RELATING COMPLEX ORIENTATION TO CHERN CLASSES AND FORMAL GROUP LAWS

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Abstract. In this expository paper, we discuss complex orientation of cohomology theories. We motivate and justify its definition by constructing Chern classes and Connor-Floyd-Chern classes. Afterwards, we introduce the relationship of complex orientation to formal group laws. We include a proof of the structure of the Lazard ring, and briefly introduce the reader to the complex cobordism spectrum MU.

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Introduction

Stable homotopy theory began around 1937 with Freudenthal's suspension theorem, which states that if q is small relative to n, then $\pi_{n+q}(S^n)$ is independent of n. It rose to prominence as a branch of algebraic topology with Frank Adams' use of stable phenomena to resolve the Hopf invariant one problem, and continued its rise with the work of Thom, who reduced the problem of classifying manifolds up to cobordism to a solvable problem in stable homotopy theory. Recent successes of stable homotopy theory include the resolution of the Kervaire invariant one problem in 2009 by Hopkins, Hill, and Ravanel.

An important subfield of stable homotopy theory that played a crucial role in the resolution of the Kervaire invariant one problem is chromatic homotopy theory, an approach to stable homotopy theory from the "chromatic" viewpoint, which traces

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back to Quillen's work on the relationship between complex-orientable cohomology theories and formal groups. Indeed, for a novice who wants to begin learning chromatic homotopy theory, one of the first notions encountered is that of complex orientation of generalized cohomology theories.

This article has two goals. The first is to motivate complex orientation through the most basic example of a complex-oriented cohomology theory: singular cohomology. Singular cohomology admits a theory of Chern classes, invariants which help us understand complex vector bundles and, by naturality, are classes in the singular cohomology of $BU(n)$, the classifying space of the unitary group $U(n)$. Retracing the theory of Chern classes reveals the essence of the complex-orientation condition and what it gives us, namely a generalization of Chern classes called Connor-Floyd-Chern classes. The second goal is to introduce the reader to the relation of complex orientation to formal group laws. While a full account of this relationship is beyond the scope of this article, the hope is to acquaint the reader with the very basic notions that are related to the very beginnings of the chromatic story.

In section 1, we provide a cursory review of generalized cohomology theories and spectra, and in section 2 we introduce the notion of complex orientation. In sections 3-5 we review the classical theory of Chern classes. In section 6, we construct the Connor-Floyd-Chern classes for complex-oriented cohomology E, and show that every complex bundle admits an E-orientation. In section 7 we discuss formal group laws, how a formal group law is associated to a complex-oriented E, and prove Lazard's theorem. In section 8, we introduce the complex cobordism spectrum MU as a central player in recovering a complex-oriented E from its associated formal group law via Landweber's exact functor theorem.

This article is intended for those who are at least familiar with Chapters 1-19 of Peter May's *A Concise Course in Algebraic Topology* [8] and vector bundles and spectral sequences at the level of Chapter 3 and 4 of [9]. Some familiarity with generalized (co)homology theories and the Atiyah-Hirzebruch spectral sequence may help, but is not strictly necessary for the conceptual punchline of this article. Lastly, assume that all vector bundles we refer to are numerable or have a paracompact base, so that they admit a metric.

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1. Generalized Cohomology Theories and Spectra

Complex-oriented cohomology theories are generalized cohomology theories with additional structure. While the rest of the paper is devoted to motivating and exploring implications of this additional structure, in this section we discuss generalized cohomology theories. Furthermore we include a brief discussion of spectra since, although much of this paper does not involve spectra, the most important example of a complex-oriented cohomology is the complex cobordism spectrum MU.

Recall that singular cohomology satisfies the Eilenberg-Steenrod axioms. Removing the dimension axiom yields the generalized cohomology theories. Our interest in them was legitimized by, for example, the discovery of topological K-theory.

Definition 1.1 (Generalized cohomology theory)**.** A generalized cohomology theory h^* is a sequence of contravariant functor $h^i : \mathbf{CWpairs} \to \mathbf{Ab}$ for $i \geq 0$ along with natural transformations $\delta^i : h^i(A) \to h^{i+1}(X, A)$ satisfying the following axioms:

- (Exactness) For any $(X, A) \in \mathbf{CWpairs}$, there is a long exact sequence
  $\cdots \to h^{n-1}(A) \to\to h^n(X, A) \to h^n(X) \to h^n(A) \to h^{n+1}(X, A) \to \cdots$
- (Homotopy) If two morphisms $f, g : (X, A) \to (Y, B)$ are homotopic, then
  $$f^* = g^* : h^n(Y, B) \to h^n(X, A)$$
  for every $n \geq 0$.
- (Excision) For $(X, A), (Y, B) \in \mathbf{CWpairs}$, the inclusion $k : (A, A \cap B) \to (A \cup B, B)$ induces an isomorphism
  $$k^* : h^n(A \cup B, B) \to h^n(A, A \cap B).$$

Often, algebraic topologists prefer working with reduced generalized cohomologies. This is because the suspension axiom holds in all degrees.

A based CW-complex is an object $(X, *) \in \mathbf{CWpairs}$, where the base point $*$ is in the 0th-skeleton of X. The full subcategory of based CW-complexes is denoted $\mathbf{CW}_*$, where morphisms and homotopies in $\mathbf{CW}_*$ being base-point preserving.

Definition 1.2 (Reduced generalized cohomology theory)**.** A reduced generalized cohomology theory is a sequence of contravariant functors $\tilde{h}^i : \mathbf{CW}_* \to \mathbf{Ab}$ along with natural transformations $\sigma^i : \tilde{h}^i \to \tilde{h}^{i+1} \circ \Sigma$ satisfying the following axioms:

- (Exactness) For $A \subset X \in \mathbf{CW}_*$, the sequence
  $$\tilde{h}^i(X/A) \to \tilde{h}^i(X) \to \tilde{h}^i(A)$$
  is exact.
- (Homotopy) If two morphisms $f, g : X \to Y$ in $\mathbf{CW}_*$ are homotopic, then
  $$f^* = g^* : \tilde{h}^i(Y) \to \tilde{h}^i(X)$$
  for every i.
- (Suspension) For $X \in \mathbf{CW}_*$, the homomorphism
  $$\sigma^i(X) : \tilde{h}^i(X) \to \tilde{h}^{i+1}(\Sigma X)$$
  is an isomorphism for every i.

To relate reduced and unreduced theories, consider the functor
$$\pi : \mathbf{CWpairs} \to \mathbf{CW}_*$$
where $\pi : (X, A) \mapsto (X/A, *)$. For any reduced cohomology theory $\tilde{h}^*$, the composition $\tilde{h}^* \circ \pi$ is an unreduced cohomology theory. On the other hand, the map $(X, A) \to (X/A, *)$ factors as the composition
$$(X, A) \to^i (X \cup CA, CA) \to^j (X/A, *)$$

where i is inclusion into the bottom of the cone and j collapses the cone. Note j is a homotopy equivalence. Then for any unreduced cohomology h^*, the excision and homotopy axiom imply $h^*(X, A) \cong h^*(X/A, *)$. In other words:

Proposition 1.3. *For any reduced cohomology theory $\tilde{h}^*$ on $\mathbf{CW}_*$, $h^* := \tilde{h}^* \circ \pi$ is a cohomology theory on* **CWpairs**. *Conversely, any cohomology theory h^* can be written as $\tilde{h}^* \circ \pi$ for some reduced cohomology $\tilde{h}^*$ on $\mathbf{CW}_*$.*

A very special fact about singular cohomology is that it is representable by the Eilenberg-Maclane spaces $K(G, n)$. In other words, $\tilde{H}^n(X; G) \cong [X, K(G, n)]$. By the suspension axiom, we have $\tilde{H}^n(X; G) \cong \tilde{H}^{n+1}(\Sigma X; G)$. Thus,

$$[X, K(G, n)] \cong [\Sigma X, K(G, n+1)] \cong [X, \Omega K(G, n+1)],$$

which means that $K(G, n) \cong \Omega K(G, n+1)$. This leads us to the following notion.

Definition 1.4 (Ω-spectrum)**.** An Ω-spectrum is a sequence of spaces $\{E_n\}$ in $\mathbf{CW}_*$ along with structure maps $\tilde{\sigma_n} : E_n \to \Omega E_{n+1}$ that are pointed homeomorphisms.

By Brown's representability theorem, if $\tilde{h}^*$ is a reduced cohomology theory, there exist an Ω-spectrum $\{E_n, \tilde{\sigma}_n\}$ such that $\tilde{h}^n(X)$ is naturally isomorphic to $[X, E_n]$. Any Ω-spectrum represents a reduced cohomology theory.

The category of Ω-spectra is a first step towards the stable homotopy category[1]. However, Ω-spectra are not sufficient for the goals of homotopy theorists. They do not admit a usable theory of cofibration sequences, nor suitable point-set level or homotopical work [3]. A more suitable generalization is spectra. There are different models for the category of spectra, but all pass to the same stable homotopy category. In this section, we introduce coordinate-free Lewis-May spectra as our model for spectra. We follow section 1 of [3].

A *universe* U is a real inner product space isomorphism to the sum $\mathbb{R}^\infty$ of countably many copies of $\mathbb{R}$. For $V \subset W$, we write $W - V$ for the orthogonal complement of V in W.

Definition 1.5 (Spectrum)**.** A spectrum assigns to every finite dimensional subspace $V \subset U$ a based space EV, with structure maps

$$\tilde{\sigma}_{V,W} : EV \to \Omega^{W-V} EW$$

for all pairs $V \subset W$, where $\Omega^{W-V} EW$ is the function space $F(S^{W-V}, EW)$, where S^{W-V} is the one-point compactification of $W - V$. The structure maps must be homeomorphisms and transitive, i.e. for $V \subset W \subset Z$, we must have $\tilde{\sigma}_{W,Z} \circ \tilde{\sigma}_{V,W} = \tilde{\sigma}_{V,Z}$.

A morphism $f : E \to E'$ of spectra is a collection of pointed maps $f_V : EV \to E'V$ such that the diagrams

$$\begin{tikzcd} EV \arrow[r, "\tilde{\sigma}_{V,W}"] \arrow[d, "f_V"'] & \Omega^{W-V} EW \arrow[d, "\Omega^{W-V} f_W"] \\ E'V \arrow[r, "\tilde{\sigma'}_{V,W}"'] & \Omega^{W-V} E'W \end{tikzcd}$$

[1]For a light introduction to the stable homotopy category and its properties, see [6]. Afterwards, we recommend [3] for a proper introduction to stable homotopy foundations. The first three pages of [3] also provide intuition on how the category of spectra relates to the stable homotopy category.

commute. This gives us the category of spectra. Dropping the condition that structure maps be homeomorphisms gives the category of prespectra. An important fact is that the forgetful functor $\ell : \mathbf{Spectra} \to \mathbf{PreSpectra}$ has a left adjoint L, the **spectrification functor**. Here are some important examples of spectra:

Example 1.6 (Suspension Spectrum)**.** Let X be a based space. The suspension prespectrum $\prod^\infty X$ is the prespectrum such that $\prod^\infty X(V) = \Sigma^V X := X \wedge S^V$, with the obvious structure maps coming from $X \wedge S^{W-V} \wedge S^V \cong X \wedge S^W$. Let $\Sigma^\infty X$ denote the suspension spectrum $L \prod^\infty X$. We can be quite explicit about what the V-th space of the suspension spectrum is. Let $QX = \bigcup \Omega^V \Sigma^V X$, where the union is taken over

Note $\Sigma X \cong \Sigma X$ has an adjoint map $X \to \Omega\Sigma X$. Then this yields $\Omega\Sigma X \to \Omega^2\Sigma^2 X$. Iterating, we obtain a sequence

$$X \to \Omega\Sigma X \to \Omega^2\Sigma^2 X \to \cdots$$

We can obtain sequences of this form where the indices are no longer on integers but finite dimensional vector spaces $V \subset U$. Let $QX = \bigcup \Omega^V \Sigma^V X$ where the union is taken over such sequences of inclusions. Then $\Sigma^\infty X(V) = Q(\Sigma^V X)$. This defines a functor Σ^∞ from based spaces to spectra that is left adjoint to the functor sending a spectra E to $E(\{0\})$.

Example 1.7 (Shift Desuspension Spectrum)**.** Fix a subspace $V \subset U$. Then define $\prod_V^\infty X$ as the prespectrum sending $W \mapsto \Sigma^W X$ if $V \subseteq W$, and to the point otherwise. Then define $\Sigma_V^\infty X := L \prod_V^\infty X$. This functor is left adjoint to the functor sending spectra E to its V-th space EV.

In order to define homotopy groups of spectra, we need to define the notion of a homotopy and the sphere spectrums. Let X be a based space and E a prespectrum. Then $E \wedge X$ is the prespectrum such that $E \wedge X(V) = EV \wedge X$. If E is a spectrum, then the induced structure maps of $E \wedge X$ are not necessarily homeomorphisms. Thus, if E is a spectrum, we abuse notation and define $E \wedge X$ to be $L(\ell(E) \wedge X)$. A *homotopy in the category of spectra* is a map $E \wedge I_+ \to E'$. Let $[E, E']$ denote the set of homotopy classes of maps of spectra $E \to E'$. This set has the structure of an abelian group (see page 4 of [6]).

Take $U = \mathbb{R}^\infty$, and let $\Sigma_n^\infty = \Sigma_{\mathbb{R}^n}^\infty$. For $n \geq 0$, the sphere spectrum $\mathbb{S}^n$ is $\Sigma^\infty S^n$. For $n > 0$, the sphere spectrum $\mathbb{S}^{-n}$ is $\Sigma_n^\infty S^0$. We write $\mathbb{S}$ for the suspension spectrum of S^0. Given spectrum E, we define its **n-th homotopy group** to be

$$\pi_n(E) := [\mathbb{S}^n, E]$$

for all $n \in \mathbb{Z}$.

In practice, one is often given a (pre)spectrum indexed only on nonnegative integers. Indeed, in section 8, we will describe MU as a prespectrum indexed on integers. The reader may find this confusing so we make a few clarifications:

- The above discussion with respect to coordinate-free Lewis May (pre)spectra applies analogously to prespectra and spectra indexed on integers. Furthermore, by proposition 2.4 of [7], the functor from spectra indexed on U to spectra indexed on nonnegative integers which "forgets" the extraneous indices has a left adjoint, and together these functors give an equivalence of categories. Thus, we do not lose anything homotopically when we consider spectra indexed on nonnegative integers.

- The reader may find many expository sources that define a spectrum to be this paper's definition of a prespectrum indexed on integers. While useful for different purposes, both the coordinate-free Lewis-May model and the category of prespectra indexed on integers serve as valid models for the category of spectra. Both descend to the same stable homotopy category. There are even maneuvers that promote a prespectrum to a spectrum that preserve homotopical information.
  - Given a prespectrum D indexed on integers, applying the "cylindrical functor" K sends D to a Σ-cofibrant prespectrum KD (structure maps are adjoint to based cofibrations). Applying the spectrification functor, one obtains a spectrum LKD such that $\pi_n(E) = \varinjlim_q \pi_{n+q} D_q$. If D is an Ω-spectrum, then LKD is a spectrum representing the same cohomology theory (page 10 of [7]). In general, a lack of cofibrancy in a prespectrum can mean a total loss of control of what happens homotopically after spectrification. The cylindrical functor "thickens" a prespectrum D, and the map $KD \to D$ is a space-wise homotopy equivalence.

  The essential point is that when discussing MU, its homotopy groups, and the (co)homology it represents, it suffices for the purposes of this paper to describe MU as a prespectrum indexed on integers.

This leads us to our next point, which is that prespectra represent (co)homology theories. Let E be a prespectrum indexed on nonnegative integers, and X a space. Then

$$\tilde{E}^n(X) = [\Sigma^{-n} \prod^{\infty} X, E] \text{ and } \tilde{E}_n(X) = \pi_n(E \wedge X) = [\prod^{\infty} S^n, E \wedge X]. \tag{1.8}$$

The unreduced versions are given by $E^n(X) = \tilde{E}^n(X_+)$, and $E_n(X) = \tilde{E}_n(X_+)$.

A final comment to wrap up this section. A generalized cohomology theory is called **multiplicative** if each graded abelian group $\tilde{E}^*(X)$ has the structure of a graded ring. Multiplicative cohomology theories are represented by *ring spectra*, which are monoids in the stable homotopy category. The fact that E-cohomology inherits a graded ring structure from E's monoidal structure is due to the functor π_*, which sends spectra to their homotopy groups, being lax monoidal (page 10 of [6]). The notion of a ring spectrum is not pertinent to the conceptual focus of this paper, however, and the reader should just think of it as a spectrum equipped with some multiplicative structure so that it represents multiplicative cohomology theories.

2. What is complex orientation?

Definition 2.1. A multiplicative cohomology theory E is **complex-orientable** if the map $\widetilde{E}^2(\mathbb{CP}^\infty) \to \widetilde{E}^2(\mathbb{CP}^1)$ induced by inclusion $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$ is surjective.

This additional structure on a multiplicative cohomology theory is called its **complex orientation**. It is a very simple condition. What does it buy us? What is the relevance of $\mathbb{CP}^\infty$ and $\mathbb{CP}^1$? Why a surjection of $\tilde{E}^2$-cohomologies? To explain the motivation and implications of complex orientation, we look back on the classical story of Chern classes, a special type of characteristic classes.

3. Characteristic Classes

The theory of characteristic classes allows one to study the complicated geometry of vector bundles[2] with algebra. In spaces, we have important contravariant functors

$$Vect_{n,\mathbb{C}}(-): \text{ Spaces } \to \text{ Set}$$

sending X to the set of complex n-dimensional bundles over X. The functor $Vect_{n,\mathbb{C}}$ is both homotopy invariant (theorem 17.9 of [9]) and representable by $BU(n)$, the classifying space of the unitary group $U(n)$. More precisely, we have the universal principal $U(n)$-bundle $\xi_n : EU(n) \downarrow BU(n)$ which is universal in the following sense:

Proposition 3.1. *For any CW-complex, there is a canonical bijection*

$$[X, BU(n)] \cong Vect_{n,\mathbb{C}}(X)$$

where $f \in [X, BU(n)]$ is mapped to $f^*(\xi_n)$, the pullback of the principal $BU(n)$-bundle with respect to $f : X \to BU(n)$.

An explicit description of $\xi_n : EU(n) \downarrow BU(n)$ can be found at the end of section 19 of [9]. We are much more interested in its universality though.

Definition 3.2. A complex characteristic class c with values in $H^k(-;\mathbb{Z})$ is a natural transformation $c : Vect_{n,\mathbb{C}}(-) \to H^k(-;\mathbb{Z})$.

In other words, over space X, c assigns to each complex n-plane bundle a cohomology class in $H^n(X;\mathbb{Z})$. If we have a map $f : X \to Y$, and ξ is a bundle over Y, then $c(f^*\xi) = f^*(c(\xi))$. As a consequence of the Yoneda lemma, since $Vect_{n,\mathbb{C}}(-)$ is representable by $BU(n)$, we have

$$\text{n.t.}(Vect_{n,\mathbb{C}}(-), H^k(-;\mathbb{Z})) \cong H^k(BU(n);\mathbb{Z}).$$

But these natural transformations are exactly all the possible integral complex characteristic classes. Thus,

$$H^*(BU(n);\mathbb{Z}) \cong \{ \text{ integral complex characteristic classes for n-plane bundles} \}.$$

Thus, if we hope to understand complex bundles through characteristic classes, we should study the cohomology ring of $BU(n)$. It turns out that the cohomology ring is completely described as a polynomial algebra by the most fundamental type of integral complex characteristic classes: Chern classes.

Theorem 3.3. *Let $\xi : E \downarrow B$ be a complex n-plane bundle. The Chern classes are a unique family of characteristic classes such that $c_k^{(n)}(\xi) = H^{2k}(B;\mathbb{Z})$, satisfying the following axioms:*

(1) $c_0(\xi) = 1$
(2) $c_1(\xi) = -e(\xi)$
(3) (Whitney Sum) $c_k(\xi \oplus \eta) = \sum_{i+j=k} c_i(\xi) \cup c_j(\eta)$*, for complex p-plane bundle ξ, and complex q-plane bundle η.*

The reader may notice that we are letting c_k denote the k-th chern class for complex n-plane bundles for any n. This is standard notation in the literature, and is so for the following reason: consider the trivial bundle $k\epsilon$ over X. This is the pullback of the bundle $\mathbb{C}^k$ over the point $*$, along the map $X \to *$. By naturality of Chern classes, $c_i(k\epsilon) = 0$ for $i > 0$. Then by Whitney Sum, $c_n(\xi \oplus$

[2]For a review of vector bundles, see [9] or page 13 of [10].

$k\epsilon) = c_n(\xi)$. Furthermore, if ξ_n is the universal n-plane bundle, under the map $BU(n) \to BU(n+1)$ that classifies bundle $\xi_n \oplus \epsilon$, the cohomology class $c_k(\xi_{n+1})$ is sent to $c_k(\xi_n)$ for $k \leq n$ and 0 otherwise.

In the next two sections, we will construct the Chern classes, and show that $H^*(BU(n);\mathbb{Z})$ is a polynomial algebra generated by the Chern classes. The reader should have also noted the presence of the Euler class $e(\xi)$ of a complex bundle in the axioms of Chern classes. Constructing the Euler class and seeing how it relates to Chern classes will provide insight on complex orientation later on.

4. Thom Class

To construct the Euler class, we need to discuss the Thom class of a vector n-plane bundle. In this section only we work with real n-plane bundles. From the sources cited at the beginning of the next section, the existence of Euler and Thom classes for real k-bundles implies their existence of complex $2k$-plane bundles.

We will need to impose the additional structure of an orientation on our bundles. An orientation of a real vector space V is an equivalence class of bases, where two bases $\{v_1, \cdots, v_{\dim V}\}$ and $\{v'_1, \cdots, v'_{\dim V}\}$ are equivalent if and only if the matrix $\{\alpha_{ij}\}$ defined by $v'_i = \sum \alpha_{ij} v_j$ has positive determinant. This implies there are two possible orientations.

An orientation of a real n-plane bundle $\xi : E \downarrow_\pi B$ is an assignment of orientations to each fiber F, along with the following local compatibility condition. For every point p in the base space $B(\xi)$, there exists an open neighborhood U_p and local trivialization $h_p : U_p \times \mathbb{R}^n \to \pi^{-1}(U_p)$, such that for every fiber in $\pi^{-1}(U_p)$, the map $(x, v) \mapsto h_p(x, v)$ from $\{x\} \times \mathbb{R}^n$ to $\pi^{-1}(x)$ is orientation preserving.

Let F_0 denote the fiber with the zero-point removed. An orientation assigns a preferred generator $\mu_F \in H^n(F, F_0;\mathbb{Z})$. The local compatibility condition implies there is a cohomology class $\mu_p \in H^n(\pi^{-1}(U_p), \pi^{-1}(U_p)_0;\mathbb{Z})$ such that its restriction over (F, F_0) for each fiber $F \subseteq \pi^{-1}(U_p)$ is equal to μ_F. We assert the following theorem.

Theorem 4.1. *Let $\xi : E \downarrow B$ be an oriented n-plane bundle. Then the cohomology group $H^i(E, E_0;\mathbb{Z})$ is zero for $i < n$ and $H^n(E, E_0;\mathbb{Z})$ contains a unique cohomology class whose restriction over each fiber (F, F_0) equals the preferred generator μ_F. Furthermore, the map $H^k(E;\mathbb{Z}) \to H^{k+n}(E, E_0;\mathbb{Z})$ given by $x \mapsto x \cup \mu$ is an isomorphism for every integer k.*

This theorem motivates us to define the notion of a Thom space and Thom class. Let $\xi : E \downarrow B$ be a real n-plane bundle. There are several equivalent expressions for its **Thom space** $Th(\xi)$.

- Let $Sph(\xi)$ be the space obtained by forming one-point compactification of each of its fibers. Let B_∞ denote the new "points at infinity" of $Sph(\xi)$. Then $Th(\xi) = Sph(\xi)/B_\infty$.
- Choose a metric on ξ. Let $\mathbb{D}(\xi)$ denote the unit disk bundle, and $\mathbb{S}(\xi)$ the unit sphere bundle. So $\mathbb{S}(\xi) = \partial\mathbb{D}(\xi)$. Then $Th(\xi) = \mathbb{D}(\xi)/\mathbb{S}(\xi)$. Note $(\mathbb{D}(\xi), \mathbb{S}(\xi)$ is homotopy equivalent to $(E(\xi), E(\xi)\backslash Z)$, where Z is the image of the zero-section.

Example 4.2. Suppose $\xi : B \downarrow B$ is the bundle with fiber $\{*\}$. Then $Th(\xi) = B_+$. Suppose ξ is the n-plane bundle over a point $*$. Then $Th(\xi) = D^n/\partial D^n = S^n$.

How does the Thom space construction behave with products of bundles? Since

$$\partial(D^a \times D^b) = (\partial D^a \times D^b) \cup (D^a \times \partial D^b),$$

we have $Th(\xi \times \eta) \cong Th(\xi) \wedge Th(\eta)$. If η is the n-plane bundle $n\epsilon$ over a point, then $Th(\xi \times n\epsilon) \cong Th(\xi) \wedge S^n$.

The Thom space construction is natural, in that if $f : X \to X'$ is covered by a map between bundles $\xi \to \xi'$, we get a canonical pointed map $Th(\xi) \to Th(\xi')$. Furthermore, we can show that $\overline{H}^*(Th(\xi))$ has the structure of a module over graded ring $H^*(B)$. Consider the following diagram.

$$\begin{tikzcd}
E(\xi) \arrow[r] \arrow[d] & E(0 \times \xi) \arrow[r] \arrow[d] & \xi \arrow[d] \\
B \arrow[r, "\Delta"] & B \times B \arrow[r, "\pi_2"] & B
\end{tikzcd}$$

The left-hand square is the pullback along the diagonal, and the right-hand square is the pullback along projection onto the second factor. The diagram induces a map

$$Th(\xi) \to Th(0 \times \xi) = Th(0) \wedge Th(\xi) = B_+ \wedge Th(\xi),$$

which induces a map

$$H^*(B) \otimes \tilde{H}^*(Th(\xi)) \to \tilde{H}^*(Th(0) \wedge Th(\xi)) \to \tilde{H}^*(Th(\xi)).$$

Let R be a commutative ring, and ξ a real n-plane bundle over B. We say ξ is R-**oriented** if there exists a **Thom class** $\mu \in \tilde{H}^n(Th(\xi); R) \cong H^n(\mathbb{D}(\xi), \mathbb{S}(\xi); R)$ such that for every $b \in B$, upon inclusion of a fiber $F_b \hookrightarrow E(\xi)$, μ restricts to the preferred generator of $\tilde{H}^n(S^n_b; R) \cong H^n(D^n_b, S^{n-1}_b; R)$. Note the inclusion $F_b \hookrightarrow E(\xi)$ can also be thought of as $(D^n_b, S^{n-1}_b) \hookrightarrow (\mathbb{D}(\xi), \mathbb{S}(\xi))$. This leads us to the following fundamental theorem.

Theorem 4.3 (Thom Isomorphism). *Let R be a commutative ring, and let ξ be an R-oriented real n-plane bundle over B. Let $\mu \in \tilde{H}^n(Th(\xi); R)$ be a Thom class. Then*

$$- \cup \mu : H^*(B) \to \tilde{H}^{*+n}(Th(\xi))$$

is an isomorphism.

Proof. Utilize the relative Serre spectral sequence. Suppose $p : E \to B$ is a fibration (and fiber bundle), together with subbundle $p_0 : E_0 \to B$. Then there is a spectral sequence

$$E_2^{s,t} = H^s(B; H^t(p^{-1}(-), p_0^{-1}(-)) \implies H^{s+t}(E, E_0).$$

Now apply this to the fiber bundle pair $(\mathbb{D}(\xi), \mathbb{S}(\xi))$ over B. The cohomology of the fibers are zero unless the dimension is n. Then the local coefficient system is zero in dimension $\neq n$. Furthermore, the local coefficient system is trivial. This is because the edge homomorphism

$$H^n(\mathbb{D}(\xi), \mathbb{S}(\xi)) \to E_\infty^{0,n} \hookrightarrow E_\infty^{0,2} = H^0(B; H^n(p^{-1}(-), p_0^{-1}(-))$$

is induced by the inclusion of fiber. But $E_2^{0,*}$ is the $\pi_1(B)$-invariant subgroup, and since ξ has R-orientation this edge homomorphism is surjective. So all elements of

$H^n(D^n, S^{n-1})$ are fixed by $\pi_1(B)$. Thus, $E_2^{s,t} = H^s(B; H^t(D^n, S^{n-1}))$. Since our spectral sequence only has the n-th row, we have an isomorphism

$$H^s(B; H^n(D^n, S^{n-1}) \cong H^s(B; R) \otimes H^n(D^n, S^{n-1}) \mapsto H^{s+n}(\mathbb{D}(\xi), \mathbb{S}(\xi); R) \cong \tilde{H}^{s+n}(Th(\xi)),$$

which tells us $- \cup \mu : H^s(B) \to \tilde{H}^{s+n}(Th(\xi))$ is indeed an isomorphism. The multiplicative structure of the E_2 page implies the isomorphism of $H^*(B)$-modules. □

We are now ready to discuss the **Euler class**. Suppose ξ is an R-oriented real n-plane bundle. Consider the composition $B \to \mathbb{D}(\xi) \to Th(\xi)$, where the first map is the zero section and the second is the collapse map. The Euler class $e(\xi) \in \tilde{H}^n(B) \cong H^n(B)$ is the pullback of the R-orientation $\mu \in \tilde{H}^n(Th(\xi))$ under this map. Naturality follows from the naturality of the Thom space construction.

In the next section, we will discuss Chern classes, in which the Euler class existence is implicit. However, before we move on, we discuss an equivalent formulation of the Euler class which is computationally useful.

Suppose $\xi : E \downarrow^p B$ is an R-oriented real n-place bundle. Apply the cohomological Serre spectral sequence to $S^{n-1} \to E \to^p B$. As noted in the proof of Theorem 4.3, the local coefficient system is trivial. Then we have

$$E_2^{s,t} = H^s(B; H^t(S^{n-1})) \implies H^{s+t}(E).$$

There are only two nonzero rows: rows 0 and $n-1$. Inspecting the n-th page, where the only potentially nonzero differentials can occur, yields exact sequences

$$0 \to E_\infty^{s-n,n-1} \to H^{s-n}(B) \to H^s(B) \to E_\infty^{s,0} \to 0$$

for all $s \geq 0$. Since this spectral sequence converges, $E_\infty^{p,q} = gr_p H^{p+q}(E)$. Since we only have two rows, we have short exact sequences

$$0 \to E_\infty^{s,0} \to H^s(E) \to E_\infty^{s-n+1,n-1} \to 0$$

for every $s \geq 0$. Stitching these exact sequences together yields the *cohomology Gysin sequence*

$$\cdots \to H^{s-1}(E) \to H^{s-n}(B) \to H^s(B) \to H^s(E) \to \cdots.$$

The maps $H^s(B) \to H^s(E)$ are induced by $p : E \to B$, which follows from a standard edge homomorphism argument[3]. The maps $p_* : H^{s-1}(B) \to H^{s-n}(B)$ are called "Umkher maps." The multiplicative structure of the spectral sequence implies that p_* is a module homomorphism for graded algebra $H^*(B)$:

$$p_*((p^*(x))y) = xp_*(y).$$

We are most interested in the maps $H^{*-n}(B) \to H^*(B)$. The R-orientation of ξ provides us with a distinguished class $\sigma \in E_2^{0,n-1} = H^0(B; H^{n-1}(S^{n-1}))$, such that $E_2^{*,n-1}$ is a free $E_2^{*,0}$-module generated by σ. Let $e = d_n(\sigma) \in E_2^{n,0} \cong H^n(B)$. The map $H^{*-n}(B) \to H^*(B)$ sends

$$x \in H^{s-n}(B) \text{ to } d_n(x \cup \sigma) = \pm ex$$

by the Leibniz formula. This e turns out to coincide with $e(\xi)$ as defined earlier, up to a sign. In the interest of brevity, we assume this is true and point the curious reader to lemma 35.3 on page 131 of [9] for further details.

[3] If the reader is confused about this, see page 87 of [9] and apply the same principles to edge homomorphisms for cohomological spectral sequences.

5. Constructing Chern Classes

There is subtlety to how the usual notion of orientation of a vector bundle relates to the notion of R-orientation as discussed in the previous section. We defer to the reader to page 120 of [9] and page 67 of [2] for helpful discussions. There are also details involving the underlying real vector bundle of a complex vector bundle having a canonical preferred orientation. We defer the reader to page 155 of [10] for details. All of this is to say: for any complex n-plane bundle $\xi : E \downarrow B$, there is a well-defined Euler class $e(\xi) \in H^{2n}(B;\mathbb{Z})$. We are now ready to construct the Chern classes. In the process, we will see how the Euler and Thom class relate to Chern classes, and gain insight on the cohomology ring of $BU(n)$.

Theorem 5.1. *There exist classes $c_i \in H^{2i}(BU_n)$ for $1 \leq i \leq n$ such that:*

(1) $H^*(BU(n)) = \mathbb{Z}[c_1, \cdots, c_n]$
(2) The map $H^*(BU(n)) \to H^*(BU(n-1))$, induced by $BU(n-1) \to BU(n)$, maps c_i to c_i if $i < n$, and 0 otherwise.
(3) $c_n = (-1)^n e(\xi_n)$.

Proof. Proceed by induction. We know that

$$H^*(BU(1)) \cong H^*(\mathbb{CP}^\infty) \cong \mathbb{Z}\langle e \rangle,$$

where $|e| = 2$. Assuming our claim holds for $n-1$, consider the spherical fibration $S^{2n-1} \to BU(n-1) \to BU(n)$. By our inductive hypothesis, $H^*(BU(n-1))$ vanishes in odd dimension and is a polynomial algebra. Combining this fact with the cohomology Gysin sequence and properties of the Umkher maps yields short exact sequences

$$0 \to H^{s-n}BU(n) \to H^s BU(n) \to H^s BU(n-1) \to 0$$

for every s. These short exact sequences yield the following facts:

(1) The Euler class $e(\xi_n)$ is a nonzero divisor.
(2) $H^*(BU(n)) \to H^*(BU(n-1))$ is a surjection, and $H^*(BU(n))/(e(\xi_n)) \cong H^*(BU(n-1))$.
(3) For $s < 2n$, $H^s(BU(n)) \cong H^s(BU(n-1))$.
(4) $H^*(BU(n))$ vanishes in odd dimension.

Define $c_n = (-1)^n e(\xi_n)$. The above facts along with a filtration argument show that $H^*(BU(n)) \cong H^*(BU(n-1))[c_n]$. □

In section 4, we established a bijection between integral complex characteristic classes for n-plane bundles and elements of $H^*(BU(n);\mathbb{Z})$. Our calculation of $H^*(BU(n);\mathbb{Z})$ underlines the fundamental nature of Chern classes. Understanding all integral complex characteristic classes requires only understanding of Chern classes. Naturally, we should have techniques for computing Chern classes. Luckily, calculations for vector bundles can be *reduced to calculations for line bundles*, which are much easier to work with.

Let ξ be a complex n-plane bundle, and consider its associated projectivization bundle $\mathbb{P}(\xi) := P(\xi) \times_{GL_n(\mathbb{C})} \mathbb{CP}^{n-1}$. We claim that the map $i^* : H^*(\mathbb{P}(\xi)) \to H^*(\mathbb{CP}^{n-1})$ induced by fiber inclusion is surjective. The cohomology ring $H^*(\mathbb{CP}^{n-1})$ is generated by the Euler class $e(\lambda') \in H^2(\mathbb{CP}^{n-1})$ of the canonical line bundle λ' over $\mathbb{CP}^{n-1}$. Let λ' and λ denote the canonical line bundles over $\mathbb{CP}^{n-1}$ and $\mathbb{P}(\xi)$, respectively. The inclusion of $\mathbb{CP}^{n-1}$ into $\mathbb{P}(\xi)$ corresponds to the inclusion of $E(\lambda')$

into $E(\lambda)$. Naturality of the Euler class implies i^* is surjective. By Leray-Hirsch, we have

$$H^*(\mathbb{P}(\xi)) = H^*(B(\xi)) \otimes H^*(\mathbb{CP}^{n-1}) = H^*(B(\xi)) < 1, e, e^2, \cdots, e^{n-1} >,$$

where $e = e(\lambda) \in H^2(\mathbb{P}(\xi))$. The essential point is that $H^*(B(\xi))$ injects into $H^*(\mathbb{P}(\xi))$.

Consider again the associated projectivization $\mathbb{P}(\xi)$ of ξ. Consider the tautologous line bundle λ over $E(\mathbb{P}(\xi))$. Then λ embeds canonically into $\pi^*(E)$. Endowing $\pi^*(E) \downarrow E(\mathbb{P}(\xi))$ with a metric yields $\pi^*(E) = \lambda_1 \oplus \lambda_1^\perp \downarrow E(\mathbb{P}(\xi))$. Then we obtain the Whitney sum of a line bundle and a $(n-1)$-plane bundle over $E(\mathbb{P}(\xi))$. Furthermore, $H^*(B) \to H^*(E(\mathbb{P}(\xi)))$ is injective by Leray-Hirsch. Iterating this process yields the splitting principle.

Lemma 5.2 (Splitting Principle)**.** *For a complex n-plane bundle $\xi : E \downarrow B$, there exists a space F_ξ and a map $f : F_\xi \to B$ such that $f^*(\xi) = \lambda_1 \oplus \cdots \oplus \lambda_n$ over F_ξ. Furthermore, $H^*(B) \to H^*(F_\xi)$ is monic.*

The splitting principle is not only useful, but also implies uniqueness of Chern classes. If there were two families of characteristic classes satisfying the Chern class axioms, then they would agree on the universal line bundle of $BU(1) \cong \mathbb{CP}^\infty$. Then by naturality, they agree on all line bundles and thus, by the splitting principle, they agree on any bundle.

6. Back to Complex Orientation: Connor Floyd Chern classes and E-orientation

We saw that singular cohomology admitted a theory of Chern classes, which provided a very nice description of the cohomology ring of $BU(n)$. Is there an analog for generalized cohomology theories? At the bare minimum, ordinary cohomology is multiplicative–a fact which our constructions implicitly relied on–so we should restrict our attention to multiplicative cohomology theories. Recall that in order to construct Chern classes for singular cohomology we needed the notion of a Thom class. We can extend this notion in the following way:

Definition 6.1 (E-orientable)**.** Let E be a multiplicative cohomology theory. A complex n-plane bundle ξ is E-orientable if there exist $\mu \in \tilde{E}^n(Th(\xi))$ which maps to the canonical generator $\sigma_n \in \tilde{E}^n(S^n)$ upon restriction to fiber.

In the case of ordinary cohomology, a Thom class is unique up to a sign if it exists. For any given E, not every complex bundle admits an E-orientation. If it does, then the orientation μ may not be unique.

Recall that by the splitting principle, Chern classes of a complex n-bundle are determined by Chern classes on line bundles, and these are determined by the Chern classes of the universal line bundle $\xi_1 : EU(1) \downarrow BU(1)$. The Chern classes on line bundles are determined by their Euler classes. Then by naturality, we need only consider the Euler class of the universal line bundle, which is determined by its Thom class. This suggests that we may have a theory of Chern classes for E-cohomology if E is complex-orientable.

Definition 6.2 (Complex Orientation)**.** A multiplicative cohomology theory E is complex-orientable if the map $\tilde{E}^2(\mathbb{CP}^\infty) \to \tilde{E}^2(\mathbb{CP}^1)$ induced by inclusion $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$ is surjective.

We will now show that complex-orientable E admit an analog of Chern classes, called Connor-Floyd-Chern classes. After constructing Connor-Floyd-Chern classes, we show that for complex-orientable E, every complex bundle admits an E-orientation.

The calculations that follow will heavily employ the cohomological and homological Atiyah-Hirzebruch spectral sequence (AHSS). These are generalizations of the Serre spectral sequences. The reader can find a detailed treatment of these two spectral sequences and their pairing in section 4.2 of [4]. For clarity, there are two computational themes that the reader should keep in mind.

- The E-(co)homology of the CW complexes we consider will look analogous to their ordinary (co)homology.
- A common technique for computing the E-(co)homology of CW complexes is to compute the E-(co)homology of the k-th skeleton and then take a limiting argument. Homology commute with direct limits, and if the inverse system of skeletal cohomologies satisfies the Mittag-Leffler condition, then the cohomology is isomorphic to the colimit of the skeletal cohomologies. See section 4.2 of [4] for further discussion.

We begin with some standard facts.

Proposition 6.3. *Let E^* be a complex-oriented cohomology theory, and E_* its corresponding homology theory. Let $\mu \in E^2(\mathbb{CP}^\infty)$ denote its complex orientation. Let ι_n denote the inclusion $\mathbb{CP}^n \hookrightarrow \mathbb{CP}^\infty$.*

(1) $E^*(\mathbb{CP}^n) \cong \pi_*(E)[\iota_n^*(\mu)]/(\iota_n^*(\mu)^{n+1})$;
(2) $E^*(\mathbb{CP}^\infty) \cong \pi_*(E)[[\mu]]$;
(3) $E_*(\mathbb{CP}^n) \cong \pi_*(E)\{\alpha_0, \cdots, \alpha_n\}$ *where α_k is the dual basis element of $\iota_n^*(\mu)^k$ under the pairing $E^*(\mathbb{CP}^n) \otimes E_*(\mathbb{CP}^n) \to \pi_*(E)$;*
(4) $E_*(\mathbb{CP}^\infty) \cong \pi_*(E)\{\alpha_k | k \geq 0\}$ *where α_k is the dual basis element of μ^k under the pairing $E^*(\mathbb{CP}^\infty) \otimes E_*(\mathbb{CP}^\infty) \to \pi_*(E)$;*
(5) $E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)$

Proof. (1) Consider the AHSS for $E^*(\mathbb{CP}^n)$:

$$E_2^{s,t} = H^s(\mathbb{CP}^n) \otimes E^t(*) \implies E^{s+t}(\mathbb{CP}^n).$$

Recall $H^*(\mathbb{CP}^n) \cong \mathbb{Z}[x]/(x^{n+1})$. Then E_2 looks like $\pi_*(E)[x]/(x^{n+1})$. Note that $x \in E_2^{2,0}$ is an infinite cycle representing $\iota_n^*(\mu) \in E^2(\mathbb{CP}^n)$. This implies that the spectral sequence collapses on the second page. Thus, $E^*(\mathbb{CP}^n) \cong \pi_*(E)[\iota_n^*(\mu)]/(\iota_n^*(\mu)^{n+1})$.

(2) For $m < n$, let $i_{m,n} : \mathbb{CP}^m \hookrightarrow \mathbb{CP}^n$ denote the inclusion map. The inverse system $i_{m,n}^* : E^*(\mathbb{CP}^n) \to E^*(\mathbb{CP}^m)$ satisfies the Mittag-Leffler condition. Thus

$$E^*(\mathbb{CP}^\infty) \cong \varprojlim_n E^*(\mathbb{CP}^n) \cong \varprojlim_n \pi_*(E)[\iota_n^*(\mu)]/(\iota_n^*(\mu)^{n+1}) \cong \pi_*(E)[[\mu]].$$

(3) Consider the AHSS for $E_*(\mathbb{CP}^n)$:

$$E_{s,t}^2 = H_s(\mathbb{CP}^n) \otimes E_t(*) \implies E_{s+t}(\mathbb{CP}^n).$$

Write $H_*(\mathbb{CP}^n) \cong \mathbb{Z}\langle a_0, \cdots, a_n \rangle$, where a_k is the dual basis element to x^k under the identification $H_*(\mathbb{CP}^n) \cong \mathrm{Hom}(H^*(\mathbb{CP}^n), \mathbb{Z})$. Pairing the AHSS for $E^*(\mathbb{CP}^n)$ and $E_*(\mathbb{CP}^n)$), it follows that the a_k are infinite cycles and the spectral sequence for $E_*(\mathbb{CP}^n)$ collapses on the second page. This stable page can be identified with $E_*(\mathbb{CP}^n)$. The pairing of the two AHSS induces a pairing of their stable pages. Therefore, the pairing $E^*(\mathbb{CP}^n) \otimes E_*(\mathbb{CP}^n) \to \pi_*(E)$ is nonsingular. Thus, $E_*(\mathbb{CP}^n)$ is dual to $E^*(\mathbb{CP}^n)$ as $\pi_*(E)$-modules. The identification of α_k with $(\iota_n^*(\mu))^k$ is immediate.

(4) This is immediate by homology commuting with colimits.

(5) Let $\pi_i : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ denote projection onto the i-th factor. Define $\mu_i \in E^2(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)$ to be $\pi_i^*(\mu)$. Let $j_{m,n} : \mathbb{CP}^m \times \mathbb{CP}^n \hookrightarrow \mathbb{CP}^\infty \times \mathbb{CP}^\infty$ denote the inclusion map. Since

$$H^*(\mathbb{CP}^m \times \mathbb{CP}^n) \cong H^*(\mathbb{CP}^m) \otimes H^*(\mathbb{CP}^n) \cong \mathbb{Z}[x_1, x_2]/(x_1^{m+1}, x_2^{n+1})$$

we can argue as in the proof of item (1) to show

$$E^*(\mathbb{CP}^m \times \mathbb{CP}^n) \cong \pi^*(E)[j_{m,n}^*(\mu_1), j_{m,n}^*(\mu_2)]/(j_{m,n}^*(\mu_1)^{m+1}, j_{m,n}^*(\mu_2)^{n+1}).$$

Applying Mittag-Leffler finishes the claim.

□

Once we compute $E_*(BU(n))$, we use the pairings of AHSS to compute $E^*(BU(n))$ and arrive at the Connor-Floyd-Chern classes.

Proposition 6.4. *Let E^* be a complex-oriented cohomology theory, and E_* its corresponding homology theory.*

(1) The maps $BU(1)^n \to BU(n)$ induce maps

$$E_*(BU(1))^{\otimes n} \to E_*(BU(n)).$$

Using this product, $E_*(BU(n))$ is the free $\pi_*(E)$-module with basis

$$\{\alpha_{k_1} \cdots \alpha_{k_t} | 1 \le k_1 \le \cdots \le k_t \text{ and } t \le n\}.$$

(2) There are ***Connor-Floyd-Chern classes*** *$cf_k \in E^{2k}(BU(n))$ for $1 \le k \le n$ such that*

$$E^*(BU(n)) \cong \pi_*(E)[[cf_1, \cdots, cf_n]].$$

Proof. The main idea of item (1)'s proof is that the E-homology of $BU(n)$ is analogous to the ordinary homology of $BU(n)$, which falls out from its homological AHSS.

The main idea of item (2)'s proof is that one can express the Chern classes generating $H^*(BU(n))$ as particular elements in $\mathrm{Hom}(H_*(BU(n), \mathbb{Z})$. Letting $BU(n)_{(k)}$ denote the k-th skeleton, $H^*(BU(n)_{(k)})$ and $H_*(BU(n)_{(k)})$ are nice subgroups whose generators are cleanly expressed in terms of the generators of $H^*(BU(n))$ and $H_*(BU(n))$. Pairing the homological and cohomological AHSS on $BU(n)_{(k)}$, the E-(co)homology of $BU(n)_{(k)}$ falls out as being analogous to the ordinary (co)homology of $BU(n)_{(k)}$. Mittag-Leffler and taking $k \to \infty$ finishes the claim. Here are the proof sketches.

(1) We use induction on $n \ge 1$. In proposition 6.3, we computed $E_*(BU(1)) \cong E_*(\mathbb{CP}^\infty)$. Assume that $n \ge 2$. Consider the AHSS

$$E^2_{k,t} = H_k(BU(n)) \otimes E_t(*) \implies E_{k+t}(BU(n)).$$

By proposition 2.4.1 on page 48 of [4], the map $BU(1)^n \to BU(n)$ allows us to express $H_*(BU(n))$ as a free abelian group with basis

$$\{a_{k_1} \cdots a_{k_t} | t \le n\}.$$

The map $BU(1) \times BU(n-1) \to BU(n)$ induces a pairing from the tensor product of the two AHSS for $E_*(BU(1))$ and $E_*(BU(n-1))$ to the AHSS for $E_*(BU(n))$. Each product $a_{k_1}\cdots a_{k_t}$ is an infinite cycle representing $\alpha_{k_1}\cdots\alpha_{k_t}$. The AHSS of $E_*(BU(n))$ then collapses on the second page. Since the stable page is a free $\pi_*(E)$-module, we have that $E_*(BU(n))$ is a free $\pi_*(E)$-module with basis $\alpha_{k_1}\cdots\alpha_{k_t}$.

(2) Fix a positive integer k. Let $BU(n)_{(k)}$ denote the k-th skeleton of $BU(n)$. Consider the pairing of the homological and cohomological AHSS

$$E_2^{s,t} = H^s(BU(n)_{(k)}) \otimes E^t(*) \implies E^{s+t} \tag{6.5}$$

and

$$E^2_{s,t} = H_s(BU(n)_{(k)}) \otimes E_t(*) \implies E_{s+t}(BU(n)_{(k)}) \tag{6.6}$$

Note $H_*(BU(n)_{(k)})$ has a basis consisting of the set of all $a_{k_1}\cdots a_{k_t}$ of degree at most k. Then spectral sequence 6.6 is a subspectral sequence of the spectral sequence used in the previous item. Therefore, it collapses on the second page. Thus,

$$E_*(BU(n)_{(k)}) \cong \pi_*(E)\langle \alpha_{k_1}\cdots\alpha_{k_t} | t \le n \text{ and } k_1+\cdots+k_t \le k/2\rangle.$$

Note that $H^*(BU(n)) \cong \mathbb{Z}[c_1,\cdots,c_n]$ and under the identification $H^*(BU(n)) \cong \mathrm{Hom}(H_*BU(n),\mathbb{Z})$, the k-th chern class c_k is the dual basis element to a_1^k. Furthermore, $H^*(BU(n)_{(k)})$ is the subgroup of $\mathbb{Z}[c_1,\cdots,c_n]$ with basis all monomials in terms of the c_k of degree at most k. Consider the pairing of spectral sequences 6.5 and 6.6. The c_k are all infinite cycles. Multiplicativity of spectral sequence 6.5 implies that it collapses on the second page. Then the pairing induced by the pairing of stable pages is nonsingular. Therefore,

$$E^*(BU(n)_{(k)}) \cong \mathrm{Hom}_{\pi^*(E)}(E_*(BU(n)_{(k)}), \pi_*(E)).$$

Let $cf_k = (a_1^k)^*$, which projects to $c_k \in E_\infty^{2k,0}$. Then $E^*(BU(n)_{(k)})$ is the free $\pi_*(E)$-module generated by all monomials in terms of $cf_1,\cdots,cf_n$ of degree at most k. Since the inverse system of the $E^*(BU(n)_{(k)})$ satisfies the Mittag-Leffler condition,

$$E^*(BU(n)) \cong \varprojlim E^*(BU(n)_{(k)}) \cong E^*[[cf_1,\cdots,cf_n]].$$

□

The Connor-Floyd-Chern classes satisfy axioms analogous to the axioms (3.3) of the ordinary Chern classes. Now that we have constructed them and understand the E-cohomology of $BU(n)$, we can show that all complex n-plane bundles admit an E-orientation. By naturality, it suffices to show that the universal n-plane bundle $\xi_n : EU(n) \downarrow BU(n)$ admits an E-orientation.

Recall the spherical fibration $S^{2n-1} \to BU(n-1) \to BU(n)$. Note

$$BU(n) \cong \mathbb{D}(\xi_n) \text{ and } BU(n-1) \cong \mathbb{S}(\xi_n).$$

Then

$$\tilde{E}^*(Th(\xi_n)) \cong E^*(\mathbb{D}(\xi), \mathbb{S}(\xi)) \cong E^*(BU(n), BU(n-1)).$$

Write $\theta : BU(n-1) \to BU(n)$. Then $\theta^* : E^*(BU(n)) \to E^*(BU(n-1))$ is surjective with kernel $E^*(BU(n), BU(n-1))$ generated by cf_n as a $\pi_*(E)$-module.

Proposition 6.7. *The class $cf_n \in E^{2n}(BU(n), BU(n-1))$ is an E-orientation of the universal n-plane bundle $\xi_n : EU(n) \downarrow BU(n)$.*

Proof. By the splitting principle (5.2), we have a map $f : F_{\xi_n} \to BU(n)$ that pulls back ξ_n to a direct sum of line bundles. Note that $F_{\xi_n} \cong BT^n \cong BU(1)^n$. The image of cf_n in $\tilde{E}^*(Th(f^*\xi_n))$ is

$$t_1 \cdots t_n \in \pi_*(E)[[t_1, \cdots, t_n]] \cong E^*(BU(1)^n).$$

But note that the total space of $f^*\xi_n$ is isomorphic to $\bigoplus_{i=1}^n \pi_i^* EU(1)$, where

$$\pi_i : BU(1)^n \to BU(n)$$

is the i-th projection map. We have reduced our claim to $n = 1$, and we are done since E is complex-orientable. □

The actual value of Connor-Floyd-Chern classes depend on both the cohomology theory and choice of complex orientation. While we have developed a valuable theory shared by all complex-oriented cohomologies, we have yet to develop a way of studying and differentiating them. The next two sections will introduce the beginning of this story.

7. Formal Group Laws and Lazard Ring

One way of differentiating between complex-oriented cohomologies is by studying how their Connor-Floyd-Chern classes behave with respect to tensor products of line bundles. Let E be a complex-oriented cohomology theory. Let ℓ_1, ℓ_2 be complex line bundles over X, and let $g_1, g_2 : X \to \mathbb{CP}^\infty$ denote their respective classifying maps. The classifying map of $\ell_1 \otimes \ell_2$ factors through the map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ that classifies $\pi_1^*(\xi_1) \otimes \pi_2^*(\xi_1)$, in the following way:

$$\begin{tikzcd}
\ell_1 \otimes \ell_2 \arrow[r] \arrow[d] & \pi_1^*(\xi_1) \otimes \pi_2^*(\xi_1) \arrow[r] \arrow[d] & \xi_1 \arrow[d] \\
X \arrow[r, "g_1 \times g_2"'] & \mathbb{CP}^\infty \times \mathbb{CP}^\infty \arrow[r] & \mathbb{CP}^\infty
\end{tikzcd}$$

This classifying map from $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$ to $\mathbb{CP}^\infty$ induces a map

$$\pi_*(E)[[z]] \cong E^*(\mathbb{CP}^\infty) \to E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong \pi_*(E)[[x, y]]$$

where $z \mapsto F^E(x, y)$. By naturality of the Connor-Floyd-Chern classes, we have the following.

Proposition 7.1. *There is a power series $F^E(x, y) \in \pi_*(E)[[x, y]]$ such that*

$$cf_1(\ell_1 \otimes \ell_2) = F^E(cf_1(\ell_1), cf_1(\ell_2)).$$

Because tensor products of line bundles are unital, associative, and commutative, the power series $F^E(x, y)$ is an example of a **formal group law**.

Definition 7.2. A **formal group law** over a graded commutative ring R is a homogeneous power series $F(x, y) \in R[[x, y]]$ of degree two such that

$$F(x, 0) = x, F(0, y) = y$$

$$F(x, F(y, z)) = F(F(x, y), z)$$

and

$$F(x, y) = F(y, x).$$

Example 7.3. The additive formal group law $G_a(x,y)$ is $x+y$ and the multiplicative formal group law $G_m(x,y)$ is $x+y+vxy$ for some $|v|=-2$.

Proposition 7.4. *The functor $\mathcal{F}: GCR \to Set$ sending a graded commutative ring R to the set of formal group laws over R is co-representable. In other words, there is a ring L and a formal group law G over L, such that any formal group law F over R can be obtained from G over L by applying a unique ring homomorphism $L \to R$.*

Proof. Let $S = \mathbb{Z}[a_{ij}]$, for all pairs of natural numbers (i,j) except for $(0,0)$. Each a_{ij} is an indeterminate.

A formal group law $F(x,y) = x+y+\sum_{i,j>0} a_{ij}x^iy^j$ over R introduces relations between the a_{ij} appearing in its summation. We must have $a_{10} = a_{01} = 1$. Commutativity implies $a_{ij} = a_{ji}$. More tediously, there is associativity: $F(x, F(y,z)) = F(F(x,y),z)$. Expanding both sides yields

$$\sum_{ij>0} a_{ij}(x+y+\sum_{m,k>0} a_{mk}x^my^k)^iz^j = \sum_{i,j>0} a_{ij}x^i(y+z+\sum_{m,j>0} a_{mk}y^mz^k)^j.$$

Let L denote the ring S modulo these relations. We define the desired formal group law G over L to be $G(x,y) = x+y+\sum_{ij} a_{ij}x^iy^j$.

Suppose $F(x,y) = x+y+\sum_{i,j>0} c_{ij}x^iy^j$ is a formal group law over R. Let $\phi : L \to R$ be the ring homomorphism defined by sending $a_{ij} \to c_{ij}$. Applying $\phi : L \to R$ maps G to F. □

While the **Lazard ring** L appears to be large and complicated, we have the following description.

Theorem 7.5 (Lazard's Theorem)**.** *$L \cong \mathbb{Z}[x_1, x_2, \cdots]$, where $|x_i| = -2i$.*

The rest of this section will be devoted to proving this. The first step is to show theorem 7.10, which states that $L \otimes \mathbb{Q} \cong \mathbb{Q}[x_1, x_2, \cdots,]$. A preliminary notion we need is morphisms and isomorphisms between formal group laws.

Definition 7.6. Let R be a graded commutative ring, and F, G formal group laws over R. Define

$$\mathrm{Hom}_R(F,G) = \{\phi \in (R[[X]])^2 : \phi(0) = 0, \phi F(X,Y) = G(\phi X, \phi Y)\}.$$

Furthermore, we require every $\phi \in \mathrm{Hom}_R(F,G)$ to be homogeneous of degree two.

Note that any $\phi \in \mathrm{Hom}_R(F,G)$ is of the form $a_0x + a_1x^2 + \cdots$ where $|a_i| = -2i$. In $R[[X]]$, any $\phi = a_0x + a_1x^2 + \cdots$ is invertible if a_0 is a unit in R. Then $\phi \in Hom_R(F,G)$ is an isomorphism if and only if a_0 is a unit in R. We say that ϕ is a **strict isomorphism** if $a_0 = 1$.

Example 7.7. Let invertible $\phi = x + a_1x^2 + \cdots$ be homogeneous of degree two. Let F be a formal group law over R. Define

$$^{\phi}F(X,Y) := \phi F(\phi^{-1}X, \phi^{-1}Y) \text{ and } F^{\phi}(X,Y) := \phi^{-1}F(\phi X, \phi Y).$$

Both $^{\phi}F(X,Y)$ and $F^{\phi}(X,Y)$ are formal group laws over R by properties of ϕ, and ϕ can be considered as a map in both $\mathrm{Hom}_R(F, ^{\phi}F)$ and $\mathrm{Hom}_R(F^{\phi}, F)$.

To show theorem 7.10, we will consider a map from the Lazard ring to $\mathbb{Q}[m_1, m_2, \cdots]$ which classifies a special formal group law. This formal group law is the following.

Proposition 7.8. *Let $R = \mathbb{Z}[m_1, m_2, \cdots]$ where $|m_i| = -2i$. Let*

$$\log X := X + m_1 X^2 + m_2 X^3 + \cdots .$$

Then $G_a^{\log}(X,Y) = \log^{-1}(\log X + \log Y)$ is the universal formal group law for group laws strictly isomorphic to the additive group law.

Proof. Suppose R' is a graded commutative ring, and F a formal group law over R strictly isomorphic to $G_a = X + Y$. Then there exist $\phi = X + a_1 X^2 + \cdots$ such that $\phi F = \phi X + \phi Y$. Then $F = \phi^{-1}(\phi X + \phi Y)$. Define a map from R to R' where $m_i \mapsto a_i$. This defines a graded ring homomorphism that maps $G_a^{\log}$ to F. □

The inverse to the desired map $L \otimes Q \to \mathbb{Q}[m_1, m_2, \cdots]$ will fall out from the $\mathbb{Q}$-algebra structure.

Proposition 7.9. *Over a $\mathbb{Q}$-algebra, any formal group law is strictly isomorphic to the additive group law. Denote $\log_F : F \to G_a$ as an isomorphism, and $\exp_F$ its inverse. By construction, $\log_F$ and $\exp_F$ are unique.*

Proof. Suppose we have a strict isomorphism ϕ between formal group law F and G_a. Then $\phi F = \phi_X + \phi_Y$. Differentiating with respect to Y and setting $Y = 0$ yields $\phi'(X)F_2(X,0) = 1$, where $F_2 = \frac{\partial}{\partial Y}F(X,Y)$. Then $\phi(X) = \int_0^X \frac{1}{F_2(t,0)}dt$, and ϕ exists if R is a $\mathbb{Q}$-algebra. □

Combining Propositions 7.9 and 7.8 yields

Theorem 7.10. *$L \otimes \mathbb{Q} \cong \mathbb{Q}[m_1, m_2, \cdots]$, where $|m_i| = -2i$*

Proof. We have a unique map $L \to R \otimes \mathbb{Q}$ classifying the universal additive group law $G_a^{\log}$. This maps factors through a map $L \otimes \mathbb{Q} \to R \otimes \mathbb{Q}$. But since $L \otimes \mathbb{Q}$ is a $\mathbb{Q}$-algebra, there is a map from $R \otimes \mathbb{Q} \to L \otimes \mathbb{Q}$ classifying its group law. These maps are inverses. □

With theorem 7.10, we just need proposition 7.11 to prove Lazard's theorem (7.5). For any abelian A and positive integer n, consider the graded commutative ring $\mathbb{Z} \oplus A[2-2n]$, where $\mathbb{Z}$ is in degree 0 and A is in degree $2-2n$. Define the multiplication to be $(n,a)(m,b) = (nm, nb + am)$.

Consider the set of formal group laws $\mathcal{F}(\mathbb{Z} \oplus A)$. By proposition 7.4, $\mathcal{F}(\mathbb{Z} \oplus A)$ is in bijection with a subset of $Hom(L, \mathbb{Z} \oplus A)$, which consists of all graded commutative ring homomorphisms that map the degree zero generator of L to the degree zero generator of $\mathbb{Z} \oplus A$. Consider the map $L \to \mathbb{Z}$ classifying the additive group law over $\mathbb{Z}$. Let I be the kernel of this map. There is a canonical projection $\mathbb{Z} \oplus A \to \mathbb{Z}$, with kernel A. Any map $L \to \mathbb{Z} \oplus A[2-2n]$ preserves the aforementioned kernels, and induces a map $I \to A$. By the multiplication we defined on $\mathbb{Z} \oplus A[2-2n]$, $A^2 = 0$. Hence, this map factors through $(I/I^2)^{2-2n} \to A$. This implies

$$\mathcal{F}(\mathbb{Z} \oplus A[2-2n]) = Hom(L, \mathbb{Z} \oplus A[2-2n]) = Ab((I/I^2)^{2-2n}, A).$$

We claim the following is true.

Proposition 7.11. *$\mathcal{F}(\mathbb{Z} \oplus A[2-2n]) \cong A$*

For clarity, we assume proposition 7.11 is true and first prove theorem 7.5 and prove proposition 7.11 afterwards.

Proof of Lazard's theorem. By proposition 7.11, $Ab((I/I^2)^{2-2n}, A) \cong A$, for all abelian A. By the structure theorem of finitely generated abelian groups, this implies $(I/I^2)^{2-2n} \cong \mathbb{Z}$. In degree $2-2i$, pick generator $x_{i-1} \in L$ which maps to the generator of $(I/I^2)^{2-2n}$. Then we have a surjection $\mathbb{Z}[x_1, x_2, \cdots] \to L$. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathbb{Z}[x_1, x_2, \cdots] \arrow[r] \arrow[d] & L \arrow[d] \\
\mathbb{Q}[x_1, x_2, \cdots] \arrow[r] & L \otimes \mathbb{Q}
\end{tikzcd}$$

The left map is injective because $\mathbb{Z}[x_1, x_2, \cdots]$ is torsion free. The bottom row is an isomorphism. Thus, $\mathbb{Z}[x_1, x_2, \cdots] \to L$ is injective. □

Thus, all that remains is to prove proposition 7.11. What does a formal group law over $\mathbb{Z} \oplus A[2-2n]$ look like? It looks like $F(X,Y) = X + Y + P(X,Y)$, where

$$P(X,Y) := \sum_{i,j \neq 0, i+j=n} a_i X^i Y^j.$$

This is due to the grading. The formal group law properties of F impose conditions on $P(X,Y)$. Symmetry implies $P(X,Y) = P(Y,X)$. Associativity is more complicated. Expanding $F(X, F(Y,Z)) = F(F(X,Y),Z)$ yields

$$X + Y + P(Y,Z) + \sum_{i+j=n} a_i X^i (Y + Z + P(Y,Z))^j$$

$$= X + Y + P(X,Y) + Z + \sum_{i+j=n} a_i (X + Y + P(X,Y))^i Z^j.$$

This implies

$$P(Y,Z) + P(X, Y+Z) = P(X,Y) + P(X+Y, Z). \tag{7.12}$$

This follows from the multiplication of $\mathbb{Z} \oplus A[2-2n]$. Now for $a \in A$, define $\Phi_a = X + aX^n$. We leave it to the reader to check that $\Phi_a^{-1} = \Phi_{-a}$. Thus, ${}^{\Phi_a}G_a = \Phi_a(\Phi_{-a}X + \Phi_{-a}Y)$ is a formal group law. Expanding, we obtain

$${}^{\Phi_a}G_a(X,Y) = X - aX^n + Y - aY^n + a(X - aX^n + Y - aY^n)^n$$

$$= X + Y + a[(X+Y)^n - X^n - Y^n]$$

We assert the following without proof.

Lemma 7.13. *Let $\epsilon_n := \gcd\{\binom{n}{i} | 1 \leq i \leq n-1\}$. Then $\epsilon_n = p$ if n is a power of p, and $\epsilon_n = 1$ otherwise.*

Define $C_n(X,Y) = \frac{1}{\epsilon_n}[(X+Y)^n - X^n - Y^n]$, which is in $\mathbb{Z}[X,Y]$. Since ${}^{\Phi_a}G_a(X,Y)$ is a formal group law, $X + Y + aC_n$ is a formal group law as well.

We are now ready to begin our *proof of proposition 7.11*. We claim the map $A \to \mathcal{F}(\mathbb{Z} \oplus A[2-2n])$, where $a \mapsto X + Y + aC_n(X,Y)$ is a bijection. If we denote the set of $P(X,Y)$ as $\mathcal{P}_n$, this is equivalent to showing $A \to \mathcal{P}_n$, where $a \mapsto aC_n(X,Y)$, is bijective. Injectivity follows from the fact that $C_n(X,Y)$ is a primitive polynomial. Surjectivity remains to be proven.

Letting nonnegative $i + j + k = n$, expanding equation (7.12) yields

$$\sum_{i=0} a_j Y^j Z^k + \sum a_i \binom{j+k}{j} X^i Y^j Z^k = \sum_{k=0} a_i X^i Y^j + \sum a_{i+j} \binom{i+j}{i} X^i Y^j Z^k.$$

Comparing coefficients of $XY^{m-1}Z^{n-m}$ for $1 \leq m \leq n-1$, we have

$$a_1 \binom{n-1}{m-1} = m a_m \tag{7.14}$$

We also assert the following lemma without proof.

Lemma 7.15. *$C_{pk}(X,Y) = uC_k(X^p, Y^p) \mod p$ for some unit u.*

We now begin our casework.

(1) $[A = \mathbb{Z}/p]$ Choose arbitrary $P(X,Y) \in \mathcal{P}_n$. Suppose p does not divide n. Consider $P - a_1 \frac{e_n}{n} C_n = \sum_i a'_i X^i Y^j$. This has leading term $a'_1 = 0$. Then by the above relation, $a'_m = 0$ for m not divisible by p. Consider the terms a'_m such that m is divisible by p. Then by symmetry, $a'_m = a'_{n-m} = 0$, where p cannot divide $n-m$. Thus, $P = a_1 \frac{e_n}{n} C_n$.

Suppose that $p = n$. Consider $P - a_1 C_n$. The leading term is $a'_1 = 0$. By equation (7.14), we have that all other terms must be zero.

Suppose that p divides n, but $p < n$. Let $n = pk$. By equation (7.14), $a_1 \binom{n-1}{p-1} = pa_p = 0$. This means $a_1 = 0$, so for all m divisible by p, $a_m = 0$. Then the leftover terms to consider are those with index divisible by p. Then $P(X,Y) = Q(X^p, Y^p)$, where Q satisfies equation (7.12). We can proceed by induction until k either equals p or is not divisible by p. In which case, our previous work shows that $Q = aC_k(X,Y)$. Then by lemma 7.15, $P(X,Y) = aC_k(X^p, Y^p) = auC_pk(X,Y)$, as desired.

(2) $[A = \mathbb{Z}/p^n]$ Proceed by induction on the diagram

$$\begin{tikzcd}
0 \arrow[r] & pA \arrow[r] \arrow[d, dashed, "\cong"] & A \arrow[r] \arrow[d] & A/pA \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathcal{P}_n(pA) \arrow[r] & \mathcal{P}_n(A) \arrow[r] & \mathcal{P}_n(A/pA) \arrow[r] & 0
\end{tikzcd}$$

(3) $[A = \mathbb{Q}]$ We want to show that the map $\mathbb{Q} \to \mathbb{Z} \oplus \mathbb{Q}[2-2n]$ is a bijection. Since $\mathbb{Z} \oplus \mathbb{Q}$ is a $\mathbb{Q}$-algebra, by proposition 7.9, any formal group law F is strictly isomorphic to the additive group law. Then $F = \theta(\theta^{-1}X + \theta^{-1}Y)$ where $\theta = X + aX^n$. But we've seen this construction before! F will be of the form $X + Y + a[(X+Y)^n - X^n - Y^n]$, and thus is of the form $X + Y + a\epsilon_n C_n$.

(4) $[A = \mathbb{Z}]$ This follows from the $A = \mathbb{Q}$ case because C_n is a primitive polynomial.

(5) $[A$ is a finitely generated abelian group$]$ This follows from our previous casework and the commutativity of the following diagram.

$$\begin{tikzcd}
 & A \oplus B \arrow[dl, "\cong"'] \arrow[dr, dashed] & \\
\mathcal{P}_n(A) \oplus \mathcal{P}_n(B) \arrow[rr, "\cong"] & & \mathcal{P}_n(A \oplus B)
\end{tikzcd}$$

(6) [A is an arbitrary abelian group] Suppose we have $P = \sum a_i X^i Y^j$. Then consider the abelian group finitely generated by all a_i. This is a subset of A. By our previous work, $P = aC_n$ for some $a \in A$.

8. Towards MU

Recall from proposition 7.1 that every complex-oriented E has an associated formal group law F^E. The assignment

$$\{\text{complex-oriented cohomology theories}\} \to \{\text{formal group laws}\}$$

is a fruitful way of studying complex-oriented cohomologies. Understanding exactly why this is a useful assignment is beyond the scope of this paper, but the rough idea is that formal group laws are both reasonably tractable objects and remember a great deal about the cohomology theory. In fact, under mild hypotheses, one can reconstruct E from its formal group law. The 'universal complex-oriented cohomology theory' MU plays a central role in this reconstruction.

Definition 8.1. Define the complex cobordism prespectrum MU in the following way. For the spaces, let

$$MU_{2n} = MU(n) := Th(\xi_n) \text{ and } MU_{2n+1} = \Sigma MU(n).$$

For the structure maps, consider the classifying map $BU(n) \to BU(n+1)$ of $\xi_n \oplus \epsilon$. This induces a map on Thom spaces

$$Th(\xi_n \oplus \epsilon) \to Th(\xi_n)$$

which we define to be the structure map of $\Sigma^2 MU(n) \to MU(n+1)$. Furthermore, we let $\Sigma MU_{2n} = \Sigma MU(n) \to MU_{2n+1}$ be the identity map.

It turns out that MU-cohomology is complex-oriented. Furthermore, it is the universal one in the following sense.

Theorem 8.2. *Let E be a complex-oriented cohomology theory. Let $x^{MU} \in \tilde{MU}^2(\mathbb{CP}^\infty)$ denote the canonical complex orientation of MU. The map*

$$[\phi : MU \to E] \mapsto \phi(x^{MU})$$

determines a bijection between ring spectrum maps $MU \to E$ and complex orientations of E.

Proof. See lecture 6 of [5] for justifications. □

Given a formal group law F^E, how does MU help recover E? The canonical complex orientation of MU determines a formal group law $f(x, y) \in \pi_*(MU)[[x, y]]$ by proposition 7.1. By proposition 7.4, this formal group law is given by a map of graded rings $L \to \pi_*(MU)$. In fact, Quillen proved the following celebrated theorem.

Theorem 8.3 (Quillen)**.** *The map $L \to \pi_*(MU)$ is an isomorphism of graded rings.*[4] [5]

[4]A proof can be found on page 75 of [1].

[5]A corollary is that the homotopy groups of MU are zero at odd degrees, by theorem 7.5.

This observation of Quillen's leads one to construct an inverse (not a true inverse) to the assignment discussed at the beginning of this section. Namely, suppose we have a "suitable" commutative ring R and a formal group law $F(x, y) \in R[[x, y]]$, classified by a map $\phi : L \to R$. We can attempt to define a new cohomology theory E by $E^*(X) = MU^*(X) \otimes_L R$ for finite complexes X. This construction does not always satisfy the generalized cohomology theory axioms, but Landweber's exact functor theorem (LEFT) gives a purely algebraic criterion on ϕ to guarantee when E^* is a cohomology theory.

While, for example, complex K-theory was discovered geometrically, interesting cohomology theories such as elliptic cohomology were first constructed using LEFT. Although LEFT is very useful, it should be noted that it is far from a complete description of the correspondence between complex-oriented cohomologies theory and formal group laws. For example, the Brown-Peterson spectrum and Morava K-theories are important and interesting ring spectra whose complex-oriented cohomologies cannot be reconstructed from their associated formal group laws.

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