TEICHMÜLLER THEORY AND HYPERBOLIZATION OF THREE-MANIFOLDS

SIWEI LIANG

Abstract. We first give an introduction to Teichmüller theory. Then we develop other essential tools. Finally, we present McMullen’s proof of Thurston’s Hyperbolization Theorem for Haken 3-manifolds that do not fiber over the circle, with a few topological arguments omitted.

Contents

1. Introduction 2
2. Quasiconformal Maps 3
2.1. Differentiable quasiconformal maps 3
2.2. Geometric function theory 4
2.3. General definitions and properties 6
3. Teichmüller Theory 7
3.1. Teichmüller spaces 7
3.2. Quadratic differentials 9
3.3. Teichmüller extremal maps 11
3.4. Analytic structure of Teichmüller spaces 12
3.5. The infinitesimal Teichmüller metric and cometric 15
3.6. Quasiconformal deformation of Kleinian groups 18
4. Geometric Limits of Quadratic Differentials 19
4.1. Compact spaces of quadratic differentials 19
4.2. Mass distribution of quadratic differentials 21
5. The Theta Operator 23
5.1. A dichotomy on contraction 23
5.2. Amenability: graphs, groups, and coverings 24
5.3. The amenable case 25
5.4. The nonamenable case 27
5.5. Refinements in the geometric case 30
6. The Hyperbolization Theorem 32
6.1. The gluing problem 32
6.2. From topology to pointwise contraction 34
6.3. Inefficiency over the thin part 37
6.4. Uniform contraction for acylindrical manifolds 42
6.5. Iteration towards toroidality 44
Appendix A. Fuchsian and Kleinian Manifolds 46
Acknowledgements 48
References 48

Date: November 25, 2022.
Please send any questions and corrections to mathsway683@gmail.com.
1. Introduction

Classically, Teichmüller theory refers to the analytic approach to Teichmüller spaces, a particular class of moduli space of surfaces. Over more than eighty years after its birth, the theory has enjoyed vast development with derivative theories in different branches of mathematics.

In this paper, we first introduce what modern mathematics regards as classical Teichmüller theory, and then demonstrate how Teichmüller theory paves the way for Thurston’s Hyperbolization Theorem, a remarkable theorem relating 3-manifold topology to hyperbolic geometry. The general Hyperbolization Theorem states that a compact, irreducible, atoroidal 3-manifold always admits a complete hyperbolic metric of finite volume. We will only prove this theorem for Haken 3-manifolds that do not fiber over the circle.

The Hyperbolization Theorem is understood as the hyperbolic part of the Geometrization Theorem: first a compact 3-manifold is topologically decomposed into pieces along certain embedded spheres, real projective planes, tori, and Klein bottles via prime decomposition and geometric decomposition respectively, and then the Geometrization Theorem states that every piece admits one of the eight geometries depending on its topology. Each piece is compact and irreducible in its own right, while the extra topological condition for being hyperbolic is atoroidality. Haken 3-manifolds allow further topological decomposition into simple Kleinian manifolds, and that is where Teichmüller theory comes in. However, without the Haken condition, all the known proofs resort to hard geometric analysis, but that is a whole other story.

The paper is organized as follows. In Section 2, we introduce quasiconformal maps as the analytic foundation. In Section 3, we proceed to study Teichmüller spaces from the classical viewpoint, following [Ahl, McM5, Hub], and end with the quasiconformal deformation of Kleinian groups, which links Teichmüller theory to hyperbolic geometry. In Section 4, we sketch the theory of geometric limits. In Section 5, we use geometric limits to derive the essential results on the Theta operator, following [McM1]. In Section 6, we first introduce basic 3-manifold topology for the Hyperbolization Theorem, and reduce the theorem to a fixed point problem on the Teichmüller space, using the tools in Section 3. Then through the topology of Kleinian manifolds, we incorporate the previous work on geometric limits and the contraction of Theta operators to solve the fixed point problem, following [McM2, Ota1]. In Appendix A, we list some preliminaries on hyperbolic geometry.

The proof we present, due to McMullen [McM1, McM2, McM3], differs from Thurston’s original one [Mor1, Kap] only in the fixed point problem part. Besides the original papers, a more complete account of McMullen’s proof is given in Otal’s survey [Ota1] which restricts to closed surfaces and compact 3-manifolds. We will address the subtlety arising with punctures and cusps. Some simplifications are made, while a few topological arguments are omitted.

To understand the paper, the reader should be familiar with the basic theory of complex analysis, measure theory, and manifolds. Familiarity with basic notions of hyperbolic geometry is recommended as well.
2. Quasiconformal Maps

Notations 2.1. Write $\mathbb{C}$ for the complex plane, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ for the Riemann sphere, $\mathbb{H}$ for the upper-half plane or the hyperbolic plane, $\mathbb{R} = \partial \mathbb{H}$ the extended real axis, $\mathbb{D}$ for the open unit disk or the Poincaré disk, and $\mathbb{C}^*, \mathbb{D}^*$ for being punctured at the origin. All maps between Riemann surfaces are assumed to be continuous and orientation-preserving unless otherwise stated.

2.1. Differentiable quasiconformal maps. We begin with analysis of differentiable functions. Let $f : \Omega \to \Omega'$ be a $C^1$-homeomorphism between regions in $\mathbb{C}$. Write $w = f(z)$ in coordinate forms. Then the Jacobian $J_f = |f_z|^2 - |f_\bar{z}|^2 > 0$, so $|f_z| < |f_{\bar{z}}|$ and

$$|(f_z - |f_z|)|dz| \leq |dw| \leq (|f_z| + |f_{\bar{z}}|)|dz|,$$

where both limits are attainable. Therefore, on each tangent space, say at $z \in \Omega$, $f$ maps circles about the origin to ellipses with the same eccentricity

$$D_f := \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|} \geq 1,$$

called the dilation of $f$ at $z$. We are also interested in the complex dilation defined by $\mu_f := f_{\bar{z}}/f_z$. Clearly, $|\mu_f| < 1$ and $D_f = (1 + |\mu_f|)/(1 - |\mu_f|)$. This $\mu_f$ is related to the Beltrami differential $\mu := \mu_f \, dz/d\bar{z}$ given by $f$, which is a differential of type $(-1, 1)$ on $\Omega$ and independent of the conformal parametrization (see Remark 2.19).

Say $f$ is $K$-quasiconformal for some constant $K \in [1, \infty)$, if $D_f \leq K$ uniformly; equivalently, $|\mu_f| \leq k$ uniformly, where $k := (K - 1)/(K + 1)$. Say $f$ is quasiconformal if it is $K$-quasiconformal for some $K$.

Example 2.2 (Composition and Teichmüller distance). It is natural to consider how quasiconformal maps behave under composition. Let $f, g$ be quasiconformal maps. Set $\xi = f(z), h = g \circ f$. Then

$$\mu_{h \circ f} \circ f = \mu_g \circ f = \frac{f_z \mu_{g \circ f} - \mu_f}{f_z \mu_f - \mu_{g \circ f}} = \frac{f_z \mu_h - \mu_f}{f_z \mu_f - \mu_{g \circ f}}.$$

In particular, we have

$$|\mu_f| = |\mu_f| |f^{-1}|;$$

$$\mu_{g \circ f} = \mu_f,$$ if $g$ is conformal;

$$\mu_g \circ f = \left(\frac{f}{|f|}\right)^2 \mu_{g \circ f},$$ if $f$ is conformal.

More importantly, by passing to absolute values in (2.3),

$$\log (D_{h \circ f \circ f} \circ f) = d_\mathbb{D}(\mu_h, \mu_f),$$

where $d_\mathbb{D}$ is the Poincaré metric on $\mathbb{D}$. In particular, we conclude that the composition of a $K_1$-quasiconformal and a $K_2$-quasiconformal map is $K_1 K_2$-quasiconformal. It is then natural to define a distance between two quasiconformal maps $f$ and $h$ as $\sup_z d(\mu_h, \mu_f)$. This is called the Teichmüller distance.
2.2. Geometric function theory. Geometric function theory motivates general interpretation of quasiconformal maps. The idea of “relaxed” conformal maps originated from Grötzsch’s problem of finding the most nearly conformal map among all homeomorphisms from a certain rectangle to another.

Example 2.5 (Grötzsch’s problem). Let \( R, R' \) be two rectangles with sides \( a, b \) and \( a', b' \) respectively, and \( f : R \to R' \) be a \( C^1 \)-homeomorphism that maps \( a \)-sides into \( a' \)-sides and \( b \)-sides into \( b' \)-sides. To minimize \( \sup_z D_f \), we have

\[
\iint_R D_f \, dx \, dy = \frac{\text{Area}(R')}{a'b'} \iint_R \frac{|f_z| + |f_{ar{z}}|}{|f_z| - |f_{ar{z}}|} \, dx \, dy
\]

\[
= \frac{1}{a'b'} \iint_R (|f_z|^2 - |f_{ar{z}}|^2) \, dx \, dy \iint_R \frac{|f_z| + |f_{ar{z}}|}{|f_z| - |f_{ar{z}}|} \, dx \, dy
\]

\[
\geq \frac{1}{a'b'} \left[ \iint_R (|f_z| + |f_{ar{z}}|) \, dx \, dy \right]^2
\]

\[
\geq \frac{1}{a'b'} \left[ \int_0^b \left( \int_0^a |df(x + iy)| \right) \, dy \right]^2 \geq \frac{a'b'^2}{b'},
\]

i.e.

\[(2.6) \quad \frac{1}{ab} \iint_R D_f \, dx \, dy \geq \frac{a'}{b'} : \frac{a}{b}.
\]

We may assume \( a/b \leq a'/b' \) from the start. The moduli of the rectangles \( R, R' \) (with an orientation) are defined by \( m = a/b, m' = a'/b' \) respectively. The modulus is a conformal invariant. By (2.6), the affine map

\[ f(z) = \frac{1}{2} \left( \frac{a}{a'} + \frac{b}{b'} \right) z + \frac{1}{2} \left( \frac{a}{a'} - \frac{b}{b'} \right) \bar{z} \]

attains both the least maximal and the least average dilation. We conclude that there exists a \( K \)-quasiconformal diffeomorphism from \( R \) to \( R' \) (under boundary correspondence) if and only if

\[ \frac{1}{K} \leq \frac{m'}{m} \leq K. \]

The result motivates a geometric definition of quasiconformality by constraining the deformation on quadrilaterals.

The concept of moduli fits into a general context.

Definition 2.7. Let \( X \) be a Riemann surface. By a Borel metric \( \rho \) we mean a metric with local form \( \rho = \rho(z)|dz| \) for some Borel measurable function \( \rho(z) \geq 0 \). Denote \( \rho \)-length of a rectifiable path \( \gamma \) and \( \rho \)-area of \( X \) respectively by

\[ L_\rho(\gamma) := \int_\gamma \rho, \quad A_\rho(X) := \int_X \rho^2 = \iint_X \rho(z)^2 \, dx \, dy. \]

Then for a family \( \Gamma \) of paths\(^1\) on \( X \), set \( L_\rho(\Gamma) := \inf_{\gamma \in \Gamma} L_\rho(\gamma) \) and define the extremal length of \( \Gamma \) as

\[ \lambda(\Gamma) := \sup_\rho \frac{L_\rho(\Gamma)^2}{A_\rho(X)}. \]

\(^1\)We henceforth only choose the rectifiable paths.
where $\rho$ ranges over all Borel metrics on $X$ with finite positive $\rho$-area. Clearly $\lambda(\Gamma)$ is a conformal invariant for $(X, \Gamma)$. Say $\rho$ is extremal, if it realizes this supremum.

**Theorem 2.8** (Beurling’s criterion). If the measure $\rho^2$ on $X$ lies in the closed convex hull of the measures $\{\rho_\gamma : \gamma \in \Gamma, L_\rho(\gamma) = L_\rho(\Gamma)\}$, then $\rho$ is extremal for $\Gamma$.

**Proof.** Denote $\langle \cdot, \cdot \rangle$ as the pairing of functions and measures. For any other Borel metric $\alpha$, normalize both $\alpha, \rho$ with total area 1. Then for any $\gamma \in \Gamma$, we have

$$L_\alpha(\gamma) \leq \int_\gamma \alpha = \langle \alpha/\rho, \rho |_\gamma \rangle.$$

Since the probability measure $\rho^2$ lies in the closed convex hull of measures of the form $(\rho |_\gamma)/L_\rho(\Gamma)$, we have

$$2L_\alpha(\Gamma) \leq \langle \alpha/\rho, \rho^2 \rangle \leq \left( \int_X \alpha^2 \int_X \rho^2 \right)^{1/2} = 1,$$

by the Cauchy-Schwarz inequality. Hence, $\rho$ is extremal. $\square$

**Example 2.9.** A quadrilateral $Q \subset \mathbb{C}$ is a Jordan domain with 4 marked points on $\partial Q$ and a distinguished pair of opposite sides. Denote $Q^*$ as the same quadrilateral with the other pair of opposite sides distinguished. Note that $Q$ is conformally equivalent to a unique rectangle $R(a) := [0, a] \times [0, 1]$ with the unit-length sides distinguished. By definition, $a = \text{mod}(R(a)) = \text{mod}(Q) = 1/\text{mod}(Q^*)$.

Denote $\Gamma(Q)$ as the family of paths joining the distinguished sides in $Q$. Let $\rho$ be the Euclidean metric on $R(a)$. Since the $\rho$-geodesics in $\Gamma(R(a))$ are exactly the horizontal segments, by Beurling’s criterion $\rho$ is extremal and hence $\lambda(\Gamma(Q)) = \text{mod}(Q)$.

**Example 2.10.** A doubly connected region, or a Riemann surface $X$ with cyclic fundamental group, is conformally equivalent to one of the following: $\mathbb{C}^*, \mathbb{D}^*$, or an annulus $A(r) := \{ z : 1 < |z| < r \}$ for some unique $r > 1$. Their moduli are defined by $\text{mod}(\mathbb{C}^*) = \text{mod}(\mathbb{D}^*) = \infty$ and $\text{mod}(A(r)) = \log(r)/(2\pi)$. Let $\Gamma$ be the family of paths joining the two components of $\partial A(r)$ in $A(r)$. Since the radial ones are geodesic for the cylindrical metric $\rho = |dz|/|z|$, by Beurling’s criterion $\rho$ is extremal, and thus

$$\lambda(\Gamma) = \frac{1}{2\pi} \log(r) = \text{mod}(A(r)).$$

Let $\Gamma^*$ be the family of simple essential loops in $A(r)$. Then

$$\lambda(\Gamma^*) = \frac{1}{\lambda(\Gamma)} = \frac{1}{\text{mod}(A(r))}.$$

**Example 2.11.** Given a complex torus $X = \mathbb{C}/(\mathbb{Z} \oplus \tau \mathbb{Z})$ where $\tau \in \mathbb{H}$, denote $\Gamma[1,0]$ as the family of loops on $X$ in the homotopy class $[1,0]$. Consider the flat metric $\rho = |dz|$. Since the horizontal loops are $\rho$-geodesics of unit length, by Beurling’s criterion $\rho$ is extremal and we find that

$$\lambda(\Gamma[1,0]) = \frac{1}{3\tau},$$

$^2$The calculation goes through even if $\alpha/\rho = \infty$ somewhere.

$^3$In Example 2.5, this distinction arises as the orientation of a rectangle.
where $\Im \tau$ denotes the imaginary part of the complex number $\tau$.

Let $A$ be an annulus in $X$ that lies in the homotopy class $[1, 0]$. Let $\Gamma^*$ be the family of essential loops in $A$. Then $\lambda(\Gamma^*) = 1/\text{mod}(A)$ by Example 2.10. Moreover, as $\Gamma^* \subset \Gamma[1, 0]$, we have $\lambda(\Gamma[1, 0]) \leq \lambda(\Gamma^*)$ by definition of the extremal length. Hence, $\text{mod}(A) \leq \Im \tau$.

The extremal length argument turns out very useful. We will encounter it again in Section 3 and Section 6. We will also need the following basic fact.

**Proposition 2.12.** Let $\Gamma_1, \Gamma_2$ be two families of paths. If $\Gamma_1, \Gamma_2$ are supported in disjoint measurable sets $E_1, E_2$ respectively, then $\lambda(\Gamma_1 \cup \Gamma_2)^{-1} \geq \lambda(\Gamma_1)^{-1} + \lambda(\Gamma_2)^{-1}$.

**Proof.** Denote $\Gamma = \Gamma_1 \cup \Gamma_2$, and $X$ as the ambient space. WLOG, $\lambda(\Gamma) \neq 0$. Consider a Borel metric $\rho$ with $L_\rho(\Gamma) > 0$, $\rho|_{E_1} = \rho_1$, $\rho|_{E_2} = \rho_2$, and $\rho = 0$ outside. Then $L_\rho(\Gamma) \leq L_{\rho_1}(\Gamma_1)$, $L_\rho(\Gamma) \leq L_{\rho_2}(\Gamma_2)$, and $A_\rho(X) = A_{\rho_1}(X) + A_{\rho_2}(X)$. Thus,

\[
\frac{A_\rho(X)}{L_\rho(\Gamma)^2} \geq \frac{A_{\rho_1}(X)}{L_{\rho_1}(\Gamma_1)^2} + \frac{A_{\rho_2}(X)}{L_{\rho_2}(\Gamma_2)^2}.
\]

By varying $\rho$, we conclude that $\lambda(\Gamma)^{-1} \geq \lambda(\Gamma_1)^{-1} + \lambda(\Gamma_2)^{-1}$. \qed

2.3. **General definitions and properties.** For a less differentiable map, we may either look into its a.e. derivatives and proceed as in Section 2.1, or consider its geometric deformation by putting the ideas of Example 2.5 and Example 2.9 into precise terms. This leads to two equivalent definitions; see [Ahl] for the equivalence.

**Definition 2.13.** Let $\Omega, \Omega'$ be regions of $\mathbb{C}$, and $f : \Omega \to \Omega'$ be a homeomorphism onto its image. Write $w = f(z)$ in complex coordinates. Given a constant $K \in [1, \infty)$, set $k := (K - 1)/(K + 1)$. Say the homeomorphism $f$ is $K$-quasiconformal if it satisfies one of the following equivalent conditions:

1. $f$ has locally integrable distributional derivatives that satisfy $|f_\tau| \leq k|f_z|$;
2. $f$ deforms the modulus of every quadrilateral in $\Omega$ by a multiple at most $K$; in other words, the moduli of quadrilaterals are $K$-quasi-invariant.

**Remark 2.14.** (1) The definition (1) implies that quasiconformality is a local property, so we have the general definition of a quasiconformal map between Riemann surfaces, which is required to be a homeomorphism by definition.

(2) By either definition, the class of $K$-quasiconformal maps is invariant under conformal maps, and if a homeomorphism is a uniform limit of $K$-quasiconformal maps, then it is $K$-quasiconformal as well.

(3) A 1-quasiconformal map is conformal. In general, given a homeomorphism $f : \Omega \to \Omega'$ of Riemann surfaces such that $f_\tau = 0$ a.e., if $f$ is either ACL or has integrable distributional derivatives, then $f$ is conformal. See [Ahl].

**Theorem 2.15.** If a $K$-quasiconformal map sends $\Gamma$ to $\Gamma'$, then

\[\frac{1}{K} \lambda(\Gamma) \leq \lambda(\Gamma') \leq K \lambda(\Gamma).\]

**Proof.** See [LV, §3.3]. \qed

**Theorem 2.16** (Mori). Let $f : \mathbb{D} \to \mathbb{D}$ be a $K$-quasiconformal map normalized by $f(0) = 0$. Then for distinct $z_1, z_2 \in \mathbb{D}$,

\[|f(z_1) - f(z_2)| < 16|z_1 - z_2|^{1/K},\]

where the constant 16 is best possible.
Proof. See [Ahl, III]. □

Corollary 2.17. Every quasiconformal map of $\mathbb{D}$ extends to a homeomorphism of the closed unit disk $\overline{\mathbb{D}}$.

Corollary 2.18 (Compactness property). Under a suitable normalization (e.g. fix 3 points for a family of homeomorphisms of $\hat{\mathbb{C}}$), the family of $K$-quasiconformal maps is compact on compact sets.

Given a quasiconformal map $f: \Omega \rightarrow \Omega'$, the pullback of the complex structure on $\Omega'$ via $f$ gives another complex structure on $\Omega$. We may signify the new complex structure by the Beltrami differential of $f$, denoted by

$$\mu(f) := \mu(z) \frac{dz}{d\overline{z}} := \frac{\overline{\partial f}}{\partial f}.$$ 

Generally on a Riemann surface $X$, we call a $L^\infty$ measurable ($-1,1$)-form a Beltrami differential. They form a Banach space denoted by $M(X) = L^\infty(X, d\overline{z}/dz)$.

Remark 2.19. We give two interpretations of Beltrami differentials. Let $f: \Omega \rightarrow \Omega'$ be a quasiconformal map with Beltrami differential $\mu$.

(1) Geometrically, $\mu$ corresponds to an ellipse field that is the pullback of the circle field on $\Omega'$, measuring the infinitesimal difference of two complex structures on $\Omega$. The major axes of the ellipse field determines an a.e defined 1-distribution on $\Omega$, which we will refer to as the linefield of $\mu$.

(2) The ($-1,1$)-form $\mu$ is also interpreted as an antilinear map $T\Omega \rightarrow T\Omega'$ as follows. For $w(z)\partial_z \in T\Omega$, there is a pairing

$$\mu(z) \frac{dz}{d\overline{z}} \cdot w(z)\partial_z = \mu(z)\overline{w(z)} \partial_z,$$

where applying $d\overline{z}$ to $w(z)\partial_z$ gives $\overline{w(z)}$, with $\partial_z$ left at the end.

Whenever $f: X \rightarrow Y$ is quasiconformal, we have $\mu(f) \in M(X)_1$, the open unit ball in $M(X)$. The following fundamental theorem, due to cumulative work of Morrey [Mor2], Ahlfors, and Bers [AB], says that we can go from $\mu$ to $f$.

Theorem 2.20 (Measurable Riemann Mapping Theorem). Given a Beltrami differential $\mu \in M(\hat{\mathbb{C}})_1$, there is a unique quasiconformal map $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ such that $f_* = \mu f_z$, normalized so that $f$ fixes the points 0, 1, $\infty$.

Furthermore, the normalized solution $f^t$ with dilation $t\mu$, $t \in \mathbb{D}$ or $t \in \mathbb{R}$, varies analytically with respect to the parameter $t$.

3. Teichmüller Theory

Classical Teichmüller theory studies the Teichmüller spaces of surfaces. In this section, using the theory of quasiconformal maps, we give a complex analytic approach to Teichmüller spaces. A thorough treatment can be found in [Hub].

3.1. Teichmüller spaces. Suppose $X$ is a hyperbolic Riemann surface, hence uniformized by $\mathbb{H}$. Say $X$ is of finite type $(g,n)$, if it is a closed Riemann surface of genus $g \geq 0$ with $n \geq 0$ punctures. Then $\chi(X) = 2 - 2g - n < 0.$
Definition 3.1. Consider all pairs \((X_0, f_0)\), where \(f_0 : X \to X_0\) is a quasiconformal map of Riemann surfaces. We call \(f_0\) a marking and \(X_0\) a marked surface. Say two pairs \((X_1, f_1), (X_2, f_2)\) are equivalent if there is an isomorphism \(g : X_1 \to X_2\) such that the map \(f_2^{-1} \circ g \circ f_1 : X \to X\) is Teichmüller trivial, where a quasiconformal map \(f : X \to X\) is **Teichmüller trivial** if it lifts to some map \(\mathbb{H} \to \mathbb{H}\) that extends to the identity on \(\mathbb{R}\). The **Teichmüller space** \(T(X)\) of \(X\) is the space of equivalence classes of all these pairs.

Remark 3.2. If \(X\) is of finite type, a quasiconformal map \(f : X \to X\) is Teichmüller trivial if and only if \(f \simeq \text{id}_X\), as the punctures are essentially fixed. Substituting this equivalent characterization into the preceding definition, we can thus define the **Teichmüller space** of a torus or the Riemann sphere. By uniformization, the Teichmüller space of \(\mathbb{C}\) is a single point.

Remark 3.3. If \(X\) is of finite hyperbolic type, then \(T(X)\) is equivalent to:

1. the space of isotopy \(^4\) classes of complete, finite-area, hyperbolic metrics on \(X\), i.e. metrics of constant sectional curvature \(-1\); and
2. the deformation space of the Fuchsian group \(\pi_1(X)\).

This is because both the complex and hyperbolic structures on \(\mathbb{H}\) are essentially unique and share the same automorphism group. In particular, the notions of punctures and parabolic cusps are interchangeable when describing a finite-type end, depending on which structure is being emphasized.

Remark 3.4. For a Riemann surface, we will confuse its complex structure with its conformal structure, as they are directly interchangeable via isothermal coordinates.

In Example 2.2, we define the Teichmüller distance of two quasiconformal maps. This leads to the **Teichmüller metric** on \(T(X)\) defined by

\[
d_T(t_1, t_2) := \frac{1}{2} \log K, \quad \text{for } t_1, t_2 \in T(X),
\]

where \(K\) is the infimum such that \(f_1 \circ f_2^{-1}\) is \(K\)-quasiconformal for some \((X_1, f_1) \in t_1, (X_2, f_2) \in t_2\). By Example 2.2 and Corollary 2.18, \(d_T\) is a metric.

Proposition 3.5. The **Teichmüller metric** \(d_T\) is complete.

Proof. For a Cauchy sequence in \(T(X)\), lift the markings to \(\mathbb{D}\), and extend to a normalized family \(\pi_1(X)\)-equivariant, \(K\)-quasiconformal maps of \(\mathbb{D}\), for some large constant \(K\). By Corollary 2.18, the sequence subconverges to a \(\pi_1(X)\)-equivariant quasiconformal map of \(\mathbb{D}\), which descends to an accumulation point of the Cauchy sequence in \(T(X)\). \(\square\)

Definition 3.6. The **mapping class group** \(\text{Mod}(X)\) consists of all homotopy classes of orientation-preserving homeomorphisms \(h : X \to X\). There is a natural action of \(\text{Mod}(X)\) on \(T(X)\) by \(h \cdot (X_0, f_0) = (X_0, f_0 \circ h^{-1})\), which clearly preserves the Teichmüller metric. The **moduli space** of \(X\) is defined by the quotient space of this action \(\mathcal{M}(X) := T(X)/\text{Mod}(X)\), i.e. by forgetting markings.

Example 3.7 (The Teichmüller space of a torus). Let \(X\) be a complex 1-torus. Fix a basepoint, so \(X\) can be uniquely normalized as \(\mathbb{C}/(\mathbb{Z} \oplus \tau \mathbb{Z})\) for some \(\tau \in \mathbb{H}\).

\(^4\)For surfaces, homotopic homeomorphisms are isotopic. This was proved early in 1966 [Eps].
Then there is an obvious identification $\mathcal{T}(X) \cong \mathbb{H}$. We claim that for $\tau_1, \tau_2 \in \mathbb{H}$, $d_T(\tau_1, \tau_2) = d_\mathbb{H}(\tau_1, \tau_2)/2$. On one hand, the natural affine map

$$f : \mathbb{C} \to \mathbb{C}, f(z) = \frac{\tau_2 - \tau_1}{\tau_1 - \tau_1} z + \frac{\tau_1 - \tau_2}{\tau_1 - \tau_1}$$

descends to a quasiconformal map $f : \tau_1 \to \tau_2$ with

$$\log(D_f) = \cosh^{-1} \left( \frac{1 + |\mu|²}{1 - |\mu|²} \right) = \cosh^{-1} \left[ 1 + \frac{|\tau_1 - \tau_2|^²}{2(3\tau_1)(3\tau_2)} \right] = d_\mathbb{H}(\tau_1, \tau_2).$$

Thus, $d_T(\tau_1, \tau_2) \leq d_\mathbb{H}(\tau_1, \tau_2)/2$. On the other hand, by Example 2.11 the extremal length of the family of $[1, 0]$-class loops on the torus $\tau_j$ is $\lambda(\Gamma_j[1,0]) = 1/3\tau_j$. Denote $K(f)$ as the maximal dilation of $f$. If $f : \tau_1 \to \tau_2$ is a quasiconformal homeomorphism, then $f$ sends $\Gamma_1[1,0]$ to $\Gamma_2[1,0]$, and by Theorem 2.15 we have

$$\log K(f) \geq |\log \lambda(\Gamma_1[1,0])| = |\log 3\tau_1 - \log 3\tau_2| = d_\mathbb{H}(H_1(\infty), H_2(\infty)),$$

where $H_j(\xi)$ is defined as the hyperbolic horocycle centered at $\xi \in \mathbb{R}$ through $\tau_j \in \mathbb{H}$. Likewise, replace the class $[1,0]$ by $[p,q]$, and it follows that

$$\log K(f) \geq d_\mathbb{H}(H_1(p/q), H_2(p/q)).$$

Taking $p/q \in \mathbb{R}$ arbitrarily close to an endpoint of the hyperbolic geodesic through $\tau_1, \tau_2$, eventually we have $\log K(f) \geq d_\mathbb{H}(\tau_1, \tau_2)$, so $d_T(\tau_1, \tau_2) \geq d_\mathbb{H}(\tau_1, \tau_2)/2$.

We conclude that for $\mathcal{T}(X) \cong \mathbb{H}$, $d_T = d_\mathbb{H}/2$ is complete, and $\text{Mod}(X)$ acts isometrically as $\text{SL}_2(\mathbb{Z})$ on $\mathbb{H}$.

### 3.2. Quadratic differentials

Let $X$ be a Riemann surface. As quasiconformal maps deform the conformal structure, the Beltrami differentials give an infinitesimal characterization. Dually, a quadratic differential $\phi$ on $X$ is a holomorphic section of the square $2K \in Pic(X)$ of the canonical bundle $K$ on $X$. Locally, $\phi = \phi(z) \, dz^2$ for some holomorphic function $\phi(z)$.

Denote $\Omega(X)$ as the Banach space of $L^1$-integrable quadratic differentials, where the $L^1$-norm is defined by

$$\|\phi\| := \int_X |\phi| < \infty,$$

where $|\phi|$ is the measure locally given by $|\phi(z)||dz|^2$ whenever $\phi = \phi(z) \, dz^2$. If $X$ is of finite hyperbolic type $(g,n)$, then $\Omega(X)$ consists of those quadratic differentials with at worst simple poles at each puncture, as $\int |dz^2/z|^d = \int r^{1-d} \, dr \, d\theta < \infty$ around $z = 0$ if and only if $d < 2$. By the Riemann-Roch Theorem,

$$\dim \Omega(X) = 3g - 3+n.$$  

There is a geometric interpretation of $\phi$ on $X$: $|\phi|$ determines a singular flat metric with total area $\|\phi\|$, and a singular measured foliation $\mathcal{F}(\phi)$ tangent to the vectors $v$ with $\phi(v) > 0$. If $\phi(p) \neq 0$, then $\sqrt{\phi}$ is a holomorphic 1-form around $p$, and $z(x) = \int_p^x \sqrt{\phi}$ gives a local chart where $\phi = dz^2$; such a chart is unique up to transformations $z \mapsto \pm z + z_0$. The metric $|\phi|$ is flat around $p$, and $\mathcal{F}(\phi)$ is the foliation by horizontal lines with transverse measure $|dy| = |3\sqrt{\phi}|$.

If $\phi(p) = 0$, then there is a local chart where $\phi = z^d \, dz^2$. Thus, the metric $|\phi|$ is singular at $p$ with a total cone angle $(d + 2)\pi$, and $\mathcal{F}(\phi)$ looks locally like $d + 2$ rays emitting from $p$, with each pair of neighboring leaves at angle $\pi$. It is also helpful to

---

5Thus, when addressing them as quadratic differentials, we mean the holomorphic ones.
consider $\mathcal{F}(-\phi)$. The two foliations $\mathcal{F}(\phi)$ and $\mathcal{F}(-\phi)$ share the same singularities and have orthogonal leaves. Together they form coordinate lines on $X$.

Given a foliation $\mathcal{F}$ with transverse measure $\alpha_\mathcal{F}$, we are interested in the conformal data it encodes, so we may apply the technique of moduli and extremal lengths in Section 2.2. Given any measurable metric $\rho$ on $X$, form the measure $\rho \times \alpha_\mathcal{F}$, which is locally the product of $\rho$-length along leaves and transverse measure $\alpha_\mathcal{F}$. We thus define the $\rho$-length of $\mathcal{F}$ as

$$L_\rho(\mathcal{F}) := \int_X \rho \times \alpha_\mathcal{F}.$$ 

Write $\mathcal{F} \sim \mathcal{F}'$ if the two measured foliations are related by a homeomorphism isotopic to the identity. Also write $A_\rho$ as the $\rho$-area. The extremal length of the measured foliation $\mathcal{F}$ on $X$ is defined as

$$\lambda_X(\mathcal{F}) := \sup_{\rho} \frac{\inf_{\mathcal{F} \sim \mathcal{F}'} L_\rho(\mathcal{F}')^2}{A_\rho(X)},$$

where $\rho$ ranges over all measurable conformal metrics on $X$.

**Theorem 3.9.** If $X$ is of finite type and $0 \neq \phi \in \Omega(X)$, then $\lambda_X(\mathcal{F}(\phi)) = \|\phi\|$, and $\rho = |\phi|^{1/2}$ is the unique extremal metric for $\mathcal{F}(\phi)$ up to scale.

**Proof.** Write $\mathcal{F} = \mathcal{F}(\phi)$. Note that $\alpha_\mathcal{F} = |dy|$ is induced by restricting $\rho = |\phi|^{1/2}$ to orthogonals of $\mathcal{F}$, and since $\rho$ is conformal on $X$ we have

$$L_\rho(\mathcal{F}) = \int_X \rho \times \rho = A_\rho(X) = \|\phi\|.$$

On one hand, let $\mathcal{F}' = f(\mathcal{F})$ for some $f \simeq \text{id}_X$. Consider the $\rho$-nearest point projection from each leaf $L'$ of $\mathcal{F}'$ to the $\rho$-geodesic $L := f^{-1}(L')$. Since $\rho$ is negatively curved at singularities and flat elsewhere, this projection does not increase distances. Thus $\int_\mathcal{F}, \rho \geq \int_{\mathcal{F}} \rho$ and

$$\lambda_X(\mathcal{F}) \geq \frac{\inf_{\mathcal{F} \sim \mathcal{F}'} L_\rho(\mathcal{F}')^2}{A_\rho(X)} \geq \frac{L_\rho(\mathcal{F})^2}{A_\rho(X)} = \|\phi\|.$$

On the other hand, for any other conformal metric $\sigma$,

$$L_\sigma(\mathcal{F})^2 = \left( \int_X \rho \times \sigma \right)^2 \leq \int_X \rho \times \rho \int_X \sigma \times \sigma = A_\rho(X) A_\sigma(X),$$

by the Cauchy-Schwarz inequality, so $\rho$ is extremal for $\mathcal{F}$. Therefore,

$$\lambda_X(\mathcal{F}) = \sup_{\sigma} \frac{\inf_{\mathcal{F} \sim \mathcal{F}'} L_\sigma(\mathcal{F}')^2}{A_\sigma(X)} \leq \sup_{\sigma} \frac{L_\sigma(\mathcal{F})^2}{A_\sigma(X)} \leq \|\phi\|.$$

We conclude that $\lambda_X(\mathcal{F}(\phi)) = \|\phi\|$. If $\sigma$ is also extremal for $\mathcal{F}$, then the Cauchy-Schwarz inequality in (3.10) takes equality, so $\sigma$ is a constant multiple of $\rho$. □

---

6 This means $\rho$ is compatible with the conformal structure of $X$, which aligns with our motivation to study the conformal data.
3.3. Teichmüller extremal maps. The compactness property of quasiconformal maps indicates that there must be at least an extremal map in a fixed homotopy class. Teichmüller studied them and gave a beautiful account for Riemann surfaces of finite hyperbolic type. We also restrict to this case.

Definition 3.11. A Teichmüller map between hyperbolic Riemann surfaces $X_1, X_2$ is a quasiconformal map $f : X_1 \rightarrow X_2$ such that

$$\mu_f = \frac{\partial f}{\partial \overline{f}} = k \frac{\overline{\phi}}{\phi}$$

for some $\phi \in \mathcal{Q}(X_1)$ and constant $k \in [0, 1]$, hence of dilation $K = (1 + k)/(1 - k)$. When $X_1, X_2 \in \mathcal{T}(X)$, we require that $f$ respect the markings up to homotopy.

Remark 3.12. There are local charts away from zeros of $\phi$, where $\phi = dz^2$ and $f(x + iy) = \sqrt{K}x + iy/\sqrt{K}$, resembling the case of pseudo-Anosov maps and measured laminations. It was the fact that Teichmüller and pseudo-Anosov maps share geometric and dynamical behaviors that led Bers [Ber3] to another proof of Thurston’s classification theorem on $\text{Mod}(X)$.

For our purpose, we prefer local charts where $\phi = dz^2$ and $f(x + iy) = Kx + iy$. In this way, the map $f$ sends $\mathcal{F}(\phi)$ to $\mathcal{F}(\psi)$ for some $\psi \in \mathcal{Q}(X_2)$, by stretching each leaf of the foliation $\mathcal{F}(\phi)$ by a factor $K$ while preserving the transverse measure.

Call $\phi, \psi$ the initial, terminal quadratic differentials of $f$ respectively up to scale. The initial, terminal quadratic differentials of $f^{-1}$ are given by $-\psi, -\phi$ respectively.

Theorem 3.13 (Teichmüller). Let $X$ be a Riemann surface of finite hyperbolic type $(g, n)$, and $X_1, X_2 \in \mathcal{T}(X)$ be marked. Then

(a) there exists a unique Teichmüller map $f : X_1 \rightarrow X_2$ between $X_1, X_2$; and
(b) if $f : X_1 \rightarrow X_2$ is a Teichmüller map, then $f$ is the unique extremal quasiconformal map in its homotopy class. In particular,

$$d_T(X_1, X_2) = \frac{1}{2} \log K(f).$$

Proof. We first prove (b). As in Remark 3.12, let $\phi_1 \in \mathcal{Q}(X_1), \phi_2 \in \mathcal{Q}(X_2)$ be such that $\mu_f = k \frac{\overline{\phi_1}}{\phi_1}$ and $f(\mathcal{F}_1) = \mathcal{F}_2$ where $\mathcal{F}_j := \mathcal{F}(\phi_j)$. By Theorem 3.9, the conformal metric $\rho_j := |\phi_j|^{1/2}$ is extremal for $\mathcal{F}_j$ on $X_j$. Furthermore, for any other quasiconformal map $g \simeq f$, we thus have

$$K(f) \|\phi_1\| = \|\phi_2\| = \lambda_{X_1}(\mathcal{F}_2) = \frac{L_{\rho_2}(\mathcal{F}_2)^2}{A_{\rho_2}(X_2)} \leq \frac{L_{\rho_2}(g(\mathcal{F}_1))^2}{A_{\rho_2}(X_2)}.$$

The pullback metric $\gamma := g^*\rho_2$ is not conformal, but by shortening the major axis on each tangent space, we obtain the largest conformal metric $\sigma$ below $\gamma$ with

$$\sigma \leq \gamma \leq K(g)\sigma.$$

Therefore, again by Theorem 3.9,

$$\frac{L_{\rho_2}(g(\mathcal{F}_1))^2}{A_{\rho_2}(X_2)} = \frac{L_{\gamma}(\mathcal{F}_1)^2}{A_{\gamma}(X_1)} \leq K(g) \frac{L_{\sigma}(\mathcal{F}_1)^2}{A_{\sigma}(X_1)} \leq K(g) \frac{L_{\rho_1}(\mathcal{F}_1)^2}{A_{\rho_1}(X_1)} = K(g) \|\phi_1\|.$$

Together (3.14) and (3.15) show that $K(f) \leq K(g)$, so $f$ is extremal. If $g$ is also extremal, then we have equality everywhere in (3.15), so the process of rounding down $\gamma$ to $\sigma$ must preserve the length of $\mathcal{F}_1$ and decrease the area of $X_1$ exactly by factor $K(g)$, which shows that the major axes of $g$ are a.e. tangent to $\mathcal{F}_1$. Since now
\(K(f) = K(g)\) a.e., they have the same Beltrami differential and hence \(f^{-1} \circ g \simeq \text{id}\) is conformal. But \(X\) is of finite type, so \(f = g\). This proves (b).

Now for (a), it suffices to show that for any \(Y \in \mathcal{T}(X)\), there exists a Teichmüller map from \(X\) to \(Y\). Denote \(\Omega(X)_1\) as the open unit ball in \(\Omega(X)\). For each \(\phi \in \Omega(X)_1\), there is a \((Y, f) \in \mathcal{T}(X)\) such that \(f : X \to Y\) is Teichmüller with \(\mu(f) = k\phi / |\phi|\) where \(k := ||\phi|| < 1\). This can be shown either by solving the Beltrami equation, or by creating new charts for \(Y\) by stretching along the foliation \(\mathcal{F}(\phi)\).

We define the map \(\pi : \Omega(X)_1 \to \mathcal{T}(X)\) by \(\pi(\phi) = (Y, f)\). By the Measurable Riemann Mapping Theorem, \(\pi\) is continuous. We claim that \(\pi\) is proper. In fact, for any sequence \(\pi(\phi_n) = (Y_n, f_n)\) converging to some \((Z, h) \in \mathcal{T}(X)\), the sequence lies in some compact subset, so there is a large \(K > 1\) such that \(f_n\) is \(K\)-quasiconformal for every \(n\). In particular, \(\|\phi_n\| \leq (K - 1)/(K + 1) < 1\), so the sequence \(\{\phi_n\}\) lies in a compact subset of \(\Omega(X)_1\) and thus subconverges to some \(\psi \in \Omega(X)_1\). As \(\pi\) is continuous, \(\pi(\psi) = (Z, h)\). Furthermore, by the uniqueness property in (b), \(\pi\) is also injective. Now by the Fenchel-Nielsen parametrization [Wol], \(\mathcal{T}(X)\) is a manifold of real dimension \(2(3g - 3 + n)\), same as \(\Omega(X)\) by (3.8). Since \(\pi : \Omega(X)_1 \to \mathcal{T}(X)\) is a proper injective continuous map between simply connected manifolds of the same dimension, \(\pi\) must be a homeomorphism by invariance of domain, proving (a). \(\square\)

3.4. Analytic structure of Teichmüller spaces. Constructed in the preceding proof, the homeomorphism \(\pi : \Omega(X)_1 \to \mathcal{T}(X)\) raises basic questions on analytic structures of \(\mathcal{T}(X)\). While \(\Omega(X)_1\) is a complex manifold and \(\mathcal{M}(X)_1\) is a complex Banach manifold, the association \(\phi \mapsto \|\phi\|\phi/|\phi|\) is not complex analytic in general, so we need other approaches to study the analytic structure. Clearly, quadratic and Beltrami differentials will play important roles.

In this part, \(X\) is a hyperbolic Riemann surface, not necessarily of finite type.

**Notations 3.16.** The \(L^\infty\)-norm on quadratic differentials on \(X\) with the hyperbolic metric \(\rho = \rho(z)|dz|\) is given by

\[||\phi||_\infty = \sup_{X} |\phi| / \rho^2.\]

Denote \(\Omega^\infty(\mathbb{H}), \Omega^\infty(\mathbb{H}^*)\) as the corresponding Banach spaces of \(L^\infty\) quadratic differentials. For a Fuchsian group \(\Gamma\), denote \(\Omega(\mathbb{H}/\Gamma), \Omega^\infty(\mathbb{H}/\Gamma), \Omega(\mathbb{H}^*/\Gamma), \Omega^\infty(\mathbb{H}^*/\Gamma)\) as the corresponding subspaces of \(\Gamma\)-invariant quadratic differentials. If \(X = \mathbb{H}/\Gamma\), they are naturally identified with \(\Omega(X), \Omega^\infty(X), \Omega^\infty(X^*), \Omega^\infty(X^*)\) respectively.

**Definition 3.17.** The Schwarzian derivative of a holomorphic function \(f(z)\) of one variable is defined by

\[Sf := \left[ \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2 \right] dz^2.\]

**Remark 3.18.** Geometrically, the Schwarzian derivative measures the infinitesimal change of cross-ratios under \(f\); see e.g. [Thu1] for a discussion. In fact, \(f\) is a Möbius transformation if and only if \(Sf \equiv 0\).

Schwarzian derivatives behave well under composition

\[(3.19) \quad S(f \circ g) = (Sf) \circ g + Sg.\]

\(^7\)Here \(\mathbb{H}^*\) denotes the lower-half plane, to be differentiated from the star superscript in \(C^*, \mathbb{D}^*\) which denotes a puncture at 0.
Theorem 3.20 (Nehari). Let $U$ be a round disk in $\hat{\mathbb{C}}$. If $f : U \to \mathbb{C}$ is injective and holomorphic, then $\|Sf\|_\infty \leq 3/2$.

Proof. Given (3.19), we may set $f(z) = e + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$. By the area theorem, $f$ being injective holomorphic implies that $|a_1| \leq 1$. Moreover, we can compute that

$$Sf(z) = \left(\frac{6a_1}{z^4} + o(z^{-4})\right)dz^2,$$

which gives

$$\frac{|Sf|}{\rho^2}(\infty) = \frac{3}{2}|a_1| \leq \frac{3}{2}. \quad \Box$$

Construction 3.21 (Ahlfors-Weill). Given $\phi = \phi(z)dz^2 \in \Omega^\infty(\mathbb{H}^*)$ with $\|\phi\|_\infty < 1/2$, define a Beltrami differential

$$\mu_\phi(z) := \begin{cases} \frac{2\mu^2 \phi(\overline{z}) d\overline{z}}{dz} , & \text{if } z \in \mathbb{H}, \\ 0 , & \text{if } z \in \mathbb{H}^*. \end{cases}$$

Then $\|\mu_\phi\| = 2\|\phi\|_\infty < 1$, and the normalized solution $f^{\mu_\phi}$ is conformal on $\mathbb{H}^*$. The theorem of Ahlfors and Weill states that $Sf^{\mu_\phi} = \phi$. See [AW] for a proof.

Morally, $\mathcal{T}(X)$ is a quotient of the complex Banach manifold $\mathcal{M}(X)_1$ by the action of Teichmüller trivial maps, and the analytic structure will be inherit. We will make this precise by constructing an atlas for $\mathcal{T}(X)$.

Fix the basepoint $X = \mathbb{H}/\Gamma$. For $\mu \in \mathcal{M}(X)_1$, extend $\mu$ by $0$ to $\mathbb{C}$ and denote the normalized solution as $f^\mu$. Theorem 3.20 guarantees that we can define the map

$$\tilde{\Phi}_X : \mathcal{M}(X)_1 \to \Omega^\infty(\mathbb{H}^*/\Gamma), \quad \mu \mapsto Sf^{\mu|_{\mathbb{H}^*}},$$

which is analytic by the Measurable Riemann Mapping Theorem.

The Beltrami differential $\mu \in \mathcal{M}(X)_1$ can also be extended by symmetry to $\tilde{\mu} \in \mathcal{M}(X)_1$ for which we denote the normalized solution on $\mathbb{C}$ by $f_\mu$. As $f_\mu(z), \overline{f_\mu(z)}$ satisfy the same Beltrami equation and normalization, they must coincide by the Measurable Riemann Mapping Theorem. This means $f_\mu$ preserves $\mathbb{H}$ and induces a quasiconformal deformation of Fuchsian groups $\Gamma \to \Gamma_\mu := f_\mu \Gamma (f_\mu)^{-1}$, which descends to a quasiconformal map $g_\mu : X \to \mathbb{H}/\Gamma_\mu$. Define

$$(3.22) \quad \Pi_X : \mathcal{M}(X)_1 \to \mathcal{T}(X), \quad \mu \mapsto (g_\mu : X \to \mathbb{H}/\Gamma_\mu).$$

Observe that $\Pi_X$ is independent of the choice of $\Gamma$, and that $\Pi_X(\mu_1) = \Pi_X(\mu_2)$ if and only if $f_{\mu_1}|_\mathbb{R} = f_{\mu_2}|_\mathbb{R}$.

Lemma 3.23. $f^{\mu_1|_{\mathbb{H}^*}} = f^{\mu_2|_{\mathbb{H}^*}}$ if and only if $f_{\mu_1}|_\mathbb{R} = f_{\mu_2}|_\mathbb{R}$ or $\Pi_X(\mu_1) = \Pi_X(\mu_2)$.

Proof. If $f^{\mu_1|_{\mathbb{H}^*}} = f^{\mu_2|_{\mathbb{H}^*}}$, then $f^{\mu_1}(\mathbb{H}) = f^{\mu_2}(\mathbb{H})$, so

$$f^{\mu_1} \circ (f_{\mu_1})^{-1} = f^{\mu_2} \circ (f_{\mu_2})^{-1}$$

for both are normalized conformal maps from $\mathbb{H}$ to the same region.
If \( f_{\mu_1} |_{\hat{\mathbb{H}}} = f_{\mu_2} |_{\hat{\mathbb{H}}} \), consider the following map

\[
F := \begin{cases} 
  f_{\mu_2} \circ (f_{\mu_2})^{-1} \circ f_{\mu_1} \circ (f_{\mu_1})^{-1}, & \text{if } z \in f_{\mu_1}(\mathbb{H}), \\
  f_{\mu_2} \circ (f_{\mu_1})^{-1}, & \text{if } z \in f_{\mu_1}(\mathbb{H}^*).
\end{cases}
\]

Then \( F \) is conformal in both \( f_{\mu_1}(\mathbb{H}) \) and \( f_{\mu_1}(\mathbb{H}^*) \), and \( F \) is a normalized homeomorphism of \( \mathbb{C} \) by construction, so \( F = \text{id} \).

\begin{remark}
By (3.22) and Lemma 3.23, once the Fuchsian representation \( X = \mathbb{H}/\Gamma \) is fixed, \( \mathcal{T}(X) \) is naturally identified with the quasiconformal deformation space of the Fuchsian group \( \Gamma \), as promised in Remark 3.3.
\end{remark}

\begin{corollary}
The map in Corollary 3.25 induces a natural map

\[
\Phi_X : \mathcal{T}(X) \to \mathcal{Q}^\infty(\mathbb{H}^*/\Gamma),
\]

which is called the Bers embedding with respect to the basepoint \( X \).
\end{corollary}

\begin{remark}
For \( (Y, f) \in \mathcal{T}(X) \), we can use the quasiconformal conjugation to transport the basepoint and get the Bers embedding with respect to \( Y \in \mathcal{T}(X) \)

\[
\Phi_Y : \mathcal{T}(X) \to \mathcal{Q}^\infty(Y^*),
\]

where \( \mathcal{T}(X) \) and \( \mathcal{T}(Y) \) are naturally identified via \( f \). Note that when \( Y = X \), the Bers embedding does not depend on the choice of \( f \).
\end{remark}

\begin{proposition}
Near \( Y \in \mathcal{T}(X) \), \( \Phi_Y \) is open and a local homeomorphism. This gives rise to an atlas \( \{ \Phi_Y : U_Y \to V_Y \}_{Y} \) with analytic coordinate transformations.
\end{proposition}

\begin{proof}
Set \( V_Y := \mathcal{Q}^\infty(Y^*)_{1/2} \) as the open ball of radius \( 1/2 \), and \( U_Y := \Phi_Y^{-1}(V_Y) \). Define as in the Ahlfors-Weill Construction

\[
\sigma_Y : V_Y \to \mathcal{M}(\mathbb{H}/\Gamma_Y)_1, \quad \phi = \phi(z) \, dz^2 \mapsto 2y^2 \phi(z) \, d\sigma.
\]

Then \( \sigma_Y \) is a local section of \( \Phi_Y : \mathcal{M}(Y)_1 \to \mathcal{Q}^\infty(Y^*) \). It follows that

\[
\Phi_Y \circ (\Pi_X \circ \sigma_Y) = \Phi_Y \circ \sigma_Y = \text{id}.
\]

Since \( \Phi_Y \) is injective by Corollary 3.25, \( \Phi_Y : U_Y \to V_Y \) is a homeomorphism.

Under natural identification of differently based Teichmüller spaces, the following coordinate transformation is thus analytic:

\[
\Phi_{X_2} \circ \Phi^{-1}_{X_1} = \Phi_{X_2} \circ \Pi_X \circ \sigma_{X_1} = \Phi_{X_2} \circ \sigma_{X_1}.
\]

In conclusion, there is a unique complex Banach manifold structure on \( \mathcal{T}(X) \) such that the map \( \Pi_X : \mathcal{M}(X)_1 \to \mathcal{T}(X) \) is analytic. In particular, when \( X \) is of finite type \( (g, n) \), then \( \mathcal{T}(X) \) is a complex manifold of dimension \( 3g - 3 + n \). Moreover, all \( \sigma_Y \circ \Phi_Y \) where \( Y \in \mathcal{T}(X) \) give local right inverses of \( \Pi_X \), showing that \( \Pi_X \) an analytic split submersion. Lastly, our work above applies to infinite-dimensional Teichmüller spaces, in comparison with previous subsections.
3.5. The infinitesimal Teichmüller metric and cometric. A Finsler metric, or an infinitesimal metric on a Banach manifold, is a continuous function on the tangent bundle that restricts to a norm equivalent to the Banach structure on each tangent space. A Finsler metric defines lengths via integration and thus induces a metric on the manifold. There turns out to be a Finsler metric on $\mathcal{T}(X)$ that induces the Teichmüller metric, i.e. the infinitesimal Teichmüller metric. To characterize it, we start with the finite type case.

Construction 3.29. Denote $\Delta$ as the open unit disk with the metric $|dz|/(1-|z|^2)$ of constant curvature $-4$. Let $X$ be a hyperbolic Riemann surface of finite type $(g,n)$. Fix $\phi \in \Omega(X)$ and set $\mu := \zeta \phi/|\phi| \in \mathcal{M}(X)$. Define a map $F : \Delta \to \mathcal{T}(X)$ by $F(\zeta) = \Pi_X(\mu)$ as in (3.22). In other words, the Beltrami differential $\mu$ induces some $(Y,f) \in \mathcal{T}(X)$ with $\mu(f) = \mu$, and $F(\zeta) = (Y,f)$.

Proposition 3.30. $F$ is a holomorphic isometric embedding.

Proof. By Section 3.4, $F$ is holomorphic. Since each Beltrami differential $\zeta \phi/|\phi|$ gives a unique Teichmüller map, $F$ is injective and hence an embedding. It suffices to show that $F$ is isometric. Each $\zeta_j \in \Delta$, $j = 1, 2$ is associated with $f_j : X \to X_j$ with $\mu_j = \zeta_j \phi/|\phi|$. By (2.3),

$$\mu_{f_2 f_1^{-1}}(w) = \frac{\mu_2 - \mu_1}{1 - \mu_1 \mu_2} (f_1)_\circ \circ f_1^{-1}(w) = \frac{\zeta_2 - \zeta_1}{1 - \zeta_1 \zeta_2} \phi/|\phi| \circ f_1^{-1}(w).$$

We claim that this is the Beltrami differential associated to a Teichmüller map. In fact, taking $f_2 = \text{id}$ shows $\mu(f_1^{-1})$ is associated to the Teichmüller map $f_1^{-1}$. Since the Beltrami differential in (3.31) is a rescale of $\mu(f_1^{-1})$ and has norm less than 1, it is also associated to some Teichmüller map, which must be $f_2 \circ f_1^{-1}$. We conclude that $F$ is an isometric embedding, as by (2.4) we have

$$d_T(X_1, X_2) = \frac{1}{2} \log K(f_2 \circ f_1^{-1}) = d_\Delta(\zeta_1, \zeta_2).$$

Remark 3.32. The image of $F$ is a Teichmüller disk, defined as a closed one-dimensional complex submanifold of $\mathcal{T}(X)$ which is isometric to $\Delta$. In particular, a Teichmüller disk contains the geodesic through any two of its points.

By a dimension count, as $(\zeta, |\phi|)$ varies in $\Delta \times \mathcal{P}\Omega(X)$, the Beltrami differentials $\zeta \phi/|\phi|$ give rise to all the infinitesimal quasiconformal deformations of $X$. Furthermore, for any Beltrami differential $\varepsilon \mu$ representing an infinitesimal quasiconformal deformation $f : X \to X'$ with $\mu(f) = \varepsilon \mu$, we have

$$d_T(X, X') = \frac{1}{2} \log \frac{1 + \varepsilon \|\mu\|}{1 - \varepsilon \|\mu\|} = \varepsilon \|\mu\| + o(\varepsilon).$$

By Construction 3.29, if we identify $T_X \mathcal{T}(X)$ with the set of $\zeta \phi/|\phi|$, then the $L^\infty$-norm on Beltrami differentials gives the infinitesimal Teichmüller metric. This set of Beltrami differentials is not a vector space, though.

Recall the map $\Pi_X : \mathcal{M}(X) \to \mathcal{T}(X)$ in (3.22). Denote $\mathcal{M}_0(X) := \text{Ker}(d\Pi_X|_0)$ as the subspace inducing infinitesimally trivial deformations on $X$ up to isotopy. Since $\Pi_X$ is a split submersion of complex Banach manifolds, we have a natural identification $T_X \mathcal{T}(X) \cong \mathcal{M}(X)/\mathcal{M}_0(X)$. Now by Teichmüller’s theorem, each $\zeta \phi/|\phi|$ induces the Teichmüller map in the class, so $\zeta \phi/|\phi|$ uniquely minimizes the $L^\infty$-norm in its coset in $\mathcal{M}(X)/\mathcal{M}_0(X)$. Hence, the infinitesimal Teichmüller metric on $T_X \mathcal{T}(X)$ is exactly the quotient norm on $\mathcal{M}(X)/\mathcal{M}_0(X)$ of the $L^\infty$-norm of $\mathcal{M}(X)$.
The Teichmüller cometric also has a nice description. With the Bers embedding \( \Phi_X : T(X) \to \mathcal{Q}^\infty(X^*) \) as a chart, there is a natural identification \( T^*_X T(X) \cong \mathcal{Q}(X) \) by duality. Moreover, we have a natural pairing

\[
\langle \cdot, \cdot \rangle : M(X) \times \mathcal{Q}(X) \to \mathbb{R}, \quad \langle \mu, \phi \rangle = \Re \int_X \mu \phi = \Re \int_X \mu(z) \phi(z) |dz|^2,
\]

which measures the synchronization of the linefields of \( \mu \) and \( \phi \). Here, the linefield of \( \phi \) is the 1-distribution tangent to the singular foliation of \( \phi \); also recall Remark 2.19.

Observe that the \( L^\infty \)-norm on \( \zeta/|\phi| \) coincides with the dual norm of the \( L^1 \)-norm on \( \mathcal{Q}(X) \). We claim that the cometric is exactly the \( L^1 \)-norm on \( \mathcal{Q}(X) \).

**Lemma 3.34.** Let \( X = \mathbb{H}/\Gamma \). Then \( \mu \in M_0(\mathbb{H}/\Gamma) \) if and only if \( \mu = \bar{\partial} v \) for some \( \Gamma \)-invariant quasiconformal vector field \( v \) on \( \mathbb{H} \) which extends to 0 along \( \mathbb{R} \).

**Proof.** Recall that Lemma 3.23 relates the two extensions of \( \mu \in M(\mathbb{H}/\Gamma)_1 \) to \( \mathbb{C} \).

If \( \mu \in M_0(\mathbb{H}/\Gamma) \), consider a small curve \( \mu_t = t \mu + o(t) \). Extend \( \mu_t \) by zero to \( \mathbb{C} \) to obtain the normalized solution \( f^{\mu_t} \). By the Measurable Riemann Mapping Theorem, \( t \mapsto f^{\mu_t} \) is analytic, so we may write \( f^{\mu_t}(z) = z + tv(z) + o(t) \), where \( v(z) \) is quasiconformal and \( \Gamma \)-invariant over \( \mathbb{C} \). As \( d\Pi_X(\mu_t)/dt|_{t=0} = 0 \), we have \( v|_{\mathbb{H}^*} = 0 \) by Lemma 3.23. Expanding \( \bar{\partial} f^{\mu_t} = \mu_t \bar{\partial} f^{\mu_t} \) with respect to \( t \) yields

\[
tv(z) + o(t) = (t\mu(z) + o(t))(1 + tv(z) + o(t)).
\]
The linear terms give \( \mu(z) = v(z) \), with \( v \) vanishing along \( \mathbb{R} \).

Conversely, suppose \( \mu = \bar{\partial} v \) where \( v \) extends to 0 along \( \mathbb{R} \). Then both \( \mu \) and \( v \) can be extended by zero to \( \mathbb{C} \). Solving \( \bar{\partial} f^{\mu_t} = \mu_t \bar{\partial} f^{\mu_t} \) yields \( f^{\mu_t}(z) = z + o(t) \) in \( \mathbb{H}^* \). By Lemma 3.23, we have \( d\Pi_X(\mu_t)/dt|_{t=0} = 0 \) and thus \( \mu \in M_0(\mathbb{H}/\Gamma) \). \( \square \)

**Proposition 3.35.** \( M_0(X) = \mathcal{Q}(X)^\perp \), and \( M(X)/M_0(X) \cong \mathcal{Q}(X)^* \) are isometric.

**Proof.** Suppose \( \mu = \bar{\partial} v \in M_0(\mathbb{H}/\Gamma) \) for some quasiconformal \( v \) vanishing on \( \mathbb{R} \). For any \( \phi \in \mathcal{Q}(\mathbb{H}/\Gamma) \), by Stokes’ formula we have

\[
\int_{\mathbb{H}} \mu \phi = - \int_{\mathbb{R}} v \bar{\partial} \phi + \int_{\mathbb{R}} v \phi = 0.
\]

Hence, the pairing in (3.33) induces a pairing \( T_X T(X) \times \mathcal{Q}(X) \to \mathbb{C} \) whose nondegeneracy follows from either the Ahlfors-Weill Construction or the \( L^\infty \)-norm minimizers \( \zeta/|\phi| \), so \( M_0(X) = \mathcal{Q}(X)^\perp \). The quotient norm on \( M(X)/\mathcal{Q}(X)^\perp \) equals the dual norm of the \( L^1 \)-norm on \( \mathcal{Q}(X) \). \( \square \)

**Remark 3.36.** A perspective from the Kodaira-Spencer deformation theory can be found in [DH, EE]. A deformation of the complex structure on a complex manifold \( M \) is moving charts relative to one another, so modulo isotopy the infinitesimal deformation space is identified with the Čech cohomology group \( H^1(M, \mathcal{X}) \), where \( \mathcal{X} \) is the sheaf of holomorphic vector fields on \( M \).

Assume \( X \) is closed hyperbolic. Then \( \mathcal{X} = -K \), where \( K \) is the canonical bundle. By Serre duality, the cotangent space of \( T(X) \) is identified with \( H^1(X, -K)^* \cong H^0(X, 2K) = \mathcal{Q}(X) \). Under the identification \( H^1(X, -K) \cong H^{0,1}(X; -K) \), a trivial infinitesimal deformation corresponds to a 1-coboundary in the cochain group \( C^{0,1}(X; -K) \), hence of the form \( \bar{\partial} v \), as an interpretation of Lemma 3.34.

---

8More rigorously, approximate \( v \) by smooth vector fields of compact support on \( \mathbb{H} \) in the weak topology, because \( \phi \) is not controlled on \( \mathbb{R} \), but the argument works anyway.
Remark 3.37. The Ahlfors-Weill Construction gives a section \( \sigma : \Omega(X^*) \to M(X) \), \( \sigma(\phi) = 2y^2\overline{\phi} \). The image of \( \sigma \) consists of harmonic Beltrami differentials, which are the \( L^2 \)-norm minimizers in each coset. In view of Remark 3.36, they are exactly the harmonic representations of each class in \( H^1(X, -K) \). In comparison, the section \( \sigma' : \Omega(X) \to M(X) \), \( \sigma'(\phi) = ||\phi||_Q/|\phi| \) from Construction 3.29 gives the \( L^\infty \)-minimizers in each coset.

Now for the general case. By lifting to \( \mathbb{H} \), our arguments in Lemma 3.34 and Proposition 3.35 become applicable to infinite-type hyperbolic Riemann surfaces as well. The only obstruction is that Construction 3.29, which argues that the infinitesimal metric induces the Teichmüller metric, does not generalize to infinite-type \( X \) because Teichmüller’s theorem does not generalize. In fact, some analytic work and a result on Finsler manifolds will suffice, which we refer to [Hub, GL, Gar].

By varying the basepoint of \( T(X) \), we conclude the following theorem.

Theorem 3.38. Let \( X \) be a hyperbolic Riemann surface. Then
\[
T_Y T(X) \cong M(Y)/\Omega(Y)^+,
\]
for each \( Y \in T(X) \). The infinitesimal Teichmüller metric is the quotient \( L^\infty \)-norm on \( M(Y)/\Omega(Y)^+ \), and the cometric is the \( L^1 \)-norm on \( \Omega(Y) \).

Finally, we inspect the Banach geometry of \( \Omega(X) \) for Section 5. Given a unit-norm \( \phi \in \Omega(X) \), the Beltrami differential \( \mu = \overline{\phi}/|\phi| \) is the unique unit-norm element in \( M(X) \) satisfying \( (\mu, \phi) = 1 \). Geometrically, this says that the unit sphere in \( \Omega(X) \) has no sharp corners: through each point passes a unique supporting hyperplane. The following lemma studies the flex of the unit sphere in \( \Omega(\mathbb{D}) \) at a special point. Were \( \Omega(X) \) a Hilbert space, the lemma would hold for \( \alpha = 1/2 \).

Lemma 3.39. Denote \( \psi = dz^2/\pi \in \Omega(\mathbb{D}) \). If \( \phi \in \Omega(\mathbb{D}) \) with \( ||\phi|| = 1 \) such that \( \langle \overline{\psi}/|\psi|, \phi \rangle = 1 - \varepsilon \), then \( ||\psi - \phi|| < O(\varepsilon^\alpha) \) for some constant exponent \( \alpha > 0 \).

Proof. Throughout the proof we work in coordinates, and integrate with respect to the unit-area measure \( |\psi| = |dz|^2/\pi \) on \( \mathbb{D} \). Write \( \phi = \phi(z)\psi \), where \( \phi(z) \) is a holomorphic function on \( \mathbb{D} \). From the condition \( \langle \overline{\psi}/|\psi|, \phi \rangle = 1 - \varepsilon \) we have
\[
\int_\mathbb{D} |\phi(z)| = 1, \quad \int_\mathbb{D} \Re \phi(z) = 1 - \varepsilon, \quad \int_\mathbb{D} |\Im \phi(z)| = O(\varepsilon^{1/2}).
\]
Since \( \Re \phi(z), \Im \phi(z) \) are both harmonic functions, by averaging over \( \mathbb{D} \) we have
\[
\Re \phi(0) = 1 - \varepsilon, \quad |\Im \phi(z)| = O\left(\varepsilon^{1/2} \cdot d(z, \partial \mathbb{D})^{-2}\right),
\]
where \( d(\cdot, \cdot) \) is the Euclidean metric. Now \( \phi \) maps the ball \( B(0, 1 - d/2) \) into a narrow horizontal strip. Apply the Schwarz Lemma to the holomorphic map
\[
\exp(i\phi(z)) : B(z, d/2) \to \{ z : |z| < O\left(\exp(\varepsilon^{1/2}d^{-2})\right) \}
\]
for \( z \in B(0, 1 - d) \), and then on \( B(0, 1 - d) \) we have
\[
|1 - \phi(z)| \leq O\left(\varepsilon + \varepsilon^{1/2}d^{-2}\log(1/d)\right).
\]
Finally we control the boundary behavior. Since \( |\phi(z)| \) is subharmonic, we have
\[
|\phi(0)| \leq (1 - d)^{-\frac{1}{2}} \int_{B(0,1-d)} |\phi(z)|,
\]

---

\[9\text{Technically, the cotangent space here stands for the predual of the tangent space.}\]
but $|\phi(0)| \leq 1 - \varepsilon$ and $|\phi|$ has unit total mass, so

$$\int_{1-d<|z|<1} |\phi(z)| \leq O(d + \varepsilon).$$

Combining this with (3.40), we have

$$\int_{D} |1 - \phi(z)| \leq O\left(d + \varepsilon + \varepsilon^{1/2}d^{-2}\log(1/d)\right) = O\left(\varepsilon^{1/6}\log(1/\varepsilon)\right),$$

where we set $d = \varepsilon^{1/6}$. Thus the theorem holds for any $\alpha \in (0, 1/6)$.

3.6. **Quasiconformal deformation of Kleinian groups.** Proceeding with the observation that the quasiconformal deformation space of a Fuchsian group $\Gamma$ is naturally identified with $T(\mathbb{H}/\Gamma)$ (see Remark 3.24), we move up one dimension and consider Kleinian groups. Some backgrounds are given in Appendix A. We will not give any proofs; see [Thu2, Mar2] for more.

Let $\Gamma$ be a torsion-free Kleinian group. A quasiconformal deformation of $\Gamma$ is denoted by an isomorphism of Kleinian groups $\theta : \Gamma \rightarrow f\Gamma f^{-1}$, where $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ is a quasiconformal map normalized to fix $0, 1, \infty$, such that $f \circ \gamma(z) = \theta(\gamma) \circ f(z)$ for every $\gamma \in \Gamma$ and $z \in \hat{\mathbb{C}}$. In fact, it is necessary and sufficient that the Beltrami differential of $f$ is invariant under $\Gamma$, i.e. $\mu_{f \circ \gamma} = \mu_f$ for every $\gamma \in \Gamma$. This follows directly from (3.31). Hence, given such a normalized $f$, we define for every $\gamma \in \Gamma$ the Möbius transformation $f\gamma f^{-1}$ by specifying the images of $0, 1, \infty$ as $f(\gamma(0)), f(\gamma(1)), f(\gamma(\infty))$ respectively. Then the quasiconformal deformation of $\Gamma$ by $f$ is just $\Gamma \rightarrow f\Gamma f^{-1}$.

The quasiconformal deformation space or the Teichmüller space of $\Gamma$, denoted by $T(\Gamma)$, is the space of all quasiconformal deformations of $\Gamma$, which is an open subset of $\text{Hom}(\Gamma, PSL(2, \mathbb{C}))$ equipped with the compact-open topology, or the algebraic topology. In particular, $T(\Gamma)$ is a complex manifold.

Let $\Lambda, \Omega$ be the limit set and domain of discontinuity, respectively. The convex core of $\Gamma$ is defined as $CC(\Gamma) := CH(\Lambda)/\Gamma$, i.e. the quotient of its convex hull. Say $\Gamma$ is geometrically finite, if some $\varepsilon$-neighborhood of $CC(\Gamma)$ has finite volume. Morally, geometric finiteness means “essential” compactness. In particular, if $\Gamma$ is geometrically finite, then it is necessarily finitely generated. For a clearer picture, we list several equivalent characterizations.

**Theorem 3.41** (Marden [Mar2]). Let $N$ be the Kleinian manifold of a nonelementary torsion-free Kleinian group $\Gamma$. Then the following conditions are equivalent:

1. $\Gamma$ is geometrically finite.
2. $\Gamma$ has a Dirichlet domain with finitely many faces.
3. The thick part of $N^o$ is compact.
4. There is a neighborhood of each end of $N^o$ intersecting no closed geodesics.
5. There is a neighborhood of each end of $N^o$ which does not intersect with the convex core $CC(N)$.

It turns out that when $\Gamma$ is finitely generated, $T(\Gamma)$ is parametrized in terms of $T(\mathbb{H}/\Gamma)$, due to the cumulative work of Marden [Mar1], Bers [Ber2], Maskit [Mas1], and Sullivan [Sul2]. For our purpose, we restrict to a simpler case.

\[ ^{10}\text{Taking this neighborhood excludes infinite Fuchsian groups, as } CC(\Gamma) \text{ always has zero volume when } \Gamma \text{ is Fuchsian.} \]
Let $\Gamma$ be a geometrically finite, torsion-free Kleinian group with the Kleinian manifold $N = (\mathbb{H}^3 \sqcup \Omega)/\Gamma$. If $N$ has finite volume, then Mostow's rigidity theorem [Mos] implies that $T(\Gamma)$ is a single point. We thus assume $N$ has infinite volume, or equivalently that $\Omega$ is nonempty. By the Ahlfors Finiteness Theorem, $\partial N$ is a finite union of Riemann surfaces, each of finite hyperbolic type. Define $T(\partial N)$ as the product of $T(X)$ over all components $X$ of $\partial N$. Then the product of infinitesimal Teichmüller metrics gives the infinitesimal Teichmüller metric on the finite-dimensional complex manifold $T(\partial N)$.

Any $\mu \in M(\partial N)_1$ lifts to a $\tilde{\mu} \in M(\hat{\Omega})_1$. By Ahlfors' theorem, $\Lambda$ has zero measure, so $\tilde{\mu}$ is a well-defined differential in $M(\hat{C})_1$. By the Measurable Riemann Mapping Theorem, there is a unique normalized quasiconformal map $f$ of $\hat{\Omega}$ with Beltrami differential $\mu$. Then $f$ induces a quasiconformal deformation $\Gamma \to \Gamma' := f\Gamma f^{-1}$. Note that $\Omega' = f(\Omega)$ is the domain of discontinuity of $\Gamma'$. Thus, $f$ descends to a quasiconformal map $g : \partial N \to Y$, and $(Y, g) \in T(\partial N)$. Moreover, whenever $f, f'$ determine the same point in $T(\partial N)$, the map $f^{-1} \circ f'$ is Teichmüller trivial and hence acts as the identity on $\partial \Omega = \Lambda$, so $f, f'$ determine the same quasiconformal deformation of $\Gamma$. We thus have a natural map $T(\partial N) \to T(\Gamma)$. The previously promised result asserts that this map is a universal covering of complex manifolds. Still, we only need the following special case.

**Theorem 3.42.** If in addition $N$ has incompressible boundary, i.e. the inclusion of each boundary component induces an injection of the fundamental groups, then the natural map defined above is a biholomorphic equivalence $T(\partial N) \cong T(\Gamma)$.

**Remark 3.43.** In dimension 2, a deformation of Fuchsian groups corresponds to a class of quasiconformal maps. This is correct but not trivial for Kleinian groups. For $\Gamma' \in T(\Gamma)$ with the Kleinian manifold $N'$, there is a quasiconformal map between Riemann surfaces $g : \partial N \to \partial N'$ as before. Then $g$ extends to a quasiconformal map $N_1 \to N_2$ of of Kleinian manifolds, which is necessarily a homeomorphism by definition. Such a homeomorphism is unique up to isotopy. See [Mar2, §3.7.2].

If in particular $\Gamma < \text{PSL}(2, \mathbb{C})$ is conjugate to a Fuchsian group, then a quasi-conformal deformation of $\Gamma$ is called a quasifuchsian group. Equivalently, $\Lambda$ is a Jordan curve and both components of $\Omega$ are $\Gamma$-invariant; see [Mas2]. Let $X = \mathbb{H}/\Gamma$ be of finite type. Then Theorem 3.42 yields $T(\Gamma) = T(X) \times T(X^*)$. This is Bers’ simultaneous uniformization theorem [Ber1].

4. **Geometric Limits of Quadratic Differentials**

In this section, we introduce the concept of geometric limits, which will facilitate counterarguments in later sections, and then use it to prove some useful results on quadratic differentials for future use.

4.1. **Compact spaces of quadratic differentials.** The geometric limit is adapted from a general type of topology called the geometric topology defined on certain spaces of based hyperbolic manifolds; see [Thu2, CEG]. For our purpose, the geometric topology will be formulated in terms of injectivity radii. We will only give a sketch and refer to [McM1, Appendix] for details.

**Definition 4.1.** Given a topological space $X$, define the Chabauty topology on the set $C$ of closed subsets of $X$ by the sub-basis consisting of sets of the form

- $\{A \in C : A \cap K = \emptyset\}$, where $K \subseteq X$ is compact, and
\[ \{ A \in \mathcal{C} : A \cup U \neq \emptyset \} , \text{ where } U \subseteq X \text{ is open.} \]

**Proposition 4.2.** A sequence of closed subsets \( C_n \) converges to \( C \) in the Chabauty topology, if and only if

1. each \( x \in C \) is a limit \( x = \lim_n x_n \) where \( x_n \in C_n \), and
2. whenever a limit \( \lim x_{n_k} = x \) exists where \( x_{n_k} \in C_{n_k} \), then \( x \in C \).

**Proposition 4.3.** For any \( X \), the space \( \mathcal{C} \) is compact under the Chabauty topology. If \( X \) is a compact metric space, the Chabauty topology agrees with the one induced by the Hausdorff distance. If \( X \) is a Lie group like \( \text{PSL}(2, \mathbb{C}) \), the subspace of closed subgroups forms a closed and thus compact subspace of \( \mathcal{C} \).

**Construction 4.4** (Space of Riemann surfaces). In the spirit of uniformization, define \( U_\kappa = \mathbb{C} \) when \( \kappa > 0 \), \( U_\kappa = \mathbb{C} \) when \( \kappa = 0 \), and \( U_\kappa = \{ z : |z| < 2/\sqrt{-\kappa} \} \) when \( \kappa < 0 \). For \( \kappa \in [-1, 1] \), the Riemann surface \( U_\kappa \) is equipped with the metric

\[ ds_\kappa := \frac{4|dz|}{4 + \kappa |z|^2} , \]

which is complete of constant curvature \( \kappa \). Denote \( \hat{v} \) as the unit vector at the origin pointing towards the positive real axis. To each pair \( (U_\kappa, \Gamma) \), where \( \kappa \in [-1, 1] \) and \( \Gamma \) is a discrete subgroup of \( \text{PSL}(2, \mathbb{C}) \) acting freely on \( U_\kappa \), associate the Riemann surface \( X = U_\kappa/\Gamma \) with baseframe \( v \in TX \) as the image of \( \hat{v} \). Conversely, given any pointed Riemann surface \( (X, v) \), if the conformal metric is scaled to a constant curvature in \([-1, 1]\), then \( (X, v) \) uniquely determines a valid pair \( (U_\kappa, \Gamma) \).

Let \( \mathcal{X} \) be the space of such pairs \( (U_\kappa, \Gamma) \) that the injectivity radius of \( X \) at \( v \) is at least 1. Equip \( \mathcal{X} \) with the geometric topology: a sequence converges geometrically if and only if \( \kappa \) converges and \( \Gamma \) converges in the Chabauty topology on discrete subgroups of \( \text{PSL}(2, \mathbb{C}) \).

We claim that \( \mathcal{X} \) is compact under the geometric topology. By Proposition 4.3, a sequence of pairs \( (U_\kappa, \Gamma_i) \) always subconverges to a pair \( (U_\kappa, \Gamma) \). By construction, the injectivity radius at \( v_i \) is at least 1 for all \( i \), so the same holds for \( (U_\kappa, \Gamma) \). Thus the subsequence converges in \( \mathcal{X} \).

**Example 4.5.** One way to visualize the geometric topology is via Hausdorff convergence of the Dirichlet domains centered at 0. We demonstrate how this picture accompanies that of surfaces. See [Har] for an exposition with figures.

If the curvature is negative and uniformly bounded away from 0, we may fix \( \kappa = -1 \). On a Riemann surface \( X \), choose a simple closed geodesic \( \gamma \) that separates \( X \) into two components \( X_1, X_2 \). Cutting \( X \) along \( \gamma \) and sewing back increasingly thick annuli \( \{1/n \leq |z| \leq 1\} \), we get a sequence converging geometrically to a surface homotopy equivalent to \( X_2 \) but with \( \gamma \) replaced by a parabolic cusp to which \( \gamma \) converges. In the Dirichlet domain picture, this process is visualized as follows: \( \gamma \) is identified with a pairing of two sides which extend to disjoint geodesics; in the limiting Dirichlet domain, these two geodesics meet at an ideal point and arise as two neighboring sides paired by the limit of \( \gamma \).

Another kind of surgery in the geometric limit is pinching off geodesics. Let \( \gamma \) be a nonseparating simple closed geodesic on \( X \). In the limiting, \( \gamma \) is pinched off so that the geodesic degenerates to two cusps. In the Dirichlet domain picture, \( \gamma \) conjugates \( \alpha \) to \( \beta \), where \( \alpha, \beta \) are two side pairings represented by the same closed geodesic which intersects \( \gamma \) exactly once; in the limiting, \( \gamma \) gradually vanishes, while \( \alpha, \beta \) converge to two nonconjugate parabolic transformations respectively.
Construction 4.6 (Spaces of finite-type surfaces). Let \( \mathcal{X}_{g,n} \subset \mathcal{X} \) be the subspace corresponding to Riemann surfaces of finite type \((g,n)\). We can compactify \( \mathcal{X}_{g,n} \) by forming its closure in \( \mathcal{X} \). We claim that
\[
\mathcal{X}_{g,0} = \mathcal{X}_{g,0} \cup \mathcal{X}_{g,1} \quad \text{if } g \geq 1;
\]
and
\[
\mathcal{X}_{g,n} = \bigcup \{ \mathcal{X}_{h,m} : 2h + m \leq 2g + n, h \leq g, m \geq 1 \} \quad \text{if } n \geq 1.
\]
The idea is that when \( U_\kappa \) is uniformly bounded away from \( \infty \), the genus does not increase in the geometric limit, which can be seen in the Dirichlet domain picture; the volume typically does not increase in the geometric limit; a geometric limit of noncompact surfaces is still noncompact; these observations show that the closure of \( \mathcal{X}_{g,n} \) is no larger than claimed. The reverse inclusion is established via rescaling and surgery as in Example 4.5.

Construction 4.7 (Universal curve). The idea is to replace every point \((U_\kappa, \Gamma)\) of \( \mathcal{X} \) by the Riemann surface \( U_\kappa/\Gamma \) it represents. More precisely, define the universal curve \( \mathcal{C} \) as the quotient space of the bundle of universal coverings \( \{(z, (U_\kappa, \Gamma)) : z \in U_\kappa \} \subset \tilde{\mathcal{C}} \times \mathcal{X} \to \mathcal{X} \) by collapsing the fiber over \((U_\kappa, \Gamma)\) by \( \Gamma \)-action. By our condition on the injectivity radii, the quotient map is a local homeomorphism.

Definition 4.8. Say a closed set \( E \subseteq \mathcal{X} \) is a geometric limit of \( E_n \subseteq \mathcal{X} \) if \( E_n \to E \) in the Chabauty topology on closed subsets of \( \mathcal{C} \). Similarly, say a sequence of continuous maps \( f_n : \mathcal{X}_n \to Z \) converges to \( f : \mathcal{X} \to Z \) in the geometric topology if their graphs converge in the Chabauty topology on closed subsets of \( \mathcal{C} \times Z \).

Construction 4.9 (Space of quadratic differentials). Let \( \mathcal{Q} \) be the space of triples \((X, v, \phi)\) where \((X, v) \in \mathcal{X} \) and \( \phi \) is a quadratic differential\(^{11}\) on \( X \). All canonical bundles can be pieced together as a continuous complex line bundle \( K \to \mathcal{X} \) with holomorphic structure along each fiber. Endow \( \mathcal{Q} \) with the geometric topology on continuous maps whose target \( Z \) is the total space of the bundle \( K \otimes K \).

Let \( \mathcal{Q}_{g,n} \) be the space of \( L^1 \)-integrable (or equivalently, with at most simple poles at each puncture) quadratic differentials living on Riemann surfaces in \( \mathcal{X}_{g,n} \), i.e. of finite type \((g,n)\). The fiber of \( \mathcal{Q}_{g,n} \) over \( X \in \mathcal{X}_{g,n} \) is canonically identified with the cotangent space \( T^*_X \mathcal{T}(X) \), once a marking is chosen; see Section 3.5.

The compactification of quadratic differentials is established by projectivization. Let \( \mathbb{P} \mathcal{Q} := \mathcal{Q}^*/\mathbb{C}^* \) be the projective space of the set of nonzero quadratic differentials, with the quotient topology. The following theorem is shown by case study.

Theorem 4.10. \( \mathbb{P} \mathcal{Q}_{g,n} \) is precompact in \( \mathbb{P} \mathcal{Q} \).

Remark 4.11. The compact space \( \overline{\mathcal{Q}}_{g,n} \) contains possibly nonintegrable quadratic differentials, because simple poles can merge to a higher order in the limiting process. Still, every pole in a limit is necessarily of finite order.

4.2. Mass distribution of quadratic differentials. Let \( \epsilon_2 \) be the Margulis constant for hyperbolic surfaces. For a Riemann surface \( X \) of finite hyperbolic type, the thick-thin decomposition is always carried out in its conformal hyperbolic metric. Fix a parameter \( \epsilon \in (0, \epsilon_2) \) for the decomposition. Denote the \( \epsilon \)-thin part of \( X \)

\(^{11}\)Still, we mean the holomorphic ones, as in Section 3.
Proposition 4.15. Fix \( \varepsilon \) for all sufficiently small \( \varepsilon \).

Proof. Suppose by contradiction that there is a sequence \( \varepsilon > 0 \) such that for every \( \varepsilon \) Riemann surface \( X \), then for every \( \phi \in \Omega(X) \),

\[
\int_{D(x,\varepsilon)} |\phi| \leq c \int_{D(x,\varepsilon)} |\phi|.
\]

Proof. Suppose by contradiction that there is a sequence \( X_n, x_n, r_n, \phi_n \) such that \( \int_{D(x_n,\varepsilon)} |\phi_n| > n \int_{D(x_n,\varepsilon)} |\phi_n| \). For each \( n \), as \( D(x_n, r_n) \) is embedded in \( X_n \), we can scale a conformal metric of constant negative curvature in \((-1,0)\), such that \( D(x_n, r_n) \) has radius at least 1 but is also uniformly bounded in terms of \((g,n)\) in the new metric. By choosing a baseframe \( v_n \) over every \( x_n \), we have a sequence \((X_n, v_n, \phi_n) \in \mathbb{P} \mathcal{L}_{g,n}\). By Theorem 4.10, we can assume \((X_n, v_n, \phi_n) \to (X, v, \phi)\) geometrically with \( \phi \neq 0 \) by passing to a subsequence and rescaling \( \phi_n \). Furthermore, by compactness of Chabauty topology, the closed sets \( D(x_n, r_n) \) subconverge to an embedded disk \( D(x, r) \) centered at the basepoint \( x \), as the radii are scaled to be uniformly bounded. By continuity, the ratio of \( \int_{D(x,\varepsilon)} |\phi| \) to \( \int_{D(x,\varepsilon)} |\phi| \) is infinite, contradicting that \( D(x,\varepsilon) \) is compact and \( \phi \neq 0 \). \( \square \)

Lemma 4.13. For every \( \delta > 0 \), there is an \( r = r(\delta) > 0 \), such that whenever \( f(z) \) is a holomorphic function on a neighborhood of \( \mathbb{D} \), then

\[
\int_{D_r} |f(z)||dz|^2 \leq \delta \int_D |f(z)||dz|^2.
\]

Proof. This follows from the mean value property of holomorphic functions. \( \square \)

Proposition 4.14. Fix \((g,n)\). Then there is a \( \delta = \delta(\varepsilon) > 0 \) with \( \delta \to 0 \) as \( \varepsilon \to 0 \), such that for every \( \varepsilon \) Riemann surface \( X \) of type \((g,n)\) and unit-norm \( \phi \in \Omega(X) \), the \(|\phi|-mass \) of \( X_{\text{cusp}}^\varepsilon \) is at most \( \delta \).

Proof. It suffices to bound the mass of each cuspidal thin neighborhood. By the Margulis Lemma, \( X_{\text{cusp}}^\varepsilon \) consists of \( n \) cuspidal thin neighborhoods. Let \( Z_0 \) be one of them generated by a parabolic transformation \( \gamma \), and \( Z_\varepsilon \) be the component of \( X_{\text{cusp}}^\varepsilon \) contained in \( Z_0 \). As \( \gamma \) conjugates to \( z \to z+1 \) on \( \mathbb{H} \), under \( z \to e^{2\pi i z} \) and rescaling we see that the Riemann surfaces \( Z_\varepsilon \subset Z_0 \) are conformally equivalent to \( \mathbb{D}_r^* \subset \mathbb{D}^* \), where \( r = r(\varepsilon) \in (0,1) \) is a continuous function of \( \varepsilon \) such that \( r \to 0 \) as \( \varepsilon \to 0 \). Moreover, \( \phi \) is carried to a meromorphic quadratic differential \( f(z) \) \( dz^2 \), defined on a neighborhood of \( \mathbb{D} \), with at most a simple pole at \( 0 \). As a double covering around the puncture resolves the simple pole, we can assume \( f \) is holomorphic. Now for any \( \delta > 0 \), Lemma 4.13 implies that \( \int_{D_r} |f| \leq \delta \int_D |f| \), or equivalently \( \int_{Z_\varepsilon} |\phi| \leq \delta \int_{Z_0} |\phi| \) for all sufficiently small \( \varepsilon \). Summing over all components, we conclude that the \(|\phi|-mass \) of \( X_{\text{cusp}}^\varepsilon \) is at most \( \delta \) for all sufficiently small \( \varepsilon \). \( \square \)

Proposition 4.15. Fix \((g,n)\). Then there is a constant \( \xi = \xi(\varepsilon) > 0 \) such that whenever a \( \varepsilon \) Riemann surface \( X \) of type \((g,n)\) has a geodesic of hyperbolic length less than \( \varepsilon/2 \), then the \(|\phi|-mass \) of \( X_{\text{cusp}}^\varepsilon \) is at least \( \xi \) for every unit-norm \( \phi \in \Omega(X) \).
Proof. Suppose otherwise that there is a sequence \( X_n, \phi_n \) such that the \(|\phi_n|-mass\) of \( (X_n)_{good}^\epsilon \) tends to 0. As the mass of the cuspidal thin part is uniformly bounded above by Proposition 4.14, and as the number of components of the \( \varepsilon/2 \)-thick part is bounded in terms of \((g, n)\), we can find a component \( Z_n \) of the \( \varepsilon/2 \)-thick part of \( X_n \) with \( \int_{Z_n} |\phi_n| \geq \nu \) for a global constant \( \nu > 0 \). Now choose a baseframe \( v_n \) over \( Z_n \) for each \( n \), and scale a conformal metric of constant curvature \(-\kappa\) for every \( X_n \), where \( 0 < \kappa < \min(1, \varepsilon^2/4) \) is a global constant. By Theorem 4.10, we can assume \( (X_n, v_n, c_n \phi_n) \to (X, v, \phi) \) geometrically for some \( \phi \neq 0 \). Moreover by continuity, \( X \) is of constant curvature \(-\kappa\) and \( X_t^{\text{thin}} \neq \emptyset \) in the hyperbolic metric. Now as \( \int_{Z_n} |\phi_n| \in [\nu, 1] \), we can assume \( c_n \equiv 1 \) by passing to a subsequence. By our assumption, this means a limiting \( \varepsilon \)-thin component has zero \(|\phi|-mass\). Now this limiting component is either geodesic or cuspidal, depending on whether the short geodesic pinches off, but both cases contradict that \( \phi \neq 0 \).

\[ \square \]

5. The Theta Operator

The Theta operator \( \Theta \) originates from the Poincaré series of modular groups. In particular, the norm \( ||\Theta|| \) has received attention from both automorphic forms and quasiconformal maps. In this section, we characterize the contraction property of \( \Theta \) in the context of quadratic differentials, in preparation for Section 6.

5.1. A dichotomy on contraction. Let \( p : Y \to X \) be a covering of hyperbolic Riemann surfaces. The \textit{Theta operator} associated to \( p \) is defined by

\[ \Theta_{Y/X} : \mathcal{Q}(Y) \to \mathcal{Q}(X), \quad \Theta_{Y/X}(\phi)|_B = \sum_{p^{-1}B \to Y} (p^{-1})^*(\phi)|_B \]

on each ball \( B \subset X \), where the summation is over all branches of \( p^{-1} : B \to Y \). We will often suppress the subscript \( Y/X \) that indicates the covering. The integrability of \( \phi \) ensures uniform convergence on compact subsets, so \( \Theta \) is well-defined.

Pullback via \( p \) induces an embedding \( \rho : \mathcal{T}(X) \hookrightarrow \mathcal{T}(Y) \). For a quasiconformal map \( f : X \to X' \) with Beltrami differential \( \mu \), the covering \( p \) induces a lift of \( f \) to \( \tilde{f} : Y \to Y' \) with Beltrami differential \( p^*\mu \). In view of Theorem 3.38, the derivative \( dp \) is pullback of Beltrami differentials, and \( \Theta \) is the coderivative \( dp^* \) of \( \rho \) such that \( \langle dp^*(\rho) \phi, Y \rangle = \langle \mu, d\rho^*(\phi) \rangle_X \) for \( \mu \in \mathcal{M}(X) \), \( \phi \in \mathcal{Q}(Y) \).

A fundamental property of Poincaré series is \textit{completeness}. This means \( \Theta \) is always surjective. Write \( B_X := \mathcal{Q}(X)_1 = \{ \phi \in \mathcal{Q}(X) : ||\phi|| < 1 \} \).

**Theorem 5.1** (Ahlfors-Bers). \( \Theta(B_Y) \supseteq \frac{1}{3} B_X \).

**Proof.** See [Gar, §4.3]. \( \square \)

The definition of \( \Theta \) resembles pushforward of measures, except that summation over different sheets may cause cancellation. Hence, \( ||\Theta|| \leq 1 \). Later, we will prove the following dichotomy.

**Theorem 5.2.** Let \( Y \to X \) be a covering of hyperbolic Riemann surfaces. Either

1. the covering is amenable, and \( \Theta(B_Y) = B_X \), or
2. the covering is nonamenable, and \( \Theta(B_Y) \subseteq B_X \).

**Corollary 5.3.** Suppose in addition that \( X \) is of finite type. Either

1. the covering is amenable, \( ||\Theta|| = 1 \), and the embedding \( \mathcal{T}(X) \hookrightarrow \mathcal{T}(Y) \) is a global isometry for the Teichmüller metric, or
Since as the norm of the coderivative of the embedding $T$ and the first part of (2) are easily verified. For the second part of (2), first note that $X$ continuously with respect to $\Theta$. Moreover, an isomorphism $f$ lifts to an isomorphism $h$ on $T(X)$. Moreover, an isomorphism $f_1 : X \to X_1$ lifts to an isomorphism $g_1 : Y \to Y_1$, and $\Theta_{Y_1/X_1} = (f_1^{-1})^* \circ \Theta_{Y/X} \circ (g_1)^*$, so we can choose such a $c$ on $\mathcal{M}(X)$.

\[ \|\Theta_{Y_1/X_1}\| < c([X_1]) < 1 \]

for each $X_1 \in T(X)$ with the covering $Y_1/X_1$ induced by pullback via the marking, where $[X_1]$ is the location of $X_1$ in the moduli space.

**Proof.** Since $X$ is of finite type, $\mathcal{Q}(X)$ is a finite dimensional Banach space, so (1) and the first part of (2) are easily verified. For the second part of (2), first note that as the norm of the coderivative of the embedding $T(X) \to T(Y)$, $\|\Theta_{Y_1/X_1}\|$ varies continuously with respect to $X_1 \in T(X)$, so we can define the function $c$ on $T(X)$. Moreover, an isomorphism $f_1 : X \to X_1$ lifts to an isomorphism $g_1 : Y \to Y_1$, and $\Theta_{Y_1/X_1} = (f_1^{-1})^* \circ \Theta_{Y/X} \circ (g_1)^*$, so we can choose such a $c$ on $\mathcal{M}(X)$.

In terms of quasiconformal maps, the preceding results answer what happens to an extremal Teichmüller map when lifted to a covering surface. In particular, this confirms the following conjecture.

**Corollary 5.4** (Kra’s Theta Conjecture). Let $X$ be a finite-type hyperbolic Riemann surface. Then $\|\Theta_{\mathbb{D}/X}\| < 1$. In particular, if $f : X_1 \to X_2$ is a nonconformal Teichmüller map for some $X_1, X_2 \in T(X)$, then any lift $\mathbb{D} \to \mathbb{D}$ of $f$ is not extremal among quasiconformal maps with the same boundary values.

### 5.2. Amenability: graphs, groups, and coverings.

Amenability arises in many areas of mathematics. Our main concern is coverings, while graphs and groups help with characterization. Although there are many equivalent definitions of amenability, we focus on the ones that suit us. Let $G$ be a simple graph. For any set $V$ of vertices, define its *border* $\partial V$ as the set of vertices of distance 1 to $V$. Say the graph $G$ is *nonamenable* if there is a constant $C > 0$ such that the isoperimetric inequality $|V| \leq C|\partial V|$ holds for any finite subset $V \subset G$; also say the number $1/C$ gives an *expansion bound* for $G$. Otherwise, $G$ is *amenable*.

A locally compact topological group $\Gamma$ is *amenable* if $L^\infty(\Gamma)$ admits a $\Gamma$-invariant mean. We will only consider discrete groups. If $\Gamma$ is finitely generated by $S$, its amenability is equivalent to the amenability of the Cayley graph of $(\Gamma, S)$.

**Remark 5.5.**

1. A manifold $M$ is *open at infinity* if there is a constant $C$ such that $\text{vol}(D) \leq C \cdot \text{vol}(\partial D)$ for every domain $D \subset M$. A nonamenable graph is the counterpart of a manifold that is open at infinity.

2. For a group $\Gamma$ finitely generated by $S$, its Cayley graph essentially conveys the combinatorial properties including amenability, independent of the choice of $S$.

**Definition 5.6.** In the same spirit, amenability is expressed in combinatorial models of a covering $p : Y \to X$ of surfaces. Fix a finite generating set $S$ for $\pi_1(X)$ and form the *coset graph*: take cosets in $\pi_1(X)/\pi_1(Y)$ as vertices, and connect two points if and only if they are related by a left multiplication by a generator.

There is also a finer model. A *net* $\mathcal{E}$ on $X$ is a collection of simply connected open subsets of $X$ whose union is connected, such that for any two $E_1, E_2 \in \mathcal{E}$, $E_1 \cap E_2$ is either empty or connected. To $\mathcal{E}$ we associate its *Čech graph*, defined as the 1-skeleton of the nerve of $\mathcal{E}$. The pullback via $p$ induces a net on $Y$, denoted by $p^*\mathcal{E}$, whose elements are components of $p^{-1}(E)$ for each $E \in \mathcal{E}$.

Say the covering $p : Y \to X$ is *amenable* if any one of the following equivalent conditions holds:
groups are amenable. In particular, abelian groups are solvable.

Examples 5.7. 1. Finite graphs, groups, and coverings are all amenable. Solvable

want to show that the open unit ball in $\mathbb{Q}^n$ such that for each

Subgroups and quotient groups of amenable groups are amenable.

is nonamenable, as the isoperimetric inequality for $C$ such that for each $x \in \Gamma$ we have (i) the word distance $d(x, f(x)) \leq 1$ in the Cayley

a subgraph of an amenable graph containing all the vertices is amenable as well.

is nonamenable, as the isoperimetric inequality for $\mathbb{Q}$.

The fundamental group of a hyperbolic Riemann surface is nonamenable.

Fuchsian group contains a free group of rank at least two and is thus nonamenable.

The fundamental group and the universal covering of a compact Riemannian

manifold satisfy the same isoperimetric inequality. See [GKPS].

The following lemma will be useful. It explicitly measures the amenability of

coverings. The proof is a calculation of Euler characteristics; see [McM1, §4].

Lemma 5.8. Let $Y/X$ be a covering of connected surfaces with finitely generated

fundamental groups $H \to G$ such that $[G:H] = \infty$. Give $G$ the standard presentation as either $\langle a_1, \ldots, a_n \rangle$ or $\langle a_1, b_1, \ldots, a_g, b_g : [a_1, b_1] \ldots [a_g, b_g] = 1 \rangle$. Then the corresponding coset graph of $G/H$ satisfies the isoperimetric inequality $\gamma|V| \leq \partial V$ for any finite set $V$ of vertices, where the expansion bound is taken as

$$
\gamma := \begin{cases} 
2|\chi(X)|, & \text{if } \chi(Y) \geq 0, \\
|\chi(X)/\chi(Y)|, & \text{otherwise}. 
\end{cases}
$$

In particular, when $\chi(X) < 0$, the coset graph is nonamenable with an expansion bound only depending on $\chi(Y)$, and $Y/X$ is nonamenable.

5.3. The amenable case. Fix an amenable covering $p : Y \to X$ of hyperbolic Riemann surfaces, and the universal covering $\mathbb{D} \to X$, so that $\pi_1(X)$ and $\pi_1(Y)$ are represented as Fuchsian groups $\Gamma_Y \subseteq \Gamma_X$ acting on $\mathbb{D}$. For $\Theta : \Omega(Y) \to \Omega(X)$, we want to show that the open unit ball in $\Omega(X)$ is contained in the image of the open unit ball in $\Omega(Y)$. In other words, for every $\psi \in \Omega(X)$, we will find an element in its preimage with norm arbitrarily close to $\|\psi\|$. By Theorem 5.1, there is a $\phi \in \Theta^{-1}(\psi)$. As amenability amounts to admitting an invariant mean, hopefully by selecting a finite number of sheets on $Y$ that convey amenability, we can average $\phi$ over these sheets to make its norm arbitrarily close to $\|\psi\|$, while its image remains unchanged. We now make this precise.

Construction 5.9. (1) We first construct a sequence of finite sets of left cosets 

$$
|\{g\Gamma_Y \in \Gamma_X/\Gamma_Y : F \cdot g\Gamma_Y \subseteq \mathcal{C}_n\}| \to 1 \text{ as } n \to \infty.
$$
Fix a finite generating set $S$ for $\Gamma_X$, so we have an algebraic norm $\|\cdot\|$ that records the minimal length of representations by generators. Filter $\Gamma_X$ by the finite sets $S_n := \{ g \in \Gamma_X : \|g\| \leq n \}$. Then for each $n \geq 1$, the coset graph of $(\Gamma_X / \Gamma_Y, S_n)$ is amenable, so there is a sequence of finite sets of cosets $\mathcal{C}_m \subseteq \Gamma_X / \Gamma_Y$ such that

$$\frac{|\{ g \Gamma_Y \in \Gamma_X / \Gamma_Y : S_n \cdot g \Gamma_Y \subseteq \mathcal{C}_m \}|}{|\mathcal{C}_m|} \rightarrow 1 \text{ as } m \rightarrow \infty.$$  

Take $\mathcal{C}_n := \mathcal{C}_m$. Then any nonempty finite set $F \subseteq \Gamma_X$ is contained in $S_n$ for all sufficiently large $n$. Since the fraction in (5.10) can only be farther away from 1 if $F$ is enlarged, we conclude that (5.10) is satisfied.

(2) For each finite $\mathcal{C}_n$, define an averaging operator $A_n : \Omega(\mathbb{D}) \to \Omega(Y)$ by

$$A_n(\xi) := \frac{1}{|\mathcal{C}_n|} \sum_{g \Gamma_Y \in \mathcal{C}_n} \Theta_{\mathbb{D}/Y}(g^* \xi),$$

where the choice of $g \in g \Gamma_Y$ is irrelevant because $\Theta_{\mathbb{D}/Y} \circ h^* = \Theta_{\mathbb{D}/Y}$ for any $h \in \Gamma_Y$. This invariance also implies that $\Theta_{Y/X} \circ A_n = \Theta_{\mathbb{D}/X}$.

For a finite approximation process, it will be convenient to have more flexibility. Consider $\Omega_D(\cdot)$, the Banach space of $L^1$ measurable quadratic differentials on a Riemann surface, i.e. $\Omega_D(\cdot) = L^1(\cdot, dz^2)$. Then $\Omega(\cdot)$ is a closed subspace of $\Omega_D(\cdot)$, and the aforementioned operators extend naturally to the space $\Omega_D(\cdot)$. Clearly,

$$\Theta_{Y/X} \circ A_n = \Theta_{Y/X} \circ \Theta_{\mathbb{D}/Y} = \Theta_{\mathbb{D}/X} : \Omega_D(\mathbb{D}) \to \Omega_D(X).$$

**Proposition 5.11.** $\|A_n(\xi)\| \to \|\Theta_{\mathbb{D}/X}(\xi)\|$ as $n \to \infty$, for all $\xi \in \Omega_D(\mathbb{D})$.

**Proof.** First, we fix a measurable fundamental domain $\Omega \subseteq \mathbb{D}$ for $\Gamma_X$, which gives a tiling of $\mathbb{D}$ by $\Gamma_X$-translates of $\Omega$. Note that those supported on finitely many tiles form a dense subset of $\Omega_D(\mathbb{D})$. Since both involved operators are bounded, it suffices to prove the statement for any $\xi \in \Omega_D(\mathbb{D})$ with $\text{supp} \xi \subseteq F^* \Omega$, where $F \subseteq \Gamma_X$ is a finite subset. We may assume $F$ contains the identity.

The tiling of $\mathbb{D}$ descends to a tiling of $Y$ such that each tile has preimage $G^* \Omega$ for some coset $G = g \Gamma_Y$. Such a tile on $Y$ supports $A_n(\xi)$ only if $F^{-1} \cdot G$ intersects $\mathcal{C}_n$. If $F^{-1} \cdot G \subseteq \mathcal{C}_n$, call it a filled tile; otherwise, it is unfilled.

On one hand, the number of unfilled tiles of $Y$ is bounded above by

$$|F| \times \left| \{ G \in \mathcal{C}_n : F^{-1} \cdot G \not\subseteq \mathcal{C}_n \} \right| = o(|\mathcal{C}_n|),$$

where the equation follows from the amenability condition (5.10). Since over an unfilled tile, the integral of $|A_n(\xi)|$ is bounded above by $\|\xi\|/|\mathcal{C}_n|$, the total contributions to $\|A_n(\xi)\|$ of the unfilled tiles tend to 0.

On the other hand, $\psi := \Theta_{\mathbb{D}/X}(\xi)$ is obtained by collecting over the finitely many tiles that support $\xi$, superimposing them and adding up the corresponding values of $\xi$. Since on a filled tile, the averaging operation pulls every supporting tile of $\xi$ into $G^* \Omega$, we have $A_n(\xi) = p^* \psi / |\mathcal{C}_n|$. Let $Y_n$ be the union of filled tiles of $Y$. Since the integral of $|p^* \psi|$ over a single tile of $Y$ is $\|\psi\|$, we have as $n \to \infty$,

$$\|A_n(\xi)\| \to \int_{Y_n} |A_n(\xi)| = \frac{|\{ G \in \Gamma_X / \Gamma_Y : F^{-1} \cdot G \subseteq \mathcal{C}_n \}|}{|\mathcal{C}_n|} \|\psi\| \to \|\psi\|. \quad \square$$

**Proof of Theorem 5.2 (1).** Let $\psi \in B_X$. By Theorem 5.1, there is a $\xi \in \Omega(\mathbb{D})$ such that $\Theta_{\mathbb{D}/X}(\xi) = \psi$. By Proposition 5.11, the sequence $\phi_n := A_n(\xi) \in \Omega(Y)$ satisfies $\Theta_{Y/X}(\phi_n) = \psi$ and $\|\phi_n\| \to \|\psi\| < 1$ as $n \to \infty$, so $\|\phi_n\| < 1$ for large $n$. Thus, $B_X \subseteq \Theta_{Y/X}(B_Y)$. The other direction of inclusion is due to $\|\Theta\| \leq 1$. \quad \square
5.4. The nonamenable case. Fix a nonamenable covering \( Y/X \) of hyperbolic Riemann surfaces. For \( \Theta : \Omega(Y) \to \Omega(X) \), we want to show that the closure of the image of the open unit ball in \( \Omega(Y) \) is contained in the open unit ball in \( \Omega(X) \).

As in the amenable case, the fiberwise behavior of \( \Theta \) is a good starting point. In fact, once we establish the fiberwise contraction of \( \Theta \) (Theorem 5.12), the completeness of Poincaré series will finish the proof of Theorem 5.2 (2).

**Theorem 5.12.** Fix a unit-norm \( \psi \in \Omega(X) \) and set \( \Omega_\psi(Y) := \Theta^{-1}(\mathbb{C}\psi) \). Then
\[
\|\Theta|_{\Omega_\psi(Y)}\| < 1.
\]

*Proof of Theorem 5.2 (2).* Suppose by contradiction that some \( \psi \in \overline{\Theta(B_Y)} \) has unit norm. Then there is a sequence \( \phi_n \in B_Y \) such that \( \Theta(\phi_n) \to \psi \). By Theorem 5.1, for sufficiently large \( n \) there are \( \phi'_n \in B_Y \) such that \( \Theta(\phi_n + \phi'_n) = \psi \) and \( \phi'_n \to 0 \), but then \( \phi_n + \phi'_n \in \Omega_\psi(Y) \) with \( \|\phi_n + \phi'_n\| \to 1 \), contradicting Theorem 5.12. \( \square \)

The idea for Theorem 5.12 starts locally, because local uniformization facilitates calculation. Roughly, local estimation is conveyed to the global via combinatorics of nonamenability. We now make this precise. As in Lemma 3.39, it is natural to compare \( \phi \in \Omega_\psi(Y) \) with \( \psi \) through the pairing \( \langle \psi/|\psi|, \phi \rangle \).

**Definition 5.13.** For a measurable set \( E \subset X \), denote
\[
\langle \mu, \phi \rangle_E = \int_E \mu \phi, \quad \|\phi\|_E = \int_E |\phi|, \quad \|\mu\|_E = \text{ess sup}_{z \in E} |\mu(z)|.
\]

The *pairing efficiency* of \( \|\mu\| = \|\phi\| = 1 \) on \( E \) is defined as \( \langle \mu, \phi \rangle_E/\|\phi\|_E \), which is bounded above by 1; the *inefficiency* on \( E \) is denoted by \( i(E) := 1 - \langle \mu, \phi \rangle_E/\|\phi\|_E \).

For any measurable partition \( \{E_i\} \) of \( X \), the global pairing is expressed as a convex combination of local pairing efficiencies:
\[
\langle \mu, \phi \rangle = \sum_i \|\phi\|_{E_i} \left( \frac{\langle \mu, \phi \rangle_{E_i}}{\|\phi\|_{E_i}} \right).
\]

**Construction 5.15** (Selecting the net). Endow \( X \) with the metric \( |\psi| \); for its geometric interpretation see Section 3.2. By nonamenability of the covering \( Y/X \), there is a finitely generated subgroup \( \Gamma < \pi_1(X) \) with nonamenable coset graph. Choose a basepoint \( x \) and simple closed smooth curves \( \alpha_1, \ldots, \alpha_k \) through \( x \) that represent a finite generating set for \( \Gamma \); we can avoid all singularities because there are only finitely many of them. By compactness, the \( |\psi| \)-injectivity radius over \( \bigcup_i \alpha_i \) is bounded below by \( 5r \) for some \( r > 0 \). Cover each \( \alpha_i \) by a finite number of embedded \( r \)-disks whose centers are on \( \alpha_i \), and collect them over \( i \) as \( \mathcal{E} \).

We claim that \( \mathcal{E} \) is a net on \( X \). In fact, any two members that meet are contained in an embedded \( 3r \)-disk, so their intersection is connected, just as in the Euclidean space; the union of all members is connected by construction; thus \( \mathcal{E} \) is a net.

Moreover, the Čech graph \( G \) of \( \mathcal{F} := p^* \mathcal{E} \) is nonamenable. To see this, identify every edge that indicates the related two disks are in juxtaposition when we cover some \( \alpha_i \) during the construction of \( \mathcal{E} \), and delete all the other edges; then we are left with a subgraph \( G' \) containing all the vertices, which is quasi-isometric to the coset graph of \( \Gamma/(\Gamma \cap \pi_1(Y)) \) by construction. Thus \( G' \) is nonamenable; so is \( G \).

**Notations 5.16.** Let \( \mathcal{A} \) be a family of measurable subsets of a Riemann surface. Then write \( \cup \mathcal{A} := \bigcup_{E \in \mathcal{A}} E \), and write \( \| \cdot \|_{\mathcal{A}} \) as the \( L^1 \)-integral of a quadratic
Lemma 5.17. Fix $\alpha \in (0,1/6)$. For each $F \in \mathcal{F}$, \( c := \pi r^2 / \|\phi\|_F \) satisfies
\[ \|c\psi - p^*\psi\|_F \leq O(i(F)^\alpha). \]

Proof. Note that each embedded $r$-disk $F \in \mathcal{F}$ is uniformized by some holomorphic map $g : \mathbb{D} \to F$ with $r^2 \, dz^2 = g^*(p^*\psi)$, so
\[ \langle \frac{dz}{dz}, g^*\phi \rangle_{\mathbb{D}} = \frac{1}{\|\phi\|_F} \left\langle \frac{\phi}{|\psi|}, \phi \right\rangle_F = 1 - i(F). \]

By Lemma 3.39, for $\alpha \in (0,1/6)$ we have
\[ \|\frac{dz^2}{\pi} - \frac{g^*\phi}{\|\phi\|_F}\|_\mathbb{D} < O(i(F)^\alpha) \).

Pulling back to $F$ via $g^{-1}$ yields the result. \( \square \)

Lemma 5.18. For any subfamily $\mathcal{F}'$ of $\mathcal{F}$, we have
\[ \|\phi\|_{\mathcal{F}'} \geq \frac{1}{d} \sum_{F \in \mathcal{F}'} \|\phi\|_F, \text{ and } \iota(\cup \mathcal{F}') \geq \frac{1}{9} \inf_{F \in \mathcal{F}'} \iota(F). \]

Proof. The first inequality follows from the fact that every point on $Y$ is covered by at most $d$ members of $\mathcal{F}'$. For the second one, note that by Construction 5.15, the center of every $r$-disk in $\mathcal{F}'$ has injectivity radius at least $5r$, so a Vitali cover argument gives a subfamily $\mathcal{V} \subseteq \mathcal{F}'$ of pairwise disjoint members such that
\[ \sum_{V \in \mathcal{V}} \|\phi\|_V = \|\phi\|_{\mathcal{F}'} \geq \frac{1}{9} \|\phi\|_{\mathcal{F}'}. \]

By disjointness, $\iota(\cup \mathcal{V})$ is a convex combination of $\iota(V)$ where $V \in \mathcal{V}$, so $\iota(\cup \mathcal{V}) \geq \inf_{F \in \mathcal{F}'} \iota(F)$. As both the inefficiency and mass of $(\cup \mathcal{F}' - \cup \mathcal{V})$ are nonnegative,
\[ \iota(\cup \mathcal{F}') \geq \iota(\cup \mathcal{V}) \|\phi\|_{\mathcal{F}'} \geq \frac{1}{9} \inf_{F \in \mathcal{F}'} \iota(F). \]

By nonamenability of $Y/X$, the differential $p^*\psi$ is not $L^1$, so $\iota(F) > 0$ for every $F \in \mathcal{F}$ by Lemma 5.17. We partition $\mathcal{F}$ by inefficiencies:
\[ \mathcal{V} := \{ F \in \mathcal{F} : \iota(F) < i_0 \}, \mathcal{W} := \{ F \in \mathcal{F} : \iota(F) \geq i_0 \}. \]

where $i_0 > 0$ is to be determined, but small enough so that $\mathcal{W}$ is nonempty. Applying Lemma 5.18 to $\mathcal{W}$ yields $\iota(\cup \mathcal{W}) \geq i_0/9$. We have
\[ 1 - \|\Theta(\phi)\| = 1 - t = \iota(Y) \geq \iota(\cup \mathcal{W}) \|\phi\|_{\mathcal{W}} \geq \frac{i_0}{9} \|\phi\|_{\mathcal{W}}, \]

\[ \iota(\mathcal{F}) \geq \iota(\cup \mathcal{W}) \|\phi\|_{\mathcal{W}} \geq \frac{i_0}{9} \|\phi\|_{\mathcal{W}} \geq \frac{i_0}{9} \|\phi\|_{\mathcal{W}}. \]
so it suffices to choose $i_0$ and bound $\|\phi\|_W$ from below. We may assume $t \geq 1/2$; otherwise we are done. Since $\Theta(\phi) = t\psi$ and $\mathcal{F} = p^* \mathcal{E}$,

$$(5.19) \quad \|\phi\|_W + \|\phi\|_Y \geq \int_{\mathcal{F}} |\phi| \geq t \int_{\mathcal{F}} |\psi| \geq \frac{1}{2} \sigma.$$ 

To exploit nonamenability of the Čech graph $G$, we need to understand the implication of edges. Whenever two members intersect, they overlap for a definite amount, but then Lemma 5.17 implies that they have comparable mass in case of efficiency. We now make this precise.

**Proposition 5.20.** Every sufficiently small $i_0$ (independent of $\phi$) satisfies that for all $F_1, F_2 \in \mathcal{F}$ that meet, as long as $\iota(F_1) < i_0$, we have

$$\left| \frac{\|\phi\|_{F_1}}{\|\phi\|_{F_2}} - 1 \right| \leq O(\iota(F_1)^\alpha + \iota(F_2)^\alpha).$$

**Proof.** Write $\iota_j = \iota(F_j)$, and $c_j = \pi r^2 / \|\phi\|_{F_j}$. By Lemma 5.17, we have

$$\|(c_1 - c_2)\phi\| \leq O(\iota_1^\alpha + \iota_2^\alpha).$$

Moreover, by Lemma 5.17 and the definition of $\delta$ (see Notations 5.16), we have

$$c_1\|\phi\|_{F_1 \cap F_2} + O(\iota_1^\alpha) \geq c_1\|\phi\|_{F_1 \cap F_2} + \|c_1 \phi - \psi\|_{F_1 \cap F_2} \geq \|p^* \psi\|_{F_1 \cap F_2} \geq \delta \pi r^2.$$ 

Therefore, by comparing the two preceding inequalities, we obtain

$$\left| \frac{\|\phi\|_{F_1}}{\|\phi\|_{F_2}} - 1 \right| = \left| \frac{c_2 - c_1}{c_1} \right| \leq \frac{O(\iota_1^\alpha + \iota_2^\alpha)}{\delta \pi r^2 - O(\iota_1^\alpha)} \leq O(\iota_1^\alpha + \iota_2^\alpha),$$

as long as $\iota_1 < i_0$ and $O(\iota_0^\alpha) < \delta \pi r^2/2$. $\square$

**Proof of Theorem 5.12.** Proceeding with (5.19), it suffices to choose $i_0$ and bound $\|\phi\|_W$ and $\|\phi\|_Y$ from below. By Proposition 5.20, when $i_0$ is sufficiently small, we have for all intersecting $F_1, F_2 \in \mathcal{F}$,

1. $\|\phi\|_{F_2} \geq O(\|\phi\|_{F_1})$, if $F_1 \in \mathcal{V}$;
2. $\|\phi\|_{F_2} \geq \lambda \|\phi\|_{F_1}$, if $F_1, F_2 \in \mathcal{V}$, where $\lambda = 1 - O(\iota_0^\alpha)$.

Now the slow variation of $|\phi|$-mass and nonamenability shall give what we need. This is essentially a combinatorial problem on graph $G$. Identify $\mathcal{F}$ with the vertex set of $G$, and define a discrete finite measure $m$ on $G$ by $m(F) := \|\phi\|_F$ for $F \in G$.

Write $V = \mathcal{V}$ and $W = \mathcal{W}$ as sets of vertices.

For adjacent $x, y \in V$, we have $m(x) \geq \lambda m(y)$, where $\lambda = 1 - O(\iota_0^\alpha) \in (0, 1)$; for $x \in G, y \in V$, we have $m(x) \geq O(m(y))$. As it is hard to control the mass of $V$ as a monolith, we filter $V$ by $V_n := \{x \in V : m(x) \geq \lambda^n\}$ where $n \in \mathbb{Z}$. Then $V_n$ is an increasing sequence of finite sets, and $V_n = \emptyset$ for sufficiently negative $n$. Set $B_n := \partial V_n \cap \partial V, v_n = |V_n|$, and $b_n = |B_n|$. As $m$ varies slowly in $V$, we have $\partial V_n \subset B_n \cup (V_{n+1} - V_n)$, so by nonamenability,

$$b_n + \Delta v_n \geq |\partial V_n| \geq \gamma v_n,$$

where $\Delta v_n := v_{n+1} - v_n$. Now $\partial V$ is filtered by the increasing sequence $\{B_n\}$ where $B_n = \emptyset$ for sufficiently negative $n$. Every $x \in B_n - B_{n-1}$ is adjacent to some $y \in V_n$, and...
so \( m(x) \geq O(\lambda^n) \) and \( m(B_n - B_{n-1}) \geq O(\lambda^n \Delta b_{n-1}) \). Summing over \( n \),

\[
m(\partial V) = \sum_n m(B_{n+1} - B_n) \geq O\left( \sum_n \lambda^{n+1} \Delta b_n \right) = O(-b_n \Delta \lambda^n)
\]

\[
\geq O\left( \sum_n (\Delta v_n - \gamma v_n) \Delta \lambda^n \right) = O\left( (\lambda + \lambda \gamma - 1) \sum_n \lambda^n \Delta v_n \right),
\]

where we have used the fact that \( \Delta \lambda^n < 0 \). On the other hand,

\[
m(V) = \sum_n m(V_{n+1} - V_n) \leq \sum_n \lambda^n \Delta v_n,
\]

because \( m(x) \leq \lambda^n \) for \( x \in V_{n+1} - V_n \). We can choose \( \epsilon_0 > 0 \) so small that it fits both Proposition 5.20 and \( \lambda + \lambda \gamma > 1 \). Combining the two inequalities above,

\[
m(W) \geq m(\partial V) \geq O(m(V)),
\]

completing the proof for the nonamenable case. \( \square \)

5.5. **Refinements in the geometric case.** We now study how \( \|\Theta\| \) can tend to 1 in the nonamenable case. Every surface covering is geometric when pulled back to a suitable finite covering (see [Sco]), while finite coverings induce isometries between Teichmüller spaces. Hence, we will focus on the geometric coverings.

**Definition 5.21.** An embedded subsurface is **incompressible**, if it is connected and the embedding induces a monomorphism on fundamental groups. A covering of connected surfaces \( Y \to X \) is **geometric**, if there is a proper incompressible subsurface \( S \subset X \) such that the \( Y/X \) is the covering of the subgroup \( \pi_1(S) \) of \( \pi_1(X) \). Since \( X \) is connected, we can always assume \( S \) to be connected.

**Proposition 5.22.** Let \( X \) be a hyperbolic Riemann surface of finite type. Then any geometric covering \( Y/X \) is nonamenable.

**Proof.** Suppose \( Y/X \) is a geometric covering for \( S \subset X \). Then \( |\pi_1(X) : \pi_1(S)| = \infty \), and \( \pi_1(X) < \pi_1(Y) = \pi_1(S) \leq 0 \). The proposition follows from Lemma 5.8. \( \square \)

**Example 5.23.** Consider the covering induced by a simple loop. Let \( X \) be a hyperbolic Riemann surface, and \( Z/X \) be the covering induced by a simple closed geodesic \( \alpha \) of hyperbolic length \( l \). A calculation of modulus (see Example 2.10) identifies \( Z \) with the annulus \( \{ z \in \mathbb{C} : 1 < |z| < \exp(2\pi^2/l) \} \). Let \( \phi = \exp|z|^2/z^2 \). Most mass of \( \phi \) lies in the thin part of \( Z \), which injects into the thin part of \( X \). The remainder causes only a small amount of cancellation, so \( 1 - \|\Theta_{Z/X}\| < O(l) \).

Since \( \mathbb{D}/Z \) is an amenable covering, by Theorem 5.2 we have \( \Theta_{\mathbb{D}/Z}(B_0) = B_Z \) and thus \( 1 - \|\Theta_{\mathbb{D}/Z}\| = 1 - \|\Theta_{Z/X}\| < O(l) \). Intuitively, as \( l \to 0 \), \( X \) degenerates towards an infinite cylinder and \( \|\Theta_{\mathbb{D}/X}\| \to 1 \). Moreover, since \( \mathbb{D}/X \) factors through any covering of \( X \), we have \( 1 - \|\Theta_{Y/X}\| < O(l) \) for any \( Y/X \).

**Assumptions 5.24.** Fix a Riemann surface \( X \) of finite hyperbolic type \((g, n)\), and a geometric covering \( p : Y \to X \) for a proper incompressible connected subsurface \( S \subset X \). Each \( X' \in \mathcal{T}(X) \) comes with a geometric covering \( Y'/X' \), which is determined by the subsurface \( S' \), the image of \( S \) under the marking.

For the thick-thin decomposition, we fix a constant \( \varepsilon > 0 \) that is smaller than the Margulis constant for hyperbolic surfaces.
Definition 5.25. The thin part $X_{\text{thin}}$ of $X$ is the locus of points where the injectivity radius is at most $\varepsilon$ in the hyperbolic metric. The liftable part $X_{\text{lift}}$ is the union of those components of $X_{\text{thin}}$ and $X - X_{\text{thin}}$ that isotope into $S$. The amenable part of $X$ is $X_{\text{am}} := X_{\text{thin}} \cup X_{\text{lift}}$. By the definition of a geometric covering, $S$ lifts isomorphically to a unique subsurface of $Y$, so $X_{\text{lift}}$ also has a unique isomorphic lift $\tau(X_{\text{lift}})$ to $Y$. Then the amenable part of $Y$ is $Y_{\text{am}} := \tau(X_{\text{lift}}) \cup p^{-1}(X_{\text{thin}})$.

Remark 5.26. For each $X' \in \mathcal{T}(X)$, there is an induced geometric covering $Y'/X'$, and the corresponding subsurface $S' \subset X'$ is the image of $S$ under the marking. Then the liftable and amenable parts of $X', Y'$ are likewise defined, respectively.

Intuitively, $Y_{\text{am}}$ is the part of $Y'$ converging to an amenable covering. Our next result shows that the only way $||\Theta||$ tends to 1 is by taking advantage of the part of $Y'$ which is becoming more amenable.

Theorem 5.27. Under Assumptions 5.24, suppose there is a sequence $(Y_n/X_n, \phi_n)$, where $X_n \in \mathcal{T}(X)$, $Y_n/X_n$ is the induced covering, and $\phi_n \in \mathcal{Q}(Y_n)$ with $||\phi_n|| = 1$, such that $||\Theta_{Y_n/X_n}(\phi_n)|| \to 1$ as $n \to \infty$. Then the $|\phi_n|$-mass of $(Y_n)_{\text{am}}$ tends to 1.

In other words, if $||\Theta(\phi)||$ is close to 1 for some unit-norm $\phi \in \mathcal{Q}(Y)$, then most mass of $|\phi|$ lies in $Y_{\text{am}}$.

Proof. Suppose by contradiction that the statement fails. By passing to a subsequence we may assume there is a component $Z_n$ of the thick part $X_n - (X_n)_{\text{thin}}$ for each $n$, such that the $|\phi_n|$-mass of $p_n^{-1}(Z_n) - (Y_n)_{\text{am}}$ is larger than $\delta > 0$, where we have used the fact that the thick part of a finite-type hyperbolic surface is compact. Write $\psi_n = \Theta_{Y_n/X_n}(\phi_n)$. Then $||\psi_n||_{Z_n} > \delta$ for sufficiently large $n$, because

\[
\left| \int_{Z_n} \psi_n - \int_{p^{-1}(Z_n)} \phi_n \right| \leq 1 - ||\psi_n|| \to 0.
\]

We will find a problematic geometric limit. Choose a baseframe $v_n$ over $Z_n$ for every $n$, and endow $X_n$ with the conformal metric of constant curvature $-\kappa$, where $\kappa \in (0, 1)$ is independent of $n$ and so small that the injectivity radius over $v_n$ is at least 1; for example, choose $\kappa < \varepsilon^2/2$ where $\varepsilon$ is the constant for the thick-thin decomposition. As $(X_n, v_n, [\psi_n]) \in \mathcal{Z}_{\Phi, \alpha, n}$, the sequence subconverges geometrically to some $(X', \psi', [\psi'])$ where $(X', \psi') \in \mathcal{Z}_{\Phi, \alpha, \delta}$ by Theorem 4.10. Since the curvature of $X_n$ are constant negative, $X$ is of finite hyperbolic type by Construction 4.6. Since $||\psi_n||_{Z_n} \in (\delta, 1]$ with $Z_n$ in the thick part of $X_n$, we can assume $(X_n, v_n, \psi_n) \to (X', \psi', [\psi'])$ geometrically, and that $Z_n$ converges geometrically to a component $Z'$ of the thick part of $X'$, by passing to a subsequence.

Next, construct a net on $Z'$ with the metric $|\psi'|$ as in Construction 5.15, and then add more $r$-disks while avoiding singularities, to get a finite net $\mathcal{E}'$ on $Z'$, such that the $|\psi'|$-mass of $Z' - \cup \mathcal{E}'$ is less than $\delta/2$. For each sufficiently large $n$, we can construct a finite net $\mathcal{E}_n$ of $r$-disks with respect to $|\psi_n|$ on $Z_n$, such that $\mathcal{E}_n \to \mathcal{E}'$ geometrically as a finite collection of closed sets, as in Definition 4.8. Then the geometric data of $\mathcal{E}_n$, i.e. Notations 5.16 (2) – (5), converge to those of $\mathcal{E}'$. Hence, by our fiberwise bounds on $1 - ||\psi_n||$, the expansion bounds $\gamma_n$ for the Čech graph $G_n$ of $p_n \mathcal{E}_n$, i.e. Notations 5.16 (1), must tend to 0.

There is a one-one correspondence between branches of $p_n^{-1}(Z_n)$ and components of the Čech graph $G_n$, for all large $n$. Each corresponding pair is associated with a quasi-isometry between the coset graph for the covering and the component of $G_n$; the coefficients for these quasi-isometries are uniformly determined by $\mathcal{E}'$. Except
for at most one, other branches of \( p_n^{-1}(Z_n) \) are all universal coverings, whose coset graphs share a uniform expansion bound given by Lemma 5.8, where we have used the fact that \( \chi(Z_n) < 0 \) in the thick-thin decomposition. Hence, the \( |\phi_n| \)-mass of all branches corresponding to these infinite components must tend to 0. In addition, as \( \gamma_n \) must tend to 0, the exceptional branch must be an isomorphic lift, i.e. \( Z_n \subset X_{\text{lift}} \); otherwise, it is a geometric covering, whose coset graph has a uniform expansion bound again by Lemma 5.8, contradicting that \( \gamma_n \to 0 \). Now the infinite components constitute \( p_n^{-1}(\cup \delta_n) - (Y_n)_{\text{am}} \), whose \( |\phi_n| \)-mass tends to 0. Yet, by construction of \( E' \) and the geometric limits, the \( |\psi_n| \)-mass of \( Z_n - \cup E_n \) is less than \( \delta/2 \) for all large \( n \), so the \( |\phi_n| \)-mass of \( p_n^{-1}(Z_n - \cup \delta_n) \) must also be less than \( \delta/2 \) for all large \( n \) because \( ||\psi_n||/||\phi_n|| \to 1 \). This contradicts the earlier assumption that the \( |\phi_n| \)-mass of \( p_n^{-1}(Z_n) - (Y_n)_{\text{am}} \) is larger than \( \delta \). \qed

6. The Hyperbolization Theorem

In this section, we prove the nonfibered case of Thurston’s Hyperbolization Theorem, with the groundwork laid in the previous sections. Preliminaries on hyperbolic manifolds are included in Appendix A.

**Theorem 6.1** (Thurston’s Hyperbolization Theorem). Suppose \( M \) is a compact, orientable, irreducible, atoroidal, Haken 3-manifold with empty or toroidal boundary. Then the interior \( M^o \) admits a complete hyperbolic metric of finite volume.

**Definition 6.2.** Let \( M \) be a compact orientable 3-manifold.

1. Say \( M \) is irreducible if every embedded 2-sphere bounds an embedded 3-ball in \( M \), or equivalently \( \pi_2(M) = 0 \).
2. Say \( M \) is atoroidal if any map \( T \to M \) from a torus to \( M \) which induces a monomorphism \( \pi_1(T) \to \pi_1(M) \) can be homotoped into \( \partial M \); we assume that \( M \) is neither the interval bundle over the torus nor the solid torus.
3. Say a properly embedded surface \( S \subset M \) is incompressible, if \( S \) is connected, orientable, and topologically neither a disk nor a sphere\(^{12} \), and the embedding induces a monomorphism \( \pi_1(S) \to \pi_1(M) \).
4. Say \( M \) is Haken, if it is irreducible and admits an incompressible surface.

**Remark 6.3.** (1) Since taking a double covering resolves nonorientability, it is reasonable to restrict to the orientable case.

2. The Hyperbolization Theorem holds without the Haken condition, following from the Geometrization Conjecture which lies outside the scope of this paper.

3. We will not deal with the case when \( M \) fibers over the circle. This part of the theorem requires a separate proof which we refer to [Sul1, McM4, Ota2]. The reason will be explained after we formulate the gluing problem.

6.1. The gluing problem. The first step is reduction to a certain fixed point problem, or a gluing problem depending on one’s perspective. Focusing on Riemann surfaces, we content ourselves with a precise formulation of the problem.

The Haken condition provides a special access to the Hyperbolization Theorem because there is a specific way to build up a Haken manifold \( H \). Starting with a Haken manifold, we can keep cutting it along incompressible surfaces and compressing disks until we get a finite union of 3-balls. By reversing this process, we can retrieve \( H \) by starting with a finite union of 3-balls and gluing along subsurfaces.

\(^{12}\) In Definition 5.21 where we define an incompressible subsurface of a surface, we allow disks.
of the boundary for a finite number of times. The surfaces along which we glue in the reversed process are incompressible because the ones along which we cut are incompressible. See [Mor1, Kap] for details.

Following this idea, we construct the hyperbolic structure inductively. The initial 3-balls are clearly Kleinian. At each inductive step, an orbifold trick reduces the gluing process to gluing some \( M \) along the entire \( \partial M \) (see [Mor1, Kap]), where \( M \) is a finite union of geometrically finite Kleinian manifolds with incompressible, quasifuchsian\(^{13}\) boundary. This gluing amounts to an orientation-reversing, fixed-point-free, topological involution \( \tau : \partial M \to \partial M \), which is called the gluing instruction. To complete the induction, it suffices to hyperbolize \( M/\tau \).

Let \( M_0 \) be a component of \( M \). Let \( GF(M_0) \) be the quasiconformal deformation space \( \mathcal{T}(\pi_1(M_0)) \) of the Kleinian group \( \pi_1(M_0) \). Since \( M_0 \) is geometrically finite with incompressible boundary, there is a natural identification \( GF(M_0) = \mathcal{T}(\partial M_0) \) by Theorem 3.42. See Section 3.6. Now \( GF(M) \) is the product of \( GF(M_0) \) over all components \( M_0 \), so \( GF(M) = \mathcal{T}(\partial M) \) is a finite-dimensional complex manifold equipped with the infinitesimal Teichmüller metric.

The idea for hyperbolization of \( M/\tau \) is to deform \( M \) to some \( N \in GF(M) \) so that by sewing the ends under \( N \to N/\tau \), the manifold \( N/\tau \) can somehow inherit that of \( N \). For simplicity we assume that \( N \) is connected. Let \( \Gamma := \pi_1(N) \), and let \( \Lambda \) and \( \Omega \) be the limit set and domain of discontinuity of \( \Gamma \), respectively. The sewing should be carried out on the convex core \( CC(N) = CH(\Lambda)/\Gamma \) so as to yield finite volume. For each component \( X \) of \( \partial N = \Omega/\Gamma \), fix a component \( U \) of \( \Omega \) that uniformizes \( X \) as \( X = U/\pi_1(X) \). Then the collar \( C_0(X) := CH(\partial U)/\pi_1(X) \) sits isometrically in \( CC(N) \). If \( \tau \) induces isometries between the collars in \( CC(N) \), then identifying them via \( \tau \) yields a complete hyperbolic manifold \( CC(N)/\tau \) which is homeomorphic to \( N/\tau \). As in Figure 1, all the rank one cusps of \( N \) join rank two cusps after gluing, so \( CC(N)/\tau \) has finite total volume and hyperbolizes \( M/\tau \). This process works the same for a possibly disconnected \( N \in GF(M) \).

To make this precise, let the skinning map \( \sigma : \mathcal{T}(\partial M) \to \mathcal{T}(\partial \bar{M}) \) record the conformal structure of the new ends that appear when we form the quasifuchsian covering spaces for each component of \( \partial M \). Meanwhile, by abuse of notation, the gluing instruction induces an isometry \( \tau : \mathcal{T}(\partial \bar{M}) \to \mathcal{T}(\partial M) \) via pullback. Note that in the discussion above, the collar \( C_0(X) \) is exactly the convex core \( CC(\pi_1(X)) \) of the quasifuchsian group \( \pi_1(X) \). Hence, isometric sewing of collars occurs if and only if \( \tau \circ \sigma(N) = N \) for some \( N \in GF(M) = \mathcal{T}(\partial M) \). In conclusion, the gluing problem amounts to finding a fixed point for the map \( \tau \circ \sigma : \mathcal{T}(\partial M) \to \mathcal{T}(\partial M) \).

We will only deal with the nonfibered case, which means some component of \( M \) is not an interval bundle over a surface. This case will then follow from:

**Theorem 6.4.** Suppose \( M \) is a finite union of geometrically finite Kleinian manifolds with incompressible, quasifuchsian boundary, and that \( M \) is nonfibered. Then the gluing problem has a solution if and only if \( M/\tau \) is atoroidal.

\(^{13}\)In our case, \( \partial M \) being quasifuchsian is equivalent to the absence of accidental parabolics in \( M \). This condition means that if \( \gamma \in \pi_1(X) \) is hyperbolic for a component \( X \) of \( \partial M \), then \( \gamma \) must be loxodromic as an element in \( \pi_1(M) \). See [Kap].
Figure 1. A caricature of sewing the ends when \( \tau \circ \sigma (X_1) = X_2 \) as Riemann surfaces of type \((3, 1)\). Color blue indicates the collars, while color red indicates the pairing of punctures supplied by the rank one cusps of \( N \). As a pair of punctures, the cylinder \( P \) is generated by the cusp of \( X_1 \), and homotopes into \( C_0(X_1) \subset QX_1 \). For \( i = 1, 2 \), consider \( P_i \) as a cylinder topologically buried in \( C_0(X_i) \) via the identification \( \tau : C_0(X_1) = C_0(X_2) \). Then for the end of \( N \) corresponding to \( X_i \), \( P_i \) pairs the puncture of \( X_i \) to the puncture of another component of \( \partial N \). Note that \( P_i \) homotopes into \( CC(N) \), so it is buried in \( N/\tau \). After gluing, the two cylinders \( P_1, P_2 \) become homotopic in \( N/\tau \), and hence concatenate to a peripheral cylinder. As \( \tau \) is an involution of \( \partial N \), such a peripheral cylinder eventually joins itself on two ends to form a peripheral torus \( T \) and becomes a rank two cusp in \( CC(N)/\tau \).

If the gluing problem has a solution \( N \), then since \( N/\tau \) is complete hyperbolic, every \( \mathbb{Z} \oplus \mathbb{Z} \) subgroup of \( \pi_1(N/\tau) \) comes from \( \partial M \), so \( M/\tau \) is atoroidal. In what follows, we prove the converse. We might as well assume \( M/\tau \) is connected.

6.2. From topology to pointwise contraction. A natural approach to the gluing problem is to find an iteration of \( \tau \circ \sigma \) that is uniformly contracting, given that \( GF(M) \) is a complete metric space. However, if \( (M = F \times [0, 1])/\tau \) fibers over the circle with monodromy \( \tau \), then \( \sigma \) is isometric, in which case a different approach is needed as mentioned in Remark 6.3 (3).

We are led to study the contraction property of \( \sigma \). Since the definition of \( \sigma \) is componentwise, we will assume \( M \) is connected in this subsection.

Notations 6.5. Let \( N \in GF(M) \), and let \( \Lambda \) and \( \Omega \) be the limit set and the domain of discontinuity of \( \pi_1(N) \), respectively. Each component \( X \) of \( \partial N \) is uniformized by a component \( \Omega(X) \) of \( \Omega \) with stabilizer \( \pi_1(X) \), and \( \Omega(\pi_1(X)) \) is the domain of discontinuity of \( \pi_1(X) \). From these data form two manifolds

\[
CX = (\mathbb{H}^3 \cup \Omega)/\pi_1(X), \quad QX = (\mathbb{H}^3 \cup \Omega(\pi_1(X)))/\pi_1(X).
\]

Then \( CX \subseteq QX \) with the same interior. Construct \( CN \) and \( QN \) by taking the disjoint union of \( CX \) and \( QX \) respectively over all components of \( \partial N \). By definition \( \partial QN = \partial N \cup \sigma(\partial N) \). Let

\[
BN := \bigcup_X (\Omega - \Omega(X))/\pi_1(X)
\]

be a union of subsurfaces of \( \sigma(\partial N) \). Then \( \partial CN = \partial N \cup BN \).
A component of $BN$ is called a spot. Every spot on $\sigma(\partial N)$ covers a component of $\partial N$ in the following way. Let a spot $U$ lie in $\sigma(X)$ for a component $X$ of $\partial N$. Then under the covering of Riemann surfaces

$$(\Omega - \Omega(X))/\pi_1(X) \rightarrow (\Omega - \Omega(N))/\pi_1(N),$$

the spot $U$ covers a component $Y$ of $\partial N$. In the following, we refer to such a covering $U/Y$ as a spot covering. See Figure 2. Each spot covers exactly one component of $\partial N$, while a component of $\partial N$ is generally covered by countably many spots.

![Figure 2](image-url)  

**Remark 6.6.** (1) The inclusion $CX \subset QX$ is proper, for otherwise either $N = QX$ and $M$ is topologically $X \times [0, 1]$, or $N$ admits a degree two covering by $QX$ and $M$ is a twisted interval bundle over the nonorientable surface that is doubly covered by $X$. As a result, every spot is a proper subsurface on a component of $\sigma(\partial N)$.

(2) Each $N \in GF(M)$ admits a quasiconformal map from $M$ by Remark 3.43, so $N$ also has incompressible, quasifuchsian boundary. It follows from the incompressibility of boundary components that $\Lambda$ is connected and every component of $\Omega$ is simply connected. As a result, every spot on $\sigma(\partial N)$ is incompressible.

(3) For a spot covering, if a multiple of a simple essential loop is lifted, then so is this loop, as guaranteed by the Cylinder Theorem.

**Proposition 6.7.** Every spot covering $U/X$ is geometric.

**Proof.** This is clear if $U$ is a disk, a punctured disk, or an annulus. Now suppose $\chi(U) < 0$. Let $U$ lie in $\sigma(Y)$ for a component $Y$ of $\partial N$. Since $\pi_1(U)$ is finitely generated, we can find nontrivial simple loops $\alpha_1, \ldots, \alpha_k$ in $U$ that generate $\pi_1(U)$. Since $U$ is incompressible in $\sigma(Y)$ by Remark 6.6, each $\alpha_i$ homotopes through $QY$ to a nontrivial simple loop $\beta_i$ in $Y$. Moreover under $U/X$, the loop $\alpha_i$ covers a nontrivial simple loop $\gamma_i$ in $X$. Then $\beta_i, \gamma_i$ must be freely homotopic in $N$ because they are both simple. By incompressibility of boundary and spots, $\beta_1, \ldots, \beta_k$ and $\gamma_1, \ldots, \gamma_k$ are subject to the same relations. Form the union of small normal neighborhoods of $\gamma_i$ in $X$, add 2-cells corresponding to the relations of these generators if necessary, and we obtain a proper subsurface of $X$ that determines $U/X$. □

**Proposition 6.8.** There are only finitely many nonsimply connected spots on $BN$.

**Proof.** As punctured disk spots correspond to punctures of $\partial N$, their number is finite. The number of spots of negative Euler characteristic is also finite. In fact, we can choose a simple essential loop in every one of these spots, because spots are...
incompressible. Among these loops, each one is homotopic to at most one of the rest. As the spots are pairwise disjoint, taking the homotopy classes of these loops yields a multicurve, which must have finite cardinality.

Now it suffices to show that there are only finitely many annular spots. Simple nontrivial loops on different annular spots are disjoint, so they either are homotopic or form a multicurve. Hence, it suffices to show that the number of annuli corresponding to the homotopy class of a single loop is finite.

We use the extremal length argument. Let $\alpha$ be an essential loop on a component $X$ of $\partial M$, and $A_i$ be the annulus spots corresponding to $\alpha$. By Example 2.10, the extremal length of the family $\Gamma_i$ of simple nontrivial loops on $A_i$ is $\lambda(\Gamma_i) = 1/\text{mod}(A_i)$. Moreover, $\lambda(\Gamma_i) = \lambda(\Gamma_i')$, where $\Gamma_i'$ is the preimage of $\Gamma_i$ under the spot covering, which consists of simple nontrivial loops in the free homotopy class $[\alpha]$. Let $G$ be the family of simple nontrivial loops in $[\alpha]$ on $X$. Then $\Gamma \supset \bigcup \Gamma_i$, so $\lambda(\Gamma) \leq \lambda(\bigcup \Gamma_i)$. Since the number of different $\Gamma_i'$ is bounded by the topology of $\partial M$, their moduli is bounded below by a constant $c > 0$. As the families $\Gamma_i$ are supported in disjoint measurable subsets of $X$, Proposition 2.12 gives

$$\lambda(\Gamma)^{-1} \geq \lambda(\bigcup \Gamma_i)^{-1} \geq \sum_i \lambda(\Gamma_i)^{-1} = \sum_i \text{mod}(A_i).$$

Since $\lambda(\Gamma)^{-1}$ is finite and $\text{mod}(A_i) \geq c$, the number of $i$ is finite, completing the proof for $M$. Since the topological data are not affected by quasiconformal deformations, the statement follows for all $N \in GF(M)$. 

It will be helpful to work at the infinitesimal level, given Theorem 3.38. Since $\Lambda$ is a null set by Ahlfors’ theorem, each $\mu \in \mathcal{M}(\partial N)$ lifts to a $\pi_1(N)$-invariant differential $\bar{\mu} \in \mathcal{M}(\mathbb{C})$. Then for every boundary component $X$ with $Y = \sigma(X)$, as $\Omega(\pi_1(X)) = \Omega(X) \cup \Omega(Y)$, $\bar{\mu}|_{\Omega(Y)}$ descends to $d\sigma_N(\mu)|_Y$ under the $\pi_1(X)$-action. This definition of $d\sigma_N(\mu)$ comes from how a Beltrami differential deforms a Kleinian group as in Section 3.6.

Another description regards $Y$ itself. As $\mu$ is lifted to $BN$ along the quasifuchsian coverings $CX \to N$, the lift is determined on $\sigma(\partial N)$, because the complement is a null set. This lift is just $d\sigma_N(\mu)$. The quantity

$$\|d\sigma_N\| = \sup_{\|\mu\|=1, \|\phi\|=1} (d\sigma_N(\mu), \phi)_{\sigma(\partial N)} = \sup_{\|\mu\|=1, \|\phi\|=1} (\mu, d\sigma_N^*\phi)_{\sigma(\partial N)}$$

measures the infinitesimal contraction of $\sigma$, where $\mu \in \mathcal{M}(\partial N)$, $\phi \in \Omega(\sigma(\partial N))$. Dually, the coderivative of $\sigma$ is componentwise given by

$$d\sigma_N^* : \Omega(\sigma(\partial N)) \to \Omega(\partial N), \quad d\sigma_N^*\phi|_X = \sum_{U \subseteq U/X} \Theta_U/X(\phi|_U)$$

where $U$ ranges over all the spots that cover $X$. It is through this expression of $d\sigma^*$ that the results in Section 5 come into use. Clearly,

$$\|d\sigma_N\| = \|d\sigma_N^*\| \leq \sup_{U/X} \|\Theta_U/X\| \leq 1.$$

**Theorem 6.9.** There is a continuous function $c : \mathcal{M}(\partial M) \to (0, 1)$ such that

$$\|d\sigma_N\| < c(|\partial N|) < 1$$

for every $N \in GF(M)$, where $[\partial N]$ is the location in the moduli space.
Proof. By Proposition 6.8 and finiteness of \( \partial M \), there are only finitely many topological types of coverings by spots. By Remark 6.6 (2) and Proposition 6.7, each covering is geometric and determined by the same topological data. Therefore by Corollary 5.3, there is a continuous function \( c_0 : \mathcal{M}(\partial M) \to (0,1) \) such that \( \| \Theta_{U/X} \| < c_0([\partial N]) \) for all \( N \in GF(M) \), components \( X \) of \( \partial N \) and spots \( U \) covering \( X \). Therefore, for all \( N \in GF(M) \),
\[
\| d\sigma_N \| \leq \sup_{U/X} \| \Theta_{U/X} \| < c([\partial N]) < 1. \tag{\Box}
\]

6.3. Inefficiency over the thin part. Theorem 6.9 and the Mumford compactness theorem [Mum] suggest that we should study short geodesics on \( \partial N \). To further study the contraction of the skinning map, we shall look into pairing inefficiencies (see Section 5.4) over the thin part. Since the gluing instruction is not involved, we continue to assume \( M \) is connected.

Convention. For clarity, we refer to the unique conformal hyperbolic metric on a Riemann surface as the Poincaré metric, and use \( \rho \) to denote the induced distance. In comparison, we refer to the hyperbolic metric induced by \( \mathbb{H}^3 \) as simply the hyperbolic metric, and use \( d \) to denote the distance. The chordal distance on \( \partial \mathbb{H}^3 = \hat{\mathbb{C}} \) is denoted by \( d_\mathcal{O} \), with respect to the center \( \mathcal{O} \) of the ball \( \mathbb{H}^3 \).

We start with the following two lemmas. The reader may skip them first and return when they are needed.

Lemma 6.10 (Ahlfors). Let \( Q = (\partial_+, \partial_-, \partial) \) be a quasifuchsian manifold. If a loxodromic\(^{14}\) element \( \gamma \in \pi_1(Q) \) has complex translation \( \tau \) on \( \mathbb{H}^3 \), and \( \gamma \) corresponds to a geodesic of length \( \ell \) in the Poincaré metric on \( \partial_+ \), then \( |\tau| \leq 2\ell_+ \).

Proof. Let \( \gamma_\pm \) be the fixed points of \( \gamma \). The torus \( (\hat{\mathbb{C}} - \{ \gamma_+, \gamma_- \})/\langle \gamma \rangle \) is identified with \( \mathbb{C}/(Z\tau \oplus \mathbb{Z}2\pi i) \). The domain of discontinuity descends to two disjoint annuli \( A_\pm \) on \( T \), whose moduli are \( \pi/\ell_\pm \) respectively (see Section 2.2). By Example 2.11,
\[ \frac{\pi}{\ell_+} = \text{mod}(A_+) \leq \left| \frac{2\pi i}{\tau} \right| \leq \frac{2\pi}{|\tau|}. \]
Hence, \( |\tau| \leq 2\ell_+ \). Likewise, \( |\tau| \leq 2\ell_- \). \( \Box \)

Lemma 6.11. Denote \( n_\gamma(s) := \{ x \in \mathbb{H}^3 : d(x, \gamma x) \leq s \} \) for a loxodromic transformation \( \gamma \in \text{Isom}^+(\mathbb{H}^3) \), where \( d \) is the hyperbolic distance.

Fix \( \delta, r > 0 \). Then all sufficiently small \( \varepsilon_0 > 0 \) satisfy: whenever a loxodromic transformation \( \gamma \) has complex translation \( \tau \) with \( |\tau| \leq 2\varepsilon_0 \), then the neighborhood about \( n_\gamma(\varepsilon_0) \) of hyperbolic radius \( r \) is contained in \( n_\gamma(\delta) \).

Proof. Normalize the coordinate so that \( \gamma(z,t) = (e^\tau z, e^{\tau r}t) \) in the upper-half space model. Denote \( \ell \) as the \( t \)-axis, which is also the axis of \( \gamma \). We begin with the observation that \( n_\gamma(s) \) is either empty or a neighborhood about \( \ell \) of certain radius. In fact, if \( x \in n_\gamma(s) \) and \( d(y, \ell) = d(x, \ell) \), then there is a transformation \( \beta \) whose axis is \( \ell \) that sends \( x \) to \( y \). As \( \beta, \gamma \) commute and both preserve \( d \), we have \( d(y, \gamma y) = d(\beta x, \gamma \beta x) = d(x, \gamma x) \leq s \), so \( y \in n_\gamma(s) \).

\(^{14}\)See the discussion after Theorem A.4 for the definition.
Now we calculate the radius \( w = w(s) \) of \( n_\gamma(s) \) explicitly. Let \( k := |z|/t \), which is constant along \( \partial n_\gamma(s) \) by our earlier observation. Then \( w = d((k, 1), (0, \sqrt{t^2 + 1})) \), so \( k = \sinh w \). Now the distance formula on \( s = d(x, \gamma x) \) for \( x \in \partial n_\gamma(s) \) gives
\[
\cosh s = 1 + \frac{1}{2} \left( e^{\Re \tau/2} - e^{-\Re \tau/2} \right)^2 + \frac{|e^\tau - 1|^2}{2 e^{\Re \tau}} (\sinh w)^2.
\]
Write \( \tau = a + ib \) where \( 0 \leq b \leq 2\varepsilon_0 \). The preceding equality then gives
\[
w(s) = \cosh^{-1} \sqrt{\frac{e^a + e^{-a} - 2 \cos b}{e^a + e^{-a} - 2 \cos b}}.
\]
Therefore, as long as \( \varepsilon_0 \) is small enough so that
\[
e^{\varepsilon_0} + e^{-\varepsilon_0} + 4\varepsilon_0^2 < 2 + e^{-2\varepsilon} (e^{\delta/2} - e^{-\delta/2})^2,
\]
we always have
\[
e^{\delta} + e^{-\delta} - 2 \cos b \geq e^{2\varepsilon} (e^{\varepsilon_0} + e^{-\varepsilon_0} - 2 \cos b).
\]
Combining with the expression of \( w(s) \), we conclude \( w(\delta) > w(\varepsilon_0) + r \). \( \square \)

**Notations 6.12.** Denote \( \varepsilon_3 > 0 \) as a constant smaller than the Margulis constants in dimensions 2 and 3, which will be used throughout the section.

For \( \varepsilon \in (0, \varepsilon_3) \), the \textit{thin part} \( \partial N_{\text{thin}}^\varepsilon \) is the locus of points where the injectivity radius is at most \( \varepsilon \) in the Poincaré metric. The \textit{geodesic thin part} \( \partial N_{\text{geod}}^\varepsilon \) is the union of the components of \( \partial N_{\text{thin}}^\varepsilon \) associated to short geodesics. For \( x \in BN \), say \( x \) lies over \( \partial N_{\text{thin}}^\varepsilon \) (resp. \( \partial N_{\text{geod}}^\varepsilon \)), if \( x \) lies in a spot \( U \) over a boundary component \( X \), such that the spot covering \( U/X \) sends \( x \) into \( \partial N_{\text{thin}}^\varepsilon \) (resp. \( \partial N_{\text{geod}}^\varepsilon \)).

Our main result is the following theorem, a substantial step in linking topology and geometry. A hint of topological obstructions is given in (1), while the infinitesimal data of the skinning map is implied in (2).

**Theorem 6.13.** There is a constant \( b > 0 \) such that all sufficiently small \( \varepsilon \) satisfy: for all \( N \in GF(M) \) and \( x \in BN \) lying over \( \partial N_{\text{geod}}^\varepsilon \), either

1. a component of \( \partial N_{\text{geod}}^\varepsilon \) lifts to the spot containing \( x \) along the corresponding spot covering, or
2. for any \( \mu \in M(\partial N), \phi \in \Omega(\sigma(\partial N)) \) with \( \|\mu\| = \|\phi\| = 1 \), there is an embedded disk \( D \) centered at \( x \) in the Poincaré metric on \( \sigma(\partial N) \), such that the inefficiency of the pairing \( \langle d\sigma_N(\mu), \phi \rangle \) on \( D \) is larger than \( b \), i.e. \( \iota(D) > b \) in the notation of Definition 5.13.

We will basically argue by contradiction. But to get a clearer geometric picture, we assume (1) does not hold and proceed to prove (2) for all sufficiently small \( \varepsilon \). We start with the following context: Assumptions 6.14 and the normalization.

**Assumptions 6.14.** Let \( N \in GF(M) \) and \( X \) be a component of \( \partial N \). Suppose a point \( x \in \sigma(X) \) lies over a component of \( \partial N_{\text{geod}}^\varepsilon \) that corresponds to a short geodesic on \( \partial N \). Let this short geodesic represent the loxodromic element \( \gamma \) in \( \pi_1(N) \). The quasifuchsian group \( \pi_1(X) \) uniformizes both \( X \) and \( Y := \sigma(X) \) with limit set \( \Lambda(X) \) and domain of discontinuity \( \Omega(X) \sqcup \Omega(Y) \). Lying over \( x \) there is a \( y \in \Omega \cap \Omega(Y) \) that is translated by \( \gamma \) by a distance at most \( \varepsilon \) in the Poincaré metric on \( \Omega \). We may need to conjugate \( \gamma \) in \( \pi_1(N) \) to obtain \( y \). Applying Lemma 6.10 to the fundamental group of the boundary component that carries \( \gamma \), we deduce that the hyperbolic translation of \( \gamma \) on \( \mathbb{H}^3 \) is less than \( 2\varepsilon \).
We have $\gamma \notin \pi_1(X)$, for otherwise a component of $\partial N^{\varepsilon}_{\text{good}}$ lifts to the spot that contains $x$ along the corresponding spot covering, which is just (1) of Theorem 6.13.

**Construction 6.15 (Normalization).** Let $\varepsilon_0 > 0$ be a constant to be determined. Via the conjugation by a Möbius transformation, we normalize the coordinate on $\mathbb{H}^3 \sqcup \hat{\mathbb{C}}$ so that the fixed points of $\gamma$ are $\gamma_- = 0$ and $\gamma_+ \in \mathbb{C}$, while $\infty \in \Lambda(X)$ and the center $O$ of the hyperbolic ball is translated the fixed distance $\varepsilon_0$ by $\gamma$. Since the translation of $\gamma$ on $\mathbb{H}^3$ is less than $2\varepsilon$, this is possible as long as $\varepsilon < \varepsilon_0/3$.

The plenty details in the context are essentially summarized in Figure 3. As $\varepsilon \to 0$, the limiting picture in mind is that on the Riemann sphere, $\Lambda(X)$ is converging towards $\infty$, while $\gamma_+ \to 0$. Geometrically, $\gamma$ is close to a parabolic transformation that moves $O$ by the distance $\varepsilon_0$.

The universal picture is determined by the parameters $\varepsilon_0, \varepsilon$. We need to show that sufficiently small choice of $\varepsilon_0, \varepsilon$ must lead to (2) of Theorem 6.13. If not, then we can extract geometric limits of $\mu$ and $\phi$ that pair efficiently. Recall that the pairing $\langle \mu, \phi \rangle$ in (3.33) measures the synchronization of the linefields of $\mu$ and $\phi$. This means the limiting linefields of an efficient pairing must perfectly align.

Lifted to the Riemann sphere, the limiting $\mu$ and $\phi$ are interpreted as two differentials invariant under $\pi_1(N)$ and $\pi_1(X)$ respectively. Now in a neighborhood of $0$, the linefield of the limiting $\mu$ must be arranged in horocyclic style, as it is invariant under the parabolic limit of $\gamma$. However, the linefield of the limiting $\phi$ is still a singular foliation. Around $0$, these two linefields must digress from each other for a definite amount as in Figure 4, which will contradict the limiting full pairing efficiency.

Essentially, we need to establish the limiting pattern in Figure 4 in a definite neighborhood of $0$ on $Y$. With the geometric picture in mind, we now put the ideas into solid arguments.

**Proposition 6.16.** For each $R > 0$ there is a constant $\varepsilon_0(R) > 0$ such that for each $\varepsilon_0 \leq \varepsilon_0(R)$, all sufficiently small $\varepsilon$ satisfy: under Assumptions 6.14 and the normalization, the disk $D_R := \{z \in \mathbb{C} : |z| < R\}$ is contained in $\Omega(Y)$.

**Proof.** Consider the convex core $C_0(X) = CH(\Lambda(X))/\pi_1(X)$ of the quasifuchsian manifold $QX$. Each component of $\partial C_0(X)$ has the intrinsic geometry of a complete
hyperbolic surface of area $-2\pi \chi(X)$, so the injectivity radius is bounded above by $L$, where $L$ is a global constant determined by the topology of $\partial M$ as there are only finitely many components. Hence, through any $z \in \partial CH(\Lambda(X))$ there is a $\beta = \beta(z) \in \pi_1(X)$ such that $d(z, \beta z) \leq 2L$.

Denote $n_\gamma(s) := \{x \in \mathbb{H}^3 : d(x, \gamma x) \leq s\}$ as in Lemma 6.11. Fix an $R_1 > 0$. Taking $\delta = \varepsilon_3$ and $r = R_1 + 3L$ in Lemma 6.11 yields an $\varepsilon_0$. Now take $\varepsilon < \varepsilon_0/3$. As the norm of the complex translation of $\gamma$ is at most $2\varepsilon$, the $(R_1 + 3L)$-neighborhood about $n_\gamma(\varepsilon_0)$ is contained in $n_\gamma(\varepsilon_3)$ by Lemma 6.11, so $B_\gamma(R_1 + 3L) \subset n_\gamma(\varepsilon_3)$, as $\mathcal{O} \in n_\gamma(\varepsilon_0)$ by normalization. We claim that the hyperbolic ball $B_\gamma(R_1)$ of radius $R_1$ centered at $\mathcal{O}$ does not meet $CH(\Lambda(X))$. Otherwise, there is a $z$ in $B_\gamma(R_1) \cap \partial CH(\Lambda(X))$ with $B_z(3L) \subset n_\gamma(\varepsilon_3)$, but then $n_\gamma(\varepsilon_3)$ and $n_{\beta_3\gamma}$ meet at $\beta z$, so $\gamma$ and $\beta$ share the same fixed points by the Margulis Lemma, which implies $\gamma \in \pi_1(X)$ and contradicts Assumptions 6.14.

Now choose $R_1$ so large that for all $z \in \mathbb{D}_R$ the geodesic joining $z$ and $\infty$ in $\mathbb{H}^3$ meets $B(R_1)$, and choose $\varepsilon_0(R) = \varepsilon_0$ as above. For all $\varepsilon < \varepsilon_0/3$, since $\infty \in \Lambda(X)$ by normalization, $\Lambda(X)$ must lie outside $\mathbb{D}_R$, i.e. $\mathbb{D}_R$ is contained in $\Omega(Y)$. □

**Proposition 6.17.** All sufficiently small $\varepsilon_0, \varepsilon$ satisfy: under Assumptions 6.14 and the normalization, the unit disk $\mathbb{D}$ injects into $Y = \Omega(Y)/\pi_1(X)$.

**Proof.** Suppose the proposition fails for every small $\varepsilon_0$. Then there is a sequence of $\varepsilon_0, \varepsilon$ tending to 0, and $(N, X, x, y, \gamma)$ under Assumptions 6.14, such that $\mathbb{D}$ does not inject into $Y$. Thus, there is a $\beta \in \pi_1(X)$ with $\beta(\mathbb{D}) \cap \mathbb{D} \neq \emptyset$ for every term. By Proposition 6.16, every $\mathbb{D}_R$ eventually lies in $\Omega(Y)$, so by passing to a subsequence, $\beta$ converges to a parabolic transformation $\beta'$ fixing $\infty$. Since $\beta'(\mathbb{D}) \cap \mathbb{D} \neq \emptyset$ by continuity, we have $d(\beta'\mathcal{O}, \mathcal{O}) < 3$ and eventually $d(\beta\mathcal{O}, \mathcal{O}) < 3$. Take $r = 3$ and $\delta = \varepsilon_3$ in Lemma 6.11. As $\mathcal{O} \in n_\gamma(\varepsilon_0)$, we have $\beta\mathcal{O} \in n_\gamma(\varepsilon_3)$ once $\varepsilon_0$ is sufficiently small. Now $n_\gamma(\varepsilon_3)$ and $n_{\beta_3\gamma}(\varepsilon_3)$ meet at $\beta\mathcal{O}$, contradicting the Margulis Lemma and that $\gamma \notin \pi_1(X)$. □

Once and for all, fix the constant $\varepsilon_0$ such that $\mathbb{D}_2 \subset \Omega(Y)$ in Proposition 6.16, and that Proposition 6.17 is satisfied.

**Proposition 6.18.** For every small $r > 0$, all sufficiently small $\varepsilon$ satisfy: under Assumptions 6.14 and the normalization, $x \in \mathbb{D}_r$.

**Proof.** Suppose by contradiction that there is a sequence of $\varepsilon \to 0$ and tuples $(N, X, x, y, \gamma)$ under Assumptions 6.14 such that $x$ lies outside $\mathbb{D}_r$ for every term. Let $x \in U$ for a component $U$ of $\Omega$, uniformized by $f : U \to \mathbb{H}$ such that $f(x) = 0$. 

---

**Figure 4.** The limiting linefields of $\mu$ (blue) and $\phi$ (red).
Let $d_e$ denote the Euclidean distance on $\mathbb{C}$. By the Koebe 1/4 Theorem on $f^{-1}$ and since $0 \in \partial U$, we have $|f'(f(w))|/4 \leq d_e(0, w)$ for $w \in U$. Hence, the Poincaré metric on $U$ is bounded below:

$$f^*(\frac{|dz|}{1 - |z|^2}) \geq 2|f'(w)| dw \geq \frac{|dw|}{4d_e(0, w)}.$$  

In particular, outside $\mathbb{D}_r$, the Poincaré metric is bounded below by $|dw|/4r$. Since the Poincaré distance $\rho(x, \gamma x) \leq \varepsilon \to 0$, the Euclidean distance between $x, \gamma x$ must tend to 0, so an accumulation point of $x$ should be fixed by any sublimit of $\gamma$. However, by normalization, $\gamma$ can only converge to parabolic transformations fixing 0, which is a contradiction. $\square$

**Notation 6.19.** Let $D(x)$ be the largest closed disk under the Poincaré metric on $\Omega(Y)$, which is centered at $x$ and contained in $\overline{\mathbb{D}}$. Then $D(x)$ is embedded in $Y$.

**Proposition 6.20.** There is a constant $t > 0$ such that all sufficiently small $\varepsilon$ satisfy: under Assumptions 6.14 and the normalization, $\mathbb{D}_t \subseteq D(x)$. 

**Proof.** Let $\rho(z)|dz|$ be the Poincaré metric on $\Omega(Y)$. As noted above, the Koebe 1/4 Theorem implies that $\rho(z) \geq 1/(4d_e(z))$, where $d_e(z)$ is the Euclidean distance between $z$ and $\partial\Omega(Y)$. Since $\mathbb{D}_2 \subseteq \Omega(Y)$, $d_e(z)$ bounded below on $\mathbb{D}$, so $d_e(z_1)/d_e(z_2)$ is bounded above for any $z_1, z_2 \in \mathbb{D}$. Moreover, the Schwarz Lemma implies that $\rho(z)|dz|$ is bounded above by the Poincaré metric on $\mathbb{D}$, i.e. $\rho(z) \leq 4/(4 - |z|^2)$. Therefore, $\rho(z)$ is bounded both above and below on $\mathbb{D}$, so there is a global constant $a' > 1$ such that $\rho(z_1)/\rho(z_2) < a'$ for any $z_1, z_2 \in \mathbb{D}$. The metric $\rho(z)|dz|$ is $a'$-Lipschitz equivalent to $|dz|$, so the statement follows from Proposition 6.18. $\square$

**Proof of Theorem 6.13.** The proofs of Proposition 6.17 and Proposition 6.18 already reflect the idea which we now develop. It will be helpful to visualize the geometric limit as pictures on $\mathbb{H}^2$. Let $i$ denote the indices. Suppose by contradiction that there is a sequence $\varepsilon_i \to 0$, $(N_i, X_i, x_i, y_i, \gamma_i)$ under Assumptions 6.14, and $\mu_i \in \mathcal{M}(\partial N_i)$, $\phi_i \in \mathcal{Q}(\sigma(\partial N_i))$ with $\|\mu_i\| = \|\phi_i\| = 1$, such that the inefficiency of the pairing $\langle \sigma_N(\mu_i), \phi_i \rangle$ on $D_i := D(x_i)$ tend to 0, i.e. $\iota(D_i) \to 0$. By passing to a subsequence, we can assume not only that each $X_i$ has the same finite type $(g, n)$ as $\partial M$ has only finitely many components, but also that $\gamma_i$ converges to a nontrivial parabolic transformation $\gamma'$ that fixes 0 as before.

Denote $\hat{v}$ as the unit vector at $0 \in \mathbb{C}$ pointing towards the positive real axis. Give $Y_i = \sigma(X_i)$ the baseframe $v_i$, defined as the image of $\hat{v}$ under the covering $\Omega(Y_i)/\pi_1(X_i)$. For sufficiently large $i$, $D_i$ is contained in $D_i$ by Proposition 6.20, where $t > 0$ is a constant. Scale a conformal metric on $Y_i$ of constant curvature in the range $[\varepsilon, 1]$, so that $\mathbb{D}_i$ has diameter at least 2. Then under this metric, $D_i$ is an immersed disk centered at the basepoint of radius at least 1, which means the injectivity radius at the basepoint is at least 1. We thus have a sequence $(Y_i, v_i, [\phi_i])$ in $\mathbb{P} \mathcal{Z}_{g,n}$, which is precompact by Theorem 4.10. By passing to a subsequence and rescaling, we obtain a limit $(Y', v', [\phi'])$ with $\phi_i \to \phi'$ geometrically. Meanwhile, we can assume that $D_i$ converge geometrically to a region $D'$ of $\mathbb{C}$, by compactness of Chabauty topology (see Proposition 4.3). Then $\mathbb{D}_i \subseteq D'$, so we can choose a closed disk $D'' \subset \mathbb{D}_i$ tangent to 0, which is $\gamma'$-invariant. Moreover, as $D''$ embeds into $Y'$, $\phi'$ is integrable over $D''$.

We now describe the problem with the limiting picture. Lift $\phi_i$ to a $\pi_1(X_i)$-invariant quadratic differential on $\Omega(Y_i)$, restrict the lift to $D_i$, and extend by 0 to
obtain a $\phi'_i \in L^1(\hat{\mathbb{C}}, dz^2)$. As $D_i$ are embedded in $Y_i$, and $D_i \to D'$ geometrically, we have $\phi'_i \to \phi'|_{D'}$ in $L^1(\hat{\mathbb{C}}, dz^2)$. Moreover, each $\mu_i \in M(\partial N_i)$ lifts to a $\pi_1(N_i)$-invariant differential $\tilde{\mu}_i$ on $\hat{\mathbb{C}}$, which descends to $d\sigma_{N_i}(\mu_i)$ under the $\pi_1(X_i)$-action. As $L^1(\hat{\mathbb{C}}, dz^2)$ is separable, the Banach-Alaoglu theorem ensures that the bounded sequence $\tilde{\mu}_i \in M(\hat{\mathbb{C}}) = L^1(\hat{\mathbb{C}}, dz^2)^*$ converges to a weak* limit $\mu' \in M(\hat{\mathbb{C}})$ passing to a subsequence. As each $\tilde{\mu}_i$ is $\gamma_i$-invariant and has norm at most 1, $\mu'$ is $\gamma'$-invariant and has norm at most 1 by continuity.

Now as the pairing inefficiency of $\langle \tilde{\mu}_i, \phi'_i \rangle$ over $\hat{\mathbb{C}}$ is 0, that of $\langle \mu', \phi'|_{D'} \rangle$ over $\hat{\mathbb{C}}$ is also 0 by continuity. This forces that $\mu' = \overline{\phi'}/|\phi'|$ over $D'$. In particular, $\langle \mu', \phi' \rangle$ must have full efficiency over $D''$. As both $\mu'$ and $D''$ are $\gamma'$-invariant by construction, so is $\phi'|_{D''}$, but this contradicts that $\int_{D''} |\phi'| < \infty$ and $\phi' \neq 0$. □

### 6.4 Uniform contraction for acylindrical manifolds

In the absence of topological obstructions, (1) of Theorem 6.13 never occurs, so (2) establishes a definite amount of inefficiency of the skinning map $\sigma$ over $\partial N_{\text{geod}}^\varepsilon$. This will lead to uniform contraction of $\sigma$, and hence solve the fixed point problem.

**Notations 6.21.** The liftable part $BN_{\text{lift}}^\varepsilon$ (resp. amenable part $BN_{\text{am}}^\varepsilon$) of $BN$ is the union of the liftable part $U_{\text{lift}}^\varepsilon$ (resp. amenable part $U_{\text{am}}^\varepsilon$) of the coverings $U/X$ over all spots $U$ on $\sigma(\partial N)$. In particular, $BN_{\text{am}}^\varepsilon$ is the union of $BN_{\text{lift}}^\varepsilon$ and the total preimage of $\partial N_{\text{thin}}^\varepsilon$. See Definition 5.25.

**Proposition 6.22.** Fix $\varepsilon \in (0, \varepsilon_3)$. Then for each $\eta > 0$, there is a constant $\delta > 0$, such that if $\phi \in Q(\sigma(\partial N))$ has unit norm and $\|d\sigma_N^\varepsilon(\phi)\| \geq 1 - \delta$, then the $|\phi|$-mass of $BN_{\text{am}}^\varepsilon$ is at least $1 - \eta$, for all $N \in GF(M)$.

**Proof.** By Proposition 6.7 and Proposition 6.8, coverings by spots are finitely many and all geometric. Hence by Theorem 5.27, there is a constant $\delta_1 > 0$ such that

$$
\frac{\|\Theta_{U/X}(\phi)\|_X}{\|\phi\|_U} \geq 1 - \delta_1 \implies \frac{\|\phi\|_{U_{\text{am}}^\varepsilon}}{\|\phi\|_U} \geq 1 - \frac{\eta}{2},
$$

for all $N \in GF(M)$ and coverings $U/X$ by spots $U$ on $BN$. Set $\delta = \delta_1 \eta/2$. Suppose $\|d\sigma_N^\varepsilon(\phi)\| \geq 1 - \delta$ for a unit-norm $\phi \in Q(\sigma(\partial N))$. As

$$
1 - \delta \leq \|d\sigma_N^\varepsilon(\phi)\| \leq \sum_{U/X} \|\phi\|_U \cdot \frac{\|\Theta_{U/X}(\phi)\|_X}{\|\phi\|_U},
$$

the $|\phi|$-mass of the union of all spots where $\|\Theta_{U/X}(\phi)\|_X/\|\phi\|_U < 1 - \delta_1$ is bounded above by $\delta/\delta_1$. Hence, summing over all the spots that suit (6.23),

$$
\|\phi\|_{BN_{\text{am}}^\varepsilon} \geq (1 - \eta/2)(1 - \delta/\delta_1) = (1 - \eta/2)^2 \geq 1 - \eta.
$$

Therefore, it suffices to control the mass over the cuspidal thin part $\partial N_{\text{cusp}}^\varepsilon$ of $\partial N$. The preliminary work is already done in Proposition 4.14.

**Notation 6.24.** The geodesic lifted part $\mathcal{L}'(BN)^\varepsilon$ is the union of all possible lifts of components of $\partial N_{\text{geod}}^\varepsilon$ and $\partial N - \partial N_{\text{geod}}^\varepsilon$ to $BN$ along the spot coverings.

**Corollary 6.25.** Fix $\eta > 0$. Then all sufficiently small $\varepsilon$ satisfy: there is a constant $\delta > 0$ such that if $\phi \in Q(\sigma(\partial N))$ has unit norm and $\|d\sigma_N^\varepsilon(\phi)\| \geq 1 - \delta$, then the $|\phi|$-mass of $\mathcal{L}'(BN)^\varepsilon$ is at least $1 - \eta$, for every $N \in GF(M)$. 
Proof. Let \( N \in GF(M), \phi \in \Omega(\partial N) \) with \( \| \phi \| = 1, \psi = d\sigma_N^*(\phi) \), and \( m = \| \psi \| \).

Fix \( \xi, \eta_1 > 0 \). Taking \((g, n)\) over the types of boundary components of \( M \) in Proposition 4.14 shows that all sufficiently small \( \varepsilon \) satisfy: the \( |\psi|\)-mass of \( \partial N_{\text{cusp}}^\varepsilon \) is bounded above by \( \xi m \), for all \( N, \phi \). As \( \|d\sigma^*\| \leq 1 \), this shows that the \( |\phi|\)-mass not over \( \partial N_{\text{cusp}}^\varepsilon \) is bounded below by \((1 - \xi)m\). Meanwhile, by taking \( \eta = \eta_1 \) in Proposition 6.22 we can choose a \( \delta > 0 \) such that if \( m \geq 1 - \delta \), then the \( |\phi|\)-mass of \( BN_{\text{ann}}^\varepsilon \) is bounded below by \( 1 - \eta_1 \). Therefore, the \( |\phi|\)-mass over \( \partial N_{\text{geod}}^\varepsilon \) is bounded below by \((1 - \xi)m - \eta_1\), for all \( N, \phi \) and sufficiently small \( \varepsilon \).

We now use inefficiencies to control the mass that does not lift. Take \( \varepsilon \) so small that Theorem 6.13 is satisfied by some \( b > 0 \). Fix a unit-norm \( \mu \in M(\partial N) \). Let \( E \) be the total preimage of \( \partial N_{\text{geod}}^\varepsilon \). By Theorem 6.13, every \( x \in E \) is the center of an embedded disk \( D_x \) in the Poincaré metric on \( \sigma(\partial N) \) where the pairing \( \langle d\sigma_N(\mu), \phi \rangle \) has inefficiency \( i(D_x) > b \). Taking \( \alpha = 5 \) in Proposition 4.12 guarantees a Vitali cover argument showing that there is a global constant \( a > 0 \) such that the pairing inefficiency over \( E \) is at least \( ab \). As the global pairing inefficiency does not exceed \( 1 - m \), the \( |\phi|\)-mass over \( \partial N_{\text{geod}}^\varepsilon \) which does not lift is bounded above by \((1 - m)/ab\).

Therefore, for all sufficiently small \( \varepsilon \), if \( N, \phi \) satisfy \( m \geq 1 - \delta \), then the \( |\phi|\)-mass of \( \mathcal{L}(BN)^\varepsilon \) is bounded below by

\[
(1 - \xi)m - \eta_1 - \frac{1 - m}{ab} = 1 - \xi - \eta_1 - \left(1 + \frac{1}{ab}\right)\delta + \xi\delta.
\]

Take \( \xi = \eta_1 = \eta/3 \) and \( \delta < \eta/(3 + 3/ab) \), and the statement follows. \( \square \)

**Assumption.** In the rest of this section, we consider the general case that \( M \) may be disconnected. Then at least one component of \( M \) is not an interval bundle over a surface.

**Definition 6.26.** An essential cylinder \( C : (S^1 \times I, S^1 \times \partial I) \to (M, \partial M) \) rests on essential curves in \( \partial M \) but does not homotope into \( \partial M \) as a map of pairs. If \( M \) has an essential cylinder, it is cylindrical; otherwise, \( M \) is acylindrical.

**Remark 6.27.** (1) Peripheral curves are not essential by definition. Hence, the pairings of punctures given by rank one cusps are not essential cylinders.

(2) Since the essentialness of cylinders is invariant under quasiconformal deformations, \( M \) is acylindrical if and only if every \( N \in GF(M) \) is.

(3) \( M \) is acylindrical if and only if fundamental groups of different boundary components only intersect peripherally (i.e. at parabolic elements), which is equivalent to the case that all spots are disks and punctured disks; in particular, no components of \( \partial N_{\text{geod}}^\varepsilon \) lift to \( BN \).

(4) If \( M \) is acylindrical, then every component of \( M \) is acylindrical as well, which therefore cannot be an interval bundle over a surface. Previous results then apply to each component of \( M \).

**Theorem 6.28.** Suppose \( M \) is acylindrical. Then \( \sigma \) is uniformly contracting. In particular, the gluing problem has a unique solution.

**Proof.** Take \( \xi, \eta, \varepsilon > 0 \) that satisfy Corollary 6.25. By acylindricity, \( \mathcal{L}(BN)^\varepsilon \) is empty, so \( \|d\sigma_N^*(\phi)\| < 1 - \delta \) for all \( N \in GF(M) \) and \( \phi \in \Omega(\sigma(\partial N)) \) with \( \| \phi \| = 1 \). Hence \( \sigma \) is uniformly contracting. Then \( \|d(\tau \circ \sigma)\| < 1 - \delta \) as \( \tau \) is isometric. \( \square \)
6.5. Iteration towards toroidality. In this part, we complete the proof. Suppose
the gluing problem has no solutions. We will show that \( M/\tau \) is toroidal.

Absence of fixed points implies that any iteration of \( \tau \circ \sigma \) do not contract uni-
formly, so it must push forward some quadratic differential very efficiently. When
we keep iterating \( \tau \circ \sigma \), the skinning map \( \sigma \) will match up short geodesics on \( \partial M \) via
essential cylinders according to (1) of Theorem 6.13, while the gluing instruction \( \tau \)
will join these cylinders one after another in \( M/\tau \). By the finiteness of \( \partial M \), some
cylinders will eventually join end-to-end as an essential torus, which will show that
\( M/\tau \) is toroidal.

Still, in practice, we need to control the mass distribution of the quadratic differ-
cential carefully so that only (1) of Theorem 6.13 can occur. This will require us
to consider a smaller part of the deformation space.

Recall that \( d_T \) is the Teichmüller distance. Denote
\[
GF(M)_L := \{ N \in GF(M) : d_T(N, \tau \circ \sigma(N)) \leq L \}.
\]
Choose \( L > 0 \) large enough so that \( GF(M)_L \) is nonempty. Then \( GF(M)_L \) is a
closed subspace and thus itself a complete metric space under \( d_T \). We will restrict
\( \tau \circ \sigma \) to \( GF(M)_L \). Note the following:

1. \( GF(M)_L \) is invariant under \( \tau \circ \sigma \), because \( \tau \circ \sigma \) does not increase distance.
2. Let \( N \in GF(M)_L \). Then the Teichmüller geodesic segment between \( N \) and
\( \tau \circ \sigma(N) \) lies in \( GF(M)_L \). This follows easily from the triangle inequality
for \( d_T \). Recall Construction 3.29 for existence of geodesic segments.

If for some \( k > 0 \), there is a uniform bound \( \|d(\tau \circ \sigma)^k\| < c < 1 \) over \( GF(M)_L \), then
under iterations of \( (\tau \circ \sigma)^k \), the forward image of the geodesic segment in \( GF(M)_L \)
between \( N \) and \( \tau \circ \sigma(N) \) has finite total length, giving rise to a fixed point of
\( (\tau \circ \sigma)^k \). By pointwise contraction of \( \tau \circ \sigma \) (Theorem 6.9), this fixed point \( N' \) is
unique, so \( \tau \circ \sigma(N') = N' \) solves the gluing problem. Therefore, by the assumption
that the gluing problem admits no solutions, all iterates of \( \tau \circ \sigma \) fail to contract
uniformly. Dually formulated, for any \( K, \delta > 0 \), there are \( N_k \in GF(M)_L \) and
\( \phi_k \in \Theta(N_k) \) such that \( \|\phi_k\| \in [1, 1 + \delta] \) for \( k = 0, 1, \ldots, K \), where \( N_{k+1} = \tau \circ \sigma(N_k) \)
and \( \phi_{k+1} = d(\tau \circ \sigma)^k(\phi_{k+1}) \). We now choose \( k, \delta \) carefully.

Convention. (1) When it comes to amounts that are related to lengths or distance
on \( \partial N \), we will implicitly use the Poincaré metric on \( \partial N \) unless otherwise stated.
For instance, we sometimes need to consider the Poincaré metric on \( BN \).

(2) When it comes to topological data (e.g. homotopy classes), we will implicitly
identify every \( N \in GF(M) \) with \( M \) via the quasiconformal deformation from \( M \) to
\( N \), which gives a homeomorphism unique up to isotopy. See Remark 3.43.

Notations 6.29. We introduce the following constants and notations.

1. Let \( C \) be the number of components of \( \partial M \).
2. Let \( S \) be an upper bound for the number of disjoint simple closed geodesics
on \( \partial M \).
3. Let \( S' \) be an upper bound for the number of disjoint simple closed geodesics
on \( BM \) in the Poincaré metric on \( BM \), as guaranteed by Proposition 6.8.
4. Let \( K := C + S + S' \).
5. For \( \epsilon \in (0, \epsilon_3) \) and a Riemann surface \( X \) with a closed geodesic \( \alpha \) shorter
than \( \epsilon/2 \) in the Poincaré metric, denote \( X(\alpha) \) as the component corre-
sponding to \( \alpha \) of \( X_{\text{thin}}^\epsilon \).
Choose $\varepsilon > 0$ small enough that satisfies the following conditions.

(a) $\log(\varepsilon) + KL < \log(\varepsilon_3)$.

(b) There is a constant $\xi > 0$, such that whenever a component $X$ of $\partial N_k$ has closed geodesics shorter than $\varepsilon/2$, then one of them, say $\alpha$, satisfies

$$\|\phi_k\|_{X(\alpha)} > \xi\|\phi_k\|_X.$$

This is guaranteed by Proposition 4.15 and finiteness of $\partial M$.

(c) There is a constant $\delta > 0$ such that whenever a nonfibered component $P$ of $N_k$ has $|\phi_k|$-mass at least $\eta$, then the $|\phi_k|$-mass of $\sigma(\partial P) - \mathcal{L}(\sigma(\partial P))^{\varepsilon/2}$ is at most $\eta$. This is guaranteed by Corollary 6.25. Here we define the constant $\eta$ as

$$\eta := \frac{\xi}{C(2K)^K}.$$

By construction, all bounds in Notations 6.29 still work if we replace $M$ by any other $N \in GF(M)_L$. The condition in (6a) comes from the following lemma.

**Lemma 6.30** (Teichmüller). Suppose $A_1, A_2 \in \text{PSL}(2, \mathbb{R})$ are hyperbolic transformations on $\mathbb{H}$ with multipliers $\lambda_1 > 1$ and $\lambda_2 > 1$ respectively. Suppose there is a $K$-quasiconformal homeomorphism $F : \mathbb{C} \to \mathbb{C}$ such that $F \circ A_1 = A_2 \circ F$. Then

$$K^{-1} \log \lambda_2 \leq \log \lambda_1 \leq K \log \lambda_2.$$

**Proof.** We use an extremal length argument. Normalize the fixed points of both $A_i$ as $(A_i)_- = 0, (A_i)_+ = \infty$ for $i = 1, 2$. Then $A_i(z) = \lambda_i z$. Consider the torus $T_i := \mathbb{C}^*/\langle A_i \rangle$. As $F \circ A_1 = A_2 \circ F$, the map $F$ descends to a $K$-quasiconformal homeomorphism $f : T_1 \to T_2$. Denote $\Gamma_i$ as the family of simple loops in the homotopy class corresponding to $A_i$ on $T_i$. Then by viewing $T_i$ as the annulus $\{z : 1 \leq |z| \leq \lambda_i\}$ with boundary identified, Example 2.10 gives $\lambda(\Gamma_i) = \log(\lambda_i)/2\pi$. Now $f$ clearly sends $\Gamma_1$ to $\Gamma_2$, so the result follows from Theorem 2.15. \qed

**Corollary 6.31.** A closed geodesic shorter than $\varepsilon$ on $\partial N_k$ has length at most $\varepsilon_3$ on $\partial N_0$. Therefore, it belongs to one of at most $S$ homotopy classes on $\partial M$.

**Proof.** The first statement follows from Lemma 6.30, while the second one follows from the Margulis Lemma. \qed

**Proposition 6.32.** For some $j \in [0, C]$, there is a closed geodesic $\alpha_j$ on $\partial N_j$ shorter than $\varepsilon/2$, such that the $|\phi_j|$-mass of $\partial N_j(\alpha_j)$ is at least $\xi/C$.

**Proof.** By the choice of $C$, some component $X$ of $\partial N_C$ has $|\phi_C|$-mass at least $1/C$. Although $X$ may lie in a fibered component, at most $C$ iterations of $d(\tau \circ \sigma)^*$ will go from $X$ to a component $Y$ of $\partial N_j$ for some $j \in [0, C]$, such that the $|\phi_j|$-mass of $Y$ is at least $1/C$. As $1/C > \eta$, by the choice in Notations 6.29 (6c), $Y$ must contain some component of $\mathcal{L}(\sigma(\partial N_j))^{\varepsilon/2}$. This component corresponds to a closed geodesic $\gamma$ shorter than $\varepsilon/2$ in the Poincaré metric on $BN_j$. As the inclusion $BN_1 \subset \sigma(\partial N_j)$ is nonincreasing under the Poincaré metric by the Schwarz Lemma, $\gamma$ homotopes to a closed geodesic $\alpha_j$ shorter than $\varepsilon/2$ on $\sigma(\partial N_k)$. \qed

**Proposition 6.33.** Suppose $\alpha_k$ is a closed geodesic on $\partial N_k$ shorter than $\varepsilon/2$, such that the $|\phi_k|$-mass of $\partial N_k(\alpha_k)$ is $m$. If $m' := m/K - \eta > 0$, then $\alpha_k$ lifts to a closed geodesic $\beta_k$ on $BN_k$ of equal length in the Poincaré metric on $BN_k$, such that the $|d\tau^*(\phi_{k+1})|$-mass of $BN_k(\beta_k)$ is at least $m'$.\qed
Proof. If \( \alpha_k \) belongs to a fibered component, then the statement follows from \( m > m' \). Suppose otherwise. As \( \|d\sigma'\| \leq 1 \), the \( |d\tau^*(\phi_{k+1})| \)-mass over \( \partial N_k(\alpha_k) \) is at least \( m \). By the choice of \( \delta \), the \( |d\tau^*(\phi_{k+1})| \)-mass outside the geodesic thin part \( \mathcal{L}(BN_k)^{\varepsilon/2} \) is at most \( C\eta \). Because \( \mathcal{L}(BN_k)^{\varepsilon/2} \) has at most \( S' \) components, one of them has \( |d\tau^*(\phi_{k+1})| \)-mass at least \( (m - C\eta)/S' \geq m/K - \eta \). Let this component correspond to a closed geodesic \( \beta_k \) on \( BN_k \). Then \( \beta_k \) covers \( \alpha_k \) at degree 1, by the Cylinder Theorem. Therefore, the length of \( \beta_k \) in the Poincaré metric on \( BN_k \) is equal to that of \( \alpha_k \) on \( \partial N_k \). \( \square \)

Notation 6.34. For simplicity of notation, relabel as \( j = 0 \) in Proposition 6.32. We will only consider \( k = 0, 1, \ldots, S - 1 \).

Construction 6.35. Construct \( \beta_k, \gamma_k, \alpha_{k+1} \) inductively from \( \alpha_k \) such that

1. \( \alpha_k \) is a closed geodesic on \( \partial N_k \) shorter than \( \varepsilon/2 \);
2. the \( |\phi_k| \)-mass of \( \partial N_k(\alpha_k) \) is at least \( \xi/C(2K)^k \);
3. \( \alpha_k \) lifts to \( \beta_k \) on \( BN_k \);
4. \( \beta_k \) homotopes through \( QN_k \) to a \( \gamma_k \) on \( \partial N_k \); and
5. \( \tau(\gamma_k) = \alpha_{k+1} \).

For \( \alpha_0 \), (1) and (2) hold by Proposition 6.32 and relabelling. Given an \( \alpha_k \) that satisfies (1), (2), apply Proposition 6.33 to yield a \( \beta_k \) that satisfies (3), and define \( \gamma_k, \alpha_{k+1} \) by (4) and (5) respectively. We now verify that (1), (2) also hold for \( \alpha_{k+1} \).

By Proposition 6.33, the \( |d\tau^*(\phi_{k+1})| \)-mass of \( BN_k(\beta_k) \) is at least \( \xi/C(2K)^{k+1} \). By the Schwarz Lemma, the inclusion \( BN_k \subset \sigma(\partial N_k) \) is nonincreasing under the Poincaré metrics. Therefore, \( \beta_k \) is also shorter than \( \varepsilon/2 \) on \( \sigma(\partial N_k) \), and homotopes to a closed geodesic \( \beta'_k \) on \( \sigma(\partial N_k) \) still shorter than \( \varepsilon/2 \). Moreover, \( BN_k(\beta_k) \subseteq \sigma(\partial N_k)(\beta'_k) \), so the \( |d\tau^*(\phi_{k+1})| \)-mass of \( \sigma(\partial N_k)(\beta'_k) \) is at least \( \xi/C(2K)^{k+1} \). Now \( \beta'_k \) homotopes through \( QN_k \) to \( \gamma_k \). As \( \tau \) is isometric, \( d\tau^* \) is just the pullback of differentials, so assertions (1), (2) for \( \alpha_{k+1} \) follow from their validity for \( \beta'_k \), completing the inductive construction.

Proof of Theorem 6.4. By Construction 6.35, we have \( S + 1 \) essential homotopy classes \( [\alpha_0], \ldots, [\alpha_S] \) on \( \partial M \), where each \( N_k \) is canonically identified with \( M \) up to isotopy. As they lie in at most \( S \) classes, two of them are equal, say \( [\alpha_i] = [\alpha_j] \) for \( i < j \). The homotopy from \( \beta_j \) to \( \gamma_k \) in Construction 6.35 (4) maps down to an essential cylinder \( C_k \) in \( M \) connecting \( [\alpha_k] \) to \( [\gamma_k] \). These \( C_k \), where \( i \leq k < j \), are joined end to end by the gluing instruction \( \tau \), and close up to a torus in \( M/\tau \), where the last gluing is given by \( \tau([\gamma_k]) = [\alpha_j] \). This torus is essential, because the meridian lies in the homotopy class of a closed geodesic, while any choice of longitude is not null-homotopic as guaranteed by the gluing instruction \( \tau \). \( \square \)

Appendix A. Fuchsian and Kleinian Manifolds

In this appendix, we list the preliminaries on hyperbolic manifolds. For thorough introduction to related topics, see [Thu2, Rat, Mar2].

In dimension 2, hyperbolic geometry is closely related to complex analysis of one variable. Two models for the 2-dimensional hyperbolic spaces are:

1. the upper-half plane \( \mathbb{H} \), with the metric \( |dz|/3z \); and
2. the open unit disk \( \mathbb{D} \), with the Poincaré metric \( 2|dz|/(1 - |z|^2) \).
Both metrics are complete of constant curvature $-1$, and generalize to higher dimensions easily. The following two classical results in complex analysis will be helpful in comparing the conformal and hyperbolic structures on Riemann surfaces.

**Lemma A.1** (Schwarz Lemma). Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map of the open unit disk. Write $w = f(z)$. Then

$$\frac{|dw|}{1 - |w|^2} \leq \frac{|dz|}{1 - |z|^2},$$

where the equality holds if and only if $f$ is a M"obius transformation of $\mathbb{D}$.

More generally, a holomorphic map of hyperbolic Riemann surfaces does not increase the hyperbolic metric.

**Theorem A.2** (Koebe 1/4 Theorem). Let $f : \mathbb{D} \to \mathbb{C}$ be an injective holomorphic function. Then $f(\mathbb{D})$ contains the open disk centered at $f(0)$ of radius $|f'(0)|/4$.

For $x \in \mathbb{H}^n$ and $\Gamma < \text{Isom}^+(\mathbb{H}^n)$, denote $\Gamma_\epsilon(x)$ as the subgroup generated by those $\gamma \in \Gamma$ that move $x$ by a distance no more than $\epsilon$.

**Lemma A.3** (Margulis Lemma). For every $n \geq 2$, there is a constant $\epsilon_n > 0$, such that whenever $\Gamma < \text{Isom}^+(\mathbb{H}^n)$ is discrete and acts freely, then for every $x \in \mathbb{H}^n$, the subgroup $\Gamma_\epsilon_n(x)$ is either trivial or elementary.

Pushing the Margulis Lemma down to the quotient, we obtain a structural result on hyperbolic manifolds. For simplicity, we describe dimension 2.

**Theorem A.4** (Thick-thin decomposition). Let $\epsilon \in (0, \epsilon_2)$, and $X$ be a hyperbolic surface. Then the thin part $X'_{\epsilon_{\text{thin}}}$, defined as the locus of points where the injectivity radius is at most $\epsilon$, is a disjoint union of neighborhoods of short simple closed geodesics and cusps. Conformally, a geodesic thin neighborhood is an annulus, while a cuspidal thin neighborhood is a punctured disk.

A discrete subgroup of $\text{Isom}^+(\mathbb{H}^2) = \text{PSL}(2, \mathbb{R})$ is called a Fuchsian group, while a discrete subgroup of $\text{Isom}^+(\mathbb{H}^3) = \text{PSL}(2, \mathbb{C})$ is called a Kleinian group. As there is a natural inclusion $\text{PSL}(2, \mathbb{R}) \subset \text{PSL}(2, \mathbb{C})$, a Fuchsian group is sometimes viewed as a Kleinian group, depending on the context. We now discuss Kleinian groups, with some notions easily carried to Fuchsian ones.

The nonidentity elements in $\text{PSL}(2, \mathbb{C})$ falls into three categories: elliptic, parabolic, and loxodromic. Each element $\gamma \in \text{PSL}(2, \mathbb{C})$, considered as a transformation of $\mathbb{H}^3$, has either one or two fixed points on $\partial \mathbb{H}^3$. Say $\gamma$ is elliptic (resp. parabolic) if it has exactly one fixed point in $\mathbb{H}^3$ (resp. on $\partial \mathbb{H}^3 = S^2_\infty$); say $\gamma$ is loxodromic if it has two fixed points.

Let $\Gamma$ be a torsion-free Kleinian group. Then $\Gamma$ contains no elliptic elements. Consider the $\Gamma$-action on $S^2_\infty$. Say $x \in S^2_\infty$ is a limit point of $\Gamma$, if there is a $y \in S^2_\infty$ and a sequence of $\gamma_n \in \Gamma$ such that $\gamma_n(y) \to x$. The limit set $\Lambda(\Gamma)$ of $\Gamma$ is the set of all limit points, while the domain of discontinuity $\Omega(\Gamma) = S^2_\infty - \Lambda(\Gamma)$. If $|\Lambda(\Gamma)| \leq 2$, then $\Gamma$ is elementary. The limit set is closed and $\Gamma$-invariant, and every $\Gamma$-orbit in $\Lambda(\Gamma)$ is dense. In comparison, $\Gamma$ acts freely and properly on $\Omega(\Gamma)$, and the quotient $\Omega(\Gamma)/\Gamma$ is a disjoint union of Riemann surfaces, whose conformal structure is inherited from $S^2_\infty = \hat{\mathbb{C}}$. In particular, we define the Kleinian manifold $N = \mathbb{H}^3 \cup \Omega(\Gamma)/\Gamma$ of $\Gamma$, with the hyperbolic structure on the interior and conformal structure on the boundary.

The following theorems are important. We will need them in Section 6.
Theorem A.5 (Ahlfors). If $\Gamma$ is a geometrically finite Kleinian group, then the limit set $\Lambda(\Gamma) \subset \mathbb{S}_\infty^2$ has either zero or full measure.

Theorem A.6 (Ahlfors Finiteness Theorem). Let $\Gamma$ be a nonelementary, torsion-free, finitely generated, Kleinian group. Then $\Omega(\Gamma)/\Gamma$ is a finite union of Riemann surfaces. Each of them is of finite hyperbolic type.

Theorem A.7 (Cylinder Theorem). Let $N$ be a Kleinian manifold. Suppose $\gamma_1, \gamma_2$ are disjoint essential simple loops on $\partial N$ that are freely homotopic in $N$ but not within $\partial N$. Then there is an essential cylinder embedded in $N$ bounded by $\gamma_1, \gamma_2$.

Suppose instead that the freely homotopic loops are not simple, but $\gamma_i \subset U_i \subset \partial N$, where $U_1, U_2$ are disjoint neighborhoods. Then there are simple loops $\gamma'_i \subset U_i$ that bound an essential cylinder in $\partial N$.

Acknowledgements

I want to sincerely thank Professor Peter May for organizing the 2022 REU at UChicago, for reading the final draft of this paper, and for his valuable comments and corrections. I also want to thank my REU mentor Daniel Mitsutani for reading the first draft and for his suggestions on mathematical writing. Special thanks to Professor Wenyuan Yang, Professor Yi Liu, and Yushan Jiang for some helpful discussions. Meanwhile, I acknowledge the financial support of the elite undergraduate training plan in pure mathematics at Peking University. Finally, I would like to thank all those who have helped make this magical overseas REU experience possible, especially my family.

References


