THE ADAMS CONJECTURE AND THE K-THEORY OF FINITE FIELDS

MATTHEW NIEMIRO AND ZHONG ZHANG

Abstract. Daniel Quillen's proof of the Adams conjecture unexpectedly provoked his invention and initial calculations of higher algebraic K-theory. For this, Quillen was awarded the Fields medal, and algebraic K-theory today is the active focus of a serious portion of research in modern mathematics. This paper is a self-contained and modernized presentation of this development. We prove the complex Adams conjecture and calculate the K-theory of finite fields.

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1. Plan of the Demonstration

Throughout, we will consider a prime $p \neq 2$ and q a power of p. We will denote by k_q a field with q elements and by k an algebraic closure of k_q.

We open in §2 with some history. We try to survey a neighborhood of Quillen's invention of K-theory within the space of mathematical history. This may help a reader get some context for and guidance through the paper. It is hopefully an enriching account of "the story."

In §3, we introduce the *Adams operations* Ψ^k. They are a basic and important part of *the Adams conjecture*, whose proof by Quillen was a first step toward his definition of higher algebraic K-theory. Later, we will need a slightly more general version of these operations; we develop them sufficiently in anticipation of this.

In §4, we define the *Brauer lift*, whose main purpose is to produce maps $BG \to BU$ from k-representations of a finite group G. We investigate in particular those Brauer lifted maps arising from k_q-representations. These will be important for the development of K-theory. We will see that if a k-representation extends a k_q-one, then its Brauer lifted map $BG \to BU$ is homotopically invariant under the endomorphism $\Psi^q : BU \to BU$ representing the q-th Adams operation.

In §5, we state the Adams conjecture and sketch Quillen's proof of it in the complex case. For us, Quillen's most important idea was to Brauer lift standard representations to obtain maps $B\mathrm{GL}_n(k_q) \to BU$ for every n and k_q. These assemble to a map $B\mathrm{GL}(k) \to BU$ that induces a mod ℓ homology equivalence for every prime $\ell \neq p$. This foreshadows Quillen's initial work on K-theory, which starts by asking how the story changes when we fix q and consider the map

$$B\mathrm{GL}(k_q) \to BU \tag{1.1}$$

constructed by the same process—for starters, it is no longer an equivalence.

In §6, we introduce a space $F\Psi^q$ as the homotopy fixed points of $\Psi^q : BU \to BU$, and equivalently as the homotopy fiber of $\Psi^q - 1$. Both points of view are important. Thinking of $F\Psi^q$ as homotopy fixed points, the idea is that if a map $BG \to BU$ is Brauer lifted from a k_q-representation, since it is homotopy fixed by $\Psi^q : BU \to BU$ we might hope to reinterpret it as a map $BG \to F\Psi^q$. This is possible under the right conditions. In particular, we may lift our map (1.1) to a map

$$\theta : B\mathrm{GL}(k_q) \to F\Psi^q, \tag{1.2}$$

the explicit construction of which is given in this section. The remarkable fact is that θ *is an integral homology equivalence.* The proof as in [Qui72] is hard.

In §7, we define the *plus construction* X^+ of a space X and define the K-theory (in positive degrees) of a ring R as the homotopy groups of $B\mathrm{GL}(R)^+$. Since the map (1.2) is an integral equivalence, it happens that on passage to the plus construction θ induces a weak homotopy equivalence $\theta^+ : B\mathrm{GL}(k_q)^+ \xrightarrow{\sim} F\Psi^q$. The homotopy groups of $F\Psi^q$ are readily computed from the long exact sequence in homotopy of a homotopy fiber, and thus the K-theory of finite fields is determined.

2. Historical Notes

In [Ada60], Adams solved the famous Hopf invariant one problem through a detailed analysis of higher operations on ordinary cohomology. He then tried this approach on the vector fields on spheres problem, which asks for the maximal number of linearly independent tangent vector fields one may put on S^n at once. Here he was less successful, with only partial results appearing in [Ada62b, received 1960]. But operations were the right avenue. In a second attempt [Ada62a, received 1961], Adams had the nascent idea to replace higher operations on H^* with simpler, primary ones on K-theory. Namely, he constructed the *Adams operations* Ψ^k and showed that these Ψ^k describe structure on K-theory with which "having too many vector fields" is incompatible. More precisely, Adams used the properties of Ψ^k to show that S^n could not admit more than a certain number $\rho(n) - 1$ of independent vector fields, and it was already known to have at least $\rho(n) - 1$.

Meanwhile in [Ati61], Atiyah defined[1] and studied a certain quotient $J(X)$ of the group $K(X)$ of vector bundles over X. These groups $J(X)$ generalize the J-homomorphism from classical homotopy theory[2] and live in the world of vector bundles and fibrations. On the tailwind of his invention of Ψ^k and considering the relation to homotopy theory, Adams saw an opportunity: one can exploit theory on the bundle side (in particular, Ψ^k) to study $J(X)$ and leverage this to understand the homotopy groups of spheres in particular. This was Adams' next project, the result a fundamental four-paper study of $J(X)$ [Ada63, Ada65a, Ada65b, Ada66].

We can oversimplify Adams' work easily enough. For X a finite CW complex, $J(X)$ is the group $K(X)$ of vector bundles over X modulo "fiber homotopy equivalence."[3] Adams plotted to compute $J(X)$ by introducing two other quotients $J'(X), J''(X)$ which would capture $J(X)$ in a diagram of epimorphisms:

$$\begin{tikzcd} & J''(X) \arrow[d] \\ K(X) \arrow[ur] \arrow[r] \arrow[dr] & J(X) \arrow[d] \\ & J'(X) \end{tikzcd}$$

One thinks of $J'(X)$ and $J''(X)$ as lower and upper bounds for $J(X)$, respectively. In particular, if $J'(X) \cong J''(X)$, then either recovers $J(X)$. The main thrust of Adams' first three papers is that $J'(X)$ and $J''(X)$ are computable and always coincide; the Adams operations Ψ^k play a basic and important role here. The fourth paper applies these results to calculate essential—and at the time, *mysterious*—structure in the homotopy groups of spheres.

Characteristically, Adams' work on $J(X)$ stimulated lots of new mathematics. There is for example *chromatic homotopy theory*, wherein the "chromatic picture" partially grew out of attempts to explain periodic phenomena in π_*^s revealed by Adams' work. Another example, the subject of this paper, is *higher algebraic K-theory*. The relation here is strongly causal. Quillen's original construction of K_* would have been inconceivable without the context provided by Adams' work, specifically *the Adams conjecture*. Let us sketch the precise story.

The Adams conjecture roughly says that Ψ^k does not change $J(X)$. It is discussed in Adams' second paper and proved in special cases. The conjecture is an essential part of Adams' work, for as k varies it produces upper bounds on $J(X)$ from which one extracts $J''(X)$ as the "best bound." This is made precise by defining $J''(X)$ as a certain quotient of $K(X)$, in fact by a subgroup of the kernel of $K(X) \to J(X)$, thus fitting $J''(X)$ into the diagram above.

As for the relation to algebraic K-theory, the proof is the pudding. In [Qui71], Quillen proved the Adams conjecture via algebraic topology, for which he constructed a "Brauer lifted" map $B\mathrm{GL}(k) \to BU$

[1] Atiyah only defined the reduced version of $J(X)$.

[2] The classical situation is recovered by taking $X = S^n$ for $n > 1$. In this case, the reduced version of $J(S^n)$ is identifiable with the image of the classical map $\pi_{n-1}(O) \to \pi_{n-1}^s$ via a clutching construction. In particular, the $J(S^n)$ determine cyclic direct summands of stable homotopy groups. Then the hope was that these "pieces" of π_s^* are calculable and interesting (they are).

[3] This means we identify $[\Psi] \sim [\eta]$ if their associated spherical bundles are equivalent as fibrations.

and showed it to be a mod ℓ homology equivalence for every prime $\ell \neq p$. This sufficed to prove the Adams conjecture. It was natural to then ask how to extend Quillen's methods to study a "Brauer lifted" map $B\mathrm{GL}(k_q) \to BU$ at the discrete levels $q = p^r$, which needs a bit more work. This was carried out in the seminal [Qui72]. This map $B\mathrm{GL}(k_q) \to BU$ is no longer an equivalence, but it turns out to induce a map $B\mathrm{GL}(k_q) \to F\Psi^q$ which is an *integral* homology equivalence. This space $F\Psi^q$ is tractable, and via the equivalence $B\mathrm{GL}(k_q) \to F\Psi^q$ one learns about $B\mathrm{GL}(k_q)$. Now for various reasons [cf. [Cla]], Quillen divined from his work a definition of the higher algebraic K-theory of k_q:

$$K_n(k_q) := \pi_n(B\mathrm{GL}(k_q)^+), \quad \text{for } n > 0, \tag{2.1}$$

and the established homology equivalence sufficed to completely determine $\pi_n(B\mathrm{GL}(k_q)^+)$ and thus $K_n(k_q)$. Here, the $+$ indicates the *plus construction*.[4] The constructions in Definition (2.1) work for any ring R in place of k_q, leading Quillen to one of the first definitions[5] of higher algebraic K-theory:

$$K_n(R) := \pi_n(B\mathrm{GL}_n(R)^+), \quad \text{for } n > 0.$$

A short while later in [Qui73], Quillen proposed a totally different definition using the *Q-construction* which he also defined. It is categorical rather than homotopical and begets important generalizations. The "$+ = Q$ theorem" states that these two definitions are equivalent—if there was doubt that Quillen's original definition was the right one, it vanished with this result. In this way, Quillen initiated the study of algebraic K-theory proper and gave it calculational and theoretical footing, for which he was awarded the Fields Medal in 1978.

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We close with an amusing remark. Quillen's proof of the Adams conjecture described above was neither the first nor the last. Quillen had sketched the first proof in 1968, modulo a nontrivial conjecture and taking a very different path through algebraic geometry. It uses the *etale homotopy theory* which Artin and Mazur had recently developed, whose rough purpose was to lift an affirmed Adams conjecture for mod p varieties to the topological setting. Friedlander completed this sketch in 1970. Sullivan then gave another highly influential proof in 1970, also using algebraic geometry and etale homotopy. At this point one should wonder about a simpler proof via algebraic topology, and this was motivation for Quillen's second proof. But a short while later in 1974, another algebro-topological proof was found by Becker and Gottlieb—much more elementary than Quillen's, taking about two pages to his fourteen, appearing as part of an overall ten-page paper. The reader now asks how K-theory would have developed had the timing been only a little different.

[4]We will review this in §7. This construction changes a space's fundamental group and preserves its (co)homology.

[5]By this time, the lower groups K_0, K_1, and K_2 had been studied for 20-30 years. Certain exact sequences and the analogy with topological K-theory begged a full-blown algebraic K-theory K_n for all $n \geq 0$, several definitions of which were proposed in the late 60's. Quillen's was the first widely-accepted definition. A good, brief account of the state of higher K-theory at the time is [Swa70].

3. The Adams Operations

The *Adams operations* Ψ^k are certain cohomology operations on K-theory. They are important to Adams' work on $J(X)$, in particular *the Adams conjecture* which we will discuss in the next section. These Ψ^k extend the exterior power operations on bundles; more generally, given any ring R with structure comparable to exterior power operations, one can construct Adams operations $\Psi^k : R \to R$. Such rings are called *λ-rings*. We will not generally develop λ-rings, but we will need to consider Adams operations on rings other than K-groups, namely complex representation rings. We introduce these operations now and record some basic properties.

Proposition 3.1. *Let K denote either real or complex K-theory. For each $k \geq 0$, there exists a unique natural transformation $\Psi^k : K(-) \to K(-)$ such that $\psi^k[L] = [L^{\otimes k}]$ for every line bundle $L \to X$. These Ψ^k are the Adams operations on K-theory.*

Unqiueness here is a consequence of the splitting principle. These operations are ring homomorphisms.

Now, consider the set $\mathrm{Rep}_{\mathbb{C}}(G)$ of complex representations of a finite group G. It is an abelian monoid under the direct sum of representations, hence we may consider its group completion $R_{\mathbb{C}}(G)$, the *complex representation ring*. Its multiplication is induced by the tensor product of representations. We may take exterior powers of representations, and this gives $R_{\mathbb{C}}(G)$ the structure necessary to possess Adams operations.

Proposition 3.2. *Consider $R_{\mathbb{C}}(-)$ as a functor on the category of finite groups. For each $k \geq 0$, there exists a unique natural transformation $\Psi^k : R_{\mathbb{C}}(-) \to R_{\mathbb{C}}(-)$ such that $\Psi^k(\rho) = [\rho^k]$ for every linear representation $\rho : G \to \mathbb{C}^\times$. These Ψ^k are the Adams operations on $R_{\mathbb{C}}(G)$.*

Uniqueness here is a consequence of Brauer's induction principle. These operations are also ring homomorphisms.

The Adams operations on both K-theory and $R_{\mathbb{C}}(G)$ may be given explicitly as Newton polynomials evaluated on exterior powers. In the case of $R_{\mathbb{C}}(G)$, the operations can be described more simply by their effect on characters (and this also suffices to define them).

Proposition 3.3. *Denote by χ_V the character of a representation V. The Adams operations Ψ^k satisfy the equation*

$$\chi_{\Psi^k V}(g) = \chi_V(g^k), \quad \forall g \in G.$$

3.1. Operations as endomorphisms of BU.

The Adams operations restrict to operations on reduced K-theory $\tilde{K}$. Thinking of the Yoneda lemma, we should like to represent the operations $\Psi^q : \tilde{K} \to \tilde{K}$ with endomorphisms $\Psi^q : BU \to BU$. However, the Yoneda lemma does not apply since K-theory is not defined for infinite-dimensional complexes such as BU. We can get around this, and in the process describe how to represent operations on $\tilde{K}$ generally, which will be useful later.

For $n \geq 1$, we may give BU^n a CW structure whose skeleta X_m have trivial odd-dimensional cohomology. This is standard, see e.g. [MS74]. The Milnor exact sequence then degenerates to yield $[BU^n, BU] \cong \varinjlim [X_m, BU]$. We get the desired correspondence

$$[BU^n, BU] \cong \{\text{natural transformations } (\tilde{K})^n \to \tilde{K}\}.$$

By this correspondence, we may represent operations such as the Adams operations $\Psi^q : \tilde{K} \to \tilde{K}$ or the difference map $(\tilde{K})^2 \to \tilde{K}$ as endomorphisms $BU \to BU$ and $BU \times BU \to BU$, respectively.

4. Brauer Lifting

In this section we describe *Brauer lifting*, a construction which lifts k-representations of a finite group G to complex ones. Since complex representations give rise to (homotopy classes of) maps $BG \to BU$, we may also think of the Brauer lift as building maps $BG \to BU$ out of k-representations of G, and this is primarily how we will use it. As we will see in §5, the lift was Quillen's main technical innovation in his proof of the Adams conjecture, where he used it to the great effect of producing a mod ℓ homology equivalence $B\mathrm{GL}(k) \to BU$. Later in §6, it will be important to understand lifted k_q-representations in particular, so we discuss those here as well.

In what follows, fix an embedding $\rho : k^\times \hookrightarrow \mathbb{C}^\times$ and a finite group G. The case in mind is $G = \mathrm{GL}_n k_q$.

4.1. Lifting k- and k_q-representations.

Let $E : G \to \mathrm{GL}_n k$ be a finite-dimensional k-representation, and let $S_E(g) \subseteq k^\times$ denote the set of eigenvalues of $E(g)$ appearing with multiplicity. Define the *Brauer character* of E, written $\chi_E^{br} : G \to \mathbb{C}$, by

$$g \mapsto \sum_{\lambda \in S_E(g)} \rho(\lambda).$$

A classical theorem of Green says that for any modular representation $E \in \mathrm{Rep}_k(G)$, there is a *unique* virtual representation $E^{br} \in R_{\mathbb{C}}(G)$ such that $\chi_{E^{br}} = \chi_E^{br}$. The Brauer character visibly satisfies $\chi_{E\oplus W}^{br} = \chi_E^{br} + \chi_W^{br}$, hence $E \mapsto E^{br}$ defines a morphism of monoids $\mathrm{Rep}_k(G) \to R_{\mathbb{C}}(G)$. This extends uniquely to virtual representations via group completion.

Definition 4.1. Let $(-)^{br} : R_k(G) \to R_{\mathbb{C}}(G)$ denote the unique homomorphism extending $E \mapsto E^{br}$. We call this the *Brauer lift* of k-representations of G.

Regarding the Brauer lift, there is something to be said about those k-representations which extend k_q-representations. By "extend" we are referring to the extension of scalars, which defines a homomorphism

$$- \otimes_{k_q} k \; : \; R_{k_q}(G) \to R_k(G).$$

We write $\overline{E}$ for $E \otimes_{k_q} k$. Recall (Proposition 3.3) that the Adams operations act straightforwardly on characters: one has $\chi_{\Psi^q E}(g) = \chi_E(g^q)$. Thus, the representations E fixed by Ψ^q are exactly those for which $\chi_E(g^q) = \chi_E(g)$. This is the case in particular if χ_E is the Brauer character of $\overline{V}$ for some $V \in R_{k_q}(G)$, since that Brauer character at g is a sum over $S_{\bar{V}}(g)$ which is stable under the Frobenius, i.e. $S_{\bar{V}}(g) = S_{\bar{V}}(g^q)$. We get the following.

Lemma 4.2. *If E extends a k_q-representation, then E^{br} is Ψ^q-invariant. In symbols, the composite $(-)^{br} \circ (- \otimes_{k_q} k)$ defines a homomorphism*

$$(-)^{br} \circ (- \otimes_{k_q} k) : R_{k_q}(G) \to R_{\mathbb{C}}(G)^{\Psi^q}.$$

We call this composition the Brauer lift of k_q-representations of G and also denote it $(-)^{br}$.

4.2. Producing maps $BG \to BU$.

A complex representation $V \in R_{\mathbb{C}}(G)$ classifies a complex bundle $EG \times_G V \to BG$. This construction is natural, additive, and multiplicative on passage to isomorphism classes, thus extends to a map

$$\beta : R_{\mathbb{C}}(G) \to \tilde{K}^0(BG) \cong [BG, BU].$$

This is called the *Borel construction* or *associated bundle*. Now we can construct maps $BG \to BU$ out of k-representations of G by way of the composition $\beta \circ (-)^{br}$. We may refer to maps produced this way as *Brauer lifted maps*.

Again, more can be said for extensions of k_q-representations. The Borel construction preserves exterior powers, hence β commutes with the Adams operations Ψ^k on $R_{\mathbb{C}}(G)$ and $\tilde{K}^0(BG)$. Noting Lemma 4.2, the composition $\beta \circ (-)^{br}$ therefore defines a homomorphism $R_{k_q}(G) \to \tilde{K}^0(BG)^{\Psi^q}$. Identifying $\tilde{K}^0(BG)^{\Psi^q} \cong [BG, BU]^{\Psi^q}$ gives the following.

Proposition 4.3. *If E extends a k_q-representation, then the Brauer lifted map $BG \to BU$ obtained from E is homotopy invariant under the endomorphism $\Psi^q : BU \to BU$.*

5. The Adams Conjecture

We now state the Adams conjecture and sketch Quillen's proof of it. The conjecture was a basic and important part of Adams' work on $J(X)$. Quillen turned it into an obstruction problem (Step I, III below), then used Brauer lifting to construct (Step II) a map $B\mathrm{GL}(k) \to BU$ which turns out to be a mod ℓ cohomology equivalence for every prime $\ell \neq p$ (Step IV), rendering the obstruction problem trivial.

We present this section as "motivation and context" for the work discussed in the rest of this paper. It is optional, but we hope it is interesting to see the evolution of ideas, especially with hindsight.

5.1. The conjecture and a reformulation.

Let X be a finite CW complex. The stable pointed spherical fibrations over X form a monoid under fiberwise smashing, the group completion of which we write $\mathrm{Sph}(X)$. Now, for any vector bundle over X, its fiberwise one-point compactification is a pointed spherical fibration. This induces a homomorphism $J : KO(X) \to \mathrm{Sph}(X)$. One similarly obtains a map $J : KU(X) \to \mathrm{Sph}(X)$.

Conjecture 5.1 (The Adams conjecture)**.** *Let K denote either real or complex K-theory. If X is a finite CW complex, k is an integer, and $E \in K(X)$, then there exists an integer $n \geq 0$ such that*

$$k^n J(\Psi^k E - E) = 0. \tag{5.2}$$

Since $\Psi^a \circ \Psi^b = \Psi^{ab}$, it suffices to check the cases where k is a prime p.

Remark 5.3. In his original paper, Adams proves the real conjecture in the case that E is a linear combination of line and plane bundles, and in the case that $X = S^{2n}$ and E underlies a complex bundle.

With the right machinery, we can give a convenient reformulation of the Adams conjecture. It is conceptually straightforward: one looks at induced maps on classifying spaces and interprets Equation (5.2) as the nullity of their composition after p-localization, for every p. We spell this out for the J map.

Proposition 5.4. *Let $F(n)$ denote the monoid of based homotopy equivalences $S^n \to S^n$. Consider the map $O(n) \to F(n)$ which sends a matrix to the one-point compactification of the linear isometry $\mathbb{R}^n \to \mathbb{R}^n$ which it describes. In the colimit, we get a map $BO \to BF$. The space BF classifies stable spherical fibrations, i.e. $[X, BF] \cong \mathrm{Sph}(X)$, and the map $BO \to BF$ represents $J : KO(X) \to \mathrm{Sph}(X)$. Similarly in the complex case, there is a map $BU \to BF$ representing $J : KU(X) \to \mathrm{Sph}(X)$.*

Conjecture 5.5. *Let B denote either BO or BU. For every prime p, the composite*

$$B \xrightarrow{\Psi^p - 1} B \xrightarrow{J} BF \longrightarrow BF[p^{-1}] \tag{5.6}$$

Is nullhomotopic.

5.2. Sketch proof.

Conjecture 5.5 is equivalent to the Adams conjecture. Quillen proves it in [Qui71]. We shall now sketch the proof for $B = BU$ following his argument, breaking it into four steps.

Step I. First one proves the Adams conjecture for complex bundles *whose structural group may be reduced to a finite group*. Those are the bundles $E \in KU(X)$ such that for some finite group G and principal G-bundle P, the bundle E lies in the image of the map $R_{\mathbb{C}}(G) \to KU(X)$ induced by $V \mapsto P \times_G V$.

This proof has two parts. Let G denote a finite group. First, one observes the corollary to the *Brauer induction theorem* [Ser77, p. 11] that every element of $R_{\mathbb{C}}(G)$ is an integral linear combination of representations induced by one-dimensional representations of subgroups of G. Second, one extends Adams' proof of his conjecture for combinations of real line and plane bundles [c.f. Remark 5.3] to combinations of bundles associated to one and two dimensional real representations of *subgroups of G*. Specializing to the complex case and considering the first fact, we have our proof.

Step II. We next construct a map $\alpha : B\mathrm{GL}(k) \to BU$ that will ferry the rest of the proof. Let $\alpha_{n,q} : B\mathrm{GL}_n(k_q) \to BU$ denote the Brauer lifted map [c.f. §4.2] obtained from the standard representation $\mathrm{GL}_n(k_q) \to \mathrm{GL}_n(k_q)$, i.e. the identity map. These maps $\alpha_{n,q}$ are compatible in n and q, and we define α to be their colimit as $n, q \to \infty$.

Step III. Let μ denote the composition (5.6). One now shows that $\mu \circ \alpha$ is nullhomotopic. It suffices to show that $\mu \circ \alpha$ restricted to any finite skeleton is nullhomotopic, since the finiteness of $BF[p^{-1}]$'s homotopy groups kills the relevant $\lim^1$ term. This occurs by virtue of Brauer lifting: if X is any finite skeleton, then the restriction of $\mu \circ \alpha$ to X classifies a bundle whose structural group may be reduced to some $B\mathrm{GL}_n(k_q)$. It is therefore nullhomotopic by the result of Step I.

Step IV. The heart of Quillen's paper is his demonstration that $\alpha : B\mathrm{GL}(k) \to BU$ induces a mod ℓ homology isomorphism for every prime $\ell \neq p$. This completes the proof for the following reason. The nullity of $\mu \circ \alpha$ gives rise to a homotopy commutative diagram

$$\begin{tikzcd} B\mathrm{GL}(k) \arrow[r, "\alpha"] & BU \arrow[r] \arrow[d, "\mu"] & C\alpha \arrow[dl, "\beta"] \\ & BF[p^{-1}] & \end{tikzcd}$$

We want to show that μ and thus β is nullhomotopic. Since the obstructions to nullhomotoping β are detected by $H^n(C\alpha, \pi_n BF[p^{-1}])$, it suffices to show that these groups are trivial. Note that $\pi_n BF[p^{-1}]$ has order coprime to p, and if α is a homology equivalence with coefficients in this group, then the cofiber long exact sequence yields $H^n(C\alpha, \pi_n BF[p^{-1}]) = 0$.

The proof that α is a mod ℓ homology equivalence pursues a basic analogy between diagonal matrices in $\mathrm{GL}_n k_q$ and maximal torii in $U(n)$. Recall the isomorphism $H^*BU(n) \xrightarrow{\sim} (H^*BT^n)^{\Sigma_n} \cong \mathbb{Z}_\ell[\sigma_1, \ldots, \sigma_n]$, where $T^n \leq U(n)$ denotes the diagonal matrices and σ_i is the i-th elementary symmetric polynomial on n copies of the universal first Chern class. One begins by showing that the restriction map defines an *injection*

$$H^*B\mathrm{GL}_n(\mathbb{F}_q) \hookrightarrow (H^*B\mathrm{Diag}_n)^{\Sigma_n} \cong \mathbb{Z}_\ell[\sigma'_1, \ldots, \sigma'_n]. \tag{5.7}$$

This is proven in [Qui71, Theorem 4.3]. Here, $\mathrm{Diag}_n \leq \mathrm{GL}_n\mathbb{F}_q$ is the subgroup of diagonal matrices and σ'_i is the i-th elementary symmetric polynomial on the n copies of the indeterminate in $H^*B\mathrm{Diag}_1 \cong \mathbb{Z}_\ell[x]$.

Next we pass to the colimit in k_q. Injectivity in (5.7) holds in the limit, so we have maps

$$H^*BU(n) \xrightarrow{\alpha_n^*} H^*B\mathrm{GL}_n(k) \hookrightarrow (H^*B\mathrm{Diag}_n)^{\Sigma_n} \cong \mathbb{Z}_\ell[\sigma'_1, \ldots, \sigma'_n].$$

By construction, the composite maps each σ_i to σ'_i, hence it is an isomorphism. Thus the injection is an isomorphism, and then α_n^* must be too. We obtain the desired isomorphism in the colimit in n.

6. The Space $F\Psi^q$

For the Adams conjecture, we Brauer lifted the standard k-representations of $\mathrm{GL}_n(k_q)$ for every n and q to maps $\alpha_{n,q} : B\mathrm{GL}_n(k_q) \to BU$ and obtained a mod ℓ homology equivalence $\alpha : B\mathrm{GL}(k) \to BU$ in the colimit. We may perfectly well fix q instead and ask about the colimiting map $\alpha_q : B\mathrm{GL}(k_q) \to BU$. This is no longer an equivalence, but there is something to be said: the standard k-representations extend the standard k_q-ones, hence by Proposition 4.3 every Brauer lifted component map $\alpha_{n,q}$ is Ψ^q-homotopy invariant. Then as one might in ordinary algebra, we would like to think of α_q as a map

$$\theta : B\mathrm{GL}(k_q) \to F\Psi^q$$

For some "space of Ψ^q-homotopy fixed points" $F\Psi^q$. As we review in this section, this is possible with the right machinery, and we conclude by constructing θ. It is a remarkable theorem that θ is an integral homology equivalence; we will use this fact to compute the K-theory of finite fields in §7. Its difficult proof can be found in [Qui72].

We first define the space $F\Psi^q$. Recall that we briefly reviewed the representation of operations on K-theory in §3.1. It will be useful to understand $F\Psi^q$ as both the *homotopy fixed points* of $\Psi^q : BU \to BU$ and as the *homotopy fiber* of the map $\Psi^q - 1 : BU \to BU$. These are general constructions with various properties, but we will not mull over details. When we invoke properties of either the homotopy fixed points or fiber, we will indicate so. A good reference is [May99].

Definition 6.1 (As homotopy fixed points)**.** Denote by $\Delta : BU^I \to BU \times BU$ the map which takes a path to its endpoints. Define the space $F\Psi^q$ as the pullback

$$\begin{tikzcd} F\Psi^q \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & BU^I \arrow[d, "\Delta"] \\ BU \arrow[r, "id \times \Psi^q"] & BU \times BU \end{tikzcd}$$

Thus, $F\Psi^q$ consists of pairs (x, γ), where $x \in BU$ and $\gamma \in BU^I$, such that $\gamma(0) = x$ and $\gamma(1) = \Psi^q(x)$.

Proposition 6.2. *The space $F\Psi^q$ is a homotopy fiber of $\Psi^q - 1 : BU \to BU$.*

Proof. Suppose as given a baspoint $b \in BU$. Let $d : BU \times BU \to BU$ represent the difference operation on $\tilde{K}$. Define $m : BU^I \to BU^I \times_{BU} \{b\}$ to send a path p to the path $t \mapsto d(p(t), p(1))$ joining $d\Delta(p)$ to b. Lastly define $n : BU^I \to BU^I \times_{BU} b \to BU$ to take a path p to its start $p(0)$.

In representing operations on $\tilde{K}$ as endomorphisms of BU, we would like to treat b as the "zero element." Using the homotopy extension property, we may choose our representatives for $\Psi^q : BU \to BU$ and $d : BU \times BU \to BU$ so that $d(x, x) = b$, $d(x, b) = x$, and $\Psi^q(b) = b$. Altogether we get the following commutative diagram.

$$\begin{tikzcd} F\Psi^q \arrow[r] \arrow[d] & BU^I \arrow[r, "m"'] \arrow[d, "\Delta"] & BU^I \times_{BU} \{b\} \arrow[d, "n"] \\ BU \arrow[r, "\mathrm{id} \times \Psi^q"] & BU \times BU \arrow[r, "d"] & BU \end{tikzcd}$$

See that the vertical maps are all fibrations with the same fiber ΩBU. It follows that $F\Psi^q$ is homotopy equivalent to the pullback of $d \circ (\mathrm{id} \times \Psi^q)$ and n, which is what we wanted to show. □

We can immediately make a simple computation.

Proposition 6.3. *$F\Psi^q$ is simple and for $i > 0$ we have $\pi_{2i}(F\Psi^q) = 0$ and $\pi_{2i-1}(F\Psi^q) = \mathbb{Z}/(q^i - 1)$.*

Proof. As in Proposition 6.2, the space $F\Psi^q$ is a homotopy fiber, hence its homotopy groups fit into a long exact sequence

$$\cdots \longrightarrow \pi_j BU \xrightarrow{(\Psi^q - 1)^*} \pi_j BU \longrightarrow \pi_{j-1} F\Psi^q \longrightarrow \cdots$$

Bott Periodicity implies that $\pi_{2j-1}(BU) = 0$ and $\pi_{2j}(BU) = \mathbb{Z}$. Also using Bott Periodicity, one can show that $(\Psi^q - 1)^*$ acts as multiplication by $q^j - 1$. From this the groups $\pi_j F\Psi^q$ are determined.

Next we show that $F\Psi^q$ is simple. Recall that a space X is simple if $\pi_1(X)$ is abelian and acts trivially on higher $\pi_i(X)$. The above computation shows that $\pi_1(F\Psi^q)$ is abelian. As for its action on higher homotopy, note that (by general theory of fibrations) the action of $\pi_1(F\Psi^q)$ on $\pi_i(F\Psi^q)$ arises from that of $\pi_1(BU)$ on $\pi_i(BU)$. Since BU is simply connected, this action is trivial, hence $F\Psi^q$ is simple. □

6.1. **Producing maps $BG \to F\Psi^q$.** Now we will reinterpret our Ψ^q-invariant map $B\mathrm{GL}(k_q) \to BU$ as a map $B\mathrm{GL}(k_q) \to F\Psi^q$. We can do this generally; let G be a finite group and let E be a k_q-representation of G. As in Proposition 4.3, the extension $\overline{E} \in R_k(G)$ gives rise to a class in $[BG, BU]^{\Psi^q}$. Now as $F\Psi^q$ is the homotopy fixed point space of Ψ^q, one might expect that if there are no "complicated" maps $X \to BU$ for a space X, then there should be an equivalence $[X, BU]^{\Psi^q} \cong [X, F\Psi^q]$. As in the next two propositions, this intuition is almost correct and the hypothesis is fulfilled in the case $G = \mathrm{GL}_n(k_q)$ we are interested in.

Proposition 6.4. *Recall that $F\Psi^q$ is a pullback and we denoted by ϕ the evident map $F\Psi^q \to BU$. If a space X is such that $[X, \Omega BU] = 0$, then $\phi_* : [X, F\Psi^q] \to [X, BU]^{\Psi^q}$ is an isomorphism.*

Proof. For surjectivity, see that by the definition as a pullback, a map $f : X \to BU$ together with a homotopy equivalence $f \simeq \Psi^q f$ is the same data as a map $g : X \to F\Psi^q$ such that $\phi \circ g = f$.

For injectivity, see that as a homotopy fiber, the space $F\Psi^q$ fits into the fiber sequence generated by $\Psi^q - 1$ (c.f. [May99, §6])

$$\cdots \to \Omega BU \to F\Psi^q \xrightarrow{\phi} BU \xrightarrow{\Psi^q - 1} BU,$$

And this induces the exact sequence of pointed sets

$$\cdots \to [X, \Omega BU] \to [X, F\Psi^q] \xrightarrow{\phi_*} [X, BU] \xrightarrow{(\Psi^q - 1)_*} [X, BU].$$

From this and our hypothesis that $[X, \Omega BU]$ is trivial, it is immediate that ϕ_* is injective. □

Proposition 6.5. *If G is a finite group, then $[BG, \Omega BU] = 0$.*

Proof. This is an application of the Atiyah-Segal completion theorem. Slightly more is said in [Qui72]. □

Definition 6.6 (Construction of θ)**.** Let $G = \mathrm{GL}_n(k_q)$. Consider the maps

$$R_{k_q}(\mathrm{GL}_n(k_q)) \xrightarrow{\beta \circ (-)^{br}} [B\mathrm{GL}_n(k_q), BU]^{\Psi^q} \cong [B\mathrm{GL}_n(k_q), F\Psi^q],$$

The isomorphism coming from the previous two propositions. Take $\theta_n : B\mathrm{GL}_n(k_q) \to F\Psi^q$ to represent the image of the standard representation of $\mathrm{GL}_n(k_q)$ under the composite. These θ_n are compatible with respect to the maps $B\mathrm{GL}_m(k_q) \to B\mathrm{GL}_{m+1}(k_q)$ induced by inclusions, hence assemble to a map $\theta : B\mathrm{GL}(k_q) \to F\Psi^q$. Furthermore, the Milnor exact sequence implies $\mathrm{colim}_n[B\mathrm{GL}_n(k_q), BU] \cong [B\mathrm{GL}(k_q), BU]$, so this uniquely defines θ up to homotopy.

The essential fact is that θ is an integral equivalence. This is hard and calculational. We refer the reader to [Qui72] for the proof.

Theorem 6.7. *The map $\theta : B\mathrm{GL}(k_q) \to F\Psi^q$ induces an integral homology isomorphism.*

We have constructed an integral homology equivalence $\theta : B\mathrm{GL}(k_q) \to F\Psi^q$ (Theorem 6.7) and now ask what it does on homotopy groups. We might like to appeal to the classical Whitehead theorem[6] and say that θ is a homotopy equivalence. However, that does not apply to non-simply connected spaces, and in any case these spaces have distinct fundamental groups with $\pi_1 F\Psi^q \cong (\pi_1 B\mathrm{GL}(k_q))^{ab}$. In a dramatic turn of events, we can address all this in one go: there is a *plus construction* $X \mapsto X^+$ which abelianizes $\pi_1 B\mathrm{GL}(k_q)$ and preserves homology, and through this construction θ induces a homology equivalence $\theta^+ : B\mathrm{GL}(k_q)^+ \to F\Psi^q$ since homology is preserved, and furthermore the *generalized Whitehead theorem* applies to $B\mathrm{GL}(k_q)^+$ to let us conclude that θ^+ is a weak homotopy equivalence. We will define the K-theory of k_q as the homotopy of $B\mathrm{GL}(k_q)^+$, so this equivalence lets us compute $K_n(k_q)$.

In what follows, all spaces are based CW complexes.

7.1. Acyclicity and the plus construction.

An *acyclic space* is one with the homology of a point. A map of connected spaces is called an *acyclic map* if it has acyclic homotopy fiber. Such maps are equivalences in a strong sense, and are in particular integral homology equivalences. Our concern for acyclic maps is tied to their role in the following construction. Recall that a group is called *perfect* if it is equal to its commutator, i.e. if it has trivial abelianization.

Definition 7.1 (Relative plus construction)**.** Let X be a connected space and let $N \leq \pi_1 X$ be perfect and normal. A *plus construction relative to N* is an acyclic map $f : X \to Y$ such that $\ker f_* = N$.

As in the following theorem, plus constructions exist, are unique up to homotopy, and are universal among maps killing their designated subgroup. One may obtain a plus construction by attaching 2-cells to alter π_1 and then attaching 3-cells to undo any effects the 2-cells had on homology. This was first devised in some form by Kervaire in [Ker69] for unrelated reasons.

Theorem 7.2. *Let N denote a perfect normal subgroup of $\pi_1(X)$.*

(1) There is a plus construction $X \to Y$ relative to N.
(2) Let $f : X \to Y$ be a plus construction relative to N. If $g : X \to Z$ is any map such that $g_*(N) = 0$, then there is a map $h : Y \to Z$ through which f factors, and h is unique up to pointed homotopy.
(3) In particular, if g is another plus construction relative to N, then h is a homotopy equivalence. Thus plus constructions are unique up to homotopy.

Every group G has a maximal perfect subgroup $P \leq G$, its *perfect radical*, and this subgroup P is always normal. We are most interested in this subgroup. We define the *plus construction* of a space X to be its plus construction relative to the perfect radical of $\pi_1(X)$ and denote the resulting space by X^+. Then the following is clear from (2) above and the fact that abelian groups have trivial perfect radicals.

Corollary 7.3. *Let $f : X \to X^+$ be a plus construction of X. If a space Z has an abelian fundamental group, then $f_* : [X^+, Z] \to [X, Z]$ is bijective. Given $g : X \to Z$, we denote by g^+ a corresponding map $X^+ \to Z$.*

7.2. The K-theory of finite fields.

For $n > 0$, Quillen defined $K_n(k_q)$ to be the homotopy of $B\mathrm{GL}(k_q)^+$:

$$K_n(k_q) := \pi_n B\mathrm{GL}(k_q)^+.$$

The perfect radical of $\mathrm{GL}(k_q)$ is the subgroup $E(k_q)$ of elementary matrices,[7] hence $K_1(k_q) = \mathrm{GL}(k_q)/E(k_q)$ which agrees with the classical definition of K_1. There is motivation for this definition coming from multiple directions, especially the homology equivalence $B\mathrm{GL}(k) \to BU$ constructed in Quillen's proof of the Adams conjecture. The motivation is an interesting subject, but a bit involved. We refer the reader to [Cla] and [Qui70] for more on this.

Now we put everything together to compute $K_n(k_q)$. Noting Corollary 7.3, the map $\theta : B\mathrm{GL}(k_q) \to F\Psi^q$ factors through the plus construction (which exists by Theorem 7.2), yielding a map

$$\theta^+ : B\mathrm{GL}(k_q)^+ \to F\Psi^q.$$

[6]There are a few statements called the Whitehead theorem. Here we refer to the following: "if a map of simply-connected CW complexes is an integral homology equivalence, then it is a weak homotopy equivalence."

[7]The *elementary matrices* are those differing from the identity matrix by exactly one off-diagonal entry.

Plus constructions are acyclic and so preserve homology. Since θ is an integral homology equivalence (Theorem 6.7), it follows that θ^+ is as well. Passing to the plus construction has, first of all, equalized the fundamental groups of our spaces. But we can say more: the space $B\mathrm{GL}(k_q)^+$ is an H-space,[8] in particular it is simple, and we know that $F\Psi^q$ is simple also (Proposition 6.3). We can invoke the following.

Theorem 7.4 (Generalized Whitehead theorem)**.** *An integral homology equivalence between simple spaces is a weak homotopy equivalence.*

Therefore θ^+ is a weak homotopy equivalence. We computed the homotopy groups of $F\Psi^q$ (Proposition 6.3), giving us those of $B\mathrm{GL}(k_q)^+$. The K-theory of finite fields is determined.

Corollary 7.5. *For $i > 0$ we have $K_{2i}(k_q) = 0$ and $K_{2i-1}(k_q) \cong \mathbb{Z}/(q^i - 1)$.*

[8] One can induce the H-space structure by defining a direct sum operation on $\mathrm{GL}(k_q)$. See also [Cam].

References

[Ada60] J. F. Adams. On the non-existence of elements of Hopf invariant one. *Annals of Mathematics*, 72(1):20–104, 1960.

[Ada62a] J. F. Adams. Vector fields on spheres. *Bulletin of the American Mathematical Society*, 68(1):39 – 41, 1962.

[Ada62b] J.F. Adams. Vector fields on spheres. *Topology*, 1(1):63–65, 1962.

[Ada63] J.F. Adams. On the groups J(X)—I. *Topology*, 2(3):181–195, 1963.

[Ada65a] J.F. Adams. On the groups J(X)—II. *Topology*, 3(2):137–171, 1965.

[Ada65b] J.F. Adams. On the groups J(X)—III. *Topology*, 3(3):193–222, 1965.

[Ada66] J.F. Adams. On the groups J(X)—IV. *Topology*, 5(1):21–71, 1966.

[Ati61] M. F. Atiyah. Thom complexes. *Proceedings of the London Mathematical Society*, s3-11(1):291–310, 1961.

[Cam] Tim Campion. Why is $BGL(R)^+$ an H-space? Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/2570695 (version: 2017-12-17).

[Cla] Dustin Clausen. Plus construction considerations. MathOverflow. https://mathoverflow.net/q/121351 (version: 2013-02-09).

[Ker69] Michel A. Kervaire. Smooth homology spheres and their fundamental groups. *Trans. Amer. Math. Soc.*, 144:67–72, 1969.

[May99] J.P. May. *A Concise Course in Algebraic Topology*. Chicago Lectures in Mathematics, Chicago, 1999.

[MS74] John W. Milnor and James D. Stasheff. *Characteristic Classes*. Princeton University Press, Princeton, 1974.

[Qui70] Daniel Quillen. Cohomology of groups. In Gauthier-Villars, editor, *Proc. International Congress of Mathematicians*, volume 2, pages 47–51, 1970.

[Qui71] Daniel Quillen. The Adams conjecture. *Topology*, 10(1):67–80, 1971.

[Qui72] Daniel Quillen. On the cohomology and K-theory of the general linear groups over a finite field. *Annals of Mathematics*, 96(3):552–586, 1972.

[Qui73] Daniel Quillen. Higher algebraic K-theory: I. In H. Bass, editor, *Higher K-Theories*, pages 85–147, Berlin, Heidelberg, 1973. Springer Berlin Heidelberg.

[Ser77] Jean-Pierre Serre. *Linear Representations of Finite Groups*. Springer New York, New York, 1977.

[Swa70] Richard G. Swan. Algebraic K-theory. In Gauthier-Villars, editor, *Proc. International Congress of Mathematicians*, volume 1, pages 191–199, 1970.