambda^{2m} \frac{\mathcal{H}^m[K \cap B(a, \delta\lambda)]}{\alpha(m)(\delta\lambda)^m}.$$

Since K has m-dimensional density one at a, taking $\delta \to 0$ gives us our first claim.

As for the second claim, as noted above, $\mathrm{Tan}^m[\phi(K), \phi(a)] \subset \mathrm{Tan}[\phi(K), \phi(a)]$. It can also be verified that K having m-dimensional density one at a implies that $\mathrm{Tan}(K, a) = \mathbb{R}^m$, which means that Lemma 4.2 implies $\mathrm{Tan}[\phi(K), \phi(a)] = \mathrm{Im}(D\phi(a))$. To prove the opposite inclusion, we will show that for all $v \in \mathbb{R}^m$, $D\phi(a)(v) \in \mathrm{Tan}^m[\phi(K), \phi(a)]$.

Recall that by Theorem 3.3, $w \in \mathrm{Tan}^m[\phi(K), \phi(a)]$ if and only if

$$\Theta^{*m}[\phi(K) \cap E(\phi(a), w, \epsilon), \phi(a)] > 0 \text{ for every } \epsilon > 0.$$

Given $\epsilon > 0$, we can choose $\eta > 0$ and $\zeta > 0$ such that

$$(|v| + \eta) \cdot \zeta + ||D\phi(a)|| \cdot \eta \leq \epsilon.$$

If we choose δ sufficiently small, then $|\phi(x) - \phi(a) - D\phi(a)(x-a)| \leq \zeta|x-a|$ for $x \in B(a, \delta/\lambda)$. We can then prove that $\phi[E(a, v, \eta) \cap B(a, \delta/\lambda)] \subset E[\phi(a), D\phi(a)(v), \epsilon]$ as follows: given $x \in \mathbb{R}^m$, $r > 0$ such that $|x - a| \leq \delta/\lambda$ and $|r(x - a) - v| < \eta$, we can see

$$\begin{aligned}
&|r(\phi(x) - \phi(a)) - D\phi(a)(v)| \\
&\quad = |r(\phi(x) - \phi(a)) - D\phi(a)(v) + rD\phi(a)(x - a) - rD\phi(a)(x - a)| \\
&\quad \leq r|\phi(x) - \phi(a) - D\phi(a)(x - a)| + |D\phi(a)(r(x - a) - v)| \\
&\quad \leq |r(x - a)|\zeta + ||D\phi(a)|| \cdot |r(x - a) - v| \\
&\quad < (|v| + \eta)\zeta + ||D\phi(a)|| \cdot \eta \leq \epsilon.
\end{aligned}$$

This implies

$$K \cap E(a, v, \eta) \cap B(a, \delta/\lambda) \subset \phi^{-1}(\phi(K) \cap E(\phi(a), D\phi(a)(v), \epsilon) \cap B(\phi(a), \delta)).$$

Theorem 5.2 then gives us

$$\begin{aligned}
&\lambda^{-2m}\mathcal{H}^m[K \cap E(a, v, \eta) \cap B(a, \delta/\lambda)]/[\alpha(m)(\delta/\lambda)^m] \\
&\quad \leq \mathcal{H}^m[\phi(K) \cap E(\phi(a), D\phi(a)(v), \epsilon) \cap B(\phi(a), \delta)]/[\alpha(m)\delta^m].
\end{aligned}$$

Our left hand side approaches $\lambda^{-2m}\mathcal{H}^m[E(a, v, \eta) \cap B(a, 1)/\alpha(m)]$ as $\delta \to 0$ since K has m-dimensional density one at a and $\mathcal{H}^m[E(a, v, \eta) \cap B(a, R)]/R^m\alpha(m)$ are equal for all $R > 0$. Since $\lambda^{-2m}\mathcal{H}^m[E(a, v, \eta) \cap B(a, 1)/\alpha(m)]$ is greater than zero, we conclude

$$\begin{aligned}
&\Theta^{*m}[\phi(K) \cap E[\phi(a), D\phi(a)(v), \epsilon], \phi(a)] \\
&\quad \geq \Theta^m_*[\phi(K) \cap E[\phi(a), D\phi(a)(v), \epsilon], \phi(a)] \\
&\quad > 0
\end{aligned}$$

and $D\phi(v) \in \mathrm{Tan}^m[\phi(K), \phi(a)$.

□

Lemma 5.7. *Let $W \subset \mathbb{R}^n$ be $(\mathcal{H}^m, m)$ rectifiable and equal $\cup_{i=1}^{\infty}\phi_i(K_i)$ except for a set of $\mathcal{H}^m$ measure zero for Lipschitz functions $\phi_i : K_i \to \mathbb{R}^n$ and bounded $K_i \in \mathbb{R}^m$. Given any $\lambda > 1$, we can find a countable collection of bounded measurable sets $\{K_i\}$ and Lipschitz functions from $\mathbb{R}^m$ to $\mathbb{R}^n$ $\{\phi_i\}$ such that*

1. *$\{\phi_i(K_i)\}$ is a countable collection of disjoint subsets of W with $\mathcal{H}^m[W \setminus \cup_{i=1}^{\infty}\phi_i(K_i)] = 0$, and*
2. *for all i,*
   (a) *$Lip(\phi_i) \leq \lambda$,*

(b) $\phi_i|K_i$ *is injective,*
(c) $Lip[(\phi_i|K_i)^{-1}] \leq \lambda$,
(d) $D\phi_i(a)$ *is injective for all* $a \in K_i$.

Proof. First notice that by Theorem 5.4, we can assume $\phi_i : \mathbb{R}^m \to \mathbb{R}^n$. Without loss of generality, we can assume that W is just $\phi(K)$ for some Lipschitz function $\phi : K \to \mathbb{R}^n$ and K bounded by the following argument:

In the general case where W is $\mathcal{H}^m$ almost everywhere a countable union of such $\phi_i(K_i)$, W can be rewritten $\mathcal{H}^m$ almost everywhere as a countable union of measurable subsets of W each contained in $\phi_i(\mathbb{R}^m)$ for some i. We then take the preimage of these subsets under their corresponding ϕ_i to be the new K_i, and let these subsets be the new $\phi_i(K_i)$. Suppose we have proven Lemma 5.7 for $W = \phi(K)$ for some Lipschitz function $\phi : K \to \mathbb{R}^n$ and K bounded, then it suffices to make the $\phi_i(K_i)$ disjoint, which we can do by replacing each K_i by $\phi_i^{-1}[\phi_i(K_i) \setminus \cup_{j=i+1}^{\infty} \phi_j(K_j)]$.

Let $W = \phi(K)$ as described above. We reason again that it suffices to prove the following claim:

Property 5.8. $P = \{x : D\phi(x) \text{ is injective}\}$ has a countable covering of measurable sets $\{E_i\}$ such that $\phi|E_i$ is injective and there exists an invertible linear function $s \in GL(m, \mathbb{R})$ such that $\text{Lip}[(\phi|E_i) \circ s^{-1}] \leq \lambda$ and $\text{Lip}[s \circ (\phi|E_i)^{-1}] \leq \lambda$.

To prove that Property 5.8 is sufficient, we argue as follows: First replace ϕ with $\phi \circ s^{-1}$ and K with $s(K)$, which would still give us a proper cover of W, then intersect K with each of the E_i to form K_i and take $\{\phi(K_i)\}$ as a countable cover of $W \cap P$. It can be verified that if $D\phi(x)$ is not injective, then $J_m\phi(x) = 0$, so by Theorem 5.5,

$$\mathcal{H}^m(\phi(\mathbb{R}^m \setminus P)) \leq \int_{\mathbb{R}^m \setminus P} J_m\phi(x) d\mathcal{L}^m = 0$$

W will actually be covered $\mathcal{H}^m$ almost everywhere by $\{\phi(K_i)\}$. Now, to obtain our desired result, we simply replace K_i by $\phi^{-1}[\phi(K_i) \setminus \cup_{j=i+1}^{\infty} \phi(K_j)]$ to make the individual $\phi(K_i)$ disjoint.

We will now prove Property 5.8 and hence Lemma 5.7. Since $\lambda > 1$, we can find $\epsilon > 0$ so that $\lambda^{-1} + \epsilon < 1 < \lambda - \epsilon$. Let S be a countable dense subset of $GL(m, \mathbb{R})$. Given $s \in S$ and $j \in \mathbb{N}$, let $Z(s, j)$ be the measurable set of $a \in \mathbb{R}^m$ such that

(1) $(\lambda^{-1} + \epsilon)|s(v)| \leq |D\phi(a)(v)| \leq (\lambda - \epsilon)|s(v)|$ for all $v \in \mathbb{R}^m$,
(2) $|\phi(b) - \phi(a) - D\phi(a)(b - a)| \leq \epsilon|s(b - a)|$ for all $b \in B(a, j^{-1})$.

Simple algebraic manipulation of the two properties above gives us that if a measurable set $A \in Z(s, j)$ has diameter less than j^{-1}. Then,

$$|\phi(b) - \phi(a)| \leq |D\phi(a)(b - a)| + \epsilon|s(b - a)| \leq \lambda|s(b) - s(a)|,$$

$$|\phi(b) - \phi(a)| \geq |D\phi(a)(b - a)| - \epsilon|s(b - a)| \geq \lambda^{-1}|s(b) - s(a)|$$

for all $a, b \in A$, which implies that $\phi|A$ is injective, $\text{Lip}[(\phi|A) \circ s^{-1}] \leq \lambda$, and $\text{Lip}[s \circ (\phi|A)^{-1}] \leq \lambda$. Taking a countable number of these sets to cover each of the $Z(s, j)$ then tells us that

$$\bigcup_{\substack{s \in GL(m,\mathbb{R}) \\ j \in \mathbb{N}}} Z(s, j)$$

has a countable cover of measurable sets with the required properties.

It then suffices to show that all $a \in P$ belong to some $Z(s, j)$. Since $D\phi(a)$ is injective, we can write it as $h \circ g$ where $g \in GL(m, R)$ and h is a map that embeds an m-dimensional vector subspace into $\mathbb{R}^n$. It can be verified that $|Df(a)(v)| = |g(v)|$ for all $v \in \mathbb{R}^m$. We then choose $s \in S$ close enough to g such that

$$||s \circ g^{-1}|| < (\lambda^{-1} + \epsilon)^{-1}$$

and

$$||g \circ s^{-1}|| < \lambda - \epsilon.$$

This implies that for any $v \in \mathbb{R}^m$,

$$|s(v)| \leq (\lambda^{-1} + \epsilon)^{-1} |g(v)|$$

and

$$|g(v)| \leq (\lambda - \epsilon)|s(v)|,$$

which fulfills (1). To prove that a also fulfills (2) for the same s and some i, we notice that $|b - a| \leq ||s^{-1}|| \cdot |s(b - a)|$. Since our ϵ is fixed, by the differentiability of ϕ at a, we get that the second condition is fulfilled for i sufficiently large, and we are done.

□

With Lemma 5.6 and 5.7, we can now finally prove our theorem that a $(\mathcal{H}^m, m)$ rectifiable set has tangent planes $\mathcal{H}^m$ almost everywhere.

Theorem 5.9. *If W is an $(\mathcal{H}^m, m)$ rectifiable, then for $\mathcal{H}^m$ almost all x in W, $Tan^m(W, x)$ is an m-dimensional vector subspace of $\mathbb{R}^n$.*

Proof. By Lemma 5.7, we can assume that W equals $\cup_{i=1}^{\infty} \phi_i(K_i)$ with the described properties $\mathcal{H}^m$ almost everywhere. First, since K_i are measurable and $\mathcal{H}^m$ measurable sets of $\mathbb{R}^m$ have m-dimensional density almost everywhere, we know that K_i have m-dimensional density almost everywhere.

Notice that Theorem 5.5 implies Lipschitz functions map sets of m-dimensional density zero to sets of m-dimensional density zero, hence $\Theta^m(W \setminus \phi_i(K_i), x) = 0$ for $\mathcal{H}^m$ almost all $x \in \phi_i(K_i)$ for all i. By Theorem 5.3, Lipschitz functions are differentiable almost everywhere. We can then infer that $\mathcal{H}^m$ almost everywhere on W, $\mathrm{Tan}^m(W, x) = \mathrm{Tan}^m(\phi_i(K_i), x)$ by the same argument that proves $\mathrm{Tan}(K, x) = \mathbb{R}^m$ if $K \subset \mathbb{R}^m$ has m-dimensional density one at x. ϕ_i and K_i given by Lemma 5.7 fulfill the hypothesis of Lemma 5.6, which then tells us that $\mathrm{Tan}^m(\phi_i(K_i), x) = \mathrm{Tan}(\phi_i(K_i), x) = \mathrm{Im}(D\phi_i(x))$, and we are done.

□

M-dimensional manifolds have tangent planes everywhere, $(\mathcal{H}^m, m)$ rectifiable sets have tangent planes $\mathcal{H}^m$ almost everywhere. This comparison therefore formalizes our intuition that a rectifiable set is a manifold almost everywhere. In fact, once we define some sense of distance between sets, we can prove that one can approximate rectifiable sets with manifolds to an arbitrarily close degree.

To conclude this paper, we make another interesting observation: Both manifolds and rectifiable sets are defined analytically (i.e. described as images of certain types of functions); yet, starting with a simple lemma about how the tangent space of a set transforms under the derivative of a function, we were able to prove geometric properties about both classes of sets (i.e. what such a set looks like locally). Additionally, since manifolds are defined based on differentiable functions and have tangent planes almost everywhere, the idea of manifolds seems to associate differentiability with tangent planes. Then from this perspective, rectifiable sets having tangent planes almost everywhere is rather natural, since just as manifolds are images of differentiable functions, rectifiable sets are images of Lipschitz functions, and Lipschitz functions are differentiable almost everywhere.

Acknowledgments

I would like to thank my mentor, Hakan Solak, for supporting me throughout the writing of this paper. I would also like to thank Professor André Neves for giving pointers and recommending textbooks that served as the main reference sources for this paper. This paper would not have been possible without the help from either one of them.

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