MORSE THEORY FOR PARTIALLY ORDERED SETS IN CONTEXT

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Abstract. This paper aims to introduce and provide intuition for Morse theory on partially ordered sets. In particular, we showcase proofs of the fundamental theorems of Morse theory due to Fernandez-Ternero et al.

Contents

1. Introduction

In 1928, Marston Morse [17] developed a tool in differential topology which analyzed the homotopy types of differentiable manifolds through the critical points their smooth functions. This tool, eponymously called Morse theory, would receive considerable use in powerful results in the second half of the century: it was used by Raoul Bott [5] to prove the Periodicity Theorem, which established periodicity in the homotopy groups of the classical groups, and by Smale [23] to prove the h-Cobordism Theorem, a fundamental result in higher manifold theory.

More specifically, Morse discovered that for "nice" smooth real-valued functions f on a differentiable manifold, the homotopy type of $f^{-1}(-\infty, t]$ for real t depends on the critical points of f. In particular, $f^{-1}(-\infty, b]$ has the same homotopy type as $f^{-1}(-\infty, a]$ if $[a, b]$ contains no critical points. Furthermore, if $[a, b]$ contains only one critical point, $f^{-1}(-\infty, b]$ has the homotopy type of $f^{-1}(-\infty, a]$ with a cell attached.

The question then became how to extend Morse theory to broader classes of spaces. This was finally solved in 1998 by Robin Forman in [8]. This paper gave birth to a version of Morse theory applied to CW-complexes. This discrete Morse theory has since found applications in computer science and applied mathematics. In particular, part of Forman's approach centered on looking at (partial) ordering between cells in the complex. Given that in 1937, Alexandroff noticed ([2]) that partially ordered sets (posets) and (T_0) topological spaces allowing arbitrary intersections are essentially the same, Forman's approach lends itself to extending Morse theory to the realm of posets.

More specifically, for posets which are the analogues of (homology) manifolds, we can define "nice" real-valued functions which are almost order-preserving. For these functions, we can make analogues of the fundamental theorems of Morse theory. Take f to be such a function on a (homologically admissible) poset X. There is some notion of critical point such that the set of all elements in or less than $f^{-1}(-\infty, a]$ has the same weak homotopy type and homology as $f^{-1}(-\infty, b]$ if $(a, b]$ contains no critical points. Additionally, if $f^{-1}(-\infty, b]$ has only one more critical point than $f^{-1}(-\infty, a]$ and $[a, b]$ contains only one value in $\mathrm{im}(f)$, we can attach the critical point at its boundary in $f^{-1}(-\infty, a]$ to get $f^{-1}(-\infty, b]$.

In Section 2, we briefly review Morse's fundamental theorems while also providing intuition through the example of the torus. For a full exposition the reader is guided to Milnor's classical reference [15].

In section 3, we introduce the basic theory of (T_0) finite topological spaces, posets, their isomorphism as categories, and further related notions which will appear as we get closer to developing our Morse theory.

Section 4 continues this study delving into the algebraic topology of finite spaces. In particular, we discuss strong and weak homotopy equivalences on finite spaces based largely on results due originally to Stong [24] and McCord [14].

Section 5 is the core of the paper: we introduce and develop Morse theory for posets. We begin by defining the class of posets which are analogous to manifolds: homologically admissible posets. Next we define Morse funtions: functions on posets which are almost order preserving. A property of homologically admissible posets allows us to pair up points where the Morse function is not entirely order-preserving. This, along with perturbing our Morse functions to make them locally injective, allows us to prove our fundamental theorems of Morse theory. In particular, we have two collapsing theorems, one weakly homotopic and one homological, and one adjunction theorem.

Convention: All functions are continuous unless otherwise mentioned, all posets and topological spaces are finite from the Section 3 onwards, and all finite spaces are T_0 (no two points share all their open sets). We sometimes refer to homotopy equivalences as strong homotopy equivalences (or types, etc.) to distinguish them from weak homotopy equivalences.

2. A Quick Detour into Classical Morse Theory

2.1. **Fundamental Theorems of Classical Morse Theory.** As mentioned before, we want to "cut" differentiable manifolds using smooth functions. Due to more technical reasons, we want the critical points and values of these functions to be "nice." Therefore, we introduce the following definition:

Definition 2.1. Let M be a differentiable manifold and $f : M \to \mathbb{R}$ be a smooth function on it. We denote by $\mathrm{Crit}(f)$ the set of *critical points*: points where $\mathrm{D}f$ vanishes. The images of critical points under f are called critical values.

We call a critical point p *degenerate* if the Hessian of f at p, $D^2f(p)$, does not have full rank.

A *Morse function* is smooth real-valued function from M to $\mathbb{R}$ with all critical points non-degenerate.

Remark 2.2. The natural question is whether any differentiable manifold admits a Morse function. In fact, Morse functions are dense in $C^\infty(M)$. Details found in [[21], Theorem 8.1.1].

We make precise what we mean by a cut of a manifold:

Definition 2.3. For a Morse function f on a differentiable manifold M we define $M_a^f = f^{-1}(-\infty, a]$. We call this the *cut* of M at a by f. We leave out f if it is clear from context.

In particular, we expect that if we have two cuts of a differentiable manifold M_a and M_b, they should have very similar topology so long as the function did not pass a critical point. This is exactly what the first fundamental theorem of Morse theory tells us (a full proof in [[15], Theorem 3.1]):

Theorem 2.4 (**First Fundamental Theorem of Classical Morse Theory**). *Let M a differentiable manifold with $f : M \to R$ a smooth function. Let $a, b \in \mathbb{R}$ such that $a < b$, $f^{-1}[a, b]$ is compact, and $\mathrm{Crit}(f) \cap [a, b] = \emptyset$. Then M_a is a strong deformation retract of M_b so that inclusion $i : M_a \to M_b$ is a homotopy equivalence.*

Remark 2.5. It turns out that M_a and M_b are diffeomorphic, and thus homeomorphic, which is, in fact, stronger than our homotopy equivalence, but this is not necessary for our purposes.

The second theorem will tell us exactly how the homotopy type changes when passing a critical value. For this we distinguish between critical points:

Definition 2.6. For a non-degenerate critical point x we define its *index* ω as the maximum dimensional subspace of TM_x where the Hessian of f at x, $D^2f(x)$, is negative definite.

In practical terms, the index is the number of directions in which f is decreasing.

This index ends up being the dimension of the cell which needs to be attached as the Morse function passes the critical point. Or, as the second fundamental theorem states in more precise terms ([[15], Theorem 3.2]):

Theorem 2.7 (**Second Fundamental Theorem of Classical Morse Theory**). *Let M a differentiable manifold with $f : M \to R$ a smooth function. Let p be a non-degenerate critical point of index ω and $f(p) = c$. For sufficiently small ϵ, if $f^{-1}[c - \epsilon, c + \epsilon]$ is compact and contains exactly one critical point, $M_{c+\epsilon}$ is homotopically equivalent to $M_{c-\epsilon}$ with an ω cell attached.*

Remark 2.8. We can always find Morse functions with only one critical point per critical value. This is a consequence of both the Morse lemma [[21], Theorem 8.2.1], which gives local coordinates of critical points, and Sard's Theorem [[10], Theorem 1.3], which states the critical points have measure zero. The requirement that $f^{-1}[a,b]$ be compact is also not stringent so long as M is closed [[21], Theorem 8.1.2].

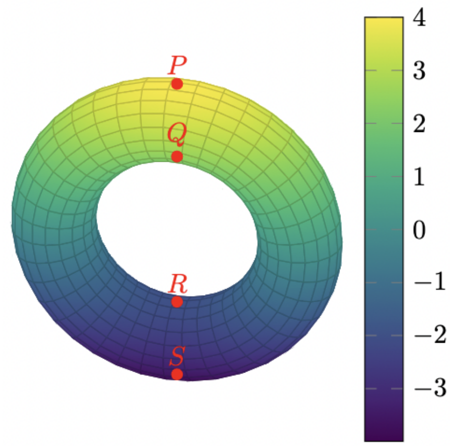


FIGURE 1. Torus and Height.

2.2. **Morse Theory Through the Torus.** We are guided by the example of the vertical torus, T, alongside the smooth function of height, $h(x) : T \to \mathbb{R}$, which is depicted above by color. The critical points P, Q, R, S are marked. For this plot, the torus has a total height of 8.

We first illustrate Theorem 2.4: let $a, b \in \mathbb{R} : h(R) < a < b < h(Q)$. Then Figure 2 show M_a and M_b:

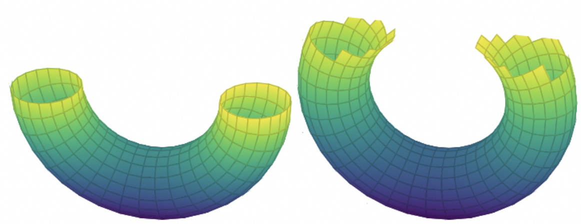


FIGURE 2. M_a and M_b.

Notice how both figures have much the same shape, both being homeomorphic to a cylinder with the caps removed, so that the two figures are also homeomorphic and, thus, homotopy equivalent.

As the lower half of the torus is the same in both cuts, we can even observe the homotopy equivalence by just looking at $h^{-1}(t)$. Figure 3 shows the tops of many cuts between both the middle of the torus and the height of Q and then between Q and P. Notice how the different $h^{-1}(t)$ are very similar (homeomorphic in fact) between critical points. Between the heights of R and Q, the top of the cut is two circular curves which indicates the deformed tubes of M_a and M_b. Between Q and P the tops of the cuts are vaguely oval shaped curved which indicate an M_t of a torus with the top removed.

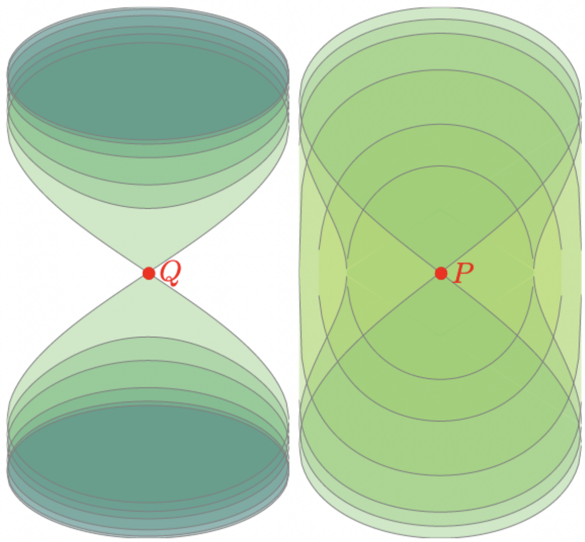


FIGURE 3. Cuts between the middle and Q, and between Q and P.

We now illustrate Theorem 2.7. In particular, the critical points P, Q, R, S have indices $2, 1, 1, 0$ and images under h of p, q, r, s respectively. Then Theorem 2.7 tells us that the cuts above and below $h^{-1}(s)$ should differ in homotopy type by a 0-cell: a point.

But all cuts below $h^{-1}(s)$ are empty so we need only check that cuts just above S have the homotopy type of a point. Figure 4 shows such a cut; it is contractible.

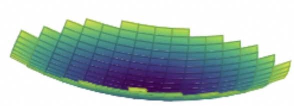


FIGURE 4. First Cut

We see the theorem in full force when we approach $h(R)$ Figure 5. We can go from the "bowl" right before R to the curved open cylinder right after by adding

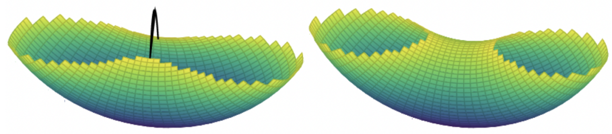


Figure 5. Attaching the first 1-cell

in a 1-cell connecting the outer edges which can then be stretched out to form the curved cylinder:

Right before we reach Q, Figure 6 we can also connect the curved cylinder to leave one hole on the top to be plugged by a 2-cell once we pass P.

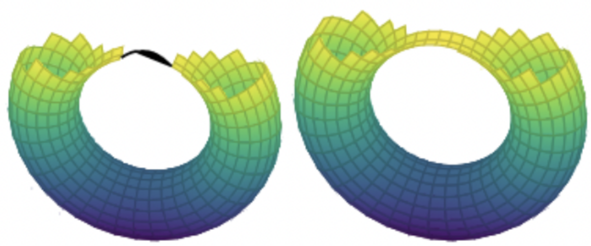


Figure 6. Attaching the second 1-cell

This allows us to see the power of Morse theory. Using only the height of the torus we recover its structure as a CW-complex: we can reconstruct the torus through cells of dimension $0, 1, 1, 2$

3. Basic Notions

3.1. **Finite Spaces.** Before we can get the discrete analogue of Morse theory, we begin by introducing finite topological spaces:

Definition 3.1. A *finite topological space* is a topological space $(X, \mathcal{T})$ with only finitely many points which satisfies the T_0 axiom (no two points share all of their open sets).

Remark 3.2. Continuous functions between finite spaces are defined in exactly the same way as for infinite topological spaces: $f^{-1}(U)$ must be open for open U. These, however, do not often align with our intuitive notion of continuity.

A surprising feature of finite topological spaces is that the closed subsets ($C = X - U$) form a T_0 topology on X as well:

Definition 3.3. For $(X, \mathcal{T})$, we denote by the *opposite topology* $(X, \mathcal{T}')$ where sets in T' are the closed sets in $\mathcal{T}$. It is straightforward to check that $\mathcal{T}'$ is also a topology and is T_0.

Definition 3.4. Let X be a finite space. For each $x \in X$, we can define a *minimal open set* containing it:

$$U_x = \bigcap \{U : x \in U\}$$

for U open in X.

Unlike in infinite topological spaces, this intersection is open as there are only finitely many possible open sets with which to form the intersection. These open sets form a basis for X and, in fact, are contained in every other basis for a finite topological space.

We can also look at the U_x's of the opposite topology, *minimal closed sets*:

$$F_x = \bigcap \{C : x \in C\}$$

for C closed in X.

Since finite spaces need only be T_0, each U_x may contain points other than x. In particular, this defines a (partial) ordering between the points.

3.2. **Posets.**

Definition 3.5. A partially ordered set (*poset*) $(X, \leq)$ is a set together with a relation (called a *partial order*) such that the relation satisfies:

- Reflexivity: $x \leq x$
- Transitivity: $x \leq y, y \leq z \implies x \leq z$
- Antisymmetry: $x \leq y, y \leq x \implies x = y$

A *sub-poset* is a subset of $(X, \leq)$ with the same inherited partial ordering between its elements.

If $y \leq x$ and $y \neq x$ we write $y < x$. We say two elements are *comparable* if $x \leq y$ or $y \leq x$.

An element x is *maximal* if x has no elements which cover it. An element y is the *maximum* if y is the unique maximal element. The notions of *minimal* and *minimum* are dually defined.

We remind that for this paper all posets have finite underlying sets.

Remark 3.6. These sets are much like familiar totally ordered sets with the exception that not every pair of elements is comparable: we need not have either $x \leq y$ or $y \leq x$.

Definition 3.7. Posets are commonly depicted with *Hasse diagrams*, denoted $\mathcal{H}(X)$ for a poset X. These consist of directed graphs where vertices correspond to points in the poset and edges correspond to covering relations. We place the covering elements on top of the covered elements rather than drawing the arrows.

Example 3.8. As an example, consider the power set of the set of three elements $\{x, y, z\}$. Inclusion forms a partial order on this set. The Hasse diagram of this poset is depicted on the left in Figure 7.

Definition 3.9. For any point x in a poset X we define sets containing all the points less than or greater than x respectively:

- $U_x = \{y \in X : y \leq x\}$
- $F_x = \{y \in X : x \leq y\}$

We also define non-inclusive versions:

- $\hat{U}_x = \{y \in X : y < x\}$

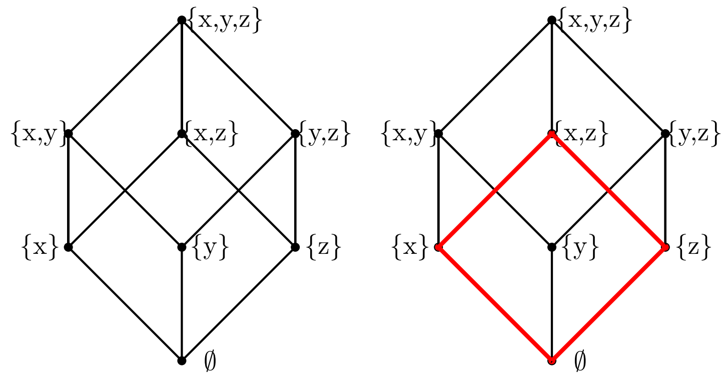


FIGURE 7. Poset by Inclusion and $U_{\{x,z\}}$

- $\hat{F}_x = \{y \in X : x < y\}$

In particular, the notation for U_x and F_x suggests connections to the corresponding notions for finite topological spaces. We explore this idea in Secion 3.3.

$U_{\{x,z\}} = \{\{x, z\}, \{x\}, \{y\}, \emptyset\}$, in our previous example, is depicted in Figure 7 on the right.

Definition 3.10. A sub-poset C of a poset X in which any two elements are comparable is called a *chain*. Similarly, sub-posets in which no two distinct elements are comparable we call *antichains*. A chain or antichain is *maximal* if no element can be added to it.

In our previous example, $\{\{x\}, \{y\}\}$ forms an antichain but not a maximal one as $\{z\}$ can be added while still remaining pairwise non-comparable. An example of a maximal chain would be $\{\{x, y, z\}, \{x, z\}, \{x\}, \emptyset\}$.

Definition 3.11. Let X be a poset.

The *height* of sub-poset A of X is the maximum length of chains in A (as measured by relations). So that say, $\{x, w\}$ with $x \leq w$, would be of length 1.

The height of an element $x \in X$ is the height of its corresponding U_x. We write $x^{(p)}$ to emphasize that x has height p.

A sub-poset is *homogeneous* of degree n if all its maximal chains are of length n. A poset X is called *graded* if for every $x \in X$, U_x is homogeneous.

3.3. **Posets and Finite Spaces.** As we have been hinting, finite topological spaces and posets are really the same objects:

Theorem 3.12. *Let $FPos$ be the category of finite posets with order-preserving maps, $FTop$ the category of finite spaces with continuous maps. Let $\mathcal{F}_P$ and $\mathcal{F}_T$ represent their forgetful functors to $FSet$ the category of finite sets with functions.*

Then there are functors $\mathcal{U}$ and $\mathcal{V}$ such that the following diagram commutes:

$$\begin{tikzcd} FPos \arrow[loop above, "id"] \arrow[d, "\mathcal{F}_P"] \arrow[dr, bend left, "\mathcal{V}"] & \\ FSet & FTop \arrow[l, "\mathcal{F}_T"] \arrow[ul, "\mathcal{U}"] \arrow[loop below, "id"] \end{tikzcd}$$

To prove this theorem there are four things we need to show:

(1) For a topological space $(X, \mathcal{T})$, the relation $\leq$ with $x \leq y$ if $x \in U_y$ defines a poset $(X, \leq)$.
(2) For a poset $(X, \leq)$, the set of $\{U_x\}$ for every x generate a T_0 topology, $\mathcal{T}$ on X.
(3) An order-preserving map, $f : X \to Y$ between posets is continuous if X and Y are taken to be finite spaces.
(4) A continuous map, $f : X \to Y$ between finite spaces is order-preserving if X and Y are taken to be posets.

Proof of (1). By definition (Definition 3.4) x is in U_x, so the relation is reflexive.

To show the other two properties, we show $x \leq y$ if and only if $U_x \subseteq U_y$:

If $x \leq y$ then $x \in U_y$, so that $U_x = U_y \cap U$ for U the intersection of all the other open sets containing x. Thus $U_x \subseteq U_y$.

If $U_x \subseteq U_y$, then $x \in U_y$ since $x \in U_x$.

As for anti-symmetry, $U_x \subseteq U_y$ and $U_y \subseteq U_x$ ensures $U_x = U_y$. The requirement that the space be T_0 ensures this implies $x = y$. Otherwise, the differentiating open set K that contains x but not y or vice versa would have both U_x and U_y as subsets. □

Proof of (2). For the U_x's to be a basis we require that:

(1) X is covered by U_x's
(2) For each $w \in U_x \cap U_y$, there is a U_z with $w \in U_z \subseteq U_x \cap U_y$.

The first statement is direct as for each point we get a U_x (which comes from reflexivity).

As for the second, we set $z = w$. Then if $w \in U_x, U_y$ we have $w \leq x, w \leq y$. If $k \in U_w$ we get have that $k \leq w$, but since partial orderings are transitive $k \leq x, k \leq y$. Thus $w \in U_w \subseteq U_x \cap U_y$.

As for T_0, assume that x, y have all their open sets the same. Then $U_x = U_y \implies U_x \subseteq U_y$ or $U_y \subseteq U_x$. Thus $x \leq y$ and $y \leq x$, but by anti-symmetry this means $x = y$. □

Proof of (3). Let V be open in Y. We show that for each x in $f^{-1}(V)$, their U_x is also in $f^{-1}(V)$. Let $w \in U_x$ then $w \leq x$ so that $f(w) \leq f(x) \implies f(w) \in U_{f(x)}$ but since $f(x)$ is in V so must its minimal open set: $U_{f(x)} \subseteq V$. □

Proof of (4). For continuous g, $U_x \subseteq g^{-1}(U_{g(x)})$ since $g(x) \in U_{g(x)}$ and $g^{-1}(U_{g(x)})$ is open. Assume $w \leq x$, then $w \in U_x \subseteq g^{-1}(U_{g(x)})$. But then, re-mapping g, $g(w) \in U_{g(x)}$ which means that $g(w) \leq g(x)$. $\square$

Remark 3.13. The opposite topology appears as switching the orientation of the ordering: $\geq$ instead of $\leq$.

4. Algebraic Topology of Finite Topological spaces

We now study the homotopy theory of finite topological spaces.

Remark 4.1. Homotopies, (strong and weak) homotopy equivalences, homotopy groups, and (singular) homology groups are defined in the same way for finite spaces as for familiar infinite Hausdorff spaces though they may be less intuitive.

Additionally, We refer to spaces which have the homotopy groups of a point as homotopically trivial. We refer to spaces which have the homology of a point as acylic.

Path-connectedness is also defined in the same way but the following results makes it much more intuitive. A proof can be found by combining [[13], Lemma 2.1.10] and [[13], Proposition 2.1.12].

Proposition 4.2. *Let X be a path-connected poset. Then for any two x, y there is a sequence $\{x_i\}$ with first term x and last term y such that x_i and x_{i+1} are comparable.*

Homotopies have a similar result which replaces the classical intuition. The details are outside of our scope but can be found in [[13], Theorem 2.2.12].

Proposition 4.3. *Let X and Y be posets. Define a partial ordering on the function space Y^X by $f \leq g$ for $f, g \in Y^X$ if $f(x) \leq g(x)$ for all $x \in X$.*

Then, $\leq$ on Y^X makes Y^X a finite space. Furthermore, $f \simeq g$ if and only if f, g are in the same path component of Y^X.

In particular $f, g : X \to Y$ are homotopic if and only if there is a finite sequence of maps $\{f_i\}$ beginning with f and ending with g such that f_i and f_{i+1} are comparable. Thus comparable maps f and g are homotopic as they have such a sequence: $\{f, g\}$ since $f \leq g$.

The rest of this section focuses on methods of removing points in posets while keeping the same (weak or strong) homotopy type. For (strong) homotopy types, this method is quite fruitful resulting in necessary and sufficient conditions for two posets to be homotopically equivalent. For weak homotopy types, this method is not sufficient. Regardless, we will require both methods for establishing weak homotopy equivalence between cuts in our posets when we define their Morse theory.

4.1. Beat Points.

The key concept in the study of strong homotopy types for finite topological spaces is the beat point:

Definition 4.4. Let X be a poset with $x \in X$. We call x a *beat point* if $\hat{U}_x$ has a maximum (*down beat point*) or $\hat{F}_x$ has a minimum (*up beat point*).

Proposition 4.5. *Let X be a poset. If X has a maximum or minimum, X is contractible.*

Proof. Without loss of generality, we consider X to have a maximum, passing to the opposite topology if needed. We take A to be the one point poset $\{a\}$ where a is the maximum of X. We claim $X \simeq A$.

Let $i : A \to X$ be the inclusion map and $r : X \to A$ be the only map from X to A. Both these maps are continuous. Moreover $\mathrm{id}_X \leq i \circ r$ so that these maps are homotopic and $r \circ i = \mathrm{id}_A$.

□

Corollary 4.6. *Let X be a poset. Then U_x and F_x are contractible for any x. Furthermore, for x in X, $\hat{U}_x$ and $\hat{F}_x$ are contractible for x a down or up beat points respectively.*

It turns out that removing beat points does not change the homotopy type of a poset:

Proposition 4.7. *Let X be a poset. If $x \in X$ is a beat point, $Y = X - \{x\}$ is a strong deformation retract of X*

This is for much the same reason: the mapping from $X \to Y$ which keeps everything constant but sends the beat point to its associate maximum or minimum of $\hat{U}_x$ or $\hat{F}_x$ is comparable to the identity (when composed with inclusion). A full proof is in [[3], Proposition 1.3.4] but we illustrate with an example which showcases the content of the proof.

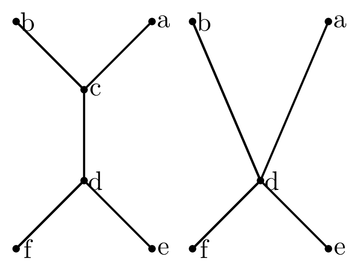


FIGURE 8. The Bowtie and the X as posets

Example 4.8. Consider the poset B which looks like a vertical bowtie (left) Figure 8. We also present the view as topological spaces with all the open sets drawn in Figure 9.

Then we see that c is a down beat point and d is an up beat point. The computations are much the same but let's consider the retraction of the bowtie, B, to $C = B - \{c\}$.

In particular, let $i : C \to B$ be inclusion, $r : B \to C$ be the identity for every point but c which gets sent to d, and $g : B \to B$ be the same map as r but with the codomain now including c.

We note that $i \circ r = g$ and $r \circ i = \mathrm{id}_C$. But $g \leq \mathrm{id}_B$ and thus g and id_B are homotopic.

As for how this looks, just drag down the c point to the d and label it d. The case with removing d would end up with a very similar picture, with an isomorphism given by relabeling d to c.

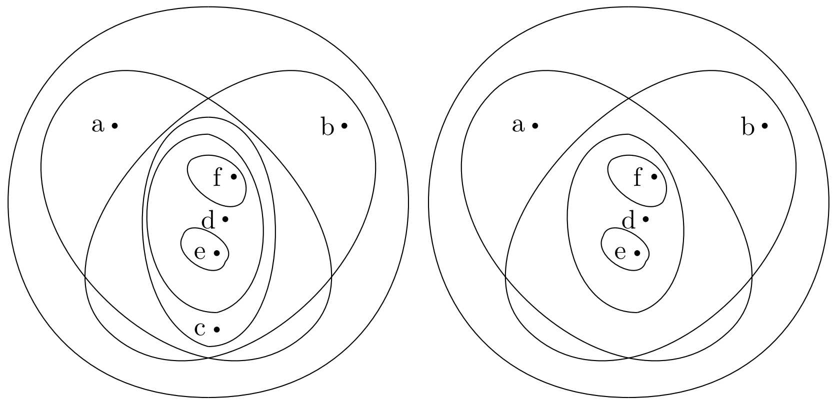


FIGURE 9. The Bowtie and the X as finite spaces

The space received by removing c from B still has b, a, f, e as beat points. However, in principle, we could apply the same procedure until we get a single point. We explore this idea:

Definition 4.9. Let X be a poset. Since X is finite, we can apply Proposition 4.7 repeatedly until we get a space with no beat points.

In particular, we call this a *minimal finite space*. We call the minimal finite space associated to a finite topological space by beat point deletion (Proposition 4.7) its *core*.

We have a remarkable fact, due to Stong ([24]), about cores of finite spaces:

Theorem 4.10. *Let X be a minimal finite space. If $f : X \to X$ is homotopic to the identity, $f = id_X$.*

Proof. We will show there exists no continuous map f comparable to id_X except id_X itself. As homotopic maps are still homotopic in the opposite topology, we need only consider the case $f \geq \mathrm{id}_X$.

We proceed by induction. If y is maximal in X then $f(y) \geq y$ requires $f(y) = y$. Thus, the maximal points are fixed. Take $x \in X$ and, inductively, assume $z \in X$ such that $z > x$ have $f(z) = z$. Since f is continuous (and therefore order-preserving):

$$z = f(z) \geq f(x) \geq x$$

If $f(x) > x$ we have that $f(x) := z_0$ has $z_0 > x$, but then

$$z \geq f(x) \implies z \geq z_0$$

so that z_0 is a minimum of $\hat{F}_x$ which would make x a beat point. This contradicts that X is a minimal finite space.

But by Proposition 4.3, this means that no other maps are homotopic to the identity. Therefore, any map homotopic to the identity is the identity.

∎

Corollary 4.11. *If X, Y are minimal finite spaces they are homotopically equivalent if and only if they are homeomorphic.*

Moreover, if X, Y are finite topological spaces then they are homotopically equivalent if and only if their cores are homeomorphic.

Proof. We start with the first statement:

Let $f : X \to Y$ be a homotopy equivalence and g be its homotopy inverse. Then $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$, but by the previous theorem we have $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. Both compositions will only be bijective if and only if both f and g are.

The second statement follows immediately since X, Y are homotopically equivalent to their respective cores. □

4.2. γ **Points.** Unlike in nice Hausdorff spaces (such as CW-complexes), weak homotopy classes and strong homotopy equivalence classes do not necessarily coincide for finite spaces. Moreover, weak homotopy types for finite spaces have no such classification theorem as there is for strong homotopy equivalences. There is, however, an analogue of beat points— γ points— for weak homotopy types, but as we will see in this section, this does not serve to fully classify spaces by weak homotopy type. Still, the method of removing γ points will prove important in establishing weak homotopy equivalences between cuts in our posets for our Morse theory.

Definition 4.12. For posets X, Y we define their *join* $X \circledast Y$ as $X \sqcup Y$ with $x \leq y$ for all $x \in X, y \in Y$ and maintaining the original orderings otherwise.

Definition 4.13. Let X be a poset with $x \in X$. Define the *link* of x to be

$$\hat{C}_x = \hat{U}_x \circledast \hat{F}_x = \{w \in X : w < x \text{ or } x < w\}$$

This is just all the points directly comparable to x not including itself.

If $\hat{C}_x$ is homotopically trivial, we call x a γ point.

The following theorem justifies us calling γ points weak beat points. The proof in full detail is somewhat extensive, technical, and is outside our primary scope so the reader is pointed to [[3], Theorem 6.2.2]. A more detailed exposition can also be found in [[27], Theorem 3.21].

Theorem 4.14. *Let X be a poset. If $x \in X$ is a γ point, inclusion $i : X - \{x\} \to X$ gives a weak homotopy equivalence.*

In fact, the converse is almost always true: most one-point reductions which are weak equivalences are given by deleting γ points (a proof found in [[3], Theorem 6.2.5]):

Theorem 4.15. *Let X be a finite topological space with x not maximal or minimal and with $i : X - \{x\} \to X$ a weak homotopy equivalence. Then, x is a γ point.*

Barmak [[3], Example 6.26] presented the following example to show a one point removal which is a weak homotopy equivalence though the removed point is not γ.

Example 4.16. Let X be a finite space which is acyclic but not simply connected. Let SX be the link of X with the two point discrete space: $X \circledast \{\{+\}, \{-\}\}$. By the definition of link (Definition 4.12), SX is just X but with all points covered by $+$ and $-$ with $+, -$ not comparable. Thus, SX is also acyclic but is simply connected. Therefore, by Hurewicz, SX is homotopically trivial.

Then the non-Hausdorff cone of X, CX, given by $X \circledast \{+\}$ is also homotopically trivial (in fact, contractible as it has a maximum). So, $i : CX \to SX$ is a weak homotopy equivalence with $\hat{C}_- = X$ not homotopically trivial.

This example is rather unsatisfactory as we have no guarantee such an X exists. Using tools from the Appendix , however, we can find such an example. One small such example is due to Björner and Lutz explored in [4].

Example 4.17. Let Y be the poset given by $\mathcal{P}(10)$, the power set of $\{1, \ldots, 10\}$, ordered by inclusion.

Take the open sub-poset X given by the union of the U_x's corresponding to the following triples:

$$\{\{1\},\{2\},\{4\}\},\{\{1\},\{2\},\{5\}\},\{\{1\},\{3\},\{6\}\},\{\{1\},\{3\},\{8\}\},$$

$$\{\{1\},\{3\},\{10\}\},\{\{1\},\{4\},\{8\}\},\{\{1\},\{4\},\{9\}\},\{\{1\},\{5\},\{7\}\},$$

$$\{\{1\},\{5\},\{10\}\},\{\{1\},\{6\},\{7\}\},\{\{1\},\{6\},\{9\}\},\{\{2\},\{3\},\{5\}\},$$

$$\{\{2\},\{3\},\{5\}\},\{\{2\},\{3\},\{7\}\},\{\{2\},\{3\},\{8\}\},\{\{2\},\{4\},\{6\}\},$$

$$\{\{2\},\{4\},\{10\}\},\{\{2\},\{6\},\{7\}\},\{\{2\},\{6\},\{8\}\},\{\{2\},\{8\},\{10\}\},$$

$$\{\{3\},\{5\},\{6\}\},\{\{3\},\{5\},\{9\}\},\{\{3\},\{7\},\{9\}\},\{\{3\},\{7\},\{10\}\},$$

$$\{\{4\},\{5\},\{6\}\},\{\{4\},\{5\},\{7\}\},\{\{4\},\{5\},\{8\}\},\{\{4\},\{7\},\{9\}\},$$

$$\{\{4\},\{7\},\{10\}\},\{\{5\},\{8\},\{9\}\},\{\{5\},\{8\},\{10\}\},\{\{6\},\{8\},\{9\}\}$$

For obvious reasons, we do not draw the Hasse diagram of this poset. However, one could check that X is not simply connected and is acyclic through brute force as the space is finite.

Barmak's proof of Theorem 4.14 relied heavily on a theorem of McCord's (Theorem 4.20) which is vital to proving any map is a weak homotopy equivalence for finite spaces. We will make use of it when proving cuts of a poset are weakly homotopic as well. A proof can be found in [[14], Theorem 6].

Definition 4.18. An open cover $\mathcal{O}$ of a poset X is called basis-like if $\mathcal{O}$ is the basis for some topology on X.

Remark 4.19. In particular, for X a finite space, $\{U_x\}$ is a basis-like open cover (since it is a basis for X).

Theorem 4.20. *Let X, Y be topological spaces with $f : X \to Y$ continuous. Assume Y admits a basis-like open cover $\mathcal{U}$ with*

$$f_{|f^{-1}(U)} : f^{-1}(U) \to U$$

a weak homotopy equivalence for all $U \in \mathcal{U}$. Then, f is a weak homotopy equivalence.

5. Morse Theory on Posets

5.1. **Admissibility.** In this section we finally introduce Morse functions on posets. We restrict our posets to those which are the analogues of manifolds in the case of classical Morse theory: admissible posets. These are posets in which the minimal open set for each point is a finite analogue of an open neighborhood on a manifold. The links to proper manifolds are explored in the appendix.

We start by introducing a new piece of notation:

Definition 5.1. If $x < y$ and there is no z such that $x < z < y$ then we say that y covers x and write $x \prec y$.

Definition 5.2. For any edge $(w, x) := w \prec x$ in the Hasse diagram of $\mathcal{H}(X)$ of a poset X, we say (w, x) is *admissible* if $\hat{U}_x - \{w\}$ is homotopically trivial. We say X is admissible if all edges in $\mathcal{H}(X)$ are admissible.

In practice, this is rather hard to compute, so we leverage the Hurewicz theorem to introduce weaker notions which will produce the same result.

Definition 5.3. For $(w, x) \in \mathcal{H}(X)$, we say (w, x) is *1-admissible* if $\hat{U}_x - \{w\}$ is simply connected. We say X is 1-admissible if all edges in $\mathcal{H}(X)$ are 1-admissible.

For $(w, x) \in \mathcal{H}(X)$, we say (w, x) is *homologically admissible* if $\hat{U}_x - \{w\}$ is acyclic. We say X is admissible if all edges in $\mathcal{H}(X)$ are homologically admissible.

Proposition 5.4. *Let X be a poset. An edge (w, x) is admissible if and only if it is 1-admissible and homologically admissible.*

Proof. This is an immediate consequence of the Hurewicz [[9], Theorem 4.37]. □

Minian states the following [[16], Definition 3.1, Definition 3.8]

Proposition 5.5. *Let X be a homologically admissible poset. Then it is graded (Definition 3.11).*

This property by itself is not of particular importance, it does, however, allow us to introduce a special case of Minian's cellular homology [[16], Section 3]. We use this cellular homology to prove two properties (Proposition 5.7 and Proposition 5.8) which we require for our Morse theory.

Definition 5.6. For a homologically admissible poset X we define its *cellular homology* with $\mathbb{Z}_2$ coefficients by defining its chain complex $(C_\star, d)$:

(1) $C_k(X)$ is a $\mathbb{Z}_2$ vector space with one generator for each $x^{(k)}$ of height k.
(2) $d : C_k(X) \to C_{k-1}(X)$ is defined by $d(x) = \sum_{w \prec x} w$.

This homology agrees with singular homology with the same coefficients. A proof can be found in [[16], Theorem 3.7]. We exploit $d^2 = 0$ to prove the two following properties for homologically admissible posets.

Proposition 5.7. *Let X be a homologically admissible poset. Any non-minimal $x \in X$ covers at least two other points. We call this property* down-wide.

Proof. For any $x^{(1)}$ (x with height 1), we have that x must cover some element w as x is non-minimal. Also, $\hat{U}_x - \{w\}$ is acyclic and thus nonempty so that there is $w' \neq w$ such that $w' \prec x$.

Inductively, assume X is down-wide for points of height less than k. Assume $w^{(k-1)}$ is unique such that $w^{(k-1)} \prec x^{(k)}$, then applying the differential map:

$$d^2(x) = d(w) = \sum_{q \prec w} q \neq 0$$

which is a contradiction.

□

Proposition 5.8. *Let X be a homologically admissible poset. Then if $x \prec y \prec z$ there is $y' \neq y$ such that $x \prec y' \prec z$. We call this property* two-wide.

Proof. Suppose X has $x \prec y \prec z$ we show that there is some $y' \neq y$ with $x \prec y' \prec z$.

We apply the differential to y and z to look at the elements they cover:

$$d(y) = x + \sum_{w \prec y, w \neq x} w$$

$$d(z) = y + \sum_{v \prec z, v \neq y} v$$

However, since $d^2 = 0$ we have

$$0 = d^2(z) = d(y) + \sum_{v \prec z, v \neq y} d(v) = x + \sum_{w \prec y, w \neq x} w + \sum_{v \prec z, v \neq y} d(v)$$

Since this expression holds, there must be some $y' \prec z$ such that

$$d(y') = x + \sum_{u \prec y', u \neq x} u$$

so that it can negate the other x in the expression of $d^2(z)$. □

5.2. **Morse Functions.**

Definition 5.9. For X a poset, we call a map $f : X \to \mathbb{R}$ a *Morse function* if it satisfies for every $x \in X$ that f is mostly order-preserving. More specifically, f is Morse if:

(1) $|\{y \in X : x \prec y \text{ and } f(x) \geq f(y)\}| \leq 1$
(2) $|\{w \in X : w \prec x \text{ and } f(w) \geq f(x)\}| \leq 1$

In other words, for every point x, we allow f to place only one $y \prec x$ higher than x and only one $x \prec w$ lower than x.

Our critical points, in analogue of the classical case, are the points where f is completely order-preserving:

Definition 5.10. For a poset X, let $f : X \to \mathbb{R}$. If $x \in X$ satisfies both of the following conditions:

(1) $|\{y \in X : x \prec y \text{ and } f(x) \geq f(y)\}| = 0$
(2) $|\{w \in X : w \prec x \text{ and } f(w) \geq f(x)\}| = 0$

we call x a *critical point*.

A point is called *regular* if it is not critical. The images of critical points and regular points under f are called critical and regular values respectively. We use *Crit*(f) for the set of critical points.

It turns out that in building our Morse theory the following result is key:

Lemma 5.11. *Let X be a homologically admissible poset with $w, y \in X$ such that $w < y$ and $w \not\prec y$. Then there are two distinct $x, \tilde{x} \in X$ such that $w \prec x, \tilde{x} < y$.*

Proof. Since $w < y$ we can find some finite sequence $\{x_i\}$ such that $w = x_1$ and $y = x_n$ such that $x_i \prec x_{i+1}$. We can obtain such a chain inductively. Start with $C_1 = \{w, y\}$. Then let $C_2 = \{w, v, y\}$ for $w < v < y$.

For C_k we construct C_{k+1} by inserting a term between the first two terms of C_k if possible. If not we insert a term between the next two terms and so on. Since the poset is finite, this process terminates.

We note that X is two-wide. Then take x_1 as x and since $w \prec x_1 \prec x_2$ by Proposition 5.8 we know there exists a $\tilde{x}$ with $\tilde{x} : w \prec \tilde{x} \prec x_2 < y$. □

Proposition 5.12. *Let X be a homologically admissible poset. Then for any Morse function $f : X \to \mathbb{R}$, for each regular x in X, one and only one of the following conditions holds.*

(1) $|\{y \in X : x \prec y \text{ and } f(x) \geq f(y)\}| = 1$
(2) $|\{w \in X : w \prec x \text{ and } f(w) \geq f(x)\}| = 1$

We call any Morse function satisfying this result exclusive*.*

Proof. Let $x \in X$ be regular. Then at least one of the two conditions holds. Assume, for the sake of contradiction, both hold.

The first condition requires x be non-maximal and the second requires x be non-minimal. Then, by our previous lemma we have some $x' = x$ such that $y > x' \succ w$. However, y is regular since $f(x) \geq f(y)$ so that y cannot cover another point with greater or equal image. If $x' \prec y$ then $f(x') < f(y)$.

Otherwise, we apply the same lemma to y and x' to get z_1 and z_1' covering x' less than y. But at least one of these must have greater f value than x' by the definition of Morse function (Definition 5.9). If y covers this one, then we are done as once again y cannot over another point with greater or equal image apart from x. Otherwise we keep applying the same procedure. Since we only have finitely many point, this terminates and we have some sequence of points z_i with starting with x' and ending with y such that $f(x_i) < f(x_{i+1})$. Regardless, $f(x') < f(y)$.

Then, the first condition applied to w gives us that $f(w) < f(x')$ which yields the following contradiction:

$$f(x) \leq f(w) < f(x') < f(y) \leq f(x)$$

□

We can split the regular points into those which satisfy Condition (1) or Condition (2). For each of these points, there is a corresponding point of the other kind. Thus we can pair up regular points. This inspires the following definition:

Definition 5.13. A *matching* $\mathcal{M}$ of a poset X is a subset of edges (w, x) of $\mathcal{H}(X)$ such that $\mathcal{M}$ contains no two edges sharing a point. A matching is *admissible, 1-admissible, or homologically admissible* when all of its edges are admissible, 1-admissible, or homologically admissible respectively.

Definition 5.14. We remind that for a poset X, its $\mathcal{H}(X)$ is a directed graph. For a matching $\mathcal{M}$ on X we define $\mathcal{H}_{\mathcal{M}}(X)$ to be $\mathcal{H}(X)$ with edges in $\mathcal{M}$ reversed in orientation. Then we call $\mathcal{M}$ a *Morse matching* if $\mathcal{H}_{\mathcal{M}}(X)$ has no directed cycles.

We call points which are not incident to edges of $\mathcal{M}$ its *critical points* denoted by $\mathrm{Crit}(\mathcal{M})$.

Morse matchings give us a way to analyze Morse functions which will be useful in developing our fundamental theorems. In particular, as mentioned before, Morse functions on homologically admissible posets pair up regular points.

In particular, the matching made from pairing regular points above is Morse (a proof in [[7], Theorem 3.15]):

Theorem 5.15. *Let X be a homologically admissible poset with $f : X \to \mathbb{R}$ Morse. Then there is a Morse matching $\mathcal{M}_f$ given by $(w \prec x) \in \mathcal{M}_f$ if $f(x) \leq f(w)$. Moreover, $Crit(\mathcal{M}_f) = Crit(f)$.*

Moreover, Morse matchings allows us to locate Morse functions on homologically admissible posets [[16], Lemma 3.12]:

Theorem 5.16. *Let X be a homologically admissible poset with $\mathcal{M}$ a Morse matching on X. Then there is a Morse function $f : X \to \mathbb{R}$ with $Crit(\mathcal{M}) = Crit(f)$.*

5.3. **Technical Prerequisites.** We are almost ready to prove our fundamental theorems of Morse theory. We start by defining our analogue to the classical cuts of differentiable manifolds:

Definition 5.17. For X a poset with Morse function $f : X \to \mathbb{R}$ and $a \in \mathbb{R}$ we define

$$X_t^f = \bigcup_{f(x) \leq t} U_x$$

We call this the *cut of X by a* and leave out the f if it is clear from context. Note that X_t is open in X.

Then we develop some results we need for our Morse theory. Firstly, we modify our Morse functions so that critical values and regular values are distinct. We also separate critical values.

Lemma 5.18. *Let X be a path-connected homologically admissible poset. Let $f : X \to \mathbb{R}$ a Morse function. Let $a, b \in \mathbb{R} : a < b$ such that $(a, b]$ contains only one value, c, in $im(f)$. Then there is $g : X \to \mathbb{R}$ Morse and exclusive such that:*

- *$X_t^g = X_t^f$ for $t < c - \epsilon$ and $t > b$.*
- *$Crit(f) = Crit(g)$.*
- *No regular point of g with value in $(a, b]$ shares that value with any critical point.*
- *g is injective on its critical points with images in $(a, b]$.*

Proof. We construct g from f. Say that the number of critical points in $f^{-1}(c)$ is k by only shifting points with image c. We let ϵ be smaller than $c - a$.

No critical point in $f^{-1}(c)$ is adjacent to any other point in $f^{-1}(c)$ for no critical point can be covered or cover by another point with the same value. Thus, we can distribute the critical values for these points between c and $c - \epsilon$ at intervals of $\frac{\epsilon}{k+1}$ apart. The image remains unchanged for regular points.

As f was Morse before, and g maintains the relative ordering of adjacent points so that g is still Morse. For the same reason, g and f have the same critical points. Since we also only adjusted points in the ϵ strip under c we still have the same cuts outside that interval.

We note that we can actually get $X^g_t = X^f_t$ for $t < c$ and $t > b$ (with no ϵ gap) so long as $c \neq b$. This is because we can start our procedure by replacing c with $c + \epsilon$ in such a case. □

We then correct injectivity for regular points.

Lemma 5.19. *Let X be a path-connected homologically admissible poset. Let $f : X \to \mathbb{R}$ an exclusive Morse function. Let $a, b \in \mathbb{R} : a < b$ such that $(a, b]$ contains at most one regular value, c, and contains no critical values. Then there is $g : X \to \mathbb{R}$ Morse with*

- *$X^g_t = X^f_t$ for $t < c - \epsilon$ and $t > b$.*
- *$Crit(f) = Crit(g)$.*
- *the restriction of g to $(a, c]$ is injective on its regular points.*

Proof. Much the same as last lemma, we build g from f by only adjusting points in $f^{-1}(c)$. Let there be k points in $f^{-1}(c)$.

Note that the path components of $f^{-1}(c)$, when considered as a sub-poset, are all chains. This is because by the definition of Morse function (Definition 5.9), no point could cover or be covered by more than one point with equal image. We rearrange the images of the different path components without altering relative order of adjacent elements.

We construct g in $(a, b]$ as follows. Pick some path component of $f^{-1}(c)$ and take its maximal element and place its image at $c - \frac{\epsilon}{k}$. Continue by placing the next largest point $\frac{\epsilon}{k}$ lower and so on. Then, repeat the same procedure with the next path component. Start by placing its image $\frac{\epsilon}{k}$ beneath the minimal element of the previous path component and then continue placing each smaller point beneath. We repeat for each path component.

Since for $w \prec z$ we already had $f(w) = f(z)$ and now we have $g(w) > g(z)$, these changes do not affect g being Morse.

As before, the ϵ is only necessary if $b = c$. □

Applying both of these procedures allows us to make our Morse functions locally injective and, with enough applications and b's taken to be not in the image, completely injective.

Additionally, we worry about the number of path components in cuts. We denote by $b_0(X)$ the number of path components.

Proposition 5.20. *For X a path-connected homologically admissible poset let $f : X \to \mathbb{R}$ be an injective Morse function. Then if $a, b \in \mathbb{R}$ such that $a < b$ and $b_0(X_a) < b_0(X_b)$ there is a critical value $c \in (a, b]$ with $b_0(X_c) = b_0(X_a) + 1$. Moreover, v, the (unique point in the) preimage of c, is minimal.*

Proof. As f is injective, we only get new path components of X_u when we hit a new point in $\mathrm{im} f$. Then let $c \in \mathrm{im} f$ be the minimum value such that $c > a$ and $b_0(X_c) = b_0(X_a) + 1$. Let $v \in f^{-1}(c)$ (it is unique by injectivity). We show that c is critical and that v is minimal.

We start with the latter. Assume, for the sake of contradiction, v is not minimal. Since we have a new path component at c, all elements covered by v must have larger images under f than c. However, X is down-wide so that if v is non-minimal we would require at least two such elements. But this contradicts the definition of a Morse function (Definition 5.9).

Now assume c is not critical. Then v is not critical and since it is minimal, v is covered by w with strictly smaller image under f than c (as f is injective). But then $v \in X_{f(w)}$. But then X_t could not have gained a new path component when reaching c as no new points were added due to f being injective and v already being in X_t prior to passing c as $c > f(w)$ □

Proposition 5.21. *Let X be a path-connected homologically-admissible poset, $f : X \to \mathbb{R}$ an injective and Morse function, and $a, b \in \mathbb{R} : a < b$. If $(a, b]$ contains only one regular value $f(p) = c$ and no critical values, then there is $q \in X_a$ with $q \prec p$ or $p \prec q$.*

Proof. By the last proposition, $b_0(X_a) \geq b_0(X_b)$ so that p is not a different path component. But since f is exclusive, regular points come in pairs. □

With these results we can finally start working on our fundamental theorems of Morse theory.

5.4. **Fundamental Theorems.** We start by stating our fundamental theorems of Morse theory for posets. We first list the collapsing theorems. These theorems are most similar to Theorem 2.4 as they show that cuts between critical values have the same weak homotopy type or homology.

Theorem 5.22 (First Collapsing Theorem of Morse Theory on Posets)**.** *Let X be a path-connected homologically admissible poset with $f : X \to \mathbb{R}$ a Morse function. Assume $(a, b]$ contains no critical values of f. Then:*

1. *if the Morse matching assigned to f is admissible, inclusion $i : X_a \to X_b$ is a weak homotopy equivalence.*
2. *additionally, if there is at most one regular value in $(a, b]$, then $X_b = X_a$ or $X_b - X_a = \{v, w\}$ where $w \prec v$ and w up beat in X_b and v γ in $X_b - \{w\}$.*

Theorem 5.23 (Second Collapsing Theorem of Morse Theory on Posets)**.** *Let X be a finite path-connected homologically-admissible poset with $f : X \to \mathbb{R}$ a Morse function. Let $a, b \in \mathbb{R}$ with $a < b$ and with $(a, b]$ containing no critical values of f. Then if the Morse matching assigned to f is homologically admissible, inclusion $i : X_a \to X_b$ induces an isomorphism in homology.*

Additionally, we also have have a theorem that explains how posets change upon passing critical points:

Theorem 5.24 (Adjunction Theorem of Morse Theory on Posets)**.** *Let X be a path-connected homologically admissible poset with $f : X \to \mathbb{R}$ a Morse function. Let x^p be the unique critical point with $g(x) \in (a, b]$ for $a < b$ with $x \in X_b$ but not X_a. Additionally, assume that $(a, b]$ contains no regular values and that $b \notin im(f)$. Then $X_b = X_a \cup_{\partial x^{(p)}} x^{(p)}$.*

The appendix, in particular Theorem 6.4, shows how adding in a point of a poset of height k affects the weak homotopy type of a point in the same way that adding a k-cell (more specifically a k-simplex).

We begin by proving a lemma common to Theorem 5.22 and Theorem 5.23

Lemma 5.25. *Let X be a path-connected homologically-admissible poset with $f : X \to \mathbb{R}$ a Morse function. Let $a, b \in \mathbb{R}$ with $a < b$ and $(a, b]$ containing at most one regular value, $f(v)$, and no critical values. Then, either $X_b = X_a$ or $X_b - X_a = \{v, w\}$ where $w \prec v$ and w is up beat in X_b.*

Proof. We can assume f is injective by Lemma 5.18 and Lemma 5.19. If $(a,b] \cap \mathrm{im}(f) = \emptyset$, then $X_a = X_b$ so we assume $(a,b]$ contains exactly one regular value $c = f(v)$. By Proposition 5.21, the other regular point paired to v must be in X_a. Since f is exclusive we get two cases depending on whether v is the upper or lower half of its regular pair.

(1) Case 1: There is $v \prec w$ with $f(w) < f(v)$.
(2) Case 2: There is $w \prec v$ with $f(w) > f(v)$.

We begin with Case 1:

We know $f(w) \leq a$ since $f(v)$ is unique in $(a,b]$. But then v must be in X_a and, by injectivity, v is the only point with image $f(v)$. So $X_b = X_a$.

Case 2: We claim $X_b - X_a = \{v, w\}$ where $w \prec v$ and w up beat in X_b.

Assume, there is $u \neq v$ with $w \prec u$. We show $u \in X_b \implies u \in X_a$.

We know $u \in X_b \iff u \leq r$ with $f(r) \leq b$. We also have $f(w) > f(v)$ and since $f(v)$ unique in $(a,b]$, we need that $f(w) > b$ and $f(u) > f(w)$ since f is Morse. This means $r \neq u$. But $f(v)$ unique in $(a,b]$ so that if v is not the only possible r, $f(r) \leq a$.

We show if v is r then there is r' with $f(r') \leq a$.

We know that $u \prec v$ since $f(w) > f(v)$ and $f(u) > f(v)$. So then there must be an intermediary point between u and v. However, that point, r', also cannot have image larger than v. So $f(r')$ is smaller than a since $f(v)$ unique in $(a,b]$.

So if there is $u \neq v$ such that $w \prec u$ we have two possibilities:

(1) If $u \in X_b$, then $u \in X_a$ so w is up beat.
(2) Otherwise, $u \notin X_b$, but then w is still up beat.

□

Then we prove the first theorem for cuts with a difference of at most one regular value.

Proposition 5.26. *Let X be a path-connected homologically admissible poset with $f : X \to \mathbb{R}$ a Morse function. Let $a, b \in \mathbb{R}$ with $a < b$ and $(a,b]$ containing at most one regular value $f(v)$ and no critical values. Then, if we have the second case of the last proposition: inclusion $i : (X_b - \{w\}) - \{v\} \to X_b - \{w\}$ is a weak homotopy equivalence if and only if the Morse matching assigned to f is admissible. Additionally, v is a γ point in $X_b - \{w\}$.*

Proof. We use Theorem 4.20 with $\{U_x\}$ for $x \in X_b - \{w\}$ as our basis-like open cover. We can assume f is injective by Lemma 5.18 and Lemma 5.19.

If $x \neq v$, both U_x and $i^{-1}(U_x)$ are contractible since they each have maximums. Then they are homotopically equivalent by $i|_{i^{-1}(U_x)} : i^{-1}(U_x) \to U_x$. In particular, $i|_{i^{-1}(U_x)}$ is a weak homotopy equivalence since it is also a homotopy equivalence.

If $x = v$, we have $i|_{i^{-1}(U_x)} : i^{-1}(U_x) \to U_x$ is the inclusion map $i : \hat{U}_v - \{w\} \to \hat{U}_v$. The codomain is once again contractible and since the matching associated to f is admissible $\hat{U}_v - \{w\} = \hat{C}_v$ is homotopically trivial. Thus, inclusion is a weak homotopy equivalence.

□

Proof of Theorem 5.22. We can substitute f by its injective counterpart by Lemma 5.18 and Lemma 5.19. We pick intervals with only one regular value and repeatedly apply Lemma 5.25, Proposition 5.26 and beat point deletion (Proposition 4.7) as needed. □

We prove the homological theorem in the case of cuts differing by at most one regular value as well:

Proposition 5.27. *Let X be a path-connected homologically-admissible poset with $f : X \to \mathbb{R}$ a Morse function. Let $a, b \in \mathbb{R}$ with $a < b$ and $(a, b]$ containing at most one regular value $f(v)$ and no critical values. Then, if $X_b - X_a = \{v, w\}$ as in Lemma 5.25, inclusion $i : X_a \to X_b$ induces an isomorphism in all homology groups.*

Proof. Note that any matching for a homologically admissible poset is itself homologically admissible.

We apply the Long Exact Sequence of Homology [[9], Theorem 2.16] to (X_b, X_a). This gives us that $i : X_a \to X_b$ induces an isomorphism in all homology groups if and only if the relative homology groups $H_\star(X_b, X_a)$ are all trivial.

We then employ the Excision Theorem [[9], Theorem 2.20], which gives us that if A, B open and cover C then $H_\star(B, A \cap B) \cong H_\star(C, A)$. We take $A = X_a, B = U_v, C = X_b$. Then, we have $H_\star(U_v, \hat{U}_v - \{w\}) \cong H_\star(X_b, X_a)$. But the matching associated to f is homologically admissible so that $\hat{U}_v - \{w\}$ is acyclic. Note that U_v is contractible and, thus, acyclic.

We apply the Long Exact Sequence of Homology again yielding $H_\star(U_v, \hat{U}_v - \{w\}) \cong H_\star(\hat{U}_v - \{w\}) \cong 0$. □

Proof of Theorem 5.23. We can assume f is injective by Lemma 5.18 and Lemma 5.19. We use repeated applications of Proposition 5.27 with intervals containing only one regular value each. □

Finally we prove the adjunction theorem:

Proof of Theorem 5.24. We can replace f by its injective counterpart by Lemma 5.18 and Lemma 5.19.

In particular, we know that if $w \prec x$ we have $f(w) < f(x)$ since x is critical. Then, since x is unique in $(a, b]$, we have $f(w) \le a$ so that $w \in X_a^g$. But then $\partial x \subseteq X_a$ so that we can safely attach x at its boundary. Hence, $X_b = X_a \cup x$ as sets. Additionally, as posets, we can make X_b from X_a by making x cover all of the points it covers in X. □

6. Appendix: Complexes and Manifolds for Posets

The question which underlies this section is where exactly the manifolds are hidden in the above theory. Along the way we will explain where the example of the mysterious poset from Example 4.17 came from. It turns out that these two pursuits are related.

The first clue in pursuit of both goals are given by the McCord functors which link posets to ordered simplicial complexes and CW-complexes. Details can be found in [14]. A good reference on abstract (and more general) simplicial complexes is [6].

Definition 6.1. For a poset X, we denote by $\mathcal{K}X$ the ordered simplicial complex whose k-simplices are chains of length k. For f an order-preserving map between posets, we denote $\mathcal{K}f$ as the associated simplicial map. We call $\mathcal{K}X$ the *order complex* of X.

For K an ordered simplicial complex, we denote by ΔK the poset of its simplices ordered by inclusion. We call this the *face poset* . Similarly, simplicial maps f are carried to order preserving maps Δf.

The face poset, as mentioned, can also be extended to cell complexes.

Definition 6.2. The *face poset* of K a CW-complex is ΔK where cells are ordered by inclusion. Cellular maps f are also carried to order preseving maps Δf.

Remark 6.3. These functors compose, according to a theorem of McCord's [[14], Theorem 3], to the first barycentric subdivision of K:

$$\mathcal{K}\Delta(K) = \mathrm{sd}(K)$$

These functors point to the existence of deep connections between ordered simplicial complexes and posets. In particular, McCord proved the following theorem:

Theorem 6.4. *Let X be a poset and K be an ordered simplcial complex*

(1) *there exists a map μ_X such that $\mu_X : \mathcal{K}(X) \to X$ is a weak homotopically equivalence.*
(2) *there exists a map μ_K such that $\mu_K : K \to \Delta K$ is a weak homotopically equivalence.*

For details on these proofs see [[3], Theorem 1.4.6] and [[3], Theorem 1.4.12]. The functors also provide motivation for the following definition:

Definition 6.5. Additionally, we call a poset X a *model* of a CW complex Y if $|\mathcal{K}(X)|$ and $|Y|$ are homotopy equivalent.

Note that this means $|Y|$ and X are weakly homotopically equivalent.

Example 4.19 came from the application of these functors to a space with particularly desirable properties. In particular, we used an ordered simplicial complex related to the Poincaré homology 3-sphere.

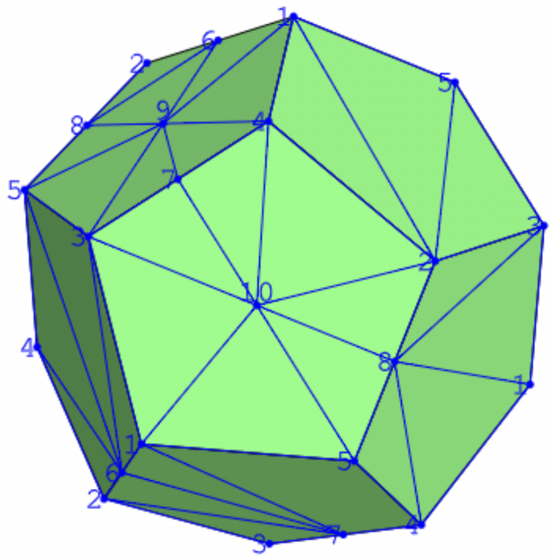
Definition 6.6. A *homology 3-sphere* is a 3-manifold with the homology groups of a sphere yet not homeomorphic to the 3-sphere.

One example of a homology 3-sphere is the the *Poincaré homology sphere* is built by taking a dodecahedron and identifying each face with its opposite face while twisting $\frac{\pi}{5}$ so as to have them line-up and then imposing the quotient topology. This presentation is due to Seifert and Threlfall [25].

In particular we used the following graph over the boundary of the (dodecahedron model of the) Poincaré homology 3-sphere:

A full exposition on why exactly it satisfies our desired conditions can be found in [4].

However, as Minian states in [[16], Remark 3.10], any triangulation of the Poincaré 3-homology sphere can be converted into such a counterexample.



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