ANALYZING RANDOM WALKS WITH ELECTRICAL NETWORKS: AN INTRODUCTION

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Abstract. We introduce ways that electrical network theory is useful for analyzing random walks. In particular, we focus on how electrical networks allow us to compute escape probabilities and commute times. We prove the Commute Time Identity and the Pólya Recurrence Theorem for random walks via electrical methods.

Contents

1. Introduction

The initial setting for our investigation is a finite, connected, undirected graph G. A particle moves randomly on the vertices of G, and we wish to understand some of the behavior of this random walk using nonstandard techniques.

The question central to our endeavors is this: Given two states x and y, how "difficult" is it for the walk to travel from x to y? We will formalize what we mean by "difficulty" with effective resistance by treating G as an electrical network. Using the notion of effective resistance, we will answer our question in two different ways: first in terms of escape probabilities (Proposition 4.2), then in terms of commute times (Theorem 6.9). Lastly, the Pólya Recurrence Theorem (Theorem 7.12) will formalize the notion that it is "infinitely difficult" for a simple random walk to "escape" to infinity without first returning to the origin in dimensions 1 and 2, but only "finitely difficult" in dimensions 3 and higher. We hope that in answering the central question, we illustrate how analyzing random walks with electrical networks is illuminating, physically intuitive, and computationally useful.

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2. Preliminaries

We will assume knowledge of the basic definitions associated with graph theory and probability theory. Although we also suspect the reader has some familiarity with Markov chains and random walks, we will casually review the introductory definitions below.

The probability $P(x, y)$ is called a *transition probability*; it is the probability of the Markov chain moving from x to y in one step. Transition probabilities of a Markov chain are decided only by the present state x at time t and by the potential future state y at time $t + 1$. We organize all such transition probabilities of a Markov chain into a *transition matrix* P where each row $P(x, \cdot)$ is a probability distribution.

A *random walk* on a graph G where G has vertices V_G is a Markov chain whose states are elements of V_G. Typically, the set of states, called the *state space*, will be denoted S and will be synonymous with V_G. A *simple random walk* moves among its neighboring states according to a uniform distribution at each step. Following convention, we write $\mathbf{P}_\mu(A)$ to denote the probability of event A occurring given that the starting state of the chain was chosen according to the probability distribution μ. Similarly, $\mathbf{P}_a(A)$ assumes that the chain starts at state a with probability 1. Analogous notation holds for the expectation $\mathbf{E}_\mu$.

A probability distribution π is *stationary* for the transition matrix P if $\pi = \pi P$, so that π in theory should characterize the long-term proportion of time spent at each state. Note that we are interested in *irreducible* chains, which have the property that, for any states x and y, it is always possible to reach x from y given enough time, similar to the connectedness of graphs. For irreducible chains, the stationary distribution exists and is unique.

Hitting times are the first time that a Markov chain achieves a certain state or enters a set of states. *First return times* are the first time that a chain achieves a state x when the chain started at x. *Stopping times* are random variables which only depend on present and past information about the chain and cannot see into the future; hitting and return times are examples of stopping times.

Definition 2.1. A *network* is a finite, undirected, connected graph G where each of the edges $\{x, y\}$ of G are associated with non-negative numbers $c(x, y)$ called *conductances*. The reciprocals $r(x, y) = 1/c(x, y)$ are called *resistances*. Since the edges of G are undirected, we may write $c(x, y) = c(y, x)$ and $r(x, y) = r(y, x)$. We may sometimes write $c(e)$ where e is an edge, and we will refer to a general network as $(G, \{c(e)\})$. We may also refer to the vertices of G as *nodes*, and we distinguish two vertices a and z, called the *source* and *sink*, respectively, which are considered the start and endpoints of a network.

Given a network, we may produce a Markov chain by definining transition probabilities as follows:

$$P(x, y) = \frac{c(x, y)}{c(x)}, \tag{2.2}$$

where $c(x) = \sum_{y:y\sim x} c(x, y)$. Hence, we may think of conductances as weights; just as the walk is more likely to traverse edges with higher weights, charges more easily traverse edges with higher conductance. The simple random walk assigns conductance 1 to each edge of the network.

The *stationary distribution* for a random walk on a network is

$$\pi(x) = \frac{c(x)}{c_G}, \tag{2.3}$$

where $c_G = \sum_{x \in V_G} c(x)$. We omit the proof of this fact, but it follows from [3, pp. 116, 13].

We now introduce laws derived from physics which hold for networks. We state these as physical laws, not mathematical claims, since these laws are seen in electrical circuits, where a and z are nodes which are hooked up to a battery of some voltage. Note that *voltage* is a function on the nodes of a network, which in physics measures the amount of work needed to move a charge between two points. *Current* is an antisymmetric function on directed edges of a network that measures the rate of charge flow. [2] provides a general reference for the physics material discussed below.

Ohm's Law: If $\{x, y\}$ is an edge, v is the voltage function, and $i(x, y)$ is the current function, then

$$v(x) - v(y) = i(x, y)r(x, y). \tag{2.4}$$

Kirchoff's Node Law: $\sum_{y:y\sim x} i(x, y) = 0$ if $x \notin \{a, z\}$. This is the "flow in equals flow out" rule.

Kirchoff's Voltage Law: If for some $x_0, ..., x_{m-1} \in V_G$ we have $\overrightarrow{e_i} = (x_{i-1}, x_i)$ where $x_0 = x_m$, then

$$\sum_{i=1}^{m} (v(x_{i-1}) - v(x_i)) = 0. \tag{2.5}$$

Definition 2.6. Define $i(a) := \sum_{y:y\sim a} i(a, y)$ and call it the *strength* of the current. We define the quantity

$$R(a \leftrightarrow z) := \frac{v(a) - v(z)}{i(a)} \tag{2.7}$$

and call it the *effective resistance* between a and z.

The idea behind the effective resistance is that an entire network of resistors between two points a and z will experience the same current and voltage drop compared to if there was just a single resistor between a and z with resistance $R(a \leftrightarrow z)$. There are ways that we can modify an electrical network and still preserve the same effective resistance. The two main simplifications are the series law and parallel law.

Series Law: Resistors in series are in a single line, and their resistances add.

Parallel Law: Resistors are in parallel when their two terminals connect to the same nodes. For resistors in parallel, their conductances add.

There are two other network modification techniques which will be useful for us but do not necessarily preserve effective resistance.

Gluing: Vertices are *glued* when we connect them with perfectly conducting wire (no resistance), at which point we can treat them as the same vertex. (If there were any edges connecting x and y, they are removed). Gluing is especially useful when there is some underlying symmetry in the network, because whenever some symmetry implies that two vertices x and y have the same voltage, we can glue x and y together without changing the currents and voltages in the network.

This is because no current flows between vertices at the same voltage. We will take advantage of this fact in Example 4.5.

Cutting: Edges are cut when they are deleted from the network. We will observe how cutting affects effective resistance when we discuss Rayleigh's Monotonicity Law.

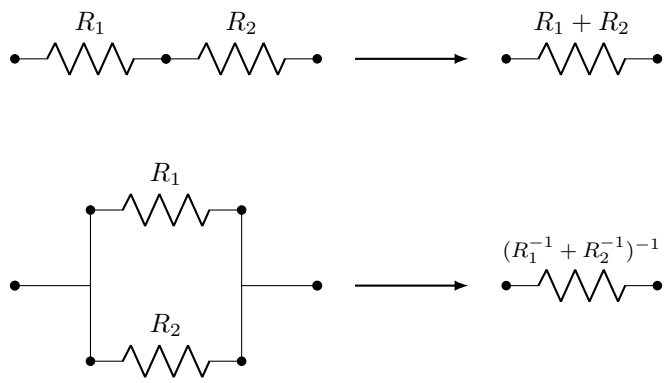

FIGURE 1. Series Law and Parallel Law.

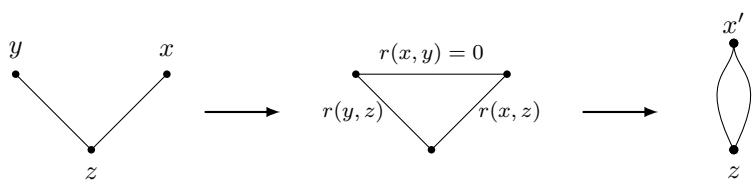

FIGURE 2. Gluing: x and y are glued together and treated as a single vertex x'.

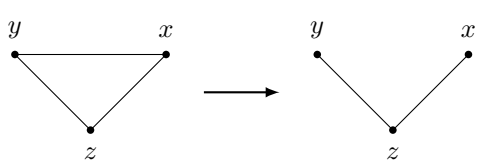

FIGURE 3. Cutting: The edge $\{x, y\}$ is deleted from the network.

Definition 2.8 (General Flows)**.** A *flow* j from a to z is a function on edges such that $j(x, y) = -j(y, x)$, $\sum_{y \in S} j(x, y) = 0$ if $x \neq a, z$, and $j(x, y) = 0$ if x and y are not adjacent.

3. Voltages Are Return Probabilities

Example 3.1 (Gambler's Ruin)**.** We consider the famous Gambler's Ruin problem to illustrate the probabilistic utility of network theory.

Greg the Gambler starts with $\$k$, and at each time step, he flips a fair coin. He gains \$1 if the coin lands heads and loses \$1 if it lands tails. Greg stops playing when he reaches $\$n$ or \$0. What is the probability that Greg reaches $\$n$ before going broke?

Let $p(k)$ be the probability that, starting at k, Greg's fortune, which is a simple random walk on $\{0, ..., n\}$, reaches n before 0. Then we have the following equations:

$$p(k) = \begin{cases} 0 & \text{if } k = 0 \\ 1 & \text{if } k = n \\ \frac{1}{2}p(k-1) + \frac{1}{2}p(k+1) & \text{if } 0 < k < n. \end{cases} \tag{3.2}$$

The third case comes from the Law of Total Probability, where we have conditioned on the first step that the random walk takes. At this point, the usual solution would be to guess that $p(k)$ is linear, then argue uniqueness of the result. However, let's see what happens if we approach this problem in a slightly different way. See Figure 4 below for our setup.

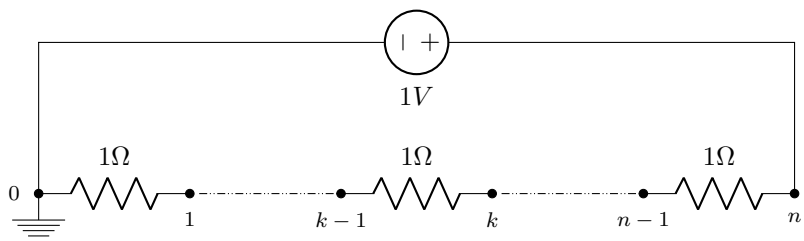

Figure 4. Circuit configuration of the Gambler's Ruin problem.

We have hooked up the endpoints 0 and n to a 1-volt battery so that $v(n) = 1$ and have grounded 0, meaning we declare 0 as the "reference point" for voltage, so $v(0) = 0$. Between each integer, we have a unit resistor. What is the voltage $v(k)$ at point k? Kirchoff's node law tells us

$$i(k, k+1) + i(k, k-1) = 0,$$

hence Ohm's Law gives

$$v(k) - v(k+1) + v(k) - v(k-1) = 0.$$

Solving for $v(k)$ gives

$$v(k) = \frac{1}{2}v(k-1) + \frac{1}{2}v(k+1).$$

To summarize, we have the following conditions for v:

$$v(k) = \begin{cases} 0 & \text{if } k = 0 \\ 1 & \text{if } k = n \\ \frac{1}{2}v(k-1) + \frac{1}{2}v(k+1) & \text{if } 0 < k < n. \end{cases} \tag{3.3}$$

Hence, intriguingly, (3.2) agrees with (3.3) precisely. Could it be possible that $v(k) = p(k)$? Could it be that when calculating certain probabilities for a random walk, it is equivalent to calculating voltages for points in the corresponding network? The answer to both of these questions is a resounding yes, under the right boundary conditions. To uncover the reason for why the correspondence between voltages and probabilities occurs, we refer to the Uniqueness Principle.

Definition 3.4. We say that a function $h : S \to \mathbb{R}$ is *harmonic at x with respect to transition matrix P* if

$$h(x) = \sum_{y \in S} P(x, y)h(y). \tag{3.5}$$

We will show that functions satisfying the harmonic property with respect to irreducible chains satisfy a property known as the Uniqueness Principle. To do so, we will first prove the Maximum Principle. The following statement of the Maximum Principle is from [3, pp. 18, Exercise 1.10], and we obtained the proof by imitating the proof of Lemma 1.16 of this same text.

Proposition 3.6 (Maximum Principle and Minimum Principle)**.** *Let P be the transition matrix of an irreducible Markov chain with state space S. Let $B \subset S$ be a non-empty subset of the state space, and assume $h : S \to \mathbb{R}$ is a function harmonic at all states $x \notin B$. Then there exist $y_1, y_2 \in B$ with $h(y_1) = \max_{x \in S} h(x)$ and $h(y_2) = \min_{x \in S} h(x)$.*

Proof. Since S is finite, there must be a state x_0 such that $h(x_0) = M$, where M is the maximum. Assume that $x_0 \notin B$. Then

$$h(x_0) = \sum_{x \in S} P(x_0, x)h(x). \tag{3.7}$$

We claim that (3.7) implies $h(z) = M$ for all $z \in S$ with $P(x_0, z) > 0$. Otherwise, we would have some z with $P(x_0, z) > 0$ and $h(z) < M$. If this were the case, then

$$\begin{aligned} M = h(x_0) &= P(x_0, z)h(z) + \sum_{y \neq z} P(x_0, y)h(y) \\ &< P(x_0, z)M + M \sum_{y \neq z} P(x_0, y) \\ &= M, \end{aligned}$$

establishing a contradiction. It follows from the irreducibility of P that there exists some sequence of states $(x_0, ..., x_{n-1}, x_n)$ such that $P(x_{i-1}, x_i) > 0$ with $x_0, ..., x_{n-1} \notin B$, and $x_n \in B$. Repeating the argument above tells us that $h(x_0) = \cdots = h(x_{n-1}) = h(x_n) = M$. Thus, h obtains the maximum in B.

The proof of the minimum principle follows from flipping inequalities; assume $h(w_0) = m$ is the minimum and there is some z with $P(w_0, z) > 0$ and $h(z) > m$. The argument is then analogous to the above. □

Corollary 3.8 (Uniqueness Principle)**.** *If $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ are harmonic functions on $S \setminus B$ and if $f(x) = g(x)$ for all $x \in B$, then $f(x) = g(x)$ for all $x \in S$.*

Proof. Define $h : S \to \mathbb{R}$, where $h(x) = f(x) - g(x)$. We have that h is harmonic on $S \setminus B$, so we may apply the maximum and minimum principles to h to conclude that, since $h(x) = 0$ on $S \setminus B$, we must have $h(x) = 0$ for all x. □

Remark 3.9. The following version of the Uniqueness Principle will be helpful. Let G be a finite subgraph of $\mathbb{Z}^d$, and let A be the vertex set of G. Then, following [4], we may define the boundary of A as follows:

$$\partial A = \{z \in \mathbb{Z}^d \setminus A : d(z, A) = 1\}, \tag{3.10}$$

or the set of all vertices not in A that neighbor some point of A. Define $S := \bar{A} = A \cup \partial A$ and let $B = \partial A$. Then Corollary 3.8 tells us that if $f, g : S \to \mathbb{R}$ are harmonic on A, and $f(x) = g(x)$ on ∂A, then $f(x) = g(x)$ for all $x \in S$. This is the version of the Uniqueness Principle that we often use.

Proposition 3.11. *If v is the voltage function on a network $(G, \{c(e)\})$ with source a, sink z, $v(a) = 1$, and $v(z) = 0$, then $v(x)$ represents the probability that a random walk starting at x reaches a before reaching z, and we call this quantity the* ***return probability****.*

Proof. We follow the calculations from [1, pp. 36]. Let $x \neq a, z$. Kirchoff's node law implies

$$\sum_{y \in S} i(x, y) = 0.$$

Hence, by Ohm's Law,

$$\sum_{y \in S} (v(x) - v(y))c(x, y) = 0,$$

therefore

$$v(x) \sum_{y \in S} c(x, y) = \sum_{y \in S} c(x, y)v(y),$$

which implies, by (2.2),

$$v(x) = \sum_{y \in S} \frac{c(x, y)}{c(x)} v(y) = \sum_{y \in S} P(x, y)v(y). \tag{3.12}$$

Hence, the voltage function is harmonic on $x \neq a, z$. Similarly, let $h(x)$ be the probability that, starting at x, the walk reaches state a before state z. Then, by the Law of Total Probability, where we condition on the next step of the walk,

$$h(x) = \sum_{y \in S} P(x, y)h(y). \tag{3.13}$$

Furthermore, $v(a) = h(a) = 1$ and $v(z) = h(z) = 0$. Therefore, the Uniqueness Principle along with (3.12) and (3.13) proves the statement. □

Proposition 3.14. *The solution to the Gambler's Ruin problem is $p(k) = \frac{k}{n}$.*

Proof. Observe that Proposition 3.11 implies that it is enough to compute the voltage at k; both v and p in Example 3.1 are harmonic functions with respect to the transition matrix given by $P(k, k+1) = P(k, k-1) = \frac{1}{2}$ for $0 < k < n$ and $P(0, 0) = P(n, n) = 1$, and both functions agree at the boundary.

To use Ohm's Law to compute $v(k)$, we first need to find the strength of the current, which equals the current passing through k. The series law implies $R(0 \leftrightarrow n) = n$. Hence, (2.7) gives $i(n) = i(k, k-1) = \frac{1}{n}$. An equivalent way to reduce the network with the series law is to have a k ohm resistor between k and 0 with

an $n-k$ ohm resistor between k and n. In this case, Ohm's Law (2.4) gives the voltage drop across the $n-k$ ohm resistor:

$$\begin{aligned} v(n) - v(k) &= 1 - v(k) \\ &= \frac{1}{n}(n-k) \\ &= 1 - \frac{k}{n}, \end{aligned}$$

so that $v(k) = \frac{k}{n}$, and the proof is complete. □

4. Escape Probabilities

We have now observed that voltages represent return probabilities, and we have applied this fact to solve a classic problem. Now, let's see what happens when we instead ask, "What is the probability that the random walk reaches z before a?"

Definition 4.1. If τ_z is the hitting time for the sink and τ_a^+ is the first return time for the source, then we define the quantity $\mathbf{P}_a(\tau_z < \tau_a^+)$, which we call an *escape probability.*

Proposition 4.2 (Escape Probability Formula)**.** *Let $a, z \in S$ be such that $a \neq z$, $v(a) = 1$, and $v(z) = 0$. Then we have*

$$\mathbf{P}_a(\tau_z < \tau_a^+) = \frac{1}{c(a)R(a \leftrightarrow z)} = \frac{C(a \leftrightarrow z)}{c(a)}. \tag{4.3}$$

Proof. By definition of effective conductance, we have

$$\begin{aligned} C(a \leftrightarrow z) = i(a) &= \sum_{y \in S} (v(a) - v(y))c(a,y) \\ &= \sum_{y \in S} (v(a) - v(y))\frac{c(a,y)}{c(a)}c(a) \\ &= c(a)(1 - \sum_{y \in S} P(a,y)v(y)) \\ &= c(a)\mathbf{P}_a(\tau_z < \tau_a^+). \end{aligned}$$

The last equality follows since the escape probability is, by definition, the complement of the return probabilty, and $v(y)$ represents the return probability for a walk starting at y. □

Remark 4.4. We have given the proof of Proposition 4.2 from [1, pp. 42], which relies on the fact that $v(a) = 1$ and $v(z) = 0$, but we could prove (as in [3]) that the escape probability formula is true for arbitrary voltage endpoint values. However, to preserve both space and the overall interpretability of the voltage function, we restrict ourselves to the case described above. In practice, it is largely inconsequential that we need to impose these boundary conditions, because we can always consider our source being attached to a 1-volt battery, and we can always declare that the voltage at the sink is 0 by grounding it.

The result from Proposition 4.2 will be fundamental to our endeavors; let us reflect for a moment to gather physical intuition for what this statement is telling us. Intuitively speaking, the effective resistance between a and z is a quantity which tells us how difficult it is for a charge to get from a to z. Similarly, the escape probability captures how difficult it is for a random walk to get from a to z (for example, a walk is less likely to escape to z if it is very far away); intuition of the network theory we have established even before learning of Proposition 4.2 should suggest that escape probabilities should be related to effective resistance somehow. The proposition solidifies the link between "difficulty" in an electrical circuit and "difficulty" in a random walk.

The following example from [1] demonstrates how escape probabilities are calculated.

Example 4.5 (Spider on a cube, part I)**.** A spider starts at point a on the corner of a cube, and performs a simple random walk on the edges of the cube until it meets the fly at point z on the same edge of the cube (see Figure 5). What is the probability that the spider reaches z before returning to a?

Connect a 1-volt battery to a and z, with the positive terminal connected to a. Because it is a simple random walk, we assign resistance 1 to each edge. Now, we make use of the *gluing* network operation as in Figure 6; by the symmetry of the cube, points c and d will be at the same voltage, and likewise for points e and f. Hence, we may glue c and d to create a single vertex c', and likewise we glue e and f to get e'. (To see how we obtain the configuration in Figure 6, imagine that, before gluing, we put a wire between c and d. Since these points are at the same voltage, no current is flowing through this wire. Therefore, if we collapse this wire, bringing d and c to a single point, we do not change any of the other currents or voltages, and the effective resistance will be the same. Repeat this reasoning for e and f.)

After gluing, the series and parallel laws allow us to compute that $R(a \leftrightarrow z) = \frac{7}{12}$. In the interest of space, we do not show these calculations here, though they are in [1, pp. 46]. Since a has 3 neighbors, it follows that $\mathbf{P}_a(\tau_z < \tau_a^+) = \frac{1}{3\cdot(7/12)} = \frac{4}{7}$.

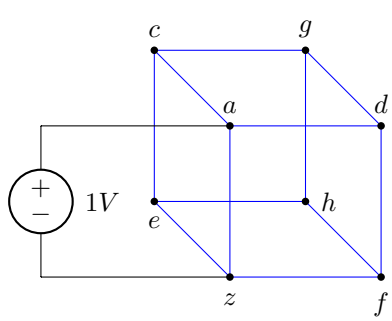

FIGURE 5. Electrical configuration for the cube problem. Each edge has unit resistance.

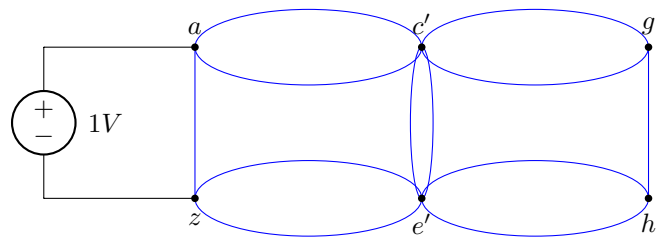

FIGURE 6. Electrical configuration after gluing.

5. Rayleigh's Monotonicity Law

Now that we have established a formula for the escape probability in terms of effective resistance, we will develop methods for bounding effective resistance and for understanding how it changes when the network is modified in various ways.

From physics, the amount of energy dissipated from a current I passing through a resistance R is I^2R. Thus, the total energy dissipation in a network is

$$E = \frac{1}{2}\sum_{x,y\in S} i^2(x,y)r(x,y) \tag{5.1}$$

$$= \frac{1}{2}\sum_{x,y\in S} i(x,y)(v(x)-v(y)). \tag{5.2}$$

We can generalize (5.1) for an arbitrary flow j by replacing i with j in the equation. In such instances, we denote the energy dissipation as E_j.

The following lemma is a general result which tells us that the energy supplied by the battery is equal to the total energy dissipated. Since this results from a routine calculation, we omit its proof, but the reader can find it and the following corollary in [1, pp. 50]. Note that a *unit flow* is a flow with strength 1.

Lemma 5.3. *Let w be any function defined on the nodes of the network and j a flow from a to z. Then*

$$(w(a)-w(z))j(a) = \frac{1}{2}\sum_{x,y\in S}(w(x)-w(y))j(x,y). \tag{5.4}$$

Proposition 5.5. *If $v(z)=0$, then the unit current flow has energy dissipation equal to $R(a\leftrightarrow z)$.*

Proof. It follows from (5.4) and the definition of $R(a\leftrightarrow z)$ that

$$v(a)i(a) = i^2(a)R(a\leftrightarrow z) = \frac{1}{2}\sum_{x,y\in S}(v(x)-v(y))i(x,y)$$
$$= \frac{1}{2}\sum_{x,y\in S} i^2(x,y)r(x,y).$$

□

The next result is fundamental. We provide the quick proof given in [1, pp. 51].

Theorem 5.6 (Thomson's Principle)**.** *The current unit flow i from a to z minimizes the total energy dissipation over all unit flows j from a to z.*

Proof. Let j be a unit flow from a to z and define $d(x,y) = j(x,y) - i(x,y)$. Then d is a flow from a to z since $d(y,x) = j(y,x) - i(y,x) = i(x,y) - j(x,y) = -d(x,y)$. Note that $d(a) = \sum_{y\in S} j(a,y) - \sum_{y\in S} i(a,y) = 1 - 1 = 0$. We have

$$\begin{aligned}\sum_{x,y\in S} j^2(x,y)r(x,y) &= \sum_{x,y\in S} (i(x,y) + d(x,y))^2 r(x,y)\\ &= \sum_{x,y\in S} i^2(x,y)r(x,y) + 2\sum_{x,y\in S}(v(x)-v(y))d(x,y)\\ &+ \sum_{x,y\in S} d^2(x,y)r(x,y),\end{aligned}$$

by expanding the sum and applying Ohm's law in the crossterm. Applying (5.4) with $w = v$ and $j = d$, we see that the crossterm is equal to $4(v(a) - v(z))d(a) = 0$. Thus, the above equality implies

$$\sum_{x,y\in S} j^2(x,y)r(x,y) = \sum_{x,y\in S} (i^2(x,y) + d^2(x,y))r(x,y) \geq \sum_{x,y\in S} i^2(x,y)r(x,y). \tag{5.7}$$

□

Remark 5.8. To summarize, $R(a \leftrightarrow z) = \min\{E_j : j \text{ is a unit flow from } a \text{ to } z\}$, and the flow that achieves the minimum is the unit current flow i. Thus, the energy dissipation of unit flows other than the unit current flow will provide upper bounds for the effective resistance.

Theorem 5.9 (Rayleigh's Monotonicity Law)**.** *The effective resistance $R(a \leftrightarrow z)$ of a network is a monotonically increasing function of the resistances along each edge of the network.*

Proof. We follow the proof from [1, pp. 54]. Let $(G, \{c(x,y)\})$ be a network with resistances $r(x,y)$ and effective resistance $R(a \leftrightarrow z)$, and let i be the unit current flow defined on this network. Next, consider this same graph G but with resistances $r'(x,y)$ where $r'(x,y) \geq r(x,y)$ for every edge $\{x,y\}$. Let $R'(a \leftrightarrow z)$ be the effective resistance of the new network, and let i' be the unit current flow. Then

$$\begin{aligned}R'(a \leftrightarrow z) &= \sum_{x,y\in S} i'^2(x,y)r'(x,y)\\ &\geq \sum_{x,y\in S} i'^2(x,y)r(x,y)\\ &\geq \sum_{x,y\in S} i^2(x,y)r(x,y)\\ &= R(a \leftrightarrow z).\end{aligned}$$

The inequalities follow from the fact that $r'(x,y) \geq r(x,y)$ for each edge and from (5.7), with i' in the place of j. The outer equalities follow from Proposition 5.5. □

The following important corollaries and their proofs come from [3, pp. 123].

Corollary 5.10. *Adding an edge does not increase the effective resistance $R(a \leftrightarrow z)$, and conversely removing an edge does not decrease $R(a \leftrightarrow z)$. If the added edge is not incident to a, then the addition does not decrease the escape probability $\mathbf{P}_a(\tau_z < \tau_a^+)$.*

Proof. Before adding an edge $\{x, y\}$ to a network, we can imagine it already existing with $c(x, y) = 0$ or $r(x, y) = \infty$. By adding the edge, we reduce $r(x, y)$ to a finite number. The opposite holds for removing an edge $\{x, y\}$; we increase $r(x, y)$ to infinity. The second half of the statement follows from (4.3). □

Corollary 5.11. *The operation of gluing vertices cannot increase effective resistance.*

Proof. When we glue vertices together, the effective resistance $R(a \leftrightarrow z)$ in Thomson's Principle is the minimum over a larger class of flows. □

Note that what was described in Corollary 5.10 and Corollary 5.11 was the Cutting Law and Shorting Law, respectively, in [1].

To finish this section, we complement our discussion of upper bounds for $R(a \leftrightarrow z)$ with a formula for lower bounds. An *edge-cutset separating a from z* is a set of edges Π in the network such that any path from a to z must contain an edge in Π. The following lemma and proposition, along with their proofs, are from [3, pp. 123-124]. We omit the proof of the lemma here.

Lemma 5.12. *If j is a flow from a to z, and Π is an edge-cutset separating a from z, then*

$$j(a) \leq \sum_{\{x,y\} \in \Pi} |j(x, y)|.$$

Proposition 5.13 (Nash-Williams Inequality)**.** *If $\{\Pi_k\}$ are disjoint edge-cutsets separating a from z, then*

$$R(a \leftrightarrow z) \geq \sum_k \left(\sum_{\{x,y\} \in \Pi_k} c(x, y) \right)^{-1}. \tag{5.14}$$

Proof. Let j be a unit flow from a to z. The Cauchy-Schwarz inequality implies that, for any k,

$$\sum_{e \in \Pi_k} c(e) \cdot \sum_{e \in \Pi_k} r(e) j^2(e) \geq \left(\sum_{e \in \Pi_k} \sqrt{c(e)} \sqrt{r(e)} |j(e)| \right)^2 = \left(\sum_{e \in \Pi_k} |j(e)| \right)^2, \tag{5.15}$$

since $r(e) = 1/c(e)$. Lemma 5.12 implies that the right side of (5.15) is bounded below by $j^2(a) = 1$. Now, applying this bound, dividing by $\sum_{e \in \Pi_k} c(e)$ and summing over all k gives

$$\sum_e r(e) j^2(e) \geq \sum_k \sum_{e \in \Pi_k} r(e) j^2(e) \geq \sum_k \left(\sum_{e \in \Pi_k} c(e) \right)^{-1}.$$

Minimizing over all unit flows from a to z proves the statement (see Remark 5.8). □

In this section, we have taken considerable effort to develop some framework for understanding how effective resistance behaves. We will use the monotonicity law and its corollaries in the following section to get simple expressions for commute times between two points in a large class of networks.

6. Commute Times on Networks

Up until now, we have only analyzed random walks on networks through the lens of escape and return probabilities. We will now balance out the discussion by looking at how electrical networks can help us compute the expectation of a certain kind of random variable. Though this section is not strictly necessary for the escape probability results in Section 4 and Section 7, we find that it is helpful for furthering our intution of random walks on graphs, especially since times are somewhat more intuitive than probabilities. The following definitions from [3] will orient us for this discussion.

Definition 6.1. Let a and b be nodes in a network. We define the *commute time between a and b* to be the expected amount of time it takes for a walk starting at a to reach b, then reach a again. The commute time is then

$$t_{a\leftrightarrow b} := \mathbf{E}_a(\tau_{a,b}) = \mathbf{E}_a(\tau_b) + \mathbf{E}_b(\tau_a), \tag{6.2}$$

where $\tau_{a,b} = \min\{t \geq \tau_b : X_t = a\}$.

Definition 6.3. The *Green's function* for a random walk stopped at a stopping time τ is defined by

$$G_\tau(a,x) = \mathbf{E}_a\left(\sum_{t=0}^{\infty} \mathbf{1}_{\{X_t = x \cap \tau > t\}}\right), \tag{6.4}$$

which counts the expected number of visits to x before time τ given that the walk started at state a.

The following lemmas from [3, pp. 120, 131] make proving the very useful Commute Time Identity quite straightforward. The first lemma will help us understand the Green's function.

Lemma 6.5. *If $G_{\tau_z}(a,x)$ is the Green's function as defined in (6.4), then*

$$G_{\tau_z}(a,a) = c(a)R(a \leftrightarrow z). \tag{6.6}$$

Proof. By definition, the probability that the walk returns to a before hitting z is $1 - \mathbf{P}_a(\tau_z < \tau_a^+)$. Once the walk has returned to a, the Markov property implies that it is as if the walk has started over again, independent of the past, and with the same return probability. Hence, the number of visits the walk makes to a before visiting z is a geometric random variable with parameter $p = \mathbf{P}(\tau_z < \tau_a^+)$. The expected value is $1/p$, and combining this fact with (4.3) gives the statement. □

Lemma 6.7. *Let (X_t) be a Markov chain with transition matrix P. Suppose that for a probability distribution μ on S, there is a stopping time τ with $\mathbf{P}_\mu(0 < \tau < \infty) = 1$ and such that $\mathbf{P}_\mu(X_\tau = \cdot) = \mu$. If ρ is the row vector*

$$\rho(x) := \mathbf{E}_\mu\left(\sum_{t=0}^{\tau-1} \mathbf{1}_{\{X_t = x\}}\right), \tag{6.8}$$

then $\rho P = \rho$. Thus, if $\mathbf{E}_\mu(\tau) < \infty$, then $\frac{\rho}{\mathbf{E}_\mu(\tau)}$ is a stationary distribution π for P.

This proof is almost identical to the proof of Proposition 1.14 of [3, pp. 11]. Since this is a lengthy calculation, we omit the proof.

The following theorem (from [3, pp. 132]) allows us to compute or estimate commute times on a wide variety of networks, allowing us to further our understanding of random walk behavior. The simplicity of the statement is particularly pleasing.

Theorem 6.9 (Commute Time Identity)**.** *Let $(G, \{c(e)\})$ be a network, and let (X_t) be the random walk on this network. For any nodes a and b in V_G,*

$$t_{a\leftrightarrow b} = c_G R(a \leftrightarrow b). \tag{6.10}$$

Proof. By the definition of $\tau_{a,b}$, $\mathbf{P}_a(X_{\tau_{a,b}} = a) = 1$, so ρ in Lemma 6.7 is equal to the Green's function $G_{\tau_{a,b}}(a, \cdot)$, and furthermore

$$\frac{G_{\tau_{a,b}}(a,a)}{\mathbf{E}_a(\tau_{a,b})} = \pi(a) = \frac{c(a)}{c_G}. \tag{6.11}$$

The statement follows from the Green's function formula given in Lemma 6.5. □

Remark 6.12. The Commute Time Identity appeared without proof in the proof sketch of Lemma 4.3 in [6], where we bounded the sums of commute times, which then led to the main result of our previous study of random walks.

Example 6.13 (Spider on a Cube, part II)**.** Consider the same spider from Example 4.5. What is the expected number of steps required for the spider to eat the fly at z, then crawl back home to a?

Since we know that $R(a \leftrightarrow z) = \frac{7}{12}$, the answer is $t_{a\leftrightarrow z} = 2 \cdot 12 \cdot \frac{7}{12} = 14$. Furthermore, referring to the diagram in Figure 5, it is a common exercise in physics to calculate that $R(a \leftrightarrow f) = \frac{3}{4}$ and $R(a \leftrightarrow h) = \frac{5}{6}$. Hence, $t_{a\leftrightarrow f} = 18$ and $t_{a\leftrightarrow h} = 20$, showing how commute times vary with more distant points.

Example 6.14. We would like to find the commute time between adjacent points for the simple random walk on the n-cycle. To find this, we hook up a 1-volt battery between the points. The setup is displayed in Figure 7 below. Using Kirchoff's Voltage Law to solve for the currents, we have $I_2 = 1$ using loop (1), and loop (2) tells us $-I_1 \cdot (n-1) + I_2 = 0$, hence $I_1 = \frac{1}{n-1}$. Hence the strength of the current is $\frac{n}{n-1}$, making $R(a \leftrightarrow b) = \frac{n-1}{n}$. Since $c_G = 2n$, we have $t_{a\leftrightarrow b} = 2n \cdot \frac{n-1}{n} = 2(n-1)$.

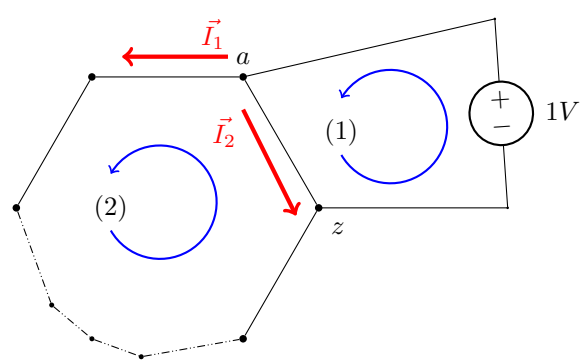

FIGURE 7. Electrical configuration for adjacent points on the n-cycle.

We now set aside the basic examples to obtain upper bounds on commute times that apply to the vast majority of networks that we will ever encounter. For the following proposition and proof, we follow [3, pp. 134]. We take the term *simple graph* to mean a graph without multiple edges between the same pair of vertices, and without any loops.

Proposition 6.15. *For the simple random walk on a simple graph G with n vertices and m edges,*

$$t_{a\leftrightarrow b} \leq 2nm \leq n^3 \quad \textit{for all } a, b. \tag{6.16}$$

For the simple random walk on a d-regular graph G on n vertices,

$$t_{a\leftrightarrow b} \leq 3n^2 - nd \quad \textit{for all } a, b. \tag{6.17}$$

Proof. First, note that $R(a \leftrightarrow b) \leq d(a, b)$, where $d(a, b)$ measures the length of the shortest path between a and b. This follows from Corollary 5.10; in the limiting case, we have that the graph is only the path connecting a and b, via the series law. If other edges not on the path are added, Corollary 5.10 tells us that $R(a \leftrightarrow b)$ cannot increase. Since $d(a, b) \leq n$ for all a, b, we have $R(a \leftrightarrow b) \leq n$ for all a, b.

Next, notice that, since G has no loops, $2m = \sum_{y\in V_G} \deg y \leq \sum_{y\in V_G} n = n^2$. Since $c_G = 2m$, Theorem 6.9 implies

$$t_{a\leftrightarrow b} \leq R(a \leftrightarrow b) \cdot 2m \leq 2mn \leq n^3. \tag{6.18}$$

Now we prove the regular graph case. Let $l := \max_{a,b\in V_G} d(a, b)$. We show $l \leq \frac{3n}{d} - 1$. Let $N(x)$ be the set of x and its nearest neighbors. Let $x, y \in V_G$ be extremal points, meaning $d(x, y) = l$, and let $x_0 = x, x_1, ..., x_l = y$ be a path such that $\{x_{i-1}, x_i\}$ is an edge. That the path is minimal implies $N(x_i) \cap N(x_j) = \varnothing$ for $j > i + 2$, hence $\sum_{i=0}^{l} |N(x_i)|$ counts each vertex in the graph at most 3 times. Thus,

$$(d + 1)(l + 1) = \sum_{i=0}^{l} |N(x_i)| \leq 3n,$$

hence $l \leq \frac{3n}{d} - 1$. Now we have $c_G = \sum_{y\in V_G} \deg y = nd$, hence Theorem 6.9 gives

$$t_{a\leftrightarrow b} = R(a \leftrightarrow b) \cdot nd \leq (\frac{3n}{d} - 1)nd = 3n^2 - nd.$$

□

Example 6.19 (The Complete Graph)**.** Consider the simple random walk on the classic example of a regular graph: the complete graph K_n with n vertices. If the walk starts at any vertex $a \neq b$, then we have that τ_b is a geometric random variable with parameter $\frac{1}{n-1}$. Hence, $\mathbf{E}_a(\tau_b) = n - 1$, and therefore by symmetry $\mathbf{E}_b(\tau_a) = n - 1$ also, so $t_{a\leftrightarrow b} = 2(n - 1)$. This example tells us that the quadratic bound given in (6.17) can be quite generous.

Since we are almost always dealing with simple graphs, Proposition 6.15 implies that, barring special cases, commute times are at most cubic in the number of vertices, and we reiterate the simplicity of the statements that have led us to these results due to network methods.

Next, we will return to the issue of escape probabilities and at last prove the Pólya Recurrence Theorem.

7. The Recurrence Theorem via Electrical Methods

We have become well-acquainted with what electrical methods can do for us in the case of finite graphs. Now, it is time we move past the finite case, and into the infinite.

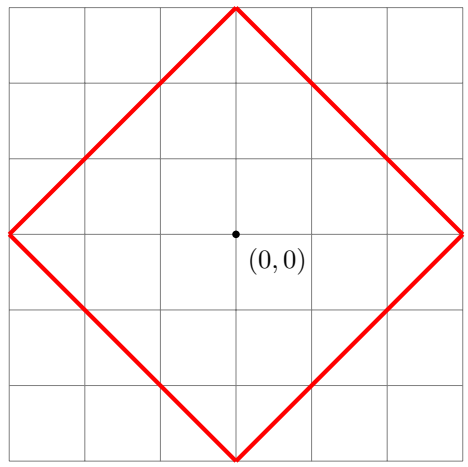

FIGURE 8. G_3 in $\mathbb{Z}^2$ is the set of all edges and vertices contained within the red boundary, plus all vertices on it.

Definition 7.1. We say that a state $a \in S$ of a random walk is *recurrent* if the walk is certain to return to a: that is, if $\mathbf{P}_a(\tau_a^+ < \infty) = 1$. Otherwise, we say the walk is *transient*.

From now on, we will be working exclusively with the simple random walk in $\mathbb{Z}^d$ starting at the vector a. Since the walk is irreducible, proving recurrence or transience of one state implies recurrence or transience of all states. This fact follows from the proposition we introduce now, which is stated and proven in [3, pp. 292]. Because the proof does not showcase electrical methods, it is omitted for brevity.

Proposition 7.2. *Suppose that P is the transition matrix of an irreducible Markov chain (X_t). Define $G(x,y) := \mathbf{E}_a(\sum_{t=0}^{\infty} \mathbf{1}_{\{X_t=y\}}) = \sum_{t=0}^{\infty} P^t(x,y)$ to be the expected number of visits to y starting from x. The following are equivalent:*

(1) $G(x,x) = \infty$ for some $x \in S$.
(2) $G(x,y) = \infty$ for all $x, y \in S$.
(3) $\mathbf{P}_x(\tau_x^+ < \infty) = 1$ for some $x \in S$.
(4) $\mathbf{P}_x(\tau_y^+ < \infty) = 1$ for all $x, y \in S$.

Definition 7.3. Let $d(a,b)$ be the shortest-path distance between a and b. Define $G_n^a := \{x \in \mathbb{Z}^d : d(a,x) \leq n\}$, and define $\partial G_n^a = \{x \in \mathbb{Z}^d : d(a,x) = n\}$. If a is the zero vector, we write G_n. (In Figure 8, we have provided an example of G_3 in $\mathbb{Z}^2$.)

We will generally write z_n to denote the vertex obtained from gluing together all points in ∂G_n to signify a unified boundary for the walk, and we ground z_n. We write $R(a \leftrightarrow z_n)$ to denote the effective resistance between the origin and the boundary ∂G_n.

Proposition 7.4. *The limit $\lim_{n\to\infty} R(a \leftrightarrow z_n)$ exists, either as a positive real number or as positive infinity.*

Proof. It is enough to show that the sequence $\{R(a \leftrightarrow z_n)\}$ is non-decreasing, but this holds since one can always obtain the graph G_{n-1}^a from G_n^a by gluing z_n with

z_{n-1}. As per Corollary 5.11, this cannot increase the effective resistance, so for any n, we have $R(a \leftrightarrow z_{n-1}) \leq R(a \leftrightarrow z_n)$. □

Remark 7.5. The proof for Proposition 7.4 came from [5], which also includes a helpful illustration for how the gluing/shorting operation works in $\mathbb{Z}^2$.

With this proposition in hand, we can now make sense of taking the effective resistance of an infinite network. We define the limit in Proposition 7.4 to be $R(a \leftrightarrow \infty)$, and we may write $C(a \leftrightarrow \infty) := R(a \leftrightarrow \infty)^{-1}$. With these limits, we can now make sense of escape probabilities in infinite networks. We can now write

$$\mathbf{P}_a(\tau_a^+ = \infty) = \lim_{n\to\infty} \mathbf{P}_a(\tau_{z_n} < \tau_a^+) = \lim_{n\to\infty} \frac{1}{c(a)R(a \leftrightarrow z_n)} = \frac{1}{c(a)R(a \leftrightarrow \infty)}. \tag{7.6}$$

Note that in $\mathbb{Z}^d$, the escape probability becomes $(2dR(a \leftrightarrow \infty))^{-1}$. We now extend Thomson's Principle using the proof from [1, pp. 111]. Say that j is a *flow from a to infinity* if j is antisymmetric and satisfies the node law at every $x \neq a$.

Proposition 7.7. *Thomson's Principle still holds for infinite networks; that is, $R(a \leftrightarrow \infty) \leq E_j$ for any unit flow j from a to infinity.*

Proof. Let j be a unit flow from a to infinity with energy dissipation

$$E_j = \frac{1}{2} \sum_{x,y\in S} j^2(x,y)r(x,y). \tag{7.8}$$

Next, we simply let j_n be the restriction of j to G_n^a, which gives a unit flow from a to ∂G_n^a. Now we can use Theorem 5.6: let i_n be the unit current flow in G_n^a from a to ∂G_n^a. Then Thomson's Principle tells us

$$\begin{aligned} R(a \leftrightarrow z_n) &= \frac{1}{2} \sum_{x,y\in V_{G_n^a}} i_n^2(x,y)r(x,y) \\ &\leq \frac{1}{2} \sum_{x,y\in V_{G_n^a}} j_n^2(x,y)r(x,y) \\ &\leq \frac{1}{2} \sum_{x,y\in S} j^2(x,y)r(x,y) \\ &= E_j. \end{aligned}$$

Taking $n \to \infty$ gives the result. □

It now follows from Proposition 7.7 that the Rayleigh Monotonicity Law also holds for infinite networks. Below, we summarize the criteria for transience in the infinite network.

Proposition 7.9. *Let $(G, \{c(e)\})$ be an infinite network. The following are equivalent:*

1. *The random walk on the network is transient.*
2. *The effective resistance $R(a \leftrightarrow \infty)$ is finite.*
3. *There exists some flow j from a to infinity with $j(a) > 0$ and $E_j < \infty$.*

Proof. That (1) and (2) are equivalent follows from (7.6). That (2) and (3) are equivalent comes from Proposition 7.7. □

Lastly, as seen in [3, pp. 294], the Nash-Williams inequality also holds in infinite networks.

Proposition 7.10. *If there exist disjoint edge-cutsets* $\{\Pi_n\}$ *that separate* 0 *from* ∞ *such that*

$$\sum_n \left(\sum_{e \in \Pi_n} c(e) \right)^{-1} = \infty, \tag{7.11}$$

then the random walk on the network $(G, \{c(e)\})$ *is recurrent.*

Proof. If (7.11) holds, then, by applying the finite case (5.15), where the cutsets separate 0 from z_n, and taking limits, we must have that $R(a \leftrightarrow z_n) \to \infty$ as $n \to \infty$. Thus, the limit in (7.6) is 0, and the walk is recurrent. □

We can now prove the recurrence theorem. The proof of the $d = 2$ case shown here was shown in [3], and the $d = 3$ case was carried out in [1].

Theorem 7.12 (Pólya Recurrence Theorem)**.** *For* $d = 1, 2$*, the simple random walk on* $\mathbb{Z}^d$ *is recurrent. For* $d > 2$*, the walk is transient.*

Proof. Throughout this proof, we take $a = 0$ without loss of generality. The $d = 1$ case can be handled directly. In this case, G_n is the path $\{-n, ..., 0, ..., n\}$. Gluing $-n$ and n together to get z_n results in two branches of resistors, with either side being equivalent to a single n ohm resistor by the series law. Now, using the parallel law, $R(0 \leftrightarrow z_n) = \frac{n}{2}$. Thus $R(0 \leftrightarrow \infty) = \infty$, and the first case is proven.

For $d = 2$, we may define the boxes $B_n = [-n, n]^2$, and the squares $S_n = \partial B_n$. Let Π_0 be defined as follows:

$$\Pi_0 = \{\{(0,0),(0,1)\},\{(0,0),(1,0)\},\{(0,0),(0,-1)\},\{(0,0),(-1,0)\}\}. \tag{7.13}$$

Next, for $n \geq 1$, define Π_n to be the set of edges which connect S_n to S_{n+1} (see Figure 9). The number of edges in Π_n is $8n + 4$, hence taking the reciprocal and summing over all n gives the sum $\sum_{n=0}^{\infty} \frac{1}{8n+4} = \infty$. Since the Π_n are disjoint and they each separate 0 from ∞, Proposition 7.10 implies that the simple random walk on $\mathbb{Z}^2$ is recurrent.

For $d = 3$, it is enough to find a unit flow with finite energy dissipation. The idea is to create a flow such that all points at the same distance from the origin have the same amount of flow passing through, in such a way that the energy dissipation is readily seen to converge. Define the flow j as follows: for all points (x, y, z) such that $x, y, z \geq 0$ with $x+y+z = n$, the flow out of (x, y, z) is $\frac{2(z+1)}{(n+3)(n+2)(n+1)}$ in the z direction, $\frac{2(y+1)}{(n+3)(n+2)(n+1)}$ in the y direction, and $\frac{2(x+1)}{(n+3)(n+2)(n+1)}$ in the x direction. To verify that j is a flow, it is enough to verify that the flow coming into (x, y, z) equals the flow coming out of (x, y, z). The definition of the flow implies that the flow out is equal to

$$\frac{2(n+3)}{(n+3)(n+2)(n+1)} = \frac{2}{(n+2)(n+1)}. \tag{7.14}$$

By construction, the flow coming into (x, y, z) comes from $(x-1, y, z)$, $(x, y-1, z)$, and $(x, y, z-1)$, and the total flow in is equal to

$$\frac{2x}{(n+2)(n+1)n} + \frac{2y}{(n+2)(n+1)n} + \frac{2z}{(n+2)(n+1)n} = \frac{2}{(n+2)(n+1)}. \tag{7.15}$$

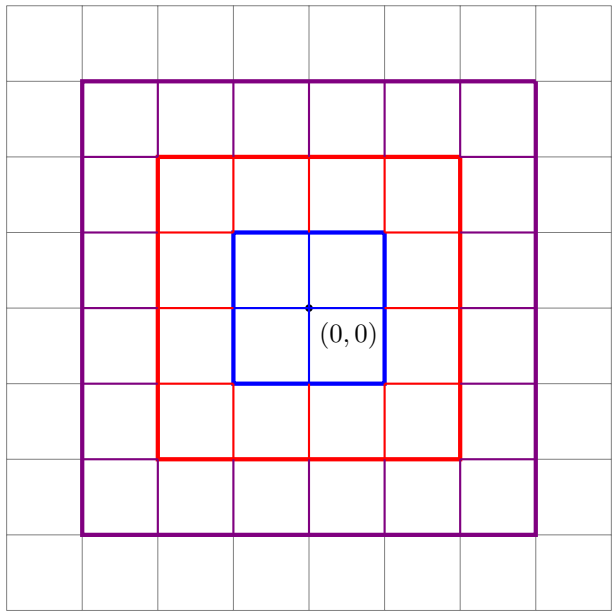

FIGURE 9. The cutsets Π_0 in blue, Π_1 in red, and Π_2 in violet with S_1, S_2, and S_3 in their respective colors.

Next, j is indeed a unit flow, since the flow out of 0 is $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1$. All that remains is to check that E_j is finite.

Observe that the flows coming out of the edges at the nth level are all bounded above by $\frac{2}{(n+1)^2}$. It now becomes a question of how many points are at the nth level, which generating functions can answer: the number of triples at the nth level is the coefficient of x^n in the polynomial $f(x)=P(x)^3$, where

$$P(x)=1+x+x^2+x^3+\cdots,$$

which equals $(1-x)^{-1}$ for algebraic purposes. Hence, we use the generating function $f(x)=(1-x)^{-3}$, and the binomial theorem extended to negative exponents implies that the coefficient of x^n in the (infinite) polynomial f is $\binom{n+2}{2}=\frac{(n+1)(n+2)}{2}$. Hence, there are

$$\frac{3}{2}(n+1)(n+2)\leq 3(n+1)^2 \tag{7.16}$$

edges coming out of the nth level. It follows that

$$E_j\leq\sum_{n=1}^{\infty}3(n+1)^2(\frac{2}{(n+1)^2})^2=12\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}<\infty, \tag{7.17}$$

thus the walk in $\mathbb{Z}^3$ is transient.

The case $d>3$ follows from the fact that the walk in $\mathbb{Z}^3$ is transient. Since $\mathbb{Z}^3\subset\mathbb{Z}^d$ for $d>3$, one constructs $\mathbb{Z}^d$ by adding edges to $\mathbb{Z}^3$; since this cannot increase the effective resistance of the infinite network by Corollary 5.10, the effective resistance of $\mathbb{Z}^d$ is finite, and the proof of the Recurrence Theorem is complete. $\square$

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