LINEAR REPRESENTATIONS OF FINITE GROUPS

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Abstract. The goal of this paper is to introduce the necessary definitions in representation theory of finite groups and develop the fundamental theory regarding characters, induced representations, and irreducibility. It is assumed the reader has experience in abstract algebra and linear algebra, but little knowledge of higher level mathematics is needed. The content follows an example based approach in order to enhance the reader's understanding of the developed theory.

Contents

1. Introduction

Representation theory was first developed in the late twentieth century with the motivation to use the well developed properties of linear algebra to better understand potentially non-linear algebraic structures. In the years following the initial creation of the theory, representations have been used from combinatorics and class field theory to quantum chemistry. This paper will focus solely on the representation theory of finite groups. We will begin with the necessary definitions and the immediate theory surrounding them, particularly the concept of irreducible representations. Then we will consider the importance of irreducible representations further by developing character theory and seeing several examples to demonstrate the utility of such representations. Finally, we will explore the properties of induced

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representations, which commonly appear in the several mathematical fields that often use representation theory.

2. Representation Theory of Finite Groups

We begin with the notion of a *linear representation* of a group. For our purposes, we will be dealing with finite groups represented in finite dimensional vector spaces over the complex field of scalars, $\mathbb{C}$.

Definition 2.1. A linear representation of a group G in a vector space V is a group homomorphism $\rho : G \to \mathrm{GL}(V)$.

If ρ_1, ρ_2 are two representations of the same group G, in V_1 and V_2 respectively, they are said to be *isomorphic* if there is an isomorphism $f : V_1 \to V_2$ of vector spaces such that,

$$f \circ \rho_1(g) = \rho_2(g) \circ f$$

for all $g \in G$.

Example 2.2. For any group G, we define $\rho : G \to \mathbb{C}^\times$. Define $\rho(g) = 1$, for all $g \in G$. This representation is called the trivial representation.

Example 2.3. Let G be a finite group acting on a finite set X, and let V be the vector space with basis elements $\{e_x\}_{x \in X}$, indexed by X. Then we define $\rho : G \to \mathrm{GL}(V)$ where $\rho(g)$ is the map such that, $e_x \mapsto e_{gx}$. This representation is referred to as the *permutation representation.* If $X = G$, then it is called the *regular representation.*

Example 2.4. Let $G = D_6$, the symmetries of an equilateral triangle. Define $\rho : G \to \mathrm{GL}_2(\mathbb{R}) \subseteq \mathrm{GL}_2(\mathbb{C})$ as the matrix transforming the plane when the equilateral triangle is embedded at the origin.

Notice that since G has finite order, $\rho(g)$ has finite order, for any representation ρ of G. Notably, this means that all eigenvalues of the linear isomorphism $\rho(g)$ are roots of unity. Another important notion, which we will now discuss, is that of *subrepresentations.*

Definition 2.5. Let V be a vector space and W be a subspace of V. If $\rho : G \to \mathrm{GL}(V)$ is a linear representation, and if W is invariant under the action of G, or $\rho(g)w \in W$ for all $g \in G$, $w \in W$, then the restriction of ρ to W, ρ^W, is a linear representation. W is then called a subrepresentation of V. There are always two trivial subrepresentations, which are the subspaces $\{0\}$ and V. If a representation has no nontrivial subrepresentations, then it is said to be *irreducible.*

We can immediately consider the criteria for irreducibility for representations of dimension 1 and 2. Dimension 1 representations are clearly always irreducible. For dimension 2 representation, they are reducible if each $\rho(g)$ leaves the same dimension one subspace invariant. This would mean each $\rho(g)$ shares an eigenspace. Thus, we have that a two dimensional representation is irreducible if all the $\rho(g)$ do not have a common eigenspace.

Example 2.6. Let $G = S_3$ and $\rho : G \to \mathrm{GL}(V)$ be the regular representation. (Example 1.3) Let $w = e_1 + e_2 + e_3$, then the subspace W spanned by w is invariant under the action of G, making ρ^W a subrepresentation. In fact, $\rho^W(g)w = w$, so ρ^W is the trivial representation. (Example 2.2)

Example 2.7. Let $G = D_6$ and $\rho : G \to \mathrm{GL}_2(\mathbb{R}) \subseteq \mathrm{GL}_2(\mathbb{C})$ be defined as in Example 1.4. The set of $\rho(g)$, for $g \in G$ has no common eigenspace. Since this is a two dimensional representation, this representation is irreducible.

It can be asked if it is sufficient to only consider irreducible representations and their properties. In order to answer this, we turn to the first theorem regarding subrepresentations. It is first attributed to Maschke in the late 19th century.

Theorem 2.8 (**Maschke**). *Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation of a finite group G in V and let W be a subrepresentation of V. Then there exists another subrepresentation U of V such that $V = U \oplus W$.*

Proof. Let U' be any subspace of V such that $V = U' \oplus W$, and let p' be the projection of V onto W along U'. We may then define a new projection, p, onto W as,

$$p = \sum_{g \in G} \rho^{-1}(g) \circ p' \circ \rho(g)$$

Let $\ker(p) = U$. For any $g \in G$, we have that $\rho^{-1}(g) \circ p \circ \rho(g) = p$. Consequently, for any $u \in U$, $(p \circ \rho(g))(u) = (\rho(g) \circ p)(u) = p(u)$. If $u \in U$, then $\rho(g)(u) \in U$, making U a subrepresentation. Since U is the kernel of a projection of V onto W, we also have that $V = U \oplus W$. $\square$

This theorem allows for the expression of any $v \in V$ as a unique sum $v = u + w$, with $u \in U$, $w \in W$. So we say that the representation V is a direct sum of the representations U and W, or the representation ρ restricted to each subspace. If $\rho^W(g)$ and $\rho^U(g)$ are given in matrix form as R_g^W and R_g^U, then the matrix form of $\rho(g)$ is given by,

$$\begin{pmatrix} R_g^W & 0 \\ 0 & R_g^U \end{pmatrix}$$

An immediate corollary of this statement is that we can decompose representations of finite groups into direct sums of irreducible representations.

Corollary 2.9. *Every representation of a finite group G in V is a direct sum of irreducible representations.*

Proof. We will prove this by induction on $\dim V$. If $\dim V = 0$, then V is the direct sum of empty representations. For $\dim V \geq 1$, if V is irreducible then we are done. Otherwise, we have that from Theorem 2.8, $V = V' \oplus V''$ where V', V'' are representations with dimension less than $\dim V$. By the induction hypothesis we have that V', V'' are the direct sum of irreducible representations, which proves the claim. $\square$

Remark 2.10. It is worth noting the limitations of the previous result. This result does not hold in the general case. For groups of any cardinality or for representations in vector spaces over fields of characteristic that divides the order of G, this result may not hold. We provide two examples below. However, for our purposes, unless explicitly stated, we assume finite groups and vector spaces over the complex field of scalars.

Example 2.11. Let $G = \mathbb{Z}$ and $\rho : G \to \mathrm{GL}_2(\mathbb{R}) \subseteq \mathrm{GL}_2(\mathbb{C})$ defined by, $\rho(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. It can be seen how the subspace spanned by the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is left unchanged and is therefore invariant under G. However, there is no complementary subspace which is also invariant under G.

Example 2.12. Let $G = \mathbb{Z}/p\mathbb{Z}$ and $\rho : G \to \mathrm{GL}_2(\mathbb{F}_p)$ be defined in the same way as example 2.11. If we consider $\rho(1)$, any one dimensional subrepresentation must be spanned by the eigenvalues of $\rho(1)$. There is only one eigenvector which spans a one dimension subspace, making the subrepresentation unique.

We can now see how irreducible representations are the 'building blocks' of representation theory of finite groups, since in many cases we can reduce the study of larger representations to the study of irreducible representations. In order to extend our understanding of irreducible representations, we will now study a very useful method of classifying the irreducible representations of finite groups, which was largely developed by Frobenius in the late nineteenth century.

3. Character Theory

Definition 3.1. For a finite group G, a finite dimensional, complex vector space V, and a linear representation $\rho : G \to V$, a *character* $\chi_\rho : G \to \mathbb{C}$ is defined as the trace of the corresponding matrix of $\rho(g)$ or,

$$\chi_\rho(g) = \mathrm{Tr}(\rho(g))$$

Some properties of characters follow immediately from the definition. First, we have that $\chi_\rho(1) = \dim V$ for any character function. Additionally, recall from linear algebra that the trace of similar matrices are equivalent. Consequently, we have that $\chi_\rho(sgs^{-1}) = \chi_\rho(g)$, for all $g, s \in G$. This also implies that equivalent representations have the same character. Less immediately, we have:

Proposition 3.2. *Let $\rho_1 : G \to \mathrm{GL}(V_1)$ and $\rho_2 : G \to \mathrm{GL}(V_2)$ be two linear representations of G, and let χ_1, χ_2 be their characters, and χ the character of the direct sum representation $V_1 \oplus V_2$. Then we have,*

(i) $\chi_1(g^{-1}) = \overline{\chi_1(g)}$

(ii) $\chi = \chi_1 + \chi_2$.

Proof. For (i), recall that if λ_i is an eigenvalue of $\rho_1(g)$, then it must be a root of unity. Consequently, for $\rho_1(g^{-1})$, we have that each eigenvalue is $\frac{1}{\lambda_i}$. Since the reciprocal of any root of unity is its complex conjugate, we have,

$$\chi_1(g^{-1}) = \Sigma\lambda_i = \Sigma\overline{\lambda_1^{-1}} = \overline{\Sigma\lambda_i^{-1}} = \overline{\chi_1(g)}$$

For (ii), consider the matrix corresponding to the direct sum representation $V_1 \oplus V_2$ for any $g \in G$. We have,

$$R_g = \begin{pmatrix} R_g^{V_1} & 0 \\ 0 & R_g^{V_2} \end{pmatrix}$$

It follows that $\mathrm{Tr}(R_g) = \mathrm{Tr}(R_g^{V_1}) + \mathrm{Tr}(R_g^{V_2})$ and thus, $\chi = \chi_1 + \chi_2$. $\square$

The rest of this section will be dedicated to demonstrating that the characters of irreducible representations form an orthonormal basis of the space of class functions. First, we prove an extremely useful result in character theory and representation theory at large. It is named for Issai Schur who first proved the lemma.

Lemma 3.3 **(Schur).** *Let $\rho_1 : G \to \mathrm{GL}(V_1)$ and $\rho_2 : G \to \mathrm{GL}(V_2)$ be two irreducible representations of G, and let $f : V_1 \to V_2$ be a linear map such that $\rho_2(g) \circ f = f \circ \rho_1(g)$ for all $g \in G$. Then if ρ_1 and ρ_2 are not isomorphic, $f = 0$, otherwise f is a homothety (A scalar multiple of the identity transformation).*

Proof. First take the case where ρ_1 and ρ_2 are not isomorphic. Let $\ker f = W$. Since $0 = \rho_2(g) \circ f(W) = f \circ ((\rho_1(g))(W))$, W is an invariant subspace of V_1. Irreducibility requires that $W = 0$ or $W = V_1$. If $W = V_1$, then $f = 0$ and we are done. Otherwise, we have that $f \circ \rho_1(g)(V_1) = f(V_1) = \rho_2(g) \circ f(V_1)$, making $f(V_1)$ an invariant subspace of V_2. Again, irreducibility means that $f(V_1) = 0$ or $f(V_1) = V_2$. But these are both contradictions as f is not an isomorphism and its kernel was shown to be zero. So we have proven the first case.

Now say ρ_1 and ρ_2 are isomorphic. Since V_1, V_2 are over $\mathbb{C}$, f must have some eigenvalue, λ. Define $f' = f - \lambda I$. Since λ is an eigenvalue, we have that $\ker f' \neq 0$. Since λI will commute with all matrices and $\rho_1 = \rho_2$, we have that $\rho_2(g) \circ f' = f' \circ \rho_1(g)$. By the same argument as case one, since the kernel is non-zero, it must be V_1. Consequently, $f' = 0$ and we have $f = \lambda I$, a homothety of ratio λ. □

This important result can be stated in a more elegant way, but first we must consider a new G-representation.

Definition 3.4. If $\rho_1 : G \to V_1$, $\rho_2 : G \to V_2$ are linear representations of G, we denote $\mathrm{Hom}_G(V_1, V_2)$ as the vector space of homomorphisms between V_1 and V_2 respecting the action of G. This is to say, if $f \in \mathrm{Hom}_G(V_1, V_2)$, then we have that $f : V_1 \to V_2$ is linear and,

$$f \circ \rho_1(g) = \rho_2(g) \circ f$$

for any $g \in G$.

Schur's lemma states that if V_1, V_2 are irreducible representations, and $V_1 \cong V_2$, then $\mathrm{Hom}_G(V_1, V_2) \cong \mathbb{C}$, since it must be a homothety. Otherwise, $\mathrm{Hom}_G(V_1, V_2) \cong \{0\}$. We may then define the representation of $\mathrm{Hom}_G(V_1, V_2)$ to be either the zero representation or the trivial representation. Finally, if χ is the character of $\mathrm{Hom}_G(V_1, V_2)$ and χ_1 of V_1 and χ_2 of V_2, then we have $\chi = \chi_1 \cdot \overline{\chi_2}$. (The proof of this can be found in Proposition 2.1 of source [2])

We will now use Schur's Lemma to demonstrate several important results in character theory, but first we define a valuable inner product for functions on a given group.

Definition 3.5. If ϕ and ψ are complex valued functions on G, define,

$$\langle \phi, \psi \rangle_G = \frac{1}{|G|} \sum_{g \in G} \phi(g)\psi(g^{-1})$$

We have that $\langle \phi, \psi \rangle_G = \overline{\langle \psi, \phi \rangle_G}$ and $\langle \phi, \psi \rangle_G$ is linear in ϕ and conjugate linear in ψ, defining an inner product.

Remark 3.6. Since we are working over $\mathbb{C}$-vector spaces we will refer to this scalar as an inner product. However, this mapping is precisely a non-degenerate Hermitian form.

Notice that when dealing with characters, we have that $\frac{1}{|G|}\sum_{g\in G}\phi(g)\psi(g^{-1}) = \frac{1}{|G|}\sum_{g\in G}\phi(g)\overline{\psi(g)}$.

Theorem 3.7. *With respect to this inner product, the characters of irreducible representations are orthonormal.*

Proof. We have that for some irreducible representations V_1, V_2, with characters χ_1, χ_2,

$$\langle\chi_1, \chi_2\rangle_G = \frac{1}{|G|}\sum_{g\in G}\chi_1(g)\overline{\chi_2(g)}$$

If χ is the character of the representation $V = \mathrm{Hom}_G(V_1, V_2)$, then $\chi = \chi_1 \cdot \overline{\chi_2}$. We then apply Schur's lemma to show that V is either the trivial representation or the zero representation. We may then directly calculate that,

$$\langle\chi_1, \chi_2\rangle_G = \frac{1}{|G|}\sum_{g\in G}\chi(g) = \begin{cases} 1 & V_1 \cong V_2 \\ 0 & V_1 \ncong V_2 \end{cases}$$

which is equivalent to an orthonormal system. □

Corollary 3.8. *Let V be a linear representation of G, with character χ, and suppose V decomposes into a direct sum of irreducible representations,*

$$V = W_1 \oplus \cdots \oplus W_n$$

Then, if W is an irreducible representation of G with character ϕ, the number of W_i isomorphic to W is the product $\langle\phi, \chi\rangle_G$.

Proof. From Proposition 3.2, we have that $\chi = \chi_1 + \cdots + \chi_n$ and thus $\langle\phi, \chi\rangle_G = \langle\phi, \chi_1\rangle_G + \cdots + \langle\phi, \chi_n\rangle_G$. By the preceding theorem this is precisely the number of W_i isomorphic to W. □

This is a very useful way of conceptualizing the inner product defined on characters and is often the most practical way of calculating said inner product. We will see later how thinking of representations as being 'built up' from irreducible representations easily complements this conceptualization.

Corollary 3.9. *Two representations with the same character are isomorphic.*

Proof. As the inner product does not depend on the chosen decomposition, we have that if characters are equivalent they contain each irreducible representation the same number of times, evidently they are isomorphic. □

The previous statement follows from Corollary 2.9, meaning that this holds only for finite dimensional $\mathbb{C}$-representations of finite groups.

Corollary 3.10. *If χ is the character of representation V, then $\langle\chi, \chi\rangle$ is a positive integer, and $\langle\chi, \chi\rangle = 1$ if and only if V is irreducible.*

Proof. Since we have the decomposition into irreducible representations as $V = m_1W_1 \oplus \cdots \oplus m_nW_n$, we have that $\chi = m_1\chi_1 + \cdots + m_n\chi_n$. This implies $\langle \chi, \chi \rangle = \Sigma m_i^2$, by Theorem 3.7. This expression is always a positive integer and equals one if and only if one m_i equals one and the remaining equal zero. □

We now turn to one of the most important results in all of character theory to conclude this section. Its consequences have a significant influence on the way many mathematicians consider representations in practice. Notably, it inspires the creation of the *character table*.

Theorem 3.11. *Let H be the space of class functions on G, namely, the set of functions $f : G \to \mathbb{C}$ such that $f(x) = f(y)$ as long as x and y are conjugate to one another in G. Then the irreducible characters of G form an orthonormal basis of H.*

Proof. Theorem 3.7 demonstrates that irreducible characters are orthonormal, so the number of irreducible characters is bounded by $\dim H$. G is finite, so the number of conjugacy classes is finite. Since every $f \in H$ takes the same value on a conjugacy class, the dimension of H is equal to the number of conjugacy classes. Thus there are a finite number of irreducible characters. It remains to be shown that the characters, $\chi_1, \cdots, \chi_n$, span the space H. It is sufficient to show that if $f \in H$ is orthogonal to $\chi_1, \cdots, \chi_n$, then it is the zero function.

Consider the linear endomorphsim $\phi(f) = \sum_{g \in G} f(g)\rho(g)$, where f is a class function and $\rho : G \to \mathrm{GL}(V)$ is a linear representation of G with character χ. We have that for any $s \in G$,

$$\rho(s^{-1})\phi(f)\rho(s) = \sum_{g \in G} f(g)\rho(s^{-1}gs) = \sum_{g \in G} f(sgs^{-1})\rho(g) = \phi(f).$$

Thus $\phi(f)$ preserves G action and so if ρ is irreducible, then Schur's Lemma applies and we have that $\phi(f)$ is a homothety of ratio λ. Consequently,

$$\dim V \cdot \lambda = \mathrm{Tr}(\phi(f)) = \sum_{g \in G} f(g)\mathrm{Tr}(\rho(g)) = |G| \cdot \langle f, \overline{\chi} \rangle = 0.$$

Hence $\phi(f) = 0$ if V is irreducible. For an arbitrary representation, we have that by direct sum decomposition $\phi(f) = 0$, for all representations, in particular the regular representation R. But if we take the image of the first basis vector of R, e_1, under $\phi(f)$, we have,

$$\phi(f) \cdot e_1 = \sum_{g \in G} f(g)\rho(g) \cdot e_1 = \sum_{g \in G} f(g)e_g = 0$$

Linear independence implies that $f(g) = 0$ for all $g \in G$. Thus we have that $f = 0$, and we have proven the claim.

□

Corollary 3.12. *The number of irreducible representations of G is equal to the number of conjugacy classes of G.*

Proof. If f is a class function on G, and there are k distinct conjugacy classes of G, we have that there are k independent choices for the values of f on G. Consequently, the vector space of class functions on G has dimension k. By Theorem 3.11, the space of class functions has a basis of all the irreducible characters of G, and by Corollary 3.9 the characters of representations are unique. Thus there are k irreducible representations of G. □

4. Character Tables

We saw in the last section how in the case of finite groups, we can reduce the study of representations to the study of their characters. This is a direct result of the fact that if two representations have the same character, then they are isomorphic by Corollary 3.9. In the previous section's conclusion, it was determined that the number of conjugacy classes equals the number of irreducible characters. This motivates the invention of the character table.

We will proceed through examples. Consider the permutation group S_3. There are 3 conjugacy classes and so from Corollary 3.11, we have that there must be three irreducible characters. If we were to create a three by three table with the columns representing conjugacy classes and the rows representing characters, we could then fill in the table with the values of the characters on those specific conjugacy classes. (In this case we use representatives for the conjugacy classes)

	(1)	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

We can observe how the first row clearly represents the trivial character, and indeed any character table will have this trivial row. Similarly, we can see that the first column displays the dimension of the irreducible representation with the given character. Let us consider another group character table in order to notice patterns that might not be immediately evident from the given example.

Take D_8, the dihedral group of order eight. First we find that the group has five conjugacy class, the identity, the rotation with order two, the rotations with order three, a reflection and its product with the reflection of order two, and the remaining two elements. We find a character table as follows,

	$\{e\}$	$\{a^2\}$	$\{a, a^3\}$	$\{x, a^2x\}$	$\{ax, a^3x\}$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

We can observe the same initial facts as with S_3. However, a few commonalities between the two tables can be seen now that are yet to be proven. We first notice that the sum of the squares of the degrees of the characters equals the order of the group. Additionally, we have that if χ is not the trivial character, then the sum of χ times the order of the conjugacy class over all classes equals zero. In order to prove these properties, let us first work to deepen our understanding of a representation we have already seen, the regular representation R.

If we denote the character of R as r, then we have $r(1) = |G|$, and $r(g) = 0$ if $g \neq 1$. The first can be seen because the basis of V is indexed by elements of G, and second because any group element that is not the identity will have no fixed points when considered as a permutation.

We can use this property to decompose the regular representation into a direct sum of irreducible representations.

Proposition 4.1. *The character of the regular representation may be written as the following linear combination of irreducible characters,*

$$r = \sum_{i=1}^{h} n_i \chi_i$$

where h is the dimension of the space of class functions on G and n_i is the dimension of the representation corresponding to χ_i.

Proof. If χ is an irreducible character, then the number of times it appears in the direct sum decomposition of r is $\langle r, \chi \rangle$. We have,

$$\langle r, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} r(g)\chi(g^{-1}) = \chi(1) = n_i.$$

Thus each irreducible character appears in the decomposition with multiplicity n_i. □

We can now prove the previously noticed properties of character tables.

Proposition 4.2. *For a finite group G, we have,*
(i) The degrees n_i of the irreducible representations of G, satisfy $\sum_{i=1}^{h} n_i^2 = |G|$, where h is the dimension of the space of class functions on G.
(ii) If χ is a nontrivial irreducible character of G, R is a set of representatives of conjugacy classes of G, and C_r denotes a conjugacy class for $r \in R$, we have that, $\sum_{i=1}^{i=|R|} \chi(r_i) \cdot |C_{r_i}| = 0$

Proof. For (i), we use Proposition 3.1 and take $|G| = r(1) = \sum_{i=1}^{i=h} n_i \chi_i(1) = \sum_{i=1}^{i=h} n_i^2$.

For (ii), since the trivial representation does not appear in any other irreducible representation,

$$0 = \langle 1, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) = \frac{1}{|G|} \sum_{i=1}^{|R|} \chi(r_i) \cdot |C_{r_i}|$$

□

These two properties alone are often sufficient in determining smaller character tables and are very useful in quick calculations.

5. Abelian Groups

One finds when studying linear representations that much more theory can be developed when concerned solely with abelian groups. This theory can then be applied to general cases when considering abelian subgroups of non-abelian groups. In this section we will develop some of the theory of abelian groups, with the goal of later applying it to the case of general representations.

If G is abelian, then we have that for any $s, g \in G$, $sgs^{-1} = g$. Consequently, each conjugacy class contains only one element.

Proposition 5.1. *G is abelian if and only if all irreducible representations have degree 1.*

Proof. From our knowledge of character theory, we have that the number of distinct irreducible representations of G equals $|G|$. However, Proposition 4.2 says that the sum of the dimensions of irreducible characters must equal $|G|$. This holds if and only if each dimension is 1. □

Corollary 5.2. *Let H be an abelian subgroup of G. The dimension of any irreducible representation of G is at most $\frac{|G|}{|H|}$.*

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation of G. Consider the restriction of ρ to H, ρ^H. Any irreducible subrepresentation W of ρ^H must have dimension 1 by Proposition 5.1. If we consider the subspace spanned by $\rho(g)W$ for all $g \in G$, we get a subrepresentation of V, which must be V, since V is irreducible. But for $g \in G$, $h \in H$, $\rho(gh)W = \rho(g)\rho(h)W = \rho(g)W$. Thus the dimension of V can be at most the number of cosets of H. □

We can immediately apply this fact that to representations which are of interest to us. For instance, any dihedral group contains a cyclic subgroup. Namely, all the rotations about the center point. This abelian group makes up precisely half the elements of the group. Thus no irreducible representation of a dihedral group can have a dimension which exceeds 2.

This proof utilized the fact that restricting the domain of a representation to a subgroup yielded a representation of its own. We study the phenomenon of expanding the domain from a subgroup, or *inducing*, in the next section. There are several interesting results on the interplay between these two connected notions of restriction and induction, some of which will be touched on later. The principle is that, on the inner product we defined previously, the two operations of restricting and inducing are *adjoints* to one another.

6. Induced Representations

The theory of induced representations is exceedingly important in representation theory at large and is used extensively in the applications of linear representations to several mathematical fields.

Definition 6.1. Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation of G, and let ρ^H be the restriction of ρ to a subgroup H. If W is a subrepresentation of ρ^H, denote the representation $\theta : H \to \mathrm{GL}(W)$ as the representation of H in W previously defined. We say that the representation ρ is induced by the representation θ if for some system of representatives R of the left cosets of H in G,

$$V = \bigoplus_{s \in R} \rho(s)W$$

This definition is possible since $\rho(g)W$ depends only on the left cosets of H in G. Indeed, if $h \in H$, $g \in G$, we have $\rho(gh)W = \rho(g)\rho(h)W = \rho(g)W$. Notice that it follows from the definition that we have $\dim V = \dim W \cdot [G : H]$.

The role of examples in internalizing this concept cannot be overstated, so we will provide several examples in the following list.

Example 6.2. Let $\theta : \{1\} \to C^\times$. For any given group G, this representation induces the regular representation of G. This can be seen as each left coset is a group element, so the induced representation is a direct sum of $|G|$ 1-dimensional subspaces. There is a natural basis consisting of one vector from each of these subspaces. G acts on the space by permuting these basis vectors as the regular representation is defined.

Example 6.3. Let $\theta : H \to C^\times$ be the trivial representation. We have that $\rho : G \to \mathrm{GL}(V)$, where there is a basis of V, $\{e_\sigma\}_{\sigma \in \frac{G}{H}}$. ρ as the permutation representation is induced by this θ since e_σ is fixed by any element of G in σ.

Example 6.4. If ρ_1 is induced by θ_1 and ρ_2 is induced by θ_2, then $\rho_1 \oplus \rho_2$ is induced by $\theta_1 \oplus \theta_2$.

Example 6.5. If the representation (V, ρ) is induced by (W, θ), and if W' is a stable subspace of W, then the subspace $V' = \sum_{s \in R} \rho(s)W'$ is stable under G, and (V', ρ) is induced by (W', θ).

Example 6.6. Let $H = \mathbb{Z}/4\mathbb{Z}$ and $G = D_8$. If $\theta : H \to W$ is an irreducible representation such that $\theta(2) = -1$, then the induced representation $\rho : G \to \mathrm{GL}_2(\mathbb{R})$ is the two dimensional irreducible representation of D_8. This special case where an induced representation is irreducible will be studied more in depth later. For dihedral groups in general, all two dimensional irreducible representations are induced from the cyclic subgroup.

We now prove the existence and uniqueness of induced representations.

Lemma 6.7. *Suppose that (V, ρ) is induced by (W, θ). Let $\rho' : G \to \mathrm{GL}(V')$ be a linear representation of G, and let $f : W \to V'$ be a linear map such that $\rho'(t)f(w) = f(\theta(t)w)$ for all $t \in H$, $w \in W$. Then there exists a unique linear map $F : V \to V'$, which extends f and has $F \circ \rho(g) = \rho'(g) \circ F$ for all $g \in G$.*

Proof. Assume there exists some map F which satisfies this criteria. Then for any $v \in V$, we have $v \in \rho(g)W$, for $g \in G$. Consequently, $F(v) = F(\rho(g)\rho(g^{-1})v) = \rho'(g)F(\rho(g^{-1})v) = \rho'(g)f(\rho(g^{-1})v)$. This equation determines F on $\rho(g)W$, for any $g \in G$, and so it determines F on V. Thus we have demonstrated uniqueness. To show existence, define a map F by the equation above. If we replace g with some gh, $h \in H$, then $\rho'(g)\rho'(h)f(\rho(h^{-1})\rho(g^{-1})v) = \rho'(g)f(\theta(h)\theta(h^{-1})\rho(g^{-1})v) = \rho'(g)f(\rho(g^{-1})v)$. Thus the value of $F(v)$ does not depend on the choice of representatives. Additionally, $\rho'(g) \circ F(v) = \rho'(g)\rho'(g')f(\rho(g'^{-1}v) = \rho'(s)\rho'(h)f(\rho(g'^{-1})v)$ $= \rho'(s)f(\theta'(h)\rho(g'^{-1})v) = \rho'(s)f(\rho(s^{-1}g)v) = F(\rho(g)v)$, for $h \in H$. This demonstrates existence, and so we have proven the claim. $\square$

Theorem 6.8. *Let (W, θ) be a linear representation of $H \leq G$. There exists a unique representation (V, ρ) of G which is induced by (W, θ)*

Proof. Without loss of generality, we assume that θ is irreducible by Example 6.4. It follows from Proposition 4.1 that θ is isomorphic to a subrepresentation of the regular representation of H. Since this can be induced to the regular representation of G, we see from Example 6.5 that θ itself may be induced, and existence has been shown.

If (ρ, V) and (ρ', V') are induced by the same representation, we can apply Lemma 6.7 to see that there is a linear map with an image of V'. Let $f : W \to V'$ be the identity. The image of F contains all $\rho'(g)W$, which means that the image of F is indeed V'. Since V and V' have that same dimension, we have uniqueness. □

The existence and uniqueness of induced representations allows us to specifically denote the representation of G induced by the representation of H in W as $\mathrm{Ind}_H^G(W)$.

We now turn to the character of an induced representation. Since $\rho = \mathrm{Ind}_H^G(W)$ is uniquely determined by (W, θ), we are able to find an explicit formula for χ_ρ from χ_θ. Indeed we have,

Theorem 6.9. *Let $\rho = \mathrm{Ind}_H^G(W)$, where ρ is induced from $\theta : H \to W$. For each $g \in G$, we have,*

$$\chi_\rho(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} \chi_\theta(x^{-1}gx)$$

Proof. First see that for any $g \in G$, $\rho(g) : \sigma W \mapsto g\sigma W$, for $\sigma \in R$, a system of representative of G/H. If we want to compute the trace of $\rho(g)$, then we only have to compute the trace of $\rho(g)$ restricted to σW such that $g\sigma \in \sigma H$. Additionally, $g\sigma W = \sigma(\sigma^{-1}g\sigma)W$, meaning $\mathrm{Tr}(\rho(g)\big|_{\sigma W}) = \chi_\theta(\sigma^{-1}g\sigma)$. So we have,

$$\mathrm{Tr}(\rho(g)) = \sum_{\substack{g\sigma \in \sigma H \\ \sigma \in R}} \mathrm{Tr}(\rho(g)\big|_{\sigma W}) = \sum_{\substack{g\sigma \in \sigma H \\ \sigma \in R}} \chi_\theta(\sigma^{-1}g\sigma) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} \chi_\theta(x^{-1}gx).$$

□

We will see induced representations appear later and study more complex properties of them in the final sections, but first we consider several useful examples.

7. Examples

We have already seen a few examples of representations and the employment of character tables to demonstrate their properties in section three. This section will be devoted to a more in depth exploration of various kinds of representations and their character tables.

7.1. **The Cyclic Group C_n.** From our study of abelian groups, we know that all the irreducible representations of the cyclic group C_n are degree 1. Denote the character of such a representation χ. We have that $\chi(g) = z$ is a complex number and $\chi(g^k) = z^k$. Additionally, since $g^n = 1$, we have that $z^n = 1$. Consequently, any z^k is some n'th root of unity. Since there are n possible choices for such, we can determine our character table for any such cyclic group by defining,

$$\chi_j(g^k) = e^{\frac{2\pi ijk}{n}}.$$

This can be seen for example in the character table for C_3,

	$\{1\}$	$\{a\}$	$\{a^2\}$
χ_0	1	1	1
χ_1	1	ζ_3	ζ_3^2
χ_2	1	ζ_3^2	ζ_3

7.2. **The Dihedral Group of order 2n D_{2n}.** This is the group G of rotations and reflections which preserve a regular polygon of n vertices. First, notice that half the group elements form a subgroup isomorphic to C_n. Let r be a generator of this subgroup. Additionally, since no reflection s is in this subgroup, the coset sR is the remaining elements of the group. Thus any element can be written in the form r^k or sr^k. We will now proceed by cases to characterize all irreducible representations of groups of this kind.

Consider the case of n being even. We will begin by constructing all of the possible degree one representations. This can be done determining all the possible kernels of a representation into $\mathbb{C}$, and then letting the remaining values be set to -1, the only other potential character value. The character table that remains has four possible representations as is explicitly written as,

	r^k	sr^k
ψ_1	1	1
ψ_2	1	-1
ψ_3	$(-1)^k$	$(-1)^k$
ψ_4	$(-1)^k$	$(-1)^{k+1}$

Notice that we have previously defined all irreducible representations of a cyclic group. We can now consider the representation induced by an irreducible representation of the cyclic subgroup. Let the irreducible characters of H be χ. For any j, we will have that $\psi_j = \mathrm{Ind}_H^G \chi_j$ is a degree two character. Additionally, in the cases where $j = 0, \frac{n}{2}, n$ we have that $\psi_j = \psi_1 + \psi_2$ or $\psi_j = \psi_3 + \psi_4$. Let us now consider the alternate case where $\rho_j = \mathrm{Ind}_H^G \phi_j$ and ϕ_j is the representation of the character χ_j. The value of ϕ is known, so we may construct the representation as,

$$\rho_j(r^k) = \begin{pmatrix} \zeta_n^{jk} & 0 \\ 0 & \zeta_n^{-jk} \end{pmatrix}, \quad \rho_j(sr^k) = \begin{pmatrix} 0 & \zeta_n^{-jk} \\ \zeta_n^{jk} & 0 \end{pmatrix}$$

We can see that $\rho_{n-j} \cong \rho_j$, and that excluding the choices of j previously analyzed, the representation is irreducible. This follows because for $\rho_j(r)$, the only stable subspaces are the coordinate axes, which are not stable under $\rho_j(s)$. We can calculate the character ψ_j to be the sum of roots of unity, meaning the imaginary portion will cancel so that,

$$\psi_j(r^k) = \zeta_n^{jk} + \zeta_n^{-jk} = 2\cos(\frac{2\pi jk}{n})$$

$$\psi_j(sr^k) = 0.$$

We can now sum the squares of the degrees of our known irreducible representations to obtain,

$$4 \cdot 1 + (\frac{n}{2} - 1) \cdot 4 = 2n$$

Consequently, by Proposition 4.2 there exist no more irreducible representations of D_{2n} with even n. Intuitively, one can see how increasing n by two increases the order of the group by four and the order of the subgroup by two. We have that there is one more induced representation up to isomorphism, meaning that the sum of square of the dimensions increases by four, exhausting all possible new irreducible representations.

Now we consider the case of n being odd. Similarly as before we can find all the possible irreducible representations of degree 1. In the case of n odd this becomes simpler as we only have one non-trivial normal subgroup. The table is as follows,

	r^k	sr^k
ψ_1	1	1
ψ_2	1	-1

Using the same induced representations as the case of n even, we can see that the same criterion for irreducibility holds. However, rather than having $\frac{n}{2} - 1$ for our number of representations, since n is odd, we will have $\frac{1}{2}(n-1)$ irreducible representations. Summing the squares of the degrees we get the same result,

$$2 \cdot 1 + \frac{1}{2}(n-1) \cdot 4 = 2n$$

Thus we can classify the irreducible representations of D_{2n} into two sets. The degree one representations, and the representations induced from the subgroup isomorphic to the cyclic group of order n.

7.3. **The Symmetric Group S_4.** For this group's character table, since conjugacy classes are determined by cycle type, we will denote a conjugacy class just by one of its representatives (i.e. the class of two disjoint transpositions as (12)(34)). Starting with degree one representations, we immediately have the trivial and sign character. To categorize the remaining irreducible representations, we will make use of a short exact sequence,

$$1 \longrightarrow K_4 \longrightarrow S_4 \xrightarrow{\pi} S_3 \longrightarrow 1$$

where K_4 is the Klein 4-group. Since π is a surjective mapping, we have that for any irreducible character χ of of S_3, then $\chi \circ \pi$ is an irreducible character of S_4. Notice that for the S_3 character table previously defined, $\chi_1 \circ \pi$ and $\chi_2 \circ \pi$ are precisely the trivial and sign representations. However, $\chi_3 \circ \pi$ is a new degree 2 representation of S_4. We have the character table now as,

	(1)	(12)	(123)	(12)(34)	(1234)
1	1	1	1	1	1
χ_{sign}	1	-1	1	1	-1
$\chi_3 \circ \pi$	2	0	-1	2	0

To determine the final two representations, we will consider the degree four permutation representation. By the same logic as in Example 2.5, we can see how the trivial representation is a subrepresentation. Let us say then that $\chi_{perm} = 1 \oplus \chi_{st}$, where χ_{st} is called the *standard representation*.

Denote the dimension three invariant subspace of the permutation representation as V. To show its irreducibility we first consider which vectors are elements of this subspace. If $v = \alpha_1 v_1 + \cdots + \alpha_4 v_4$, then the subspace isomorphic to the trivial representation has equal coefficients. Without loss of generality, there exists some

$v \in V$ such that $\alpha_1 \neq \alpha_2$. If $g = (12)$, then $v - \rho(g)v = w$ is in V, and has distinct nonzero α_1, α_2, but zeros for the other coefficients. If we repeatedly apply $\rho((1234))$ to w, we get 3 linearly independent vectors in V. Thus, the subspace generated by v under the action of G is V, and there are no invariant subspaces. The same proof can be applied to show that the general case, the standard representation of S_n, is irreducible.

We can now easily calculate the character $\chi_{st} = \chi_{perm} - 1$, since χ_{perm} is the number of fixed points. Consequently, it can be determined that the final irreducible representation must be of degree 3. This final representation is obtained by taking the tensor product of the sign character with the standard, which is referred to as *twisting* the standard representation by the sign character. The resulting character is simply the product of the two characters. Indeed, it can be checked that $\chi_{sign} \otimes \chi_{st}$ is irreducible. Thus the following character table is obtained,

	(1)	(12)	(123)	(12)(34)	(1234)
1	1	1	1	1	1
χ_{sign}	1	-1	1	1	-1
$\chi_3 \circ \pi$	2	0	-1	2	0
χ_{st}	3	1	0	-1	-1
$\chi_{sign} \otimes \chi_{st}$	3	-1	0	-1	1

7.4. **The Alternating Group A_4.** We will now classify the irreducible representations of the group of even permutations of 4 elements. For this group we will denote each class by a representative as before. First, we will begin by classifying the degree 1 irreducible representations.

There is the trivial representation, but for the others we will use a short exact sequence as before.

$$1 \longrightarrow K_4 \longrightarrow A_4 \xrightarrow{\pi} \mathbb{Z}/3\mathbb{Z} \longrightarrow 1$$

Since π is a surjective map, if χ is an irreducible character of $\mathbb{Z}/3\mathbb{Z}$, $\chi \circ \pi$ is an irreducible character of A_4. We have already categorized the irreducible representations of cyclic groups, so we can easily determine the resulting characters. Counting the sum of dimensions, we know there must exist exactly one more irreducible representation and we know it must be of degree 3. We have seen with S_4 that the three dimensional standard representation is irreducible, and the same logic applies for the case of A_4. We thus find the character table of A_4 to be,

	(1)	(12)(34)	(123)	(132)
1	1	1	1	1
$\chi_1 \circ \pi$	1	1	ζ_3	ζ_3^2
$\chi_2 \circ \pi$	1	1	ζ_3^2	ζ_3
χ_{st}	3	-1	0	0

8. Frobenius Reciprocity

Keeping with our previous notation on induced representations, where $H \leq G$, if f is a function on G, let $\mathrm{Res}f$ be the restriction of f to H. Recall that in Section

6 we derived a formula for the character of an induced representation. Now we extend this formula to class functions in general.

Definition 8.1. If f is a class function on H, then we define a new function f' on G, which is said to be induced by f as,

$$f'(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} f(x^{-1}gx)$$

The function f' is denoted $\mathrm{Ind}_H^G(f)$, and when there is no possible ambiguity, simply $\mathrm{Ind}(f)$.

In the previous section we proved that if f is a character of H, then f' is a character of G. Since class functions are the linear combinations of irreducible characters, we have that f' in general is also a class function of G.

Lemma 8.2. *For f' defined in the previous definition and R a system of representatives of the left cosets of H in G, we have,*

$$f'(g) = \sum_{\substack{r \in R \\ r^{-1}gr \in H}} f(r^{-1}gr)$$

Proof. This follows directly from the fact that f is constant on conjugacy classes, so for r and x in the same coset, $f(rgr^{-1}) = f(xgx^{-1})$. Additionally, each coset has precisely $|H|$ elements, so the multiplicity changes by $|H|$. □

Theorem 8.3 (Frobenius Reciprocity). *Let ψ be a class function on H and ϕ a class function on G. Then we have,*

$$\langle \psi, \mathrm{Res}\phi \rangle_H = \langle \mathrm{Ind}\psi, \phi \rangle_G$$

Proof. From class functions being linear combinations of character functions, we may assume without loss of generality that ψ,ϕ are characters on their respective groups. We may then directly prove the claim using constancy on conjugacy classes and Lemma 8.2,

$$\begin{aligned}
\langle \mathrm{Ind}\psi, \phi \rangle_G &= \frac{1}{|G|} \sum_{g \in G} \mathrm{Ind}\psi(g)\phi(g^{-1}) = \frac{1}{|G|} \sum_{g \in G} \sum_{\substack{r \in R \\ r^{-1}gr \in H}} \psi(r^{-1}gr)\phi(g^{-1}) \\
&= \frac{1}{|G|} \sum_{\substack{r \in R \\ h \in H}} \psi(h)\phi(h^{-1}) = \frac{1}{|G|} \cdot [G : H] \sum_{h \in H} \psi(h)\phi(h^{-1}) \\
&= \frac{1}{|H|} \sum_{h \in H} \psi(h)\phi(h^{-1}) = \langle \psi, \mathrm{Res}\phi \rangle_H .
\end{aligned}$$

□

Frobenius reciprocity can be stated in several ways. Recalling linear algebra, we can see how the theorem implies Ind_H^G and Res_H^G are *Hermitian Adjoint* operators. It could also be stated in terms of the space of H and G homomorphisms defined in Section 3.

Corollary 8.4. *For a representation V of G induced by a representation W of H, we have*

$$\mathrm{Hom}_G(\mathrm{Ind}W, V) \cong \mathrm{Hom}_H(W, \mathrm{Res}V)$$

Proof. By decomposing any map in this space into its direct sum components, there is a bijection so that in general we have,

$$\mathrm{Hom}_X(V_1 \oplus V_2, V) \cong \mathrm{Hom}_X(V_1, V) \oplus \mathrm{Hom}_X(V_2, V)$$

Thus we can decompose our space into a direct sum of its irreducible representations.

$$\mathrm{Hom}_G(\mathrm{Ind}W, V) \cong \mathrm{Hom}_G(\mathrm{Ind}W_1, V_1) \oplus \cdots \oplus \mathrm{Hom}_G(\mathrm{Ind}W_i, V_j)$$

By Schur's Lemma, for any of these constituent parts we have that either

$$\mathrm{Hom}_G(\mathrm{Ind}W_i, V_j) \cong \{0\}$$
$$\mathrm{Hom}_G(\mathrm{Ind}W_i, V_j) \cong \mathbb{C}$$

Thus we have,

$$\mathrm{Hom}_G(\mathrm{Ind}W, V) \cong \mathbb{C}^n$$

where $n = \langle \mathrm{Ind}\chi_\theta, \chi_\rho \rangle_G$ (The number of isomorphic subrepresentations). Theorem 8.3 states $\langle \mathrm{Ind}\chi_\theta, \chi_\rho \rangle_G = \langle \chi_\theta, \mathrm{Res}\chi_\rho \rangle_H$, so

$$\mathrm{Hom}_G(\mathrm{Ind}W, V) \cong \mathbb{C}^n \cong \mathrm{Hom}_H(W, \mathrm{Res}V).$$

□

9. Mackey's Irreducibility Criterion

This section focuses on a useful result of Frobenius Reciprocity. It answers the question of when an induced representation is irreducible. We begin with a lemma describing the effect of inducing from a subgroup and then restricting back to it.

Lemma 9.1. *Let $H \leq G$ and $\rho : H \to \mathrm{GL}(W)$ and let R be a system of representatives of $H\backslash G/H$. Then we have,*

$$\mathrm{Res}_H \mathrm{Ind}_H^G W \cong \bigoplus_{s \in R} \mathrm{Ind}_{H_s}^H W_s$$

where $H_s = sHs^{-1} \cap H$, a subgroup, and W_s is the representation given by $\rho^s : H_s \to \mathrm{GL}(W)$ by $\rho^s(x) = \rho(sxs^{-1})$.

Proof. Denote $V(s)$ the subspace generated by $\rho(s)W$, for $s \in R$. We have, that $\mathrm{Ind}_H^G W$ is a direct sum of the $V(s)$ over $s \in R$, since this is equivalent to a direct sum over the representative of the single cosets. Additionally, for any $h \in H$, $\rho(h)V(s) = V(s)$, so we have that $V(s)$ is a subrepresentation of $\mathrm{Res}_H \mathrm{Ind}_H^G W$.

We now must show that $V(s) \cong \mathrm{Ind}_{H_s}^H(W_s)$. Since $x \in H_s$ if and only if $\rho(xs)W = \rho(s)W$, we have that $V(s) = \oplus_{x \in R'} \rho(x)W$, for R' a system of representatives of H/H_s. Thus, $V(s) \cong \mathrm{Ind}_{H_s}^H(\rho(s)W)$. But evidently $W_s \cong \rho(s)W$, so we have that $V(s) \cong \mathrm{Ind}_{H_s}^H W_s$ and we have proven the claim. □

Theorem 9.2 (**Mackey's Criterion**). *We have that* $V = \mathrm{Ind}_H^G W$ *is irreducible if and only if* W *is irreducible and for each* $s \in G \backslash H$*,* $\langle \rho^s, \mathrm{Res}_{H_s}(\rho) \rangle_{H_s} = 0$.

Proof. We have that V is irreducible if and only if $\langle V, V \rangle_G = 1$. Applying Frobenius reciprocity and Lemma 9.1 we have,

$$\langle V, V \rangle_G = \langle W, \mathrm{Res}_H V \rangle_H = \bigoplus_{s \in R} \langle W, \mathrm{Ind}_{H_s}^H(\rho^s) \rangle_H = \bigoplus_{s \in R} \langle \mathrm{Res}_{H_s} W, \rho^s \rangle_H$$

for $s = 1$, we have that $\langle \mathrm{Res}_{H_s} W, \rho^s \rangle_H = \langle W, W \rangle_H \geq 1$. For V to be irreducible, W must be irreducible. Additionally, for each $s \notin H$, we must have $\langle \rho^s, \mathrm{Res}_{H_s}(\rho) \rangle_{H_s} = 0$. This could alternatively be phrased as for each $s \notin H$, ρ^s and $\mathrm{Res}_{H_s}(\rho)$ are *disjoint*. □

This result concludes the section and the paper. For further reading, see the bibliography.

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