HAUSDORFF DIMENSION AND FALCONER'S DISTANCE CONJECTURE

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Abstract. We begin by defining Hausdorff dimension and proving some of its basic properties. Then, using Fourier analysis, we establish a result on the Hausdorff dimension of projections of sets. Finally, we introduce Falconer's distance conjecture and show that it does not hold for the distance set under the supremum norm.

Contents

1. Hausdorff measure and dimension

1.1. Background: the Lebesgue measure.

Question 1.1. How do we formalize the notion of the "size" of an arbitrary subset $A \subset \mathbb{R}^n$?

We can begin by considering the rectangles in $\mathbb{R}^n$, i.e., the set of Cartesian products of n closed intervals $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$, with volume $v(Q) := \Pi_1^n (b_i - a_i)$. By Carathéodory's extension theorem, we can extend this idea of size to a very wide class of subsets of $\mathbb{R}^n$ as follows.

Definition 1.2. We define a function m^*, called the *Lebesgue outer measure*, that maps a subset A of $\mathbb{R}^n$ to a number in $[0, \infty]$ as

$$m^*(A) = \inf \left\{ \sum_i v(Q_i) : A \subset \bigcup_{i=1}^{\infty} Q_i,\ Q_i \text{ rectangles} \right\}. \tag{1.3}$$

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We then restrict the domain of m^* to *measurable sets* (sets $E \subset \mathbb{R}^n$ which satisfy $m^*(S) = m^*(S \cap E) + m^*(S \cap E^c)$ for all $S \subset \mathbb{R}^n$) to get a function m, which we call the *Lebesgue measure*. (All of the sets we mention in this paper will be assumed to be measurable.) If we were being precise, we would call this function m^n, as it depends on n, but most of the time the value of n is clear from context.

Remark 1.4. The n-dimensional Lebesgue measure is a very powerful tool, but it has some shortcomings. Importantly, a wide class of sets in $\mathbb{R}^n$ have measure zero (i.e., they have $m(A) = 0$), and within this class, there are some sets that are considerably larger than others. For example, both the empty set and the hyperplane $\mathbb{R}^{n-1} \times \{0\}$ have measure zero in $\mathbb{R}^n$, and both $\{0\}$ and the Cantor set C have measure zero in $\mathbb{R}$. This begs the question: are there any ways of measuring the size of a set that are sensitive to the differences between sets like these? This will be the focus of the next section.

1.2. **The Hausdorff measure.**

Definition 1.5. For $A \subset \mathbb{R}^n$, $s \leq n$, and with $d(E)$ denoting the diameter of E (i.e., $d(E) = \sup\{|x - y| : x, y \in E\}$), the *s-dimensional Hausdorff measure* $\mathcal{H}^s(A)$ is defined as

$$\lim_{\delta \to 0} \mathcal{H}^s_\delta(A), \tag{1.6}$$

where

$$\mathcal{H}^s_\delta(A) = \inf \left\{ \sum_i d(E_j)^s : A \subset \bigcup_j E_j, d(E_j) < \delta \right\}, \tag{1.7}$$

restricted to the domain of measurable sets in a manner entirely analogous to that used in Definition 1.2.

This definition holds up to multiplication by a constant dependent on s, often chosen for the integer values of s to have $\mathcal{H}^s = m$, the Lebesgue measure on $\mathbb{R}^s$. For other values, we usually choose the constant to be 1.

Definition 1.8. It is immediate from the preceding definition that for each $A \subset \mathbb{R}^n$, there is a unique real number s such that $\mathcal{H}^t(A) = \infty$ for all $t < s$ and $\mathcal{H}^r(A) = 0$ for all $r > s$. This number is called the *Hausdorff dimension* (or simply *dimension*) of A and is denoted $\dim A$.

Importantly, Definition 1.8 does not require s to be an integer. The following proposition demonstrates the existence of a set with fractional Hausdorff dimension.[1]

Proposition 1.9. *The Cantor set C has Hausdorff dimension $s = \log(2)/\log(3)$. Further, $\mathcal{H}^s(C) = 1$.*

Proof. We follow [Fal85, p. 14]. To show $\dim C \leq s$, we recall that $C = \cap_i C_i$, where $C_1 = [0, 1]$, $C_2 = [0, 1/3] \cup [2/3, 1]$, and so on. Thus, each C_i is a cover of C with 2^i intervals of length 3^{-i}, so $\mathcal{H}^s_{3^{-i}}(C) \leq 3^{-is} \cdot 2^i = 1$. Taking $i \to \infty$, we have $\mathcal{H}^s(C) \leq 1$, and therefore $\dim C \leq s$.

To show $\dim C \geq s$, it suffices to show that for any collection of intervals $\{I_j\}$ covering C, we have $\sum |I_j|^s \geq 1$. Expanding each I_j to an open set without

[1] This is where the term *fractal* comes from!

increasing its diameter by more than a factor of $(1+\epsilon)$, we can apply compactness to pass to a finite subcover. As C is totally disconnected, we can take this cover to be disjoint. Denote the sub-intervals of each C_i by $K_{i,n}$ (e.g., $K_{2,1} = [0,1/3]$ and $K_{3,4} = [8/9,1]$). Then, we can find an k such that each separate sub-interval in C_k is contained in exactly one I_j. Then, after shrinking the intervals if necessary, we note that if any I_j contains more than one sub-interval of C_k, then it must have, for some for some j_1, n_1, j_2, n_2,

$$I_j = (K_{j_1,n_1} \cap I_j) \cup L \cup (K_{j_2,n_2} \cap I_j), \tag{1.10}$$

with $L \subset C^c$ and $d(L) \geq d(I_j)/3$. Denoting $(K_{j_1,n_1} \cap I_j)$ by J_1 and $(K_{j_2,n_2} \cap I_j)$ by J_2, we have

$$\begin{aligned} |I_j|^s &= (|J_1| + |L| + |J_2|)^s \\ &\geq \left(\frac{3}{2}(|J_1| + |J_2|)\right)^s \\ &= 2\left(\frac{1}{2}|J_1| + \frac{1}{2}|J_2|\right)^s \\ &\geq |J_1|^s + |J_2|^s, \end{aligned} \tag{1.11}$$

because $3^s = 2$ and the function $f(x) = x^s$ is convex. Thus, we can pass to a subset of $\cup I_j$ in this way without increasing $\sum |I_j|^s$. We can continue to do this until we reduce $\cup I_j$ to C_i. We know that $\sum_{1 \leq i \leq n} |K_{i,n}|^s \geq 1$, and, taking $\epsilon \to 0$, we see that we never increased $\sum |I_j|^s$ to get to this point, so we have $\dim C \geq s$ and the proof is complete. □

A particularly convenient relationship between the Hausdorff dimension of a set and how the size of the set scales with distance is given by the following lemma.

Lemma 1.12 (Frostman). *Let A be a Borel set in $\mathbb{R}^n$, and let $\mathcal{M}(A)$ denote the set of finite nonzero measures with compact support contained in A. Then $\mathcal{H}^s(A) > 0$ if and only if there exists a measure $\mu \in \mathcal{M}(A)$ such that*

$$\mu(B(x,r)) \leq r^s \quad \forall x \in \mathbb{R}^n, \forall r > 0, \tag{1.13}$$

where $B(x,r)$ is the ball of radius r centered at x. Thus,

$$\dim A = \sup\left\{s : \text{there exists a } \mu \in \mathcal{M}(A) \text{ such that (1.13) holds}\right\}. \tag{1.14}$$

Proof. Following [Mat15, p. 18], we prove the lemma in the case that A is compact, though it holds whenever A is Borel. We begin with the backwards direction. If we have such a measure μ, then for any cover $\{E_j\}$ of A,

$$\sum d(E_j)^s = \sum d(B_j)^s \geq \sum \mu(B_j) \geq \sum \mu(E_j) \geq \mu(A) > 0, \tag{1.15}$$

where B_j is the ball containing E_j with $d(B_j) = d(E_j)$. This implies by definition that $\mathcal{H}^s(A) > 0$.

Now for the forward direction. We begin with a claim.

Claim 1.16. Any sequence (ν_i) of Borel measures with $\sup_i \nu_i(\mathbb{R}^n) < \infty$ contains a subsequence that *weakly converges* to a Borel measure ν, that is, we can find a subsequence (ν_{i_k}) and a Borel measure ν such that for any continuous, compactly supported φ defined on $\mathbb{R}^n$ (symbolically, any $\varphi \in C_0(\mathbb{R}^n)$),

$$\int \varphi \, d\nu_{i_k} \to \int \varphi \, d\nu, \tag{1.17}$$

as $k \to \infty$.

Proof. By the Stone-Weierstrass theorem (see [Rud76, p. 159]), $C_0(\mathbb{R}^n)$ is separable. Take a countable dense subset $\{f_n\}$ of $C_0(\mathbb{R}^n)$ with respect to the supremum norm. Then, by assumption, the set $(\int f_1 \, \mathrm{d}\nu_i)$ has a limit point in $\mathbb{R}$, so we can pass to a subsequence $(i_{k,1})$ such that $(\int f_1 \, \mathrm{d}\nu_{i_{k,1}})$ converges. Proceed inductively to find a subsequence $(i_{k,n})$ of $(i_{k,n-1})$ such that $(\int f_n \, \mathrm{d}\nu_{i_{k,n}})$ converges. Then, we define

$$T(f_n) = \lim_k \int f_n \, \mathrm{d}\nu_{i_{k,k}}, \tag{1.18}$$

which exists for all f_n in our dense set and therefore can be extended to all f. Because T is a positive linear functional, we can apply the Riesz representation theorem to get the desired ν, proving the claim. □

Back to the main event. By assumption, $\mathcal{H}^s(A) > 0$, so for any sets E_j covering A, we have

$$\sum_j d(E_j)^s > c, \tag{1.19}$$

for some $c > 0$. We will define μ as the weak limit of a sequence (μ_k), where each element of this sequence will in turn be a limit of $(\mu_{k,i})$. Break $\mathbb{R}^n$ into (countably many) dyadic cubes (denoted Q_i) of side length 2^{-k}. Then, let

$$\mu_{k,0}(E) = \sum_i c_i m(E \cap Q_i), \tag{1.20}$$

where c_i is chosen so that $\mu_{k,0}(Q_i) = d(Q_i)^s$. Then, to make $\mu_{k,n}$, we can consider $\mu_{k,n-1}$ on dyadic cubes Q_j of side length 2^{-k+n}. Then we let

$$\mu_{k,n}(E) = \sum_i d_i \mu_{k,n-1}(E \cap Q_i), \tag{1.21}$$

where

$$d_i = \begin{cases} 1 & \mu_{k,n-1}(Q_j) \le d(Q_j)^s \\ c_i & \mu_{k,n-1}(Q_j) > d(Q_j)^s \end{cases}. \tag{1.22}$$

and c_i is again chosen so that $\mu_{k,n}(Q_i) = d(Q_i)^s$. We can continue this process until we have one dyadic cube containing our original set A, and the measure we have at this point in the process will be denoted μ_k. By construction, we know that $\mu_k(Q_i) \le d(Q_i)^s$ whenever Q_i has side length at least 2^{-k}. In fact, by the definition of $\mu_{k,0}$, this holds for all dyadic cubes, and therefore $\mu_k(B) \le C(n) d(B)^s$ for all balls B, where $C(n)$ is a number dependent only on n.

We also know that each $x \in A$ is contained in a dyadic cube Q_j such that $\mu_k(Q_j) = d(Q_j)^s$, so, choosing the largest such cube for each x, we have a disjoint set of cubes that cover A and, by (1.19),

$$\mu_k(\mathbb{R}^n) = \sum_j \mu_k(Q_j) = \sum_j d(Q_j)^s > c. \tag{1.23}$$

Taking a weakly converging subsequence of μ_k, which must exist by Claim 1.16, we get a μ that satisfies the necessary properties. Our μ is clearly supported on A, satisfies (1.13) by construction (up to multiplication by a constant dependent on n), and is nonzero by (1.23). □

This result allows us to obtain an inequality that will be used to prove Theorem 3.7.

Proposition 1.24. *If $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ are Borel sets, then*

$$\dim A + \dim B \leq \dim(A \times B). \tag{1.25}$$

Proof. We follow [Mat95, p. 115]. Let $s = \dim A$ and $t = \dim B$. By Frostman's lemma, we can find a $\mu_A \in \mathcal{M}(A)$ and $\mu_B \in \mathcal{M}(B)$ such that $\mu_A(B^m(x,r)) \leq r^s$ and $\mu_A(B^n(x,r)) \leq r^t$. Then $\mu := \mu_A \times \mu_B \in \mathcal{M}(A \times B)$, and we have

$$\begin{aligned} \mu(B^{m+n}((x,y),r)) &\leq \mu(B^m(x,r) \times B^n(y,r)) \\ &= \mu_A(B^m(x,r))\mu_B(B^n(x,r)) \\ &\leq r^{s+t}, \end{aligned} \tag{1.26}$$

so, applying Frostman's lemma again, we have the result. □

2. Energy integrals and Fourier transforms

2.1. Energy integrals.

Definition 2.1. The *s-energy* of a measure μ, denoted $I_s(\mu)$, is defined as

$$I_s(\mu) = \iint |x-y|^{-s} \, \mathrm{d}\mu_x \, \mathrm{d}\mu_y. \tag{2.2}$$

Energy integrals are an indispensable tool in our study of Hausdorff dimension. The following proposition connects the two topics.

Proposition 2.3. *For any Borel set $A \subset \mathbb{R}^n$,*

$$\dim A = \sup\left\{s : \textit{there exists a } \mu \in \mathcal{M}(A) \textit{ with } I_s(\mu) < \infty\right\}. \tag{2.4}$$

Proof. We follow [Mat15, p. 19]. Using the basic formula (see, e.g., [Mat95, p. 15])

$$\int f \, \mathrm{d}\mu = \int_0^\infty \mu(\{x \in X : f(x) \geq t\}) \, \mathrm{d}t \tag{2.5}$$

and the change of variables $r = t^{-1/s}$, we get

$$\int |x-y|^{-s} \, \mathrm{d}\mu_y = s \int_0^\infty \frac{\mu(B(x,r))}{r^{s+1}} \, \mathrm{d}r. \tag{2.6}$$

So, if we have a $\mu \in \mathcal{M}(A)$ satisfying (1.13), then for $0 < t < s$ we have

$$I_t(\mu) = t \iint_0^\infty \frac{\mu(B(x,r))}{r^{t+1}} \, \mathrm{d}r \, \mathrm{d}\mu_x \leq t\mu(\mathbb{R}^n) \int_0^{d(\mathrm{spt}(\mu))} r^{s-t-1} \, \mathrm{d}r < \infty. \tag{2.7}$$

To show the other direction, assume $I_s(\mu) < \infty$. Then $\int |x-y|^{-s} \, \mathrm{d}\mu_x < \infty$ μ-almost everywhere. Thus, we can find a $M \in \mathbb{R}$ such that the set

$$A := \left\{ y : \int |x-y|^{-s} \, \mathrm{d}\mu_x < M \right\}$$

has $\mu(A) > 0$. Then, $\mu|_A(B(x,r)) := \mu(A \cap B(x,r)) \leq 2^s M r^s$, so $\mu|_A$ satisfies (1.13), and we are done. □

2.2. **Fourier transforms.** The general structure of this section will closely follow [Mat15].

Definition 2.8. We define the *Fourier transform of a function* $f \in L^1(\mathbb{R}^n)$ as

$$\widehat{f}(t) = \int f(x) e^{-2\pi i x \cdot t} \, \mathrm{d}x. \tag{2.9}$$

Similarly, the *Fourier transform of a measure* μ is defined as

$$\widehat{\mu}(t) = \int e^{-2\pi i x \cdot t} \, \mathrm{d}\mu_x. \tag{2.10}$$

Before we write on some basic properties of the Fourier transform, we need another definition.

Definition 2.11. We let $(f * g)(x)$ denote the *convolution of the functions* f *and* g, defined as

$$(f * g)(x) := \int f(x - y) g(y) \, \mathrm{d}y. \tag{2.12}$$

Similarly, the *convolution of a function* f *and a measure* μ is defined as

$$(f * \mu)(x) := \int f(x - y) \, \mathrm{d}\mu_y. \tag{2.13}$$

We also define the *Riesz kernel*, denoted k_s, to be the function

$$k_s(x) = |x|^{-s}, \tag{2.14}$$

so, by (2.2), we can write

$$I_s(\mu) = \int k_s * \mu \, \mathrm{d}\mu. \tag{2.15}$$

We begin by noting that a function can be recovered from its Fourier transform.

Theorem 2.16 (Inversion)**.** *If* $f, \widehat{f} \in L^1(\mathbb{R}^n)$*, then*

$$f(x) = \int \widehat{f}(t) e^{2\pi i x \cdot t} \, \mathrm{d}t, \tag{2.17}$$

up to a set of measure zero.

Proof. See [Mat15, p. 27]. □

We now state some elementary properties of the Fourier transform.

Proposition 2.18. *The following formulas hold. (In (3),* $\cdot$ *denotes multiplication and* $\bar{g}$ *denotes the complex conjugate of* g*.)*

1. *If* $f, g \in L^1(\mathbb{R}^n)$*, then* $\int \widehat{f} g = \int f \widehat{g}$ *(product formula).*
2. *If* $f, g \in L^1(\mathbb{R}^n)$*, then* $\widehat{(f * g)} = \widehat{f}\widehat{g}$ *(convolution formula).*
3. *If* $f, g \in L^2(\mathbb{R}^n)$*, then* $\int f \cdot \bar{g} = \int \widehat{f} \cdot \overline{\widehat{g}}$ *(Parseval's formula).*
4. *If* $f, g \in L^2(\mathbb{R}^n)$*, then* $\|f\|_2 = \|\widehat{f}\|_2$ *(Plancherel's formula).*

Proof. See [SW71, pp. 3, 8] for a proof of the first two formulas. Parseval's formula (and therefore Plancherel's formula) follows from Theorem 2.16 and the product formula; see [Mat15, p. 29] for details. □

The following corollary results from applying Theorem 2.16 to Proposition 2.18(2).

Corollary 2.19. *When $f, g, \widehat{f}, \widehat{g}, fg \in L^1(\mathbb{R}^n)$, we have $\widehat{fg} = \widehat{f} * \widehat{g}$.*

We will now extend the idea of a Fourier transform to a more general class of functions.

Remark 2.20. By Plancherel's formula, the Fourier transform is an isometry from $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. As $C_0(\mathbb{R}^n)$ is contained in $L^1(\mathbb{R}^n)$ and dense in $L^2(\mathbb{R}^n)$, we can uniquely extend the transform to a linear isometry from $L^2(\mathbb{R}^n)$ to itself, thus defining the Fourier transform for all $f \in L^1(\mathbb{R}^n) \cup L^2(\mathbb{R}^n)$, and by linearity all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ [Mat15, p. 29].

By a standard interpolation argument, we can then define the Fourier transform for all $f \in L^p$, with $1 \leq p \leq 2$.

Then, using Parseval's formula, we can further extend the notion of a Fourier transform for $f \in L^p$, $p > 2$ by associating to each $f \in L^p$ the functional T_f, defined as

$$T_f(\varphi) = \int f\varphi, \tag{2.21}$$

whenever φ is a Schwartz function.[2] Then, we define its Fourier transform as the functional $\widehat{T_f}$ that satisfies

$$\widehat{T_f}(\varphi) = T_f(\widehat{\varphi}), \tag{2.22}$$

whenever $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Lemma 2.23. *When $0 < s < n$, there is a positive constant $\gamma(n, s)$ such that, in the sense defined above, the Fourier transform of the Riesz kernel k_s is given by $\gamma(n, s)k_{n-s}$, that is,*

$$\int k_s \widehat{\varphi} = \gamma(n, s) \int k_{n-s}\varphi, \tag{2.24}$$

when $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Proof. See [Mat15, p. 35] or [Kos]. □

Theorem 2.25. *If $\mu \in \mathcal{M}(A)$ and $0 < s < n$, then*

$$I_s(\mu) = \gamma(n, s) \int |\widehat{\mu}(x)|^2 |x|^{s-n} \, \mathrm{d}x. \tag{2.26}$$

Proof. We follow [Mat15, p. 38]. Letting $k_s = |x|^{-s}$, for $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\begin{aligned} I_s(\varphi) &= \int (k_s * \varphi)\varphi \, \mathrm{d}x \\ &= \iint k_s(y - x)\varphi(x)\varphi(y) \, \mathrm{d}x \, \mathrm{d}y \\ &= \iint k_s(z)\varphi(y - z)\varphi(y) \, \mathrm{d}z \, \mathrm{d}y \\ &= \int k_s(\tilde{\varphi} * \varphi), \end{aligned} \tag{2.27}$$

[2] A *Schwartz function* is an infinitely differentiable function that, along with its derivatives, approaches zero at infinity more quickly than $|x|^{-k}$ for any $k \in \mathbb{N}$. If f is a Schwartz function, we write $f \in \mathcal{S}(\mathbb{R}^n)$.

where the second to last step results from the change of variables $z = y - x$, and $\tilde{\varphi}(x)$ denotes $\varphi(-x)$. We note that by Corollary 2.19, $\tilde{\varphi} * \varphi$ is the Fourier transform of $|\widehat{\varphi}(x)|^2$, so we can apply the previous lemma to get

$$I_s(\varphi) = \gamma(n,s) \int k_{n-s} |\widehat{\varphi}(x)|^2 = \gamma(n,s) \int |x|^{s-n} |\widehat{\varphi}(x)|^2, \tag{2.28}$$

which establishes the result for smooth measures.

To prove the general case, it suffices to use approximate identities (see [Mat15, p. 14]) and some elementary convergence theorems. The details are omitted here for the sake of brevity. □

This is all of the Fourier analysis we will need to prove a result on the dimension of the projections of sets onto lines.

2.3. Dimension of projections.

Theorem 2.29. *Let $A \subset \mathbb{R}^n$ be Borel, $\dim A = s$, and let $P_e(A)$ denote the projection of A onto $e \in S^{n-1}$. If $s \leq 1$, then*

$$\dim P_e(A) = s \tag{2.30}$$

for e almost everywhere (a.e.) on S^{n-1} with respect to the standard measure on the unit sphere, denoted σ^{n-1}. If $s > 1$, then

$$m(P_e(A)) > 0 \tag{2.31}$$

for $e \in S^{n-1}$ (σ^{n-1}-a.e.).

Proof. We follow [Mat15, p. 55]. Let $\mu \in \mathcal{M}(A)$, pick an $e \in S^{n-1}$, and define a measure $\mu_e(B)$ for $B \subset \mathbb{R}$ by

$$\mu_e(B) = \mu(P_e^{-1}(B)). \tag{2.32}$$

It is immediate that $\mu_e(B) \in \mathcal{M}(P_e(A))$. Using the Fourier transform, we establish the relation

$$\widehat{\mu_e}(r) = \int e^{-2\pi i x r} \, \mathrm{d}\mu_e = \int e^{-2\pi i (y \cdot e) r} \, \mathrm{d}\mu = \widehat{\mu}(re), \tag{2.33}$$

where y is the variable of integration in the third expression. If $s \leq 1$, then we can pick a t such that $0 < t < s$, so by Lemma 1.12 we can find a $\mu \in \mathcal{M}(A)$ such that $I_t(\mu) < \infty$. Then, we can apply the above equation, Theorem 2.25, and a change of variables to get

$$\begin{aligned}
\int I_t(\mu_e) \, \mathrm{d}\sigma_e^{n-1} &= \int_{S^{n-1}} \gamma(1,t) \int_{\mathbb{R}} |\widehat{\mu}_e(r)|^2 |r|^{t-1} \, \mathrm{d}r \, \mathrm{d}\sigma_e^{n-1} \\
&= 2\gamma(1,t) \int_{S^{n-1}} \int_0^\infty |\widehat{\mu}(re)|^2 |r|^{t-1} \, \mathrm{d}r \, \mathrm{d}\sigma_e^{n-1} \\
&= 2\gamma(1,t) \int_{\mathbb{R}^n} |\widehat{\mu}(x)|^2 |x|^{t-n} \, \mathrm{d}m \\
&= 2\gamma(1,t) \gamma(n,t)^{-1} I_t(\mu) < \infty.
\end{aligned} \tag{2.34}$$

Thus, $I_t(\mu_e)$ is finite σ^{n-1}-a.e., so we can choose a sequence $t_i \to s$ to get that $\dim P_e(A) \geq s$ (σ^{n-1}-a.e.). If $s > 1$, then we have a $\mu \in \mathcal{M}(A)$ such that $I_1(\mu) < \infty$, so, by the same logic as in the prior case, we have

$$\int_{S^{n-1}} \int_{\mathbb{R}} |\widehat{\mu_e}(r)|^2 \, \mathrm{d}r \, \mathrm{d}\sigma_e^{n-1} = 2\gamma(n,1)^{-1} I_1(\mu) < \infty. \tag{2.35}$$

This shows $\widehat{\mu_e} \in L^2(\mathbb{R})$ (σ^{n-1}-a.e.), so $\mu_e \in L^2(\mathbb{R})$ (σ^{n-1}-a.e.); see [Mat15, p. 31] for details. This implies that μ_e is absolutely continuous with respect to m, so as $\mu_e \in \mathcal{M}(P_e(A))$, we have $\mu_e(\mathbb{R}^n) > 0$. Thus, we know that $m(P_e(A)) > 0$ (σ^{n-1}-a.e.), as desired. □

3. Falconer's distance conjecture

3.1. **Background: the Steinhaus theorem.** We begin with a theorem that has been known for over a century.

Theorem 3.1 (Steinhaus)**.** *If $A \subset \mathbb{R}$ has positive measure, then its distance set $\Delta(A) := \{|x-y| : x, y \in A\}$ contains a half-open interval $[0, \epsilon)$.*[3]

Proof. Suppose 0 is not an interior point of $\Delta(A)$ in $\mathbb{R}^+$. By Definition 1.2, we can find an interval $[a,b]$ such that $\frac{A \cap [a,b]}{m([a,b])} > \frac{3}{4}$, and by supposition, we can find a $t \notin \Delta(A)$ with $|t| < \frac{m([a,b])}{4}$, so that, defining $B := A \cap [a,b]$,

$$m(B \cup B+t) = m(B) + m(B+t) > \frac{3}{2} m([a,b]), \tag{3.2}$$

which contradicts the fact that B and $B+t$ are contained in an interval of length $\frac{5}{4} m([a,b])$. □

The preceding theorem relates the measure of a set to the measure of its distance set. However, the fact that the theorem holds for all sets of positive measure indicates that this may be too coarse a method of evaluating the size of the set, and this leads us to a fundamental question.

Question 3.3. Given $A \subset \mathbb{R}^n$, how large does $\dim A$ need to be to guarantee that $\Delta(A)$ has nonzero Lebesgue measure?

Conjecture 3.4 (Falconer)**.** *If $n \geq 2$ and $A \subset \mathbb{R}^n$ is Borel with $\dim A > n/2$, then $m(\Delta(A)) > 0$.*

The following proposition is also due to Falconer.

Proposition 3.5. *Let A be a Borel subset of $\mathbb{R}^n$ with $n \geq 2$.*

(1) If $\dim A \geq (n+1)/2$, we have $m(\Delta(A)) > 0$ (in fact, the interior of $\Delta(A)$ is nonempty).
(2) If $(n-1)/2 \leq \dim A \leq (n+1)/2$, then $\dim \Delta(A) \geq \dim A - (n-1)/2$.

Proof. See [Mat95, p. 59]. □

3.2. **Falconer's conjecture with the supremum norm.** Given the difficulty of the problem with the usual Euclidean metric on $\mathbb{R}^n$, it is illuminating to consider other metrics as well.

Namely, we will consider the distance set under the supremum (or L^∞) norm, defined as

$$\|x-y\|_\infty := \max_i |x_i - y_i|, \tag{3.6}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Let $\Delta'(A) := \{\|x-y\|_\infty : x, y \in A\}$. We will show that Falconer's distance conjecture does not hold when the Euclidean distance set $\Delta(A)$ is replaced by the L^∞ distance set $\Delta'(A)$.

[3] This can be generalized to sets of positive measure in $\mathbb{R}^n$ without much added effort.

Theorem 3.7 (Counterexample for the supremum norm)**.** *There exists a Borel set* $E \subset \mathbb{R}^2$ *such that* $\dim E > 1$ *and* $m(\Delta'(E)) = 0$*.*

Lemma 3.8. *There exist Borel subsets* $A, B \subset \mathbb{R}^n$ *such that* $\dim A = \dim B = 0$*, but* $\mathcal{H}^1(A \times B) > 0$*.*

Proof. We follow [Fal85, p. 73]. Fix a sequence t_i such that $t_i \to 0$. Then find a strictly increasing integer-valued sequence m_j such that $m_0 = 0$ and

$$k_1 := (m_1 - m_0) + (m_3 - m_2) + \cdots + (m_{2j-1} - m_{2j-2}) \leq t_j m_{2j} \tag{3.9}$$

and

$$k_2 := (m_2 - m_1) + (m_4 - m_3) + \cdots + (m_{2j} - m_{2j-1}) \leq t_j m_{2j+1}. \tag{3.10}$$

Now, let A be the set of points in $[0,1]$ which have 0 in the rth decimal place for $m_i < r \leq m_{i+1}$ for all odd i, and let B be the analogue of A for even i.

Then, we split $[0,1]$ into $10^{m_{2j}}$ equal intervals, and by construction, 10^{k_1} of them cover A. Thus, for any $s > 0$,

$$\mathcal{H}^s(A) \leq 10^{k_1 - s m_{2j}} \leq 10^{t_j m_{2j} - s m_{2j}} = 10^{m_{2j}(t_j - s)}, \tag{3.11}$$

so because $(t_j - s)$ must become negative as $j \to \infty$, the last term in the above inequality must tend to 0 in the same limit. Thus, $\dim A = 0$, and by an analogous argument, $\dim B = 0$. Then, we note that we the projection of the set $A \times B$ onto the line $y = x$ is the line itself, which is of dimension 1. As projections do not ever increase the dimension of a set, this gives the result. □

Remark 3.12. We note that the set $A \times B$ would be a counterexample to the modified distance conjecture, but it does not have dimension strictly greater than 1. Lemma 3.8 does, however, demonstrate that we do not have an analogue of Proposition 3.5 for the distance set under the supremum norm. This is because we have $1/2 \leq \dim(A \times B) \leq 3/2$, but not $\dim \Delta'(A \times B) \geq \dim(A \times B) - 1/2 \geq 1/2$, by an argument similar to that used to prove Theorem 3.7.

What remains now is to modify this example to demonstrate the failure of Falconer's distance conjecture under the supremum norm.

Proof of Theorem 3.7. Consider the set A of real numbers in $[0,1]$ with decimal representation $0.a_1a_2a_3a_4...$, where $a_{4i} = 0$ for $i \in \mathbb{N}$. We aim to show $\dim A = \frac{3}{4}$ by a process similar to that which we used to prove Proposition 1.9. First, we note that $A = \cap A_j$ where A_j is the set of reals in $[0,1]$ of the form $0.a_1a_2a_3a_4...$ where $a_{4i} = 0$ for $i = 1, \ldots, j$. We can cover each A_j (and therefore A) with 10^{3j} intervals of length 10^{-4j}, so that, taking $s = \frac{3}{4}$, we have

$$\mathcal{H}^s_{10^{-4j}} \leq 10^{3j} \cdot 10^{-4js} = 1. \tag{3.13}$$

Now, we take $j \to \infty$ to get that $\dim A \leq \frac{3}{4}$.

To show the reverse inequality, we follow a method similar to that used in [Fal03, p. 60]. Define a measure μ_j supported on each A_j such that μ_j is the unique multiple of the Lebesgue measure such that $\mu_j(A_j) = 1$. By Claim 1.16, we can pass to a subsequence which weakly converges to a measure μ. Clearly, $\mu \in \mathcal{M}(A)$, so showing that μ satisfies (1.13) for $s = \frac{3}{4}$ will be sufficient to demonstrate that $\dim A \geq \frac{3}{4}$. Take an interval I with $d(I) < 1$. Then, we can find the unique $j \in \mathbb{N}$ such that

$$10^{-4(j+1)} \leq d(I) < 10^{-4j}. \tag{3.14}$$

From this, we know that I intersects with at most one of the sub-intervals of A_j. Thus,

$$\mu(I) \leq 10^{-3j} = \left(10^{-4j}\right)^{\frac{3}{4}} \leq (10 \cdot d(I))^{\frac{3}{4}}, \tag{3.15}$$

so $\dim A = \frac{3}{4}$.

Let $E = A \times A$. By Proposition 1.24 we have that $\dim E \geq \frac{3}{2}$. Further, we have $\Delta'(E) = \Delta'(A) = \Delta(A)$ from the L^∞ norm and the fact that $A \subset \mathbb{R}$.

So, it remains to show that $\Delta(A)$ has dimension strictly less than 1 and therefore is Lebesgue measure zero in $\mathbb{R}$. It is immediate that each $x \in \Delta(A)$ is of the form $\pm 0.a_1a_2a_3a_4\ldots$, with $a_{4i} = 0$ or 9 for all $i \in \mathbb{N}$. Let D_j be the numbers of the form $\pm 0.a_1a_2a_3a_4\ldots$ with $a_{4i} = 0$ or 9 for $i = 1, \ldots, j$, so $\Delta(A) = \cap_j D_j$. Observe that for any $j \in \mathbb{N}$, we can cover D_j with $10^{3j} \cdot 2^{j+1}$ intervals of length 10^{-4j}. Taking

$$s = \frac{3 + \log_{10}(2)}{4} < 1, \tag{3.16}$$

we have

$$\begin{aligned} \mathcal{H}^s_{10^{-4j}} &\leq 10^{-4js} \cdot 10^{3j} \cdot 2^{j+1} \\ &= 10^{-4js} \cdot 10^{3j} \cdot 10^{\log_{10}(2)j} \cdot 2 \\ &= 2. \end{aligned} \tag{3.17}$$

Taking $j \to \infty$, we see that $\dim \Delta(A) < 1$. Thus, the Lebesgue measure of $\Delta(A) = \Delta'(E)$ is zero and the proof is finished. $\square$

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