AN INTRODUCTION TO STACKS

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ABSTRACT. In this paper, we give an introduction to the theory of stacks and the important ideas that come with it. We keep in mind the example of the category $\mathcal{M}_g$ of families of smooth curves $\mathcal{C} \longrightarrow S$ of genus g and develop theory motivated by results about $\mathcal{M}_g$.

CONTENTS

1. INTRODUCTION

A moduli space M is a geometric object (for example, topological space, manifold, or algebraic variety) whose points are in bijection with isomorphism classes of some collection of objects. Here are a few examples of moduli spaces.

(1) The projective line $\mathbb{P}^1_{\mathbb{C}}$ parameterizes lines in $\mathbb{C}^2$.
(2) Given a smooth curve C over a field k, its Jacobian variety $\mathrm{Jac}(C)$ parameterizes degree 0 line bundles on C.
(3) (dull example) Every topological space X parameterizes its points.

In this paper, we motivate the theory of stacks by attempting to construct a scheme-like space $\mathcal{M}_g$ which parameterizes smooth curves of genus g. We see how stacks naturally provide the suitable foundations to do so, and we prove interesting properties about $\mathcal{M}_g$ and general stacks.

Since we want points in $\mathcal{M}_g$ to correspond to isomorphism classes of genus g curves, it would be a good guess to define $\mathcal{M}_g$ as a set as

$$\mathcal{M}_g := \{[C] : C \text{ is a smooth curve of genus } g \text{ over some field}\}.$$

Using Yoneda lemma, we know that studying $\mathcal{M}_g$ under this definition is equivalent to studying the functor

$$\mathrm{Mor}(-, \mathcal{M}_g) : \mathbf{Set} \longrightarrow \mathbf{Set}.$$

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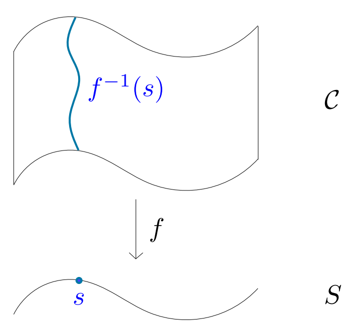
Since we are primarily interested in imposing a scheme-like geometry on $\mathcal{M}_g$, we might as well restrict our attention to morphisms $S \longrightarrow \mathcal{M}_g$ where S is a scheme. However, we do not want to consider arbitrary set morphisms $S \longrightarrow \mathcal{M}_g$ for the following reason. Say we have a set morphism $\varphi : S \longrightarrow \mathcal{M}_g$ and two points s_1 and s_2 in S that are "close" to each other. Then we want the isomorphism classes of curves $\varphi(s_1)$ and $\varphi(s_2)$ to also be "close". While the notion of being close can be made precise in S because of the underlying topological structure, there is no obvious way to measure closeness in the set $\mathcal{M}_g$. This suggests that we should define $\mathcal{M}_g$ in a different way.

Based on the functorial approach from earlier, we can try to define $\mathcal{M}_g$ as a functor

$$\mathbf{Sch} \longrightarrow \mathbf{Set}$$
$$S \mapsto \{\text{families of genus } g \text{ curves } \mathcal{C} \to S\}$$

where a family of curves $\mathcal{C} \longrightarrow S$ is defined as a smooth, proper morphism of schemes such that the fiber at each point $s \in S$ is a connected curve of genus g over the residue field of s. As a visual example, the diagram



represents a family of curves over S. By regarding curves as being embedded in some ambient geometric space, this definition of $\mathcal{M}_g$ makes precise when two curves are close to each other.

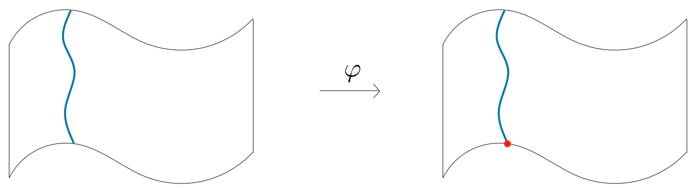
This definition of $\mathcal{M}_g$ is still not ideal. Since any two sets of the same cardinality are isomorphic, it is difficult to distinguish the sets $\{\text{families of genus } g \text{ curves } \mathcal{C} \to S\}$ from each other using just the set structure. Thus, we hope to add more structure to these sets. In particular, we want to define $\{\text{families of genus } g \text{ curves } \mathcal{C} \to S\}$ as a category for each S. To do so, we must define what morphisms between two objects are.

Since objects of $\mathcal{M}_g(S)$ are also objects in the larger category $\mathbf{Sch}/S$, we may think of defining morphisms in $\mathcal{M}_g(S)$ to be S-morphisms, i.e., defining a morphism $(\mathcal{C} \to S) \longrightarrow (\mathcal{C}' \to S)$ to be a scheme morphism $\mathcal{C} \longrightarrow \mathcal{C}'$ such that the following diagram commutes.

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{C}' \arrow[d] \\ S \arrow[r, "\mathrm{id}_S"] & S \end{tikzcd}$$

This turns out to not be the correct notion for our purposes. Consider the family of curves $\mathcal{C} \xrightarrow{f} S$ in the above diagram. Let $\tau : S \longrightarrow \mathcal{C}$ be a smooth section of f

(i.e., we have $f \circ \tau = \mathrm{id}$). Without loss of generality, we represent the topological image of τ by the bottom edge of $\mathcal{C}$. We define a morphism $\varphi : \mathcal{C} \longrightarrow \mathcal{C}$ as follows. Let x be a point in $\mathcal{C}$. Then $x \in f^{-1}(s)$ for some unique $s \in S$. We define $\varphi(x)$ to be the unique point of $f^{-1}(s)$ that intersects the bottom edge. More concisely, the morphism φ collapses a fiber of $\mathcal{C} \xrightarrow{f} S$ (represented in blue) to a single point (represented in red).



Note that φ is an S-morphism, yet it takes curves, which are the objects that we are primarily interested in, and collapses them down to a point. Since we want morphisms to take curves to curves, this notion of morphism is not suitable for our purposes. The correct definition of a morphism is the following. A morphism $(\mathcal{C} \to S) \longrightarrow (\mathcal{C}' \to S)$ of families of curves over S is a scheme morphism $\mathcal{C} \longrightarrow \mathcal{C}'$ such that the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C}' \arrow[d] \\
S \arrow[r, "\mathrm{id}_S"'] & S
\end{tikzcd}$$

is Cartesian (i.e., we have that $\mathcal{C}$ is the pullback of $\mathcal{C}'$ and S along S). However, this forces $\mathcal{C} \longrightarrow \mathcal{C}'$ to be an isomorphism. In particular, we get that $\mathcal{C} \longrightarrow \mathcal{C}'$ preserves fibers over S, so it must map curves to curves. While this definition of a morphism of family of curves seems ad-hoc, it is hoped that by the end of this paper, the reader will see how it naturally follows from defining $\mathcal{M}_g$ as a stack.

Except for Proposition 4.3, we assume very little knowledge of algebraic geometry for the first four sections. Readers who are not familiar with schemes may find it helpful to replace the word "scheme" by "manifold" in these sections. In section 5, we often use results from algebraic geometry at the level of [Vak17] and [Har77], but we try to informally state the main ideas of proofs when we do so.

2. Grothendieck topologies, sites, and prestacks

We first introduce Grothendieck topologies and sites. One usually defines a sheaf $\mathcal{F}$ (of sets) on a topological space X as a contravariant functor $\mathbf{Open}(X) \longrightarrow \mathbf{Set}$ satisfying certain gluing conditions. Sites are generalizations of this where there is an analogous notion of gluing. This allows us to define sheaves from a category that is not necessarily the category of open sets of a topological space.

Definition 2.1. A **Grothendieck topology** on a category $\mathcal{S}$ consists of the following data: for each $X \in \mathrm{Obj}(S)$, there exists a set $\mathrm{Cov}(X)$ consisting of coverings of X, i.e. collections of morphisms $\{X_i \longrightarrow X\}_{i \in I}$ such that the following conditions are satisfied.

- (identity) If $X' \longrightarrow X$ is an isomorphism, then $\{X' \longrightarrow X\}$ is a covering of X.

- (restriction) If $\{X_i \longrightarrow X\}_{i\in I}$ is a covering of X and $Y \longrightarrow X$ is a morphism, then $\{X_i \times_X Y \longrightarrow Y\}_{i\in I}$ is a covering of Y.
- (composition) If $\{X_i \longrightarrow X\}_{i\in I} \in \mathrm{Cov}(X)$ and $\{X_{ij} \longrightarrow X_i\}_{j\in J} \in \mathrm{Cov}(X_i)$ for all $i \in I$, then $\{X_{ij} \longrightarrow X_i \longrightarrow X\}_{i\in I, j\in J}$ is a covering of X.

A **site** is a category $\mathcal{S}$ equipped with a Grothendieck topology.

Since fiber products in $\mathrm{Open}(X)$ for a topological space X are just intersections, we see that $\mathbf{Open}(X)$ can be canonically endowed with a Grothedieck topology. We then see that the three conditions listed in definition 2.1 follows from the topological space axioms.

Sites were first introduced by Grothendieck to define étale cohomology. Though we will not be discussing étale cohomology in this paper, our motivating example $\mathcal{M}_g$ will be defined as a stack over the étale site.

Definition 2.2. The **étale site $\mathbf{Sch}_{\acute{e}t}$** is the category Sch where a covering of a scheme X is a collection of étale morphisms (i.e., morphisms that are smooth and unramified) $\{X_i \longrightarrow X\}$ such that $\sqcup_i X_i \longrightarrow X$ is surjective.

Note that since all open immersions are étale, we have that the étale topology is finer than the Zariski topology. We now give a more general definition of sheaves using sites. Recall that a **presheaf** $\mathcal{F}$ (of sets) on a category $\mathcal{S}$ is a contravariant functor $\mathcal{S} \longrightarrow \mathbf{Set}$. Given an object U in $\mathcal{S}$, we call elements $u \in \mathcal{F}(U)$ sections (of $\mathcal{F}$) over U. Let $V \xrightarrow{f} U$ be a morphism in $\mathcal{S}$ and let u be a section of U. We call $\mathcal{F}(f)(u) \in \mathcal{F}(V)$ the restriction of u to V and denote it by $u|_V$.

Definition 2.3. Let $\mathcal{S}$ be a site. A presheaf $\mathcal{F} : \mathcal{S} \longrightarrow \mathbf{Set}$ is a sheaf if for all $U \in \mathrm{Obj}(\mathcal{S})$ and all coverings $\{U_i \longrightarrow U\}_{i\in I}$ of U, the following hold.

(1) (sections glue) If there exist sections $x_i \in \mathcal{F}(U_i)$ such that $x_i|_{U_i\times_U U_j} = x_j|_{U_i\times_U U_j}$ for all $i, j \in I$, then there exists $x \in \mathcal{F}(U)$ such that $x|_{U_i} = x_i$ for all $i \in I$.
(2) (sections are determined locally) If $x, y \in \mathcal{F}(U)$ are sections over U such that $x|_{U_i} = y|_{U_i}$ for all $i \in I$, then $x = y$.

The two conditions in definition 2.3 are equivalent to requiring that for all coverings $\{U_i \longrightarrow U\}_{i\in I}$, the diagram

$$\mathcal{F}(U) \longrightarrow \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(U_i \times_U U_j)$$

be an equalizer diagram.

Example 2.4. (Schemes are Sheaves) Let X be a scheme. Then the functor $\mathrm{Mor}(-, X) : \mathbf{Sch} \longrightarrow \mathbf{Set}$ is a sheaf (for both the Zariski and the étale topology). This follows from the fact that morphisms of schemes glue uniquely in either topology.

Let $\mathcal{F}, \mathcal{G} : \mathcal{S} \longrightarrow \mathbf{Set}$ be presheaves. We define morphisms of presheaves to be natural transformations of functors [Aut22b]. This allows us to define the category $\mathbf{PSh}_{\mathcal{S}}$ of presheaves over $\mathcal{S}$. The category $\mathbf{Sh}_{\mathcal{S}}$ of sheaves over a site $\mathcal{S}$ is the full subcategory of $\mathbf{PSh}_{\mathcal{S}}$ where the objects are sheaves.

Given a sheaf $\mathcal{F}$ on a site $\mathcal{S}$ and an object $X \in \mathcal{S}$, we often want to consider the cases where $\mathcal{F}(U)$ has more structure than that of just a set. For example, in section 1, we considered $\mathcal{M}_g$ and regarded it as a morphism $\mathbf{Sch} \longrightarrow \mathbf{Cat}$. Motivated by this example, we might think that we should define a prestack $\mathcal{F}$ on a category $\mathcal{S}$ to be a contravariant functor $\mathcal{S} \longrightarrow \mathbf{Cat}$. However, we will give a different definition of a prestack and see how it naturally encodes a contravariant functor $\mathcal{S} \longrightarrow \mathbf{Cat}$.

Before doing so, we must fix some conventions. Given a functor of categories $p : \mathcal{X} \longrightarrow \mathcal{S}$, we say $X \in \mathrm{Obj}(\mathcal{X})$ lies over $S \in \mathrm{Obj}(\mathcal{S})$ if $p(X) = S$. Similarly, given X and X' over S and S', respectively, we say that $f \in \mathrm{Mor}_{\mathcal{X}}(X, X')$ lies over $g \in \mathrm{Mor}_{\mathcal{S}}(S, S')$ if $p(f) = g$. We represent this data as the following diagram.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, mapsto] & X' \arrow[d, mapsto] \\ S \arrow[r, "g"] & S' \end{tikzcd}$$

Definition 2.5. A **prestack** on a category $\mathcal{S}$ is a category $\mathcal{X}$ and a (covariant) functor $p : \mathcal{X} \longrightarrow \mathcal{S}$ satisfying the following two conditions.

(1) (pullbacks exist) Given a diagram

$$\begin{tikzcd} X' \arrow[r, dashed] \arrow[d, mapsto, dashed] & X \arrow[d, mapsto] \\ S' \arrow[r] & S \end{tikzcd}$$

of solid arrows, there exists an object X' of $\mathcal{X}$ lying over S' and a morphism $X' \longrightarrow X$ lying over $S' \longrightarrow S$.

(2) (pullbacks are universal) Given objects Y, X, and X' over R, S, and S', respectively, and a diagram

$$\begin{tikzcd} Y \arrow[r, dashed] \arrow[d, mapsto] \arrow[rr, bend left] & X \arrow[r] \arrow[d, mapsto] & X' \arrow[d, mapsto] \\ R \arrow[r] & S \arrow[r] & S' \end{tikzcd}$$

of solid arrows, there exists a unique morphism $Y \longrightarrow X'$ such that $Y \longrightarrow X' \longrightarrow X = Y \longrightarrow X$.

Remark 2.6. Note that (2) implies that the pullback in (1) is unique up to unique isomorphism. Therefore, we often write $X|_{S'}$ for the pullback X' of X.

Definition 2.5 should seem suspicious to the reader. After all, we claimed that prestacks would generalize the notion of presheaves, but prestacks are covariant whereas presheaves are contravariant! To see the connection between presheaves and prestacks, we must first define fiber categories.

Definition 2.7. Given a prestack $p : \mathcal{X} \longrightarrow \mathcal{S}$ and an object S of $\mathcal{S}$, we define the fiber category $\mathcal{X}(S)$ as the category whose objects are objects $X \in \mathrm{Obj}(\mathcal{X})$ that lie over S and whose morphisms are morphisms in $\mathcal{X}$ that lie over id_S.

Informally, we can think of the collection of objects in $\mathcal{X}(S)$ as the p-fiber of S and the morphisms in $\mathcal{X}(S)$ as morphisms that cannot be detected when we pass into the the category $\mathcal{S}$ through p. We will now show that the assignment $\mathcal{X}(-) : \mathcal{S} \longrightarrow \mathbf{Cat}$ of objects can be made into a contravariant functor.

Proposition 2.8. *Let $p : \mathcal{X} \longrightarrow \mathcal{S}$ be a prestack. Each morphism $g : S' \longrightarrow S$ in $\mathcal{S}$ canonically induces a covariant functor $\mathcal{X}(g) : \mathcal{X}(S) \longrightarrow \mathcal{X}(S')$. Moreover, the assignment $g \mapsto \mathcal{X}(g)$ makes $\mathcal{X}(-) : \mathcal{S} \longrightarrow \boldsymbol{Cat}$ into a contravariant functor.*

Proof. For bookkeeping reasons, we first fix pullbacks $(X|_R, \varphi^*)$ for all diagrams

$$\begin{tikzcd}
X|_R \arrow[r, dashed, "\varphi^*"] \arrow[d, dashed] & X \arrow[d] \\
R \arrow[r, "\varphi"] & T
\end{tikzcd}$$

of solid arrows such that our choice of pullbacks behaves well with composition, i.e., $(\varphi \circ \psi)^* = \varphi^* \circ \psi^*$. We also require that the pullback of X along the identity on T be X itself, with the morphism $X \longrightarrow X$ being the identity on X (for the formalists, the existence of such a choice of pullbacks can be verified using Zorn's lemma on the collections of pullbacks that behave well with composition).

Let $g : S' \longrightarrow S$ be a morphism in $\mathcal{S}$. We define $\mathcal{X}(g) : \mathcal{X}(S) \longrightarrow \mathcal{X}(S')$ as follows.

(1) For each object X, we define $\mathcal{X}(g)(X)$ to be the pullback $X|_{S'}$ of X along the morphism $S' \longrightarrow S$.

(2) Let $f : X \longrightarrow Y$ be a morphism over id_S. We then have the following diagram.

$$\begin{tikzcd}
X|_{S'} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Y \arrow[d] & Y|_{S'} \arrow[l] \arrow[d] \\
S' \arrow[r] & S \arrow[r] & S & S' \arrow[l]
\end{tikzcd}$$

Invoking the universal property on the pullback $Y|_{S'}$, we get a unique morphism $X|_{S'} \longrightarrow Y|_{S'}$ such that the following diagram commutes.

$$\begin{tikzcd}
X|_{S'} \arrow[r] \arrow[d] \arrow[rrr, dashed, bend left] & X \arrow[r] \arrow[d] & Y \arrow[d] & Y|_{S'} \arrow[l] \arrow[d] \\
S' \arrow[r] & S \arrow[r] & S & S' \arrow[l]
\end{tikzcd}$$

We define $\mathcal{X}(g)(f)$ to be the morphism $X|_{S'} \longrightarrow Y|_{S'}$.

Using the uniqueness property of pullbacks, it can easily be checked that $\mathcal{X}(g)$ is indeed a covariant functor for all morphisms g in $\mathcal{S}$. We must now verify that the assignment $\mathcal{X}(-) : \mathcal{S} \longrightarrow \mathbf{Cat}$ is a contravariant functor.

(1) Let S be any object in $\mathcal{S}$. We see that $\mathcal{X}(\mathrm{id}_S) = \mathrm{id}_{\mathcal{X}(S)}$ because of how we chose our pullbacks and the universal property of the pullback.

(2) Let $g : S' \longrightarrow S$ and $\overline{g} : S \longrightarrow S''$ be morphisms in $\mathcal{S}$. We must verify that $\mathcal{X}(\overline{g} \circ g) = \mathcal{X}(g) \circ \mathcal{X}(\overline{g})$ as functors $\mathcal{X}(S'') \longrightarrow \mathcal{X}(S)$. This also follows from how we chose our pullbacks to be compatible with composition and the universal property of the pullback.

□

Remark 2.9. The functor $\mathcal{X}(-) : \mathcal{S} \longrightarrow \mathbf{Cat}$ depends on our choice of a family of (composition compatible) pullbacks, so we cannot claim that it is canonical. However, it is almost canonical: choosing a different (composition compatible) family of pullbacks induces a canonical isomorphism of functors.

Example 2.10. (Presheaves are Prestacks) Let $\mathcal{F} : \mathcal{S} \longrightarrow \mathbf{Set}$ be a presheaf. Define the category $\mathcal{X}$ as follows. An object of $\mathcal{X}$ is an ordered pair (a, S) where

S is an object of $\mathcal{S}$ and $a \in \mathcal{F}(S)$. A morphism $(a, S) \longrightarrow (a', S')$ is a morphism $f : S \longrightarrow S'$ such that $\mathcal{F}(f)(a') = a$. We define the functor $p : \mathcal{X} \longrightarrow \mathcal{S}$ in the obvious way. It then follows that the fiber $\mathcal{X}(S)$ is just the set of sections over S.

Recall our discussion of $\mathcal{M}_g$ where we considered it as the assignment $\mathbf{Sch} \to \mathbf{Set}$ which sends a scheme S to the set $\{$families of smooth curves $\mathcal{C} \to S$ of genus $g\}$. For each scheme S, we wanted to define morphisms between elements of the set $\{$families of smooth curves $\mathcal{C} \to S$ of genus g$\}$. Our hope was that adding a categorical structure on such sets would help us in understanding $\mathcal{M}_g$. However, we saw that the "natural" definition of a morphism between $(\mathcal{C} \longrightarrow S) \longrightarrow (\mathcal{C}' \longrightarrow S)$ being an S-morphism $\mathcal{C} \longrightarrow \mathcal{C}'$ does not work for the purpose of classifying curves. Namely, it fails to exclude the S-morphisms that take curves and collapse them down to points.

The next proposition suggests that if we were somehow able to reasonably define $\mathcal{M}_g$ as a prestack $p : \mathcal{M}_g \longrightarrow \mathbf{Sch}$, then the categorical structure on the fibers $\mathcal{M}_g(S)$ would encode the properties that we desire, namely that morphisms do not lose any information about the curves.

Proposition 2.11. *Let $p : \mathcal{X} \longrightarrow \mathcal{S}$ be a prestack. Then all fibers are groupoids, i.e., for any S in $\mathcal{S}$, all morphisms in $\mathcal{X}(S)$ are isomorphisms.*

Proof. Let S be in $\mathcal{S}$ and let $f : X \longrightarrow X'$ be a morphism lying over id_S. Then using the universal property of pullbacks, we have that for the following diagram

$$\begin{tikzcd} X' \arrow[r, dashed, "g"'] \arrow[rr, bend left, "\mathrm{id}_{X'}"] \arrow[d, mapsto] & X \arrow[r, "f"'] \arrow[d, mapsto] & X' \arrow[d, mapsto] \\ S \arrow[r, "\mathrm{id}_S"] & S \arrow[r, "\mathrm{id}_S"] & S \end{tikzcd}$$

of solid squares, there exists a unique morphism $g : X' \longrightarrow X$ making the above diagram commute. That is, we have that $f \circ g = \mathrm{id}_{X'}$. We also have the following diagram.

$$\begin{tikzcd} X \arrow[r, "g\circ f"] \arrow[d, mapsto] & X \arrow[r, "f"] \arrow[d, mapsto] & X' \arrow[d, mapsto] \\ S \arrow[r, "\mathrm{id}_S"] & S \arrow[r, "\mathrm{id}_S"] & S \end{tikzcd}$$

Since $f \circ (g \circ f) = f = f \circ (\mathrm{id}_X)$ and both $g \circ f$ and id_X are morphisms over id_S, the universal property of pullbacks tells us that $g \circ f = \mathrm{id}_X$. This shows that f is an isomorphism. $\square$

3. The (2-)category of prestacks

Fix a category $\mathcal{S}$. We define the objects in the category of prestacks over $\mathcal{S}$ as prestacks $\mathcal{X} \xrightarrow{p} \mathcal{S}$. A morphism $(\mathcal{X} \xrightarrow{p} \mathcal{S}) \longrightarrow (\mathcal{Y} \xrightarrow{p'} \mathcal{S})$ is a functor $f : \mathcal{X} \longrightarrow \mathcal{Y}$ such that the following diagram commutes.

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{Y} \arrow[dl, "p'"] \\ & \mathcal{S} & \end{tikzcd}$$

Let $f, g : \mathcal{X} \longrightarrow \mathcal{Y}$ be two prestack morphisms. We define a 2-morphism $\alpha : f \longrightarrow g$ as a natural transformation such that for all X in $\mathcal{X}$, the morphism $\alpha(X) : f(X) \longrightarrow g(X)$ lies over $\mathrm{id}_{p(X)}$. We represent this visually as the diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] \arrow[rr, phantom, "\Downarrow\alpha"] \arrow[dr, "p"'] & & \mathcal{X}' \arrow[dl, "p'"] \\
& \mathcal{S} &
\end{tikzcd}$$

We can now define the category $MOR(\mathcal{X}, \mathcal{Y})$ where the objects are prestack morphisms $\mathcal{X} \longrightarrow \mathcal{Y}$ (over $\mathcal{S}$) and morphisms between two prestack morphisms are 2-morphisms. We say that a prestack morphism $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is an isomorphism if there exists a prestack morphism $g : \mathcal{Y} \longrightarrow \mathcal{X}$ and 2-isomorphisms $g \circ f \xrightarrow{\sim} \mathrm{id}_{\mathcal{X}}$ and $f \circ g \xrightarrow{\sim} \mathrm{id}_{\mathcal{Y}}$.

While it is not immediately clear why we bother with the additional 2-category structure for prestacks, it ends up being a useful tool when discussing smooth morphisms of algebraic stacks.

Proposition 3.1. *2-fiber products exist in the 2-category of prestacks over a category $\mathcal{S}$.*

We make precise the notion of the 2-fiber product in a 2-category. Given prestack morphisms $f : \mathcal{X} \longrightarrow \mathcal{Z}$ and $g : \mathcal{Y} \longrightarrow \mathcal{Z}$, the 2-fiber product is a prestack $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ together with morphisms $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \xrightarrow{\pi_{\mathcal{X}}} \mathcal{X}$ and $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \xrightarrow{\pi_{\mathcal{Y}}} \mathcal{Y}$ and a 2-morphism $\alpha : f \circ \pi_{\mathcal{X}} \xrightarrow{\sim} g \circ \pi_{\mathcal{X}}$ such that

$$\begin{tikzcd}
\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \arrow[r, "\pi_{\mathcal{Y}}"] \arrow[d, "\pi_{\mathcal{X}}"'] \arrow[dr, phantom, "\nearrow\alpha"] & \mathcal{Y} \arrow[d, "g"] \\
\mathcal{X} \arrow[r, "f"'] & \mathcal{Z}
\end{tikzcd}$$

is 2-commutative and the quadruple $(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}, \pi_{\mathcal{X}}, \pi_{\mathcal{Y}}, \alpha)$ is final with respect to this property. A construction for 2-fiber products of prestacks can be found in [Alp21]. The next proposition shows that all 2-morphisms in the category of prestacks over $\mathcal{S}$ are 2-isomorphisms, i.e., $MOR(\mathcal{X}, \mathcal{Y})$ is a groupoid for all prestacks $\mathcal{X}$ and $\mathcal{Y}$ over $\mathcal{S}$.

Proposition 3.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be prestacks over a category $\mathcal{S}$. Then $MOR(\mathcal{X}, \mathcal{Y})$ is a groupoid.*

Proof. Suppose f and g are two morphisms in $MOR(\mathcal{X}, \mathcal{Y})$ and let $\alpha : f \longrightarrow g$ be a 2-morphism. For each $X \in \mathcal{X}$, we use the universal property of the pullback on the solid diagram

$$\begin{tikzcd}
g(X) \arrow[r, dashed] \arrow[rr, bend left, "\mathrm{id}_{g(X)}"] \arrow[d, mapsto] & f(X) \arrow[r, "\alpha(X)"'] \arrow[d, mapsto] & g(X) \arrow[d, mapsto] \\
p(X) \arrow[r, "\mathrm{id}_{p(X)}"] & p(X) \arrow[r, "\mathrm{id}_{p(X)}"] & p(X)
\end{tikzcd}$$

to get a unique morphism $\beta(X) : g(X) \longrightarrow f(X)$ making the above diagram commute. That is, we have that $\alpha(X) \circ \beta(X) = \mathrm{id}_{g(X)}$. We also have the following diagram.

$$\begin{tikzcd} f(X) \arrow[r, "\beta(X)\alpha(X)"] \arrow[d, mapsto] & f(X) \arrow[r, "\alpha(X)"] \arrow[d, mapsto] & g(X) \arrow[d, mapsto] \\ p(X) \arrow[r, "\mathrm{id}_{p(X)}"] & p(X) \arrow[r, "\mathrm{id}_{p(X)}"] & p(X) \end{tikzcd}$$

Since $\alpha(X) \circ (\beta(X) \circ \alpha(X)) = \alpha(X) = \alpha(X) \circ (\mathrm{id}_{f(X)})$, the universal property of pullbacks tells us that $\beta(X) \circ \alpha(X) = \mathrm{id}_{f(X)}$. This shows that $\alpha(X)$ is an isomorphism for all X, which implies that α is a 2-isomorphism. $\square$

Observant readers may notice that the proofs for Proposition 2.11 and Proposition 3.2 are identical. The next result shows that this is not a coincidence. Let $\mathcal{S}$ be a category and $S \in \mathcal{S}$. Using Example 2.10, we may view $\mathrm{Mor}(-, S)$ as a prestack over $\mathcal{S}$. We abuse notation and denote the prestack associated to $\mathrm{Mor}(-, S)$ by S.

Proposition 3.3. *(2-Yoneda Lemma) Let $p : \mathcal{X} \longrightarrow \mathcal{S}$ be a prestack over $\mathcal{S}$ and $S \in \mathcal{S}$. Then there is an equivalence of categories*

$$MOR(S, \mathcal{X}) \longrightarrow \mathcal{X}(S)$$

Proof. Define $\varphi : MOR(S, \mathcal{X}) \longrightarrow \mathcal{X}(S)$ on objects $S \xrightarrow{f} \mathcal{X}$ by $\varphi(f) := f((\mathrm{id}_S, S)$. Given a 2-morphism $\eta : f \longrightarrow g$, we define $\varphi(\eta) := \eta((\mathrm{id}_S, S)) : f((\mathrm{id}_S, S) \longrightarrow g((\mathrm{id}_S, S)$. To show that φ is an equivalence of categories, we construct a quasi-inverse of φ, i.e., a functor $\psi : \mathcal{X}(S) \longrightarrow MOR(S, \mathcal{X})$ such that $\psi \circ \varphi$ and $\varphi \circ \psi$ are isomorphic to the identity functors on $MOR(S, \mathcal{X})$ and $\mathcal{X}(S)$, respectively.

First, for all $X \in \mathcal{X}$ and $R \xrightarrow{j} T$, we fix pullbacks $(X|_j, j^*)$ for the solid diagram

$$\begin{tikzcd} X|_j \arrow[r, dashed, "j^*"] \arrow[d, dashed, mapsto] & X \arrow[d, mapsto] \\ R \arrow[r, "j"] & T \end{tikzcd}$$

such that our choice of pullbacks behaves well with composition, i.e., $(j \circ l)^* = j^* \circ l^*$. We also require that the pullback of X along the identity on T be X itself, with the morphism $X \longrightarrow X$ being the identity on X. Let $X \in \mathcal{X}$ be an object over S. We define $\psi(X) : S \longrightarrow \mathcal{X}$ as $\psi(X)\left((R \xrightarrow{j} S, R)\right) := X|_j$ where $X|_j$ is the pullback of X along j. Let $\left(R \xrightarrow{j} S, R\right) \xrightarrow{g} \left(T \xrightarrow{l} S, T\right)$ be a morphism (i.e., we have $R \xrightarrow{g} T \xrightarrow{l} S = R \xrightarrow{j} S$).

Using this, we get a morphism $(X|_l)|_g \xrightarrow{g^*} X_l$. Since our choice of pullbacks behaves well with composition, we have $(X|_l)|_g = X|_{g \circ l} = X|_j$, which gives us a morphism $X|_j \xrightarrow{g^*} X|_l$. We define $\psi(X)(g) := g^*$. It can be shown using the prestack axioms and our choice of pullbacks that $\psi(X)$ is a functor.

Now, let $f : X \longrightarrow X'$ be a morphism over id_S. We must define a 2-morphism $\eta : \psi(X) \longrightarrow \psi(X')$. Let $\left(R \xrightarrow{j} S, R\right)$ be an object in (the prestack associated to) S. We define $\eta((R \xrightarrow{j} S, R))$ to be the unique morphism $X|_j \longrightarrow X'|_j$ making the following diagram commute.

$$\begin{tikzcd}
X|_j \arrow[r] \arrow[d, mapsto] \arrow[rrr, dashed, bend left] & X \arrow[r, "f"] \arrow[d, mapsto] & X' \arrow[d, mapsto] & X'|_j \arrow[l] \arrow[d, mapsto] \\
R \arrow[r, "j"] & S \arrow[r, "\mathrm{id}_S"] & S & R \arrow[l, "j"']
\end{tikzcd}$$

Verifying that ψ is a quasi-inverse of φ is straightforward. $\square$

4. Stacks

Given a site $\mathcal{S}$, we are now ready to define stacks over $\mathcal{S}$. As motivation, recall the following two properties of schemes:

- (Morphisms of schemes glue) Let X be a scheme and $\{U_i\}_{i\in I}$ a cover of X by open subschemes. Given a collection of scheme morphisms $f_i : U_i \longrightarrow Y$ such that $f_i|_{U_i\cap U_j} = f_i|_{U_i\cap U_j}$, there exists a unique scheme morphism $f : X \longrightarrow Y$ which restricts to f_i on U_i for all $i \in I$ (Exercise 4.4.A. in [Vak17]).
- (Schemes glue) Given a collection of schemes $\{X_i\}_{i\in I}$, open subschemes $\{X_{ij} \subseteq X_i\}_{i\in I, j\in J}$, and isomorphisms $f_{ij} : X_{ij} \longrightarrow X_{ji}$ such that the f_{ij} satisfy the cocyle condition, there exists a scheme X (unique up to unique isomorphism) with an open cover $\{U_i\}_{i\in I}$ and isomorphisms $\psi_i : U_i \longrightarrow X_i$ such that $\psi_i(U_i \cap U_j) = X_{ij}$ (Exercise 6.2.A. in [Vak17]).

We will define a stack on a site $\mathcal{S}$ as a prestack that satisfies analogous gluing properties in the corresponding Grothendieck topology.

Definition 4.1. A stack on a site $\mathcal{S}$ is a prestack $p : \mathcal{X} \longrightarrow \mathcal{S}$ such that for all $S \in \mathcal{S}$ and all coverings $\{S_i \longrightarrow S\}_{i\in I}$, the following conditions hold.

(1) (Morphisms glue) For objects X and Y over S and morphisms $\varphi_i : X|_{S_i} \longrightarrow Y$ over $S_i \longrightarrow S$ satisfying $\varphi_i|_{S_i\times_S S_j} = \varphi_j|_{S_i\times_S S_j}$, there exists a unique morphism $\varphi : X \longrightarrow Y$ lying over id_S which restricts to φ_i on S_i. Visually, for any commutative solid diagram

$$\begin{tikzcd}
 & X|_{S_i} \arrow[dr] \arrow[drr, bend left, "\varphi_i"] & & \\
X|_{S_i\times_S S_j} \arrow[ur] \arrow[dr] & & X \arrow[r, dashed] & Y \\
 & X|_{S_j} \arrow[ur] \arrow[urr, bend right, "\varphi_j"'] & &
\end{tikzcd}$$

lying over

$$\begin{tikzcd}
 & S_i \arrow[dr] \arrow[drr, bend left] & & \\
S_i \times_S S_j \arrow[ur] \arrow[dr] & & S & S \ , \\
 & S_j \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

there exists a unique morphism $X \longrightarrow Y$ filling in the diagram and lying over id_S.

(2) (Objects glue) Given objects X_i over S_i for all i and isomorphisms $\alpha_{ij} : X_i|_{S_i \times_S S_j} \longrightarrow X_j|_{S_i \times_S S_j}$ over $\mathrm{id}_{S_i \times_S S_j}$ such that $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$ (when restricted to $S_i \times_S S_j \times_S S_k$), there exists an object X over S and isomorphisms $\varphi_i : X|_{S_i} \longrightarrow X_i$ over id_{S_i} such that $\alpha_{ij} \circ (\varphi_i)|_{S_i \times_S S_j} = \varphi_j|_{S_i \times_S S_j}$ on $S_i \times_S S_j$.

The definition of a stack is a mouthful of technical jargon. However, we must think of these two properties as abstractions of the analogous properties for schemes. This is especially true for this paper, as the stacks that we will be working with will end up being closely related to schemes. We are now ready to properly define $\mathcal{M}_g$.

Definition 4.2. We define $\mathcal{M}_g$ as a prestack over **Sch**. The objects of $\mathcal{M}_g$ are families of smooth curves $\mathcal{C} \longrightarrow S$ of genus g (i.e., smooth, proper scheme morphisms $\mathcal{C} \longrightarrow S$ such that for all $s \in S$, the fiber $\mathcal{C}_s := \mathcal{C} \times_S s$ is a connected curve of genus g over the residue field of s). We define a morphism $(\mathcal{C} \longrightarrow S) \longrightarrow (\mathcal{C}' \longrightarrow S')$ to be a pair $(\mathcal{C} \longrightarrow \mathcal{C}', S \longrightarrow S')$ of morphisms such that the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C}' \arrow[d] \\ S \arrow[r] & S' \end{tikzcd}$$

is Cartesian. The functor $\mathcal{M}_g \longrightarrow \mathbf{Sch}_{ét}$ takes an object $(\mathcal{C} \longrightarrow S)$ to S and a morphism $(\mathcal{C} \longrightarrow \mathcal{C}', S \longrightarrow S')$ to $S \longrightarrow S'$.

Note that $\mathcal{M}_g$ satisfies the two prestack axioms because of the Cartesian requirement of morphisms of $\mathcal{M}_g$ and the existence of pullbacks in the category of schemes. We now show that $\mathcal{M}_g$ is a stack over $\mathbf{Sch}_{ét}$.

Proposition 4.3. *$\mathcal{M}_g \longrightarrow$ **Sch**$_{ét}$ is a stack for $g \geq 2$.*

Proof. Let $\{S_i \longrightarrow S\}$ be an étale cover of S, $\mathcal{C} \to S$ and $\mathcal{D} \to S$ families of curves of genus g over S, and $\varphi_i : (\mathcal{C}|_{S_i} \to S_i) \longrightarrow (\mathcal{D} \to S)$ be morphisms such that $\varphi_i|_{S_i \times_S S_j} = \varphi_j|_{S_i \times_S S_j}$. Note that since surjectivity and étaleness are preserved under base change, we have that $\{\mathcal{C}_i \longrightarrow \mathcal{C}\}$ is an étale cover of $\mathcal{C}$. Thus, we can use unique gluing of morphisms under the étale topology to conclude that there exists a unique scheme morphism $\varphi : (\mathcal{C}|_S \to S) \longrightarrow (\mathcal{D}|_S \to S)$ making the following diagram commute.

$$\begin{tikzcd} \mathcal{C}|_{S_i} \arrow[r] \arrow[d] \arrow[rr, bend left, "\varphi_i"] & \mathcal{C} \arrow[r, dashed, "\varphi"'] \arrow[d] & \mathcal{D} \arrow[d] \\ S_i \arrow[r] & S \arrow[r, "\mathrm{id}_S"] & S \end{tikzcd}$$

Since we already know that the left square and the composite square are Cartesian, we conclude that the right square is also Cartesian. This shows that $\mathcal{M}_g$ satisfies Definition 4.1 (1). Checking the second axiom is a bit more involved. We use Proposition 5.1.9 of [Alp21], which states that given a family $\mathcal{C} \xrightarrow{\pi} S$ of smooth genus g curves where $g \geq 2$, the line bundle $\Omega_{\mathcal{C}/S}^{\otimes 3}$ is very ample and its pushforward $\pi_*(\Omega_{\mathcal{C}/S}^{\otimes 3})$ is a vector bundle of rank $5(g-1)$.

Let $\mathcal{C}_i \xrightarrow{\pi_i} S_i$ be families of curves of genus g and let $\varphi_{ij} : (\mathcal{C}_i|_{S_i \times_S S_j} \to S_i \times_S S_j) \longrightarrow (\mathcal{C}_j|_{S_i \times_S S_j} \to S_i \times_S S_j)$ be isomorphisms that satisfy the cocycle condition. The main idea is realizing each $\mathcal{C}_i$ as a closed subscheme of some ambient projective space (independent of i) and gluing the $\mathcal{C}_i$ in this ambient space. Since $\Omega^{\otimes 3}_{\mathcal{C}_i/S_i}$ is very ample over π_i, we get a closed immersion $\mathcal{C}_i \longrightarrow \mathbb{P}((\pi_i)_*(\Omega^{\otimes 3}_{\mathcal{C}_i/S_i}))$ for all i. Moreover, we get that the φ_{ij} induce isomorphisms $\phi_{ij} : ((\pi_i)_*\Omega^{\otimes 3}_{\mathcal{C}_i/S_i})|_{S_i \times_S S_j} \longrightarrow ((\pi_j)_*\Omega^{\otimes 3}_{\mathcal{C}_i/S_i})|_{S_i \times_S S_j}$ that satisfy the cocycle condition.

Thus, we have quasi-coherent sheaves $E_i := (\pi_i)_*(\Omega^{\otimes 3}_{\mathcal{C}_i/S_i})$ over S_i for all i and isomorphisms $\phi_{ij} : E_i|_{S_i \times_S S_j} \longrightarrow E_j|_{S_i \times_S S_j}$ for all i and j that satisfy the cocycle condition. Since quasi-coherent sheaves glue via étale morphisms, we get a quasi-coherent sheaf E over S and isomorphisms $\psi_i : E|_{S_i} \longrightarrow E_i$ for all i such that $\psi_j \circ \phi_{ij}$ and ψ_i agree on triple products.

Now, consider the closed embeddings $\mathcal{C}_i|_{S_i \times_S S_j} \hookrightarrow \mathbb{P}(E_i|_{S_i \times_S S_j})$ and $\mathcal{C}_i \hookrightarrow \mathbb{P}(E_i)$ induced by $\Omega^{\otimes 3}_{\mathcal{C}_i/S_i}$ and $\Omega^{\otimes 3}_{\mathcal{C}_i/S_i}|_{S_i \times_S S_j}$, respectively. We get morphisms $\mathbb{P}(E_i|_{S_i \times_S S_j}) \longrightarrow \mathbb{P}(E_i)$ and $\mathbb{P}(E_i) \longrightarrow \mathbb{P}(E)$ such that the following diagram commutes.

$$\begin{tikzcd}
\mathbb{P}(E_i|_{S_i \times_S S_j}) \arrow[r] & \mathbb{P}(E_i) \arrow[r] & \mathbb{P}(E) \\
\mathcal{C}_i|_{S_i \times_S S_j} \arrow[u, hook] \arrow[r] \arrow[d, "\pi_i|_{S_i \times_S S_j}"] & \mathcal{C}_i \arrow[u, hook] \arrow[d, "\pi_i"] & \\
S_i \times_S S_j \arrow[r] & S_i \arrow[r] & S
\end{tikzcd}$$

Since we constructed E by gluing E_i along the ϕ_{ij}, we also have that the preimages under the morphism $\mathbb{P}(E_i|_{S_i \times_S S_j}) \longrightarrow \mathbb{P}(E)$ of the closed subschemes cut out by $\mathcal{C}_i$ and $\mathcal{C}_j$ are equal. Thus, we can glue the closed subschemes $\mathcal{C}_i$ for all i to form a closed subscheme $\mathcal{C}$ over S with isomorphisms $\beta_i : \mathcal{C}|_{S_i} \longrightarrow \mathcal{C}_i$ such that $\beta_j \circ \varphi_{ij} = \beta_i$ for all i and j. This gives us the following diagram.

$$\begin{tikzcd}
\mathbb{P}(E_i|_{S_i \times_S S_j}) \arrow[r] & \mathbb{P}(E_i) \arrow[r] & \mathbb{P}(E) \\
\mathcal{C}_i|_{S_i \times_S S_j} \arrow[u, hook] \arrow[r] \arrow[d, "\pi_i|_{S_i \times_S S_j}"] & \mathcal{C}_i \arrow[u, hook] \arrow[r] \arrow[d, "\pi_i"] & \mathcal{C} \arrow[u, hook] \arrow[d, "\pi"] \\
S_i \times_S S_j \arrow[r] & S_i \arrow[r] & S
\end{tikzcd}$$

Since each π_i is smooth and has fibers connected curves of genus g, so does π. This proves that $\mathcal{M}_g$ is a stack for $g \geq 2$. $\square$

We end the section with a brief discussion of stackification. The idea is the same as that of the sheafification of presheaves: we would like all morphisms from a prestack $\mathcal{X}$ to a stack $\mathcal{Y}$ (both over a site $\mathcal{S}$)) to factor through a fixed stack dependent only on $\mathcal{X}$.

Proposition 4.4. *For any site $\mathcal{S}$, there is a functor $\textbf{\textit{PStack}}_{/\mathcal{S}} \longrightarrow \textbf{\textit{Stack}}_{/\mathcal{S}}$ which is left-adjoint to the forgetful functor $\textbf{\textit{Stack}}_{/\mathcal{S}} \longrightarrow \textbf{\textit{PStack}}_{/\mathcal{S}}$.*

Given a prestack $\mathcal{X}$ over a site $\mathcal{S}$, we denote its stackification as $\mathcal{X}^{\mathrm{st}}$. In the context of 2-categories, the above statement means that for all prestacks $\mathcal{X}$ and all stacks $\mathcal{Y}$ over $\mathcal{S}$, the categories $MOR(X,Y)$ and $MOR(X^{\mathrm{st}},Y)$ are equivalent. The proof of the existence of the stackification of a prestack can be found in [Alp21]. We can use stackification to give a slick proof of the existence of 2-fiber products in the category of stacks: we just apply the stackification functor to the fiber product in the category of prestacks.

5. Geometry on stacks

While there is much to be said about the general theory of stacks, we will now restrict our attention to stacks that have some notion of geometry associated to them. In particular, we will study stacks that "inherit" geometry from schemes. Note that there are analogous objects in differential geometry (differentiable stacks) and topology (topological stacks).

All (pre)stacks in this section are over $\mathbf{Sch}_{\acute{e}t}$ unless stated otherwise. Many of the results in this section use ideas and techniques from algebraic geometry. Our goal is to understand the following result about the stack $\mathcal{M}_g$.

Theorem 5.1. *For $g \geq 2$, $\mathcal{M}_g$ is a smooth Deligne-Mumford stack of dimension $3g-3$ over Spec$(\mathbb{Z})$.*

Recall that given a scheme V, we associate to it a prestack by applying Example 2.10 to the sheaf $\mathrm{Mor}(-,V)$. That is, we interpret the scheme V as a functor $p:\mathcal{X}_V \longrightarrow \mathbf{Sch}_{\acute{e}t}$ where an object in $\mathcal{X}_V$ is a pair $(S \xrightarrow{f} V, S)$ for some scheme S and a morphism $(S \xrightarrow{f} V, S) \longrightarrow (S' \xrightarrow{f'} V, S')$ is a scheme morphism $S \longrightarrow S'$ such that $S \longrightarrow S' \xrightarrow{f'} V = S \xrightarrow{f} V$. We abuse notation to interpret a prestack morphism $V \longrightarrow \mathcal{Y}$ where V is a scheme to be a morphism from $\mathcal{X}_V$. It is then easy to check that $\mathrm{Spec}(\mathbb{Z})$ is the final object in the category of stacks/prestacks over $\mathbf{Sch}_{\acute{e}t}$.

Definition 5.2. A morphism $\mathcal{X} \longrightarrow \mathcal{Y}$ of prestacks is said to be **representable by schemes** if given a scheme V and a morphism $V \longrightarrow \mathcal{Y}$, the fiber product $\mathcal{X} \times_{\mathcal{Y}} V$ is a scheme. If $\mathcal{P}$ is a property of morphisms of schemes, we say that a morphism $\mathcal{X} \longrightarrow \mathcal{Y}$ representable by schemes **has property $\boldsymbol{\mathcal{P}}$** if given a scheme V and a morphism $V \longrightarrow \mathcal{Y}$, the morphism $\mathcal{X} \times_{\mathcal{Y}} V \longrightarrow V$ has property $\mathcal{P}$.

Definition 5.3. An **algebraic space** is a sheaf X on $\mathbf{Sch}_{\acute{e}t}$ such that

(1) There exists a scheme U and a surjective étale morphism $U \longrightarrow X$ representable by schemes.
(2) The diagonal morphism $X \longrightarrow X \times X$ is representable by schemes.

The morphism $U \longrightarrow X$ in (1) is called an étale presentation of X.

Definition 5.4. A morphism $\mathcal{X} \longrightarrow \mathcal{Y}$ of prestacks is said to be **representable** if given a scheme V and a morphism $V \longrightarrow \mathcal{Y}$, the fiber product $\mathcal{X} \times_{\mathcal{Y}} V$ is an algebraic space. If $\mathcal{P}$ is a property of morphisms of schemes that is étale-local, we say that a representable morphism $\mathcal{X} \longrightarrow \mathcal{Y}$ **has property $\boldsymbol{\mathcal{P}}$** if given a scheme V, a morphism $V \longrightarrow \mathcal{Y}$, and an étale presentation $U \longrightarrow \mathcal{X} \times_{\mathcal{Y}} V$, the composition $U \longrightarrow \mathcal{X} \times_Y V \longrightarrow V$ has property $\mathcal{P}$.

We now define algebraic and Deligne-Mumford stacks. These are important for two main reasons. First, they allow us to impose geometric structure on certain stacks, making their study more interesting. Second, many objects naturally arising out of scheme theory turn out not to be schemes, but stacks (for example, the quotient of scheme X by an affine group scheme acting on it is not a scheme in general). However, the general theory of stacks does not tell us enough about these objects. Thus, we specialize to algebraic/Deligne-Mumford stacks, where we have more structure.

Definition 5.5. A stack $\mathcal{X}$ is **algebraic** (respectively **Deligne-Mumford**) if

(1) The diagonal $\mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{X}$ is representable.
(2) There exists a scheme U and a surjective representable morphism $U \longrightarrow \mathcal{X}$ that is smooth (respectively étale).

We say that $U \longrightarrow \mathcal{X}$ is a smooth (respectively étale) presentation of $\mathcal{X}$

Definition 5.6. An algebraic stack $\mathcal{X}$ is **locally noetherian** if for any smooth presentation $U \longrightarrow \mathcal{X}$, the scheme U is locally noetherian.

Definition 5.7. Let $\mathcal{P}$ be a property of morphisms of schemes.

(1) If $\mathcal{P}$ is preserved under composition and base-change and is étale-local (respectively smooth-local) on the source and target, then a morphism $\mathcal{X} \longrightarrow \mathcal{Y}$ of Deligne-Mumford stacks (respectively algebraic stacks) has $\mathcal{P}$ if for all étale (respectively smooth) presentations $V \longrightarrow \mathcal{Y}$ and $U \longrightarrow \mathcal{X} \times_{\mathcal{Y}} V$, the composition $U \longrightarrow \mathcal{X} \times_{\mathcal{Y}} V \longrightarrow V$ has $\mathcal{P}$
(2) A morphism $\mathcal{X} \longrightarrow \mathcal{Y}$ of algebraic spaces representable by schemes has property $\mathcal{P}$ if given a scheme V and a morphism $V \longrightarrow \mathcal{Y}$, the morphism $\mathcal{X} \times_{\mathcal{Y}} V \longrightarrow V$ has $\mathcal{P}$.

We are interested in proving that the morphism $\mathcal{M}_g \longrightarrow \operatorname{Spec}(\mathbb{Z})$ is smooth for $g \geq 2$. Checking smoothness of specific scheme morphisms using the definitions is often painful, and since algebraic/Deligne-Mumford stacks generalize schemes, we should not expect the process to be any less painful here. We hope that there is some easier way of characterizing smooth morphisms of algebraic stacks. It turns out that there is.

Definition 5.8. Let $\mathcal{X}$ be an algebraic stack. We define the **topological space** $|\mathcal{X}|$ of $\mathcal{X}$ as

$$|\mathcal{X}| = \{x : \operatorname{Spec}(k) \longrightarrow X : k \text{ is a field}\} / \sim$$

where $(x_1 : \operatorname{Spec}(k_1) \longrightarrow \mathcal{X}) \sim (x_2 : \operatorname{Spec}(k_2) \longrightarrow \mathcal{X})$ if there exists some field k_3 containing both k_1 and k_2 such that $x_1|_{\operatorname{Spec}(k_3)}$ and $x_2|_{\operatorname{Spec}(k_3)}$ are isomorphic in $\mathcal{X}(\operatorname{Spec}(k_3))$ (interpreted using the 2-Yoneda lemma). Note that a morphism of algebraic stacks $\mathcal{X} \longrightarrow \mathcal{Y}$ naturally induces a morphism of sets $|\mathcal{X}| \longrightarrow |\mathcal{Y}|$ via composition. We define a topology on $|\mathcal{X}|$ by saying that a subset U is open if it is the image of the induced morphism of some open immersion $\mathcal{U} \longrightarrow \mathcal{X}$ (in the sense of definition 5.7 (1)).

Definition 5.9. Let $\mathcal{X} \longrightarrow \mathcal{Y}$ be a morphism of algebraic stacks.

(1) $\mathcal{X}$ is **quasi-compact** if $|\mathcal{X}|$ is a compact topological space (i.e., every open cover has a finite subcover). $\mathcal{X} \longrightarrow \mathcal{Y}$ is **quasi-compact** if for all morphisms $\operatorname{Spec}(A) \longrightarrow \mathcal{Y}$, the fiber product $\operatorname{Spec}(A) \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact.
(2) $\mathcal{X} \longrightarrow \mathcal{Y}$ is **quasi-separated** if $\mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ and $\mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}} \mathcal{X}$ are quasi-compact. $\mathcal{X}$ is **quasi-separated** if $\mathcal{X} \longrightarrow \operatorname{Spec}(\mathbb{Z})$ is quasi-separated.
(3) $\mathcal{X} \longrightarrow \mathcal{Y}$ is of **finite type** if it is locally of finite type and quasi-compact.

Definition 5.10. An algebraic stack $\mathcal{X}$ is **noetherian** if it is locally noetherian, quasi-compact, and quasi-separated.

Roughly speaking, a moduli stack is Deligne-Mumford if the automorphism groups of the objects parametrized by it are finite. As an example, consider the moduli space of all curves of genus 0 over an algebraically closed field. As a set, this consists of a single point (corresponding to $\mathbb{P}^1$). The automorphism group of the projective line is PGL_2, which is infinite. In this situation, we often rigidify the moduli problem by introducing more structure to the objects we are trying to classify. For genus 0 curves, this is done by marking n points on the curves (see [Vor01]).

Now, let's consider the moduli prestack of elliptic curves, i.e. the moduli prestack of genus 1 curves marked with a single point. While we cannot apply Proposition 4.3 here, this prestack is indeed a stack [Aut22a]. For simplicity, we restrict our attention to elliptic curves over algebraically closed fields of characteristic not equal to 2 or 3. Then the automorphism group is finite (Exercise 19.10.E. in [Vak17]). Thus, there is hope that this stack is Deligne-Mumford (and in fact it is). However, if we ignore the marked point and consider $\mathcal{M}_1$, we get curves with infinite automorphism groups (Exercise 19.10.D. in [Vak17]). Sure enough, it turns out that $\mathcal{M}_1$ is not a Deligne-Mumford stack. For $g \geq 2$, the automorphism group of a genus g curve is always finite, which suggests that $\mathcal{M}_g$ is Deligne-Mumford. This indeed turns out to be the case.

Proposition 5.11. *For $g \geq 2$, $\mathcal{M}_g$ is a noetherian Deligne-Mumford stack of finite type over Spec$(\mathbb{Z})$.*

Proposition 5.11 captures the technical results needed for proving smoothness in Theorem 5.1. Its proof is fairly involved, requiring both the development of more stack theory and results from algebraic geometry such as the existence of Hilbert Schemes. Therefore, we omit it in this paper. Readers can refer to [Ols16] or [Alp21] for a proof.

We are now ready to define the tangent space associated to a point of the topological space associated to an algebraic stack $\mathcal{X}$. Recall that given a k-scheme S and point $p \in S$ with residue field k, the tangent vectors at p may be thought of as k-scheme morphisms $\mathrm{Spec}\left(\frac{k[\epsilon]}{\epsilon^2}\right) \longrightarrow S$ that factor via a morphism $\mathrm{Spec}(k) \longrightarrow S$ whose (topological) image is equal to $\{p\}$ (Exercise 12.1.I in [Vak17]). We now extend this notion to algebraic stacks.

Definition 5.12. Let $\mathcal{X}$ be an algebraic stack and let $x : \mathrm{Spec}(k) \longrightarrow \mathcal{X}$ be a morphism. We define the tangent space of $\mathcal{X}$ at x as

$$T_{\mathcal{X},x} = \left\{ \text{2 commutative diagrams} \quad \begin{tikzcd} \mathrm{Spec}(k) \arrow[d, "\iota"'] \arrow[dr, "x"] \arrow[d, phantom, shift left=4ex, "\not\Downarrow\alpha"] & \\ \mathrm{Spec}\left(\frac{k[\epsilon]}{\epsilon^2}\right) \arrow[r, "\tau"] & \mathcal{X} \end{tikzcd} \right\} \Big/ \sim .$$

Since the objects in the diagrams remain the same, we may represent elements of $T_{\mathcal{X},x}$ more concisely as pairs (τ, α) where $\tau : \mathrm{Spec}(\frac{k[\epsilon]}{\epsilon^2}) \longrightarrow \mathcal{X}$ and $\alpha : \tau|_{\mathrm{Spec}(k)} \longrightarrow x$. We say that $(\tau, \alpha) \sim (\tau', \alpha')$ if there exists an isomorphism $\beta : \tau \longrightarrow \tau'$

in $\mathcal{X}\left(\frac{k[\epsilon]}{\epsilon^2}\right)$ (interpreted using the 2-Yoneda lemma) such that $\alpha' = \beta|_{\mathrm{Spec}(k)} \circ \alpha$ (where $\beta|_{\mathrm{Spec}(k)}$ is the 2-morphism $\tau \circ \iota \longrightarrow \tau' \circ \iota$ induced by β).

Proposition 5.13. *Let $\mathcal{X}$ be an algebraic stack and $x : Spec(k) \longrightarrow \mathcal{X}$ be a morphism where k is a field. Then we can endow $T_{\mathcal{X},x}$ with a k-vector space structure.*

Proof. Let $c \in k$ and $(\tau, \alpha) \in T_{\mathcal{X},x}$. Consider the map $\mathrm{Spec}\left(\frac{k[\epsilon]}{\epsilon^2}\right) \xrightarrow{\epsilon \mapsto c\epsilon} \mathrm{Spec}\left(\frac{k[\epsilon]}{\epsilon^2}\right)$ induced by the ring morphism $\frac{k[\epsilon]}{\epsilon^2} \xrightarrow{\epsilon \mapsto c\epsilon} \frac{k[\epsilon]}{\epsilon^2}$. We define $c\tau$ to be the composition

$$\mathrm{Spec}\left(\frac{k[\epsilon]}{\epsilon^2}\right) \xrightarrow{\epsilon \mapsto c\epsilon} \mathrm{Spec}\left(\frac{k[\epsilon]}{\epsilon^2}\right) \xrightarrow{\tau} \mathcal{X}.$$

Since $(\epsilon \mapsto c\epsilon) \circ \iota = \iota$, we get that the following diagram is 2-commutative.

$$\begin{tikzcd}
\mathrm{Spec}(k) \arrow[d, "\iota"'] \arrow[dr, "x"] \arrow[d, phantom, shift left=4ex, "\swarrow\alpha"] & \\
\mathrm{Spec}\left(\frac{k[\epsilon]}{\epsilon^2}\right) \arrow[r, "c\tau"] & \mathcal{X}
\end{tikzcd}$$

We define $c \cdot (\tau, \alpha)$ as $(c\tau, \alpha)$. Let (τ_1, α_1) and (τ_2, α_2) be elements of $T_{\mathcal{X},x}$. We will now define addition. Note that given affine scheme morphisms $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ and $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(C)$, the scheme-theoretic pushforward exists and is given by $\mathrm{Spec}(A \times_B C)$. It is also true that $\mathrm{Spec}(A \times_B C)$ satisfies the universal property of 2-cofiber coproducts in the category of algebraic stacks, but we will not prove it here. Readers can refer to the proof in [Alp21]. Consider the 2-commutative diagram

$$\begin{tikzcd}
 & \mathrm{Spec}\left(\frac{k[\epsilon]}{\epsilon^2}\right) \arrow[dl, "\pi_1"'] \arrow[rrd, bend left, "\tau_1"] & & \\
\mathrm{Spec}\left(\frac{k[\epsilon]}{\epsilon^2} \times_k \frac{k[\epsilon]}{\epsilon^2}\right) & & \mathrm{Spec}\,(k) \arrow[ul, "\iota"'] \arrow[dl, "\iota"] \arrow[r, "x"] \arrow[ur, phantom, "\Nwarrow\alpha_1"] \arrow[dr, phantom, "\swarrow\alpha_2"] & \mathcal{X} \\
 & \mathrm{Spec}\left(\frac{k[\epsilon]}{\epsilon^2}\right) \arrow[ul, "\pi_2"] \arrow[rru, bend right, "\tau_2"'] & &
\end{tikzcd}$$

where π_1 and π_2 are the pushforward morphisms in the category of affine schemes. More concretely, the morphisms π_i are the morphisms induced by projecting $\frac{k[\epsilon]}{\epsilon^2} \times_k \frac{k[\epsilon]}{\epsilon^2}$ to each factor. Using the universal property of the 2-cofiber coproduct on the above diagram, we get a unique morphism $\delta : \mathrm{Spec}\left(\frac{k[\epsilon]}{\epsilon^2} \times_k \frac{k[\epsilon]}{\epsilon^2}\right) \longrightarrow \mathcal{X}$ and 2-isomorphisms $\beta_1 : \delta\pi_1 \longrightarrow \tau_1$ and $\beta_2 : \delta\pi_2 \longrightarrow \tau_2$. Note that each β_i naturally defines a 2-morphism $\tilde{\beta}_i : \delta\pi_i\iota \longrightarrow \tau_i\iota$.

Now, consider the morphism $\pi : \mathrm{Spec}\left(\frac{k[\epsilon]}{\epsilon^2}\right) \longrightarrow \mathrm{Spec}\left(\frac{k[\epsilon]}{\epsilon^2} \times_k \frac{k[\epsilon]}{\epsilon^2}\right)$ induced by the ring morphism $\frac{k[\epsilon]}{\epsilon^2} \times_k \frac{k[\epsilon]}{\epsilon^2} \longrightarrow \frac{k[\epsilon]}{\epsilon^2}$ which sends both $(\epsilon, 0)$ and $(0, \epsilon)$ to ϵ. We can see that $\pi\iota = \pi_1\iota = \pi_2\iota$ by checking that the corresponding ring morphisms are equal. Thus, we get a morphism $\mathrm{Spec}\left(\frac{k[\epsilon]}{\epsilon^2}\right) \xrightarrow{\delta\pi} \mathcal{X}$ and a 2-morphism $\tilde{\beta}_1^{-1}\alpha_1 : x \longrightarrow \delta\pi_1\iota = \delta\pi\iota$. We define $(\tau_1, \alpha_1) + (\tau_2, \alpha_2)$ to be $(\delta\pi, \tilde{\beta}_1^{-1}\alpha_1)$. Showing that addition and scalar multiplication are well-defined and satisfy the vector space axioms is not particularly illuminating, so we omit it in this paper. $\square$

Example 5.14. Let $x : \mathrm{Spec}(k) \longrightarrow \mathcal{M}_g$ be a point in $|\mathcal{M}_g|$. Using the 2-Yoneda lemma (proposition 3.3), we have that x corresponds to a smooth family of genus g curves $C \longrightarrow \mathrm{Spec}(k)$. However, this just means that C is a smooth, connected, and proper curve of genus g over k. It is a result from deformation theory that $T_{\mathcal{M}_g,[C]} \cong H^1(C, T_C)$. Since C is projective over k, we can use Riemann-Roch to get that $\dim(T_{\mathcal{M}_g,[C]}) = -(\deg T_C + 1 - g) = -(2 - 2g + 1 - g) = 3g - 3$.

We are now ready to state an equivalent characterization of smooth morphisms of algebraic stacks. We classically think of smooth submersions of manifolds as smooth maps that induce a surjection of tangent vectors at each point. In other words, a smooth submersion has the lifting property for tangent vectors at every point. Since we now have a reasonable notion of tangent vectors of points on an algebraic stack, we hope to characterize smoothness in this way. It turns out that doing so is possible and looks exactly the same as the infinitesimal lifting criterion for smooth morphisms of schemes.

Definition 5.15. (Infinitesimal Lifting Criterion) A morphism $\mathcal{X} \longrightarrow \mathcal{Y}$ of algebraic stacks is said to satisfy the **infinitesimal lifting criterion** if given any 2-commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(A_0) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\swarrow"] & \mathcal{X} \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] & \mathcal{Y} \end{tikzcd}$$

where A and A_0 are rings and $A_0 \longrightarrow A$ is a surjective ring morphism with nilpotent kernel, there exists a lifting $\mathrm{Spec}(A) \longrightarrow \mathcal{X}$ and a 2-commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(A_0) \arrow[r] \arrow[d, hook] & \mathcal{X} \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed] & \mathcal{Y} \end{tikzcd}$$

Note that when $A_0 = \frac{k[\epsilon]}{\epsilon} \cong k$ and $A = \frac{k[\epsilon]}{\epsilon^2}$, the infinitesimal lifting criterion precisely states that for k-points of $\mathcal{Y}$ that factor through $\mathcal{X}$, the tangent vectors lift through $\mathcal{X} \longrightarrow \mathcal{Y}$. The next proposition says that this case essentially captures smoothness in nice situations.

Proposition 5.16. *Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a finite type morphism of noetherian algebraic stacks. Then f is smooth if and only if f satisfies the infinitesimal lifting criterion when $A \longrightarrow A_0$ is a surjection of local artin rings with residue field k and $\ker(A \longrightarrow A_0) \cong k$ (As an A-module).*

Proof. We assume the equivalence for when $\mathcal{X}$ and $\mathcal{Y}$ are schemes (Lemma 37.11.7 in [Sta22]).

First, we will show that smoothness implies the infinitesimal lifting criterion. Let f be a smooth morphism and say we are given a 2-commutative diagram

$$\begin{tikzcd} \operatorname{Spec}(A_0) \arrow[r, "p"] \arrow[d, hook, "i"'] & \mathcal{X} \arrow[d, "f"] \\ \operatorname{Spec}(A) \arrow[r, "t"'] \arrow[ur, phantom, "\swarrow\alpha"] & \mathcal{Y} \end{tikzcd}$$

where $A \longrightarrow A_0$ is a surjection of local artin rings with residue field k and $\ker(A \longrightarrow A_0) \cong k$. Here is our strategy for the proof: we prove that the diagram lifts under progressively weaker assumptions on f (f is smooth and representable by schemes $\rightarrow$ f is smooth and representable $\rightarrow$ f is smooth).

Assume that f is representable by schemes and smooth. By definition, this implies that the induced scheme morphism $\mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec}(A) \longrightarrow \operatorname{Spec}(A)$ is smooth. Using the universal property of the fiber product, we get a morphism $s : \operatorname{Spec}(A_0) \longrightarrow \mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec}(A)$ and 2-morphisms $\beta : \pi_{\operatorname{Spec}(A)} \circ s \longrightarrow i$ and $\gamma : \pi_{\mathcal{X}} \circ s \longrightarrow p$ such that the following diagram is 2-commutative.

$$\begin{tikzcd} \operatorname{Spec}(A_0) \arrow[dr, dashed, "s"] \arrow[drr, bend left, "p"] \arrow[ddr, bend right, "i"'] & & \\ & \mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec}(A) \arrow[r, "\pi_{\mathcal{X}}"] \arrow[d, "\pi_{\operatorname{Spec}(A)}"'] \arrow[u, phantom, "\nearrow\gamma"] \arrow[l, phantom, "\beta\swarrow"] & \mathcal{X} \arrow[d, "f"] \\ & \operatorname{Spec}(A) \arrow[r, "t"'] \arrow[ur, phantom, "\swarrow\alpha"] & \mathcal{Y} \end{tikzcd}$$

Since $\pi_{\operatorname{Spec}(A)} \circ s$ and i are morphisms of schemes (regarded as stacks over $\mathbf{Sch}_{\acute{e}t}$), the only 2-morphism between them is the identity. Thus, we have the following commutative diagram of schemes.

$$\begin{tikzcd} \operatorname{Spec}(A_0) \arrow[r, "s"] \arrow[d, "i"] & \mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec}(A) \arrow[d, "\pi_{\operatorname{Spec(A)}}"] \\ \operatorname{Spec}(A) \arrow[r, "\mathrm{id}_A"] & \operatorname{Spec}(A) \end{tikzcd}$$

Since $\pi_{\operatorname{Spec}(A)}$ is a smooth scheme morphism, we may now use the infinitesimal lifting criterion for schemes to get a scheme morphism $\operatorname{Spec}(A) \longrightarrow \mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec}(A)$ such that the following diagram commutes.

$$\begin{tikzcd} \operatorname{Spec}(A_0) \arrow[r, "s"] \arrow[d, "i"] & \mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec}(A) \arrow[d, "\pi_{\operatorname{Spec(A)}}"] \\ \operatorname{Spec}(A) \arrow[r, "\mathrm{id}_A"] \arrow[ur, dashed] & \operatorname{Spec}(A) \end{tikzcd}$$

Composing the morphism $\operatorname{Spec}(A) \longrightarrow \mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec}(A)$ with the morphism $\pi_{\mathcal{X}} : \mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec}(A) \longrightarrow \mathcal{X}$ gives us the required lifting of the original diagram.

Now, let us consider the case where f is a smooth representable morphism and we have the following 2-commutative diagram.

$$\begin{tikzcd} \mathrm{Spec}(A_0) \arrow[r, "p"] \arrow[d, hook, "i"'] \arrow[dr, phantom, "\swarrow\alpha"] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r, "t"'] & \mathcal{Y} \end{tikzcd}$$

Note that we can then use the universal property of 2-fiber products to get the following factorization of the previous diagram.

$$\begin{tikzcd} \mathrm{Spec}(A_0) \arrow[r] \arrow[d, hook, "i"'] \arrow[dr, phantom, "\swarrow\beta"] & \mathcal{X} \times_{\mathcal{Y}} \mathrm{Spec}(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow\alpha"] & \mathcal{X} \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r, "\mathrm{id}_A"'] & \mathrm{Spec}(A) \arrow[r] & \mathcal{Y} \end{tikzcd}$$

From this, we see that it suffices to produce a lift $\mathrm{Spec}(A) \longrightarrow \mathcal{X} \times_{\mathcal{Y}} \mathrm{Spec}(A)$. Thus, we may restrict our attention to the case where $\mathcal{Y} = \mathrm{Spec}(A)$ and $\mathrm{Spec}(A) \longrightarrow \mathcal{Y} = \mathrm{id}_A$. Note that the 2-morphism in this case is trivial as it is a 2-morphism between morphisms of schemes. We now want to find a morphism $\mathrm{Spec}(A) \longrightarrow \mathcal{X}$ that lifts the following diagram.

$$\begin{tikzcd} \mathrm{Spec}(A_0) \arrow[r, "p"] \arrow[d, hook, "i"'] & \mathcal{X} \arrow[dl, "f"] \\ \mathrm{Spec}(A) & \end{tikzcd}$$

First, we show lifting for the case where $\mathcal{X}$ is an algebraic space. Choose a smooth presentation $U \longrightarrow \mathcal{X}$ and a lifting $\mathrm{Spec}(k) \longrightarrow U$ of $\mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(A_0) \longrightarrow \mathcal{X}$ (doing so is possible by Exercise 4.2.3 in [Alp21], which states that a morphism Spec(k) $\longrightarrow \mathcal{X}$ lifts through some smooth presentation $U \longrightarrow \mathcal{X}$). Since $U \longrightarrow \mathcal{X}$ is a smooth morphism that is representable by schemes, we know from what we proved previously that it satisfies the infinitesimal lifting criterion. Thus, we get a lifting $\mathrm{Spec}(A_0) \longrightarrow U$ of the following solid diagram.

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[r] \arrow[d] & U \arrow[d] \\ \mathrm{Spec}(A_0) \arrow[r] \arrow[ur, dashed] & \mathcal{X} \end{tikzcd}$$

Since composition of smooth morphisms is smooth, we get that $U \longrightarrow \mathcal{X} \longrightarrow \mathrm{Spec}(A)$ is a smooth morphism between schemes. However, for schemes, this is equivalent to satisfying the infinitesimal lifting criterion. Thus, we get a lifting $\mathrm{Spec}(A) \longrightarrow U$ of the following solid diagram.

$$\begin{tikzcd} \mathrm{Spec}(A_0) \arrow[r] \arrow[d] & U \arrow[d] \\ \mathrm{Spec}(A) \arrow[r, "\mathrm{id}_A"'] \arrow[ur, dashed] & \mathrm{Spec}(A) \end{tikzcd}$$

Composing $\mathrm{Spec}(A) \longrightarrow U$ with $U \longrightarrow X$ gives us the desired lifting. Now, we show lifting for when $\mathcal{X}$ is a general (noetherian) algebraic stack. We use Exercise 4.2.3 in [Alp21] to find a smooth presentation $U \longrightarrow \mathcal{X}$ and a lifting $\mathrm{Spec}(k) \longrightarrow U$

of $\mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(A_0) \longrightarrow \mathcal{X}$. Using the universal property of $\mathrm{Spec}(A_0) \times_{\mathcal{X}} U$, we get a morphism $\mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(A_0) \times_{\mathcal{X}} U$. Consider the following diagram.

$$\begin{tikzcd}
\mathrm{Spec}(k) \arrow[r] \arrow[d] & \mathrm{Spec}(A_0) \times_{\mathcal{X}} U \arrow[d] \\
\mathrm{Spec}(A_0) \arrow[r, "\mathrm{id}_A"'] & \mathrm{Spec}(A_0)
\end{tikzcd}$$

Recall that, by definition, $\mathrm{Spec}(A_0) \times_{\mathcal{X}} U$ is an algebraic space. Using infinitesimal lifting for algebraic spaces, we get a lifting $\mathrm{Spec}(A_0) \longrightarrow \mathrm{Spec}(A_0) \times_{\mathcal{X}} U$ of the above diagram. Note that $U \longrightarrow \mathcal{X} \longrightarrow \mathrm{Spec}(A)$ is smooth as it is a composition of smooth morphisms. Now, consider the following diagram.

$$\begin{tikzcd}
\mathrm{Spec}(A_0) \arrow[r] \arrow[d] & U \arrow[d] \\
\mathrm{Spec}(A) \arrow[r, "\mathrm{id}_A"'] & \mathrm{Spec}(A)
\end{tikzcd}$$

Since U is a scheme (in particular, an algebraic space), we get a lifting $\mathrm{Spec}(A) \longrightarrow U$ of the above diagram. Composing $\mathrm{Spec}(A) \longrightarrow U$ with $V \longrightarrow \mathcal{X}$ gives us the lifting we desire.

Finally, we consider the case where f is an arbitrary smooth morphism. Using the same argument as before, we may as well restrict to the case where we have the following diagram.

$$\begin{tikzcd}
\mathrm{Spec}(A_0) \arrow[r, "p"] \arrow[d, hook, "i"'] & \mathcal{X} \arrow[dl, "f"] \\
\mathrm{Spec}(A) &
\end{tikzcd}$$

Choose a smooth presentation $U \longrightarrow \mathcal{X}$ and a lifting $\mathrm{Spec}(k) \longrightarrow U$ of $\mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(A_0) \longrightarrow \mathcal{X}$ (doing so is possible by Exercise 4.2.3 in [Alp21]). Since $U \longrightarrow \mathcal{X}$ is a smooth representable morphism, we know from what we proved previously that it satisfies the infinitesimal lifting criterion. Thus, we get a lifting $\mathrm{Spec}(A_0) \longrightarrow U$ of the following solid diagram.

$$\begin{tikzcd}
\mathrm{Spec}(k) \arrow[r] \arrow[d] & U \arrow[d] \\
\mathrm{Spec}(A_0) \arrow[r] \arrow[ur, dashed] & \mathcal{X}
\end{tikzcd}$$

Since composition of smooth morphisms is smooth, we get that $U \longrightarrow \mathcal{X} \longrightarrow \mathrm{Spec}(A)$ is a smooth morphism between schemes. However, for schemes, this is equivalent to satisfying the infinitesimal lifting criterion. Thus, we get a lifting $\mathrm{Spec}(A) \longrightarrow U$ of the following solid diagram.

$$\begin{tikzcd}
\mathrm{Spec}(A_0) \arrow[r] \arrow[d] & U \arrow[d] \\
\mathrm{Spec}(A) \arrow[r, "\mathrm{id}_A"'] \arrow[ur, dashed] & \mathrm{Spec}(A)
\end{tikzcd}$$

Composing $\mathrm{Spec}(A) \longrightarrow U$ with $U \longrightarrow X$ gives us the desired lifting.

We now prove the converse. Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ satisfy the infinitesimal lifting criterion for all surjections $A \longrightarrow A_0$ of local artininian rings with residue fields k and $\ker(A \longrightarrow A_0) \cong k$. Let $V \longrightarrow \mathcal{Y}$ and $U \longrightarrow \mathcal{X} \times_{\mathcal{Y}} V$ be smooth presentations. Using what we proved previously, the morphism $U \longrightarrow \mathcal{X} \times_{\mathcal{Y}} V$ must satisfy the infinitesimal lifting criterion. Thus, the composition $U \longrightarrow \mathcal{X} \times_{\mathcal{Y}} V \longrightarrow V$ must also satisfy the infinitesimal lifting criterion. However, $U \longrightarrow V$ is a morphism of schemes. This implies that $U \longrightarrow V$ is smooth and we conclude that $\mathcal{X} \longrightarrow \mathcal{Y}$ is smooth. □

Proposition 5.17. *$\mathcal{M}_g \longrightarrow$ Spec$(\mathbb{Z})$ is a smooth morphism for $g \geq 2$.*

Proof. Using Proposition 5.16, it suffices to show lifting for diagrams

$$\begin{tikzcd}
\mathrm{Spec}(A_0) \arrow[r, "p"] \arrow[d, hook, "i"] \arrow[dr, phantom, "\swarrow\alpha"] & \mathcal{M}_g \arrow[d, "f"] \\
\mathrm{Spec}(A) \arrow[r, "t"'] & \mathrm{Spec}(\mathbb{Z})
\end{tikzcd}$$

where $A \longrightarrow A_0$ is a surjection of local artin rings with residue field k and $\ker(A \longrightarrow A_0) \cong k$. Suppose we are given such a diagram. Consider the morphism q : $\mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(A_0)$ induced by the quotient ring morphism $A_0 \longrightarrow k$. This gives us the following diagram.

$$\begin{tikzcd}
\mathrm{Spec}(k) \arrow[r, "q"] & \mathrm{Spec}(A_0) \arrow[r, "p"] \arrow[d, hook, "i"] \arrow[dr, phantom, "\swarrow\alpha"] & \mathcal{M}_g \arrow[d, "f"] \\
& \mathrm{Spec}(A) \arrow[r, "t"'] & \mathrm{Spec}(\mathbb{Z})
\end{tikzcd}$$

Let $C_{p\circ q} \to \mathrm{Spec}(k)$ and $C_p \to \mathrm{Spec}(A_0)$ be the families of curves associated to the morphisms $p \circ q$ and q, respectively. Applying the 2-Yoneda lemma on the previous diagram gives us the following Cartesian square.

$$\begin{tikzcd}
C_{p\circ q} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C_p \arrow[d] \\
\mathrm{Spec}(k) \arrow[r] & \mathrm{Spec}(A_0)
\end{tikzcd}$$

A lifting of the original diagram corresponds to the existence of a family of curves $C_l \longrightarrow \mathrm{Spec}(A)$ and a morphism $(C_p \to \mathrm{Spec}(A_0)) \longrightarrow (C_l \longrightarrow \mathrm{Spec}(A))$ lying over $\mathrm{Spec}(A_0) \longrightarrow \mathrm{Spec}(A)$. In other words, a lifting corresponds to the existence of a family $C_l \to \mathrm{Spec}(A)$ and scheme morphism $C_p \longrightarrow C_l$ such that all squares in the following diagram are Cartesian.

$$\begin{tikzcd}
C_{p\circ q} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C_p \arrow[r, dashed] \arrow[d] \arrow[dr, phantom, "\square"] & C_l \arrow[d, dashed] \\
\mathrm{Spec}(k) \arrow[r] & \mathrm{Spec}(A_0) \arrow[r] & \mathrm{Spec}(A)
\end{tikzcd}$$

By Corollary 10.3 in [Har10], this happens if $H^2(C_{p\circ q}, T_{C_{p\circ q}})$ is trivial, which it is as $C_{p\circ q}$ is a projective curve. This shows that $\mathcal{M}_g$ is smooth over $\mathrm{Spec}(\mathbb{Z})$ for $g \geq 2$. □

In Example 5.14, we proved the tangent space of each point of $|\mathcal{M}_g|$ has dimension $3g-3$. After developing more theory, one can define the notion of dimension for points on algebraic stacks. While this definition of dimension does not agree with the dimension of the tangent space in general, it is closely related to it. For a (noetherian) moduli stack $\mathcal{X}$ and a point $x : \mathrm{Spec}(k) \longrightarrow \mathcal{X}$, we have that

$$\dim_x(\mathcal{X}) = \dim(T_{\mathcal{X},x}) - \text{error}$$

where the "error" term essentially measures how many automorphisms the objects parametrized by $\mathcal{X}$ have. Hurwitz's automorphisms theorem tells us that a curve of genus $g \geq 2$ has finitely many automorphisms, so we get that the error term is 0 and

$$\dim_{[C]}(\mathcal{M}_g) = \dim(T_{\mathcal{M}_g,[C]}) = 3g-3$$

where the last equality follows from Example 5.14. The dimension of an algebraic stack is defined as the supremum of the dimension at all points of the stack, so we conclude that $\mathcal{M}_g$ has dimension $3g-3$ for $g \geq 2$.

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