THE GENERIC VANISHING THEOREM

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Abstract. The generic vanishing theorem asserts that general topologically trivial line bundles on a compact Kähler manifold have vanishing higher cohomology. We present an algebraic proof, due to Hacon, of the generic vanishing theorem on smooth projective complex varieties.

Contents

1. Introduction

Theorems concerning vanishing of cohomologies of line bundles abound. The perhaps most famous one is the Kodaira vanishing theorem, which says that on a smooth projective complex variety, for any ample line bundle L and the canonical bundle ω_X, the higher cohomologies $H^i(X, \omega_X \otimes L)$ vanish for all $i > 0$. The theory of generic vanishing, however, investigates whether such higher cohomologies would vanish without extra conditions on the line bundle L (except it being topologically trivial) or tensoring by the canonical bundle ω_X. The theory of generic vanishing was first developed by Green and Lazarsfeld [2] using Hodge theory and other transcendental methods on compact Kähler manifolds. They proved the following generic vanishing theorem:

Theorem 1.1. *Let X be a compact Kähler manifold and let* $\mathrm{alb} : X \to \mathrm{Alb}(X)$ *be the Albanese mapping of X to its Albanese. Let*

$$S^i(X) = \{L \in \mathrm{Pic}^0(X) \mid H^i(X, L) \neq 0\}.$$

Then

$$\operatorname{codim}_{\mathrm{Pic}^0(X)} S^i(X) \geq \dim \mathrm{alb}(X) - i$$

for all integers i.

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In 2003, Hacon [3] gave an algebraic proof of the generic vanishing theorem on smooth projective varieties using the language of derived categories and the Fourier-Mukai transform. In this paper, we develop the relevant algebraic theory, including the theorem on cohomology and base change, the Picard scheme, the dual abelian variety, and the Fourier-Mukai transform. In the end we prove the generic vanishing thoerem using Hacon's method.

2. Cohomology and Base Change

The theorem on cohomology and base change is an important tool that will be repeatedly used in our discussion. In this section, we list the theorem on cohomology and base change and some of its consequences.

The following theorem can be used to deduce all the useful consequences.

Theorem 2.1. *Let $f : X \to Y$ be a proper morphism of noetherian schemes with $Y = \operatorname{Spec} A$ affine, and $\mathcal{F}$ a coherent sheaf on X that is flat over Y. Then there is a finite complex*

$$K^\bullet : 0 \to K^0 \to K^1 \to \cdots \to K^n \to 0$$

of finitely generated projective A-modules, such that for any A-algebra B, we have

$$H^p(X \times_Y \operatorname{Spec} B, \mathcal{F} \otimes_A B) \cong H^p(K^\bullet \otimes_A B)$$

functorially in B.

Proof. See the second unnumbered theorem in Mumford [7] section 5, Chapter 2. □

Corollary 2.2. *Let $X, Y, f, \mathcal{F}$ be as above except Y doesn't have to be affine. The function $Y \to \mathbf{Z}$ defined by $y \mapsto h^p(X_y, \mathcal{F}_y)$ is upper semicontinuous.*

Recall that a function $\varphi : |Y| \to \mathbf{Z}$ is upper semicontinuous if $\{y \in Y \mid \varphi(y) \geq n\}$ is a closed subset in Y for every integer n. The corollary says the fibral cohomology dimension jumps along closed sets. The notation $\mathcal{F}_y$ means the fiber $\mathcal{F} \otimes_{\mathcal{O}_Y} k(y)$.

Proof. The problem is local on Y, so we reduce to the case where $Y = \operatorname{Spec} A$ is affine and arrange that $K^\bullet$ given by the theorem is a finite complex of free A-modules. Let δ^p be the map $K^p \to K^{p+1}$, then by freeness δ^p is given by a matrix with entries in A. Tensoring by $k(y)$, we see that $\delta^p_y : K^p \otimes_A k(y) \to K^{p+1} \otimes_A k(y)$ are linear map over $k(y)$. By the theorem,

$$h^p(X_y, \mathcal{F}_y) = \dim_{k(y)} K^p \otimes_A k(y) = \dim \ker \delta^p_y - \dim \operatorname{Im} \delta^{p-1}_y.$$

This makes sense since $K^p \otimes_A k(y)$ are all finite dimensional. By rank-nullity, we have

$$\dim \ker \delta^p_y = \dim K^p \otimes_A k(y) - \dim \operatorname{Im} \delta^p_y,$$

so

$$h^p(X_y, \mathcal{F}_y) = \dim K^p \otimes_A k(y) - \dim \operatorname{Im} \delta^p_y - \dim \operatorname{Im} \delta^{p-1}_y$$

The first term is $\operatorname{rank} K^p$, which is constant. The latter terms are $-(\operatorname{rank} \delta^p_y + \operatorname{rank} \delta^{p-1}_y)$. So now it suffices to show that for any i, the set $\{y \in Y \mid \operatorname{rank} \delta^i_y \leq m\}$ is closed for every integer m. But δ^i_y are matrices over the field $k(y)$, and its rank being $\leq m$ is equivalent to all $(m+1) \times (m+1)$ minors vanishing. This is a closed condition on Y since it is described by discriminants of the minors being zero. □

The following corollary is what will be used repeatedly in practice.

Corollary 2.3. *Let $X, Y, f, \mathcal{F}$ be as above, and assume in addition that Y is reduced and connected, but doesn't have to be affine. Then for all p the following are equivalent:*

(1) $y \mapsto h^p(X_y, \mathcal{F}_y)$ is a constant function,
(2) $R^p f_*(\mathcal{F})$ is a locally free sheaf $\mathcal{E}$ on Y, and for all $y \in Y$, the natural map

$$\mathcal{E} \otimes_{\mathcal{O}_Y} k(y) \to H^p(H_y, \mathcal{F}_y)$$

is an isomorphism.

If these conditions hold, then the natural map

$$R^{p-1} f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} k(y) \to H^{p-1}(X_y, \mathcal{F}_y)$$

is an isomorphism for all $y \in Y$.

Some explanations are in order: what is the natural map $R^p f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} k(y) \to H^p(X_y, \mathcal{F}_y)$? In general, consider a base change diagram

$$\begin{tikzcd} X & X' \arrow[l, "g'"'] \arrow[d, "f'"'] \\ Y \arrow[from=u, "f"'] & Y' \arrow[l, "g"'] \end{tikzcd}$$

Let $\mathcal{F}$ be an $\mathcal{O}_X$-module (for the sake of defining the natural base change map, there is no need to assume $\mathcal{F}$ is coherent or flat over Y). Let $\mathcal{F}' = g'^* \mathcal{F}$. Recall that the higher direct image $R^p f_*(\mathcal{F})$ is the sheafification of the presheaf

$$H^p_{\text{pre}}(\mathcal{F}) : U \to H^p(f^{-1}(U), \mathcal{F}).$$

Similarly, the higher direct image $R^p f'_*(\mathcal{F}')$ is the sheafification of the presheaf

$$H'^p_{\text{pre}}(\mathcal{F}) : V \to H^p(f'^{-1}(V), \mathcal{F}').$$

There is a natural map

$$H^p_{\text{pre}}(\mathcal{F})(U) \to H'^p_{\text{pre}}(\mathcal{F})(g^{-1}(U))$$

given by pullback along $f'^{-1}(g^{-1}(U)) \to f^{-1}(U)$. (Here U is always an open set in Y, so the left side is an open set in X', while the right side is an open set in X.) This defines a morphism of presheaves $H^p_{\text{pre}}(\mathcal{F}) \to g_* H'^p_{\text{pre}}(\mathcal{F})$, and by the universal property of sheafification, we get a unique map $R^p f_*(\mathcal{F}) \to g_* R^p f'_*(\mathcal{F}')$. Now using the adjointness of g_* and g^*, this naturally corresponds to a map

$$g^* R^p f_*(\mathcal{F}) \to R^p f'_*(\mathcal{F}').$$

This is the base change map in general.

In the setup of Corollary 2.3, we have $Y' = \{y\}$, $X' = X_y$, and $\mathcal{F}' = \mathcal{F}_y$. The map g is the inclusion $\operatorname{Spec} k(y) \hookrightarrow Y$. Pulling back along this g is the same as tensoring by $\mathcal{O}_{\operatorname{Spec} k(y)}$. Hence, the left side of the above general base change map is

$$R^p f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{O}_{\operatorname{Spec} k(y)} = R^p f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} k(y).$$

The map f' is $X_y = X \times_Y \operatorname{Spec} k(y) \to \operatorname{Spec} k(y)$, and the presheaf $\underline{H}^p(\mathcal{F})$ has only one value $H^p(X_y, \mathcal{F}_y)$. This of course is a sheaf already, so here the right side of the general base change map is just $H^p(X_y, \mathcal{F}_y)$. Indeed, we recover the map stated in the corollary.

proof of Corollary 2.3. See Corollary 2 in Mumford [7] section 5, Chapter 2. □

An interesting consequence is the see-saw theorem:

Corollary 2.4 (See-saw)**.** *Let X be a complete variety over an algebraically closed field k, T any variety over k and L a line bundle on $X \times T$. Then the set*

$$T^0 = \{t \in T : L|_{X\times\{t\}} \text{ is trivial on } X \times \{t\}\}$$

is closed in T. Let $p_2 : X \times T^0 \to T^0$ be the projection. Then $L|_{X\times T^0} \cong p_2^* M$ for some line bundle M on T^0.

Recall the concept of a complete variety:

Definition 2.5. A variety X is called *complete* if for any variety Y, the projection

$$X \times Y \to Y$$

is a closed map.

proof of Corollary 2.4. First observe that a line bundle M on a complete variety X being trivial is equivalent to that $\dim H^0(X, M) > 0$ and $\dim H^0(X, M^{-1}) > 0$. Indeed, assume that

$$\dim H^0(X, M) > 0 \text{ and } \dim H^0(X, M^{-1}) > 0.$$

Then M has a non-zero section σ, which is the same as a non-zero morphism $\mathcal{O}_X \to M$ induced by σ. Similarly, M^{-1} has a non-zero section, and dualizing gives a non-zero morphsim $\tau : M \to \mathcal{O}_X$. Now $\tau \circ \sigma$ is a non-zero map from $\mathcal{O}_X \to \mathcal{O}_X$, i.e. a non-zero global section of $\mathcal{O}_X$. By completeness $\Gamma(X, \mathcal{O}_X) = k$, so such a section is an isomorphism. Hence σ and τ are isomorphisms, implying $M \cong \mathcal{O}_X$ is trivial.

Hence T_1 is the set of points t of T such that

$$\dim H^0(X \times \{t\}, L|_{X\times\{t\}}) > 0 \text{ and } \dim H^0(X \times \{t\}, L^{-1}|_{X\times\{t\}}) > 0.$$

Note that $L|_{X\times\{t\}}$ is the fiber of L at $t \in T$ under the map $X \times T \to T$ (same for L^{-1}), so the semi-continuity of fibral dimension shows that both conditions are closed conditions, so T_1 is closed in T.

For the last statement of Corollary 2.4, we may rename so that T_1 is replaced by T, and L is replaced by $L|_{X\times T_1}$. So we now want to show that $L \cong p_2^* M$ for some line bundle M on T, where $p_2 : X \times T \to T$ is the projection. By the definition of the original T_1, we have that $L|_{X\times\{t\}}$ is trivial for all $t \in T$. This in particular means the function $t \mapsto \dim_{k(t)} H^0(X \times \{t\}, L|_{X\times\{t\}})$ is a constant map. Now Corollary 2.3 says that $M = (p_2)_*(L)$ is locally free on T, and

$$M \otimes_{\mathcal{O}_T} k(t) \cong H^0(X \times \{t\}, L|_{X\times\{t\}}) = k$$

Thus M has local rank 1 everywhere, so it is a line bundle. The adjointness $\mathrm{Hom}(p_2^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}(\mathcal{G}, p_{2*}\mathcal{F})$ gives a natural map $p_2^* M \to L$ by letting $\mathcal{G} = M$ and $\mathcal{F} = L$. This natural map is an isomorphism (check on all stalks, and thus all fibers) since the fibers $X \times \{t\}$ are (geometrically) connected and $L|_{X\times\{t\}} = \mathcal{O}_{X\times\{t\}}$. The geometrically connected fiber condition implies $(p_2)_*\mathcal{O}_{X\times T} \cong \mathcal{O}_T$, so the claim follows. $\square$

Another fundamental tool is the theorem of the cube.

Theorem 2.6. *Let X, Y be complete varieties, Z any variety, and x_0, y_0, z_0 base points on X, Y, and Z. If L is any line bundle on $X \times Y \times Z$, and the restriction of L to each of $\{x_0\} \times Y \times Z$, $X \times \{y_0\} \times Z$ and $X \times Y \times \{z_0\}$ are trivial, then L is trivial.*

Corollary 2.7. *Let X be any variety and Y an abelian variety. Let $f, g, h : X \to Y$ be morphisms. Then for all $L \in \mathrm{Pic}(Y)$, we have*

$$(f+g+h)^*L \cong (f+g)^*L \otimes (f+h)^*L \otimes (g+h)^*L \otimes f^*L^{-1} \otimes g^*L^{-1} \otimes h^*L^{-1}.$$

Here $\mathrm{Pic}(Y)$ is the group of isomorphism classes of line bundles on Y.

Proof. Let $p_i : Y \times Y \times Y \to Y$ be the projection onto the i-th factor. Let $m_{ij} = p_i + p_j : Y \times Y \times Y \to Y$, and $m = p_1 + p_2 + p_3 : Y \times Y \times Y \to Y$.

Consider the line bundle

$$M = m^*L \otimes m_{12}^*L^{-1} \otimes m_{13}^*L^{-1} \otimes m_{23}^*L^{-1} \otimes p_1^*L \otimes p_2^*L \otimes p_3^*L.$$

This is a line bundle on $Y \times Y \times Y$. We attempt to show M is trivial when restricted to $Y \times Y \times \{0\}$ and use the theorem of the cube to deduce M is trivial. So let $\sigma : Y \times Y \to Y \times Y \times Y$ be defined by $\sigma(y, y') = (y, y', 0)$, and the restriction of M to $Y \times Y \times \{0\}$ is just σ^*M.

Let $q_1, q_2 : Y \times Y \to Y$ be the projections, and let $n : Y \times Y \to Y$ be the addition on the abelian variety Y. Pullback commutes with tensor product, so we can compute q^*M term by term. We have

$$\begin{aligned} q^*m^*L = (m \circ q)^*L = (p_2 + p_3)^*L = n^*L \\ q^*m_{12}^*L^{-1} = n^*L^{-1} \\ q^*m_{13}^*L^{-1} = q_1^*L^{-1} \\ q^*m_{23}^*L^{-1} = q_2^*L^{-1} \end{aligned}$$

Last three terms are easy, so

$$q^*M = n^*L \otimes n^*L^{-1} \otimes q_1^*L^{-1} \otimes q_2^*L^{-1} \otimes q_1^*L \otimes q_2^*L \otimes 0^*L \cong \mathcal{O}_{Y \times Y}$$

Therefore M restricted to $Y \times Y \times \{0\}$ is trivial, and using the same argument it is trivial when restricted to $Y \times \{0\} \times Y$ and $\{0\} \times Y \times Y$. By the theorem of cube, M is trivial on $Y \times Y \times Y$. Now we pull back M by the map (f, g, h). On one hand it is just $\mathcal{O}_X$ since M is trivial, but on the other hand it is

$$(f+g+h)^*L \otimes (f+g)^*L^{-1} \otimes (f+h)^*L^{-1} \otimes (g+h)^*L^{-1} \otimes f^*L \otimes g^*L \otimes h^*L.$$

Tensoring by inverses of various terms gives the result. $\square$

Corollary 2.8 (Theorem of the square)**.** *Let X be an abelian variety, and let $T_x : X \to X$ be translation by x. For all line bundles L on X and $x, y \in X$,*

$$(T_{x+y}^*L) \otimes L \cong T_x^*L \otimes T_y^*L.$$

Proof. We apply Corollary 2.7 with $X = Y$, f and g constant maps with images x, y, and h the identity map. Then $f + g + h = T_{x+y}$, $f + h = T_x$, $g + h = T_y$, and $f + g$ is the constant map with image $x + y$. So we have

$$T_{x+y}^*L \cong T_x^*L \otimes T_y^*L \otimes (x+y)^*L \otimes x^*L^{-1} \otimes y^*L^{-1} \otimes L^{-1}.$$

Then by choosing a trivialization at the points x, y, and $x + y$, we get

$$(T_{x+y}^*L) \otimes L \cong T_x^*L \otimes T_y^*L.$$

since the addition map commutes with pullback. $\square$

In view of the theorem of the square, for any line bundle L on X we may define a map

$$\begin{aligned} \phi_L : X &\to \mathrm{Pic}(X) \\ x &\mapsto T_x(L) \otimes L^{-1}. \end{aligned}$$

To see that ϕ_L is a group homomorphism, we check

$$\phi_L(x+y) = (T^*_{x+y}L) \otimes L^{-1} = T^*_x L \otimes T^*_y L \otimes L^{-1} \otimes L^{-1} = \phi_L(x) \otimes \phi_L(y).$$

Note that $\phi_{L_1 \otimes L_2}$ maps a point x to

$$T^*_x(L_1 \otimes L_2) \otimes (L_1 \otimes L_2)^{-1} = T^*_x L_1 \otimes L_1^{-1} \otimes T^*_x L_2 \otimes L_2^{-1} = (\phi_{L_1} + \phi_{L_2})(x),$$

so $L \mapsto \phi_L$ is a group homomorphism $\mathrm{Pic}(X) \to \mathrm{Hom}(X, \mathrm{Pic}(X))$. We denote the kernel of this map by $\mathrm{Pic}^0(X)$. Here are some properties of $\mathrm{Pic}^0(X)$:

Proposition 2.9. *Let X be an abelian variety.*

(1) $L \in \mathrm{Pic}^0(X)$ if and only if $m^*L \cong p_1^*L \otimes p_2^*L$ on $X \times X$, where p_1, p_2 are projections and m is the addition on X.

(2) If $L \in \mathrm{Pic}^0(X)$, then for all schemes S and all morphisms $f, g : S \to X$, we have $(f+g)^*L \cong f^*L \otimes g^*L$.

Proof. For 1, consider the line bundle $n^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$. For any $x \in X$, its restriction to $X \times \{x\}$ is computed by pulling back by the map $y \mapsto (y, x)$. We see that the result is $T^*_x L \otimes L^{-1}$, which is trivial because $L \in \mathrm{Pic}^0(X)$. By the seesaw theorem, $n^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$ is the pullback of a line bundle M on the second factor X. Also, $n^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$ is trivial when restricted to $\{0\} \times X$, so we have that p_2^*M is trivial when restricted to $\{0\} \times X$. This means M pulled back by the map $p_2 \circ (y \mapsto (0, y)) = \mathrm{id}$ is trivial, so M is the trivial line bundle. Thus p_2^*M is trivial too, which shows $n^*L \cong p_1^*L \otimes p_2^*L$.

For 2, pull back the isomorphism in 1 by the map $(f, g) : S \to X \times X$. □

3. The Picard Scheme and the Dual Abelian Variety

Fix an algebraically closed field k, so in the following discussion, the notions of connectedness (resp. reducedness) and geometric connectedness (resp. geometric reducedness) coincide. Considering line bundles on, say, a smooth projective variety over k as independent objects is not as effective as considering families of line bundles together and put a geometric structure on them. Namely, for some k-scheme T, we consider line bundles on $X \times_k T$. A line bundle L on $X \times_k T$ can be thought of as a family of line bundles $\{L|_{X \times \{t\}}\}$ on X. The Picard functor and the Picard scheme is a way of capturing this idea. In this section we can work in a greater generality, so X will be a reduced, connected, and proper k-scheme, possessing a rational point $e \in X(k)$. In particular, any abelian variety will satisfy this condition (they always have a rational point 0).

A problem in putting a geometric structure on families of line bundles is that line bundles have non-trivial automorphisms, so the global geometric construction would suffer from non-well-definedness in the gluing process caused by these automorphisms. To bypass this issue, we consider instead line bundles together with a trivialization:

Definition 3.1. Let T be a k-scheme. Let L be a line bundle on $X \times_k T$. A *rigidification* of L is an isomorphism $\alpha : L|_{e \times T} \cong \mathcal{O}_T$.

So our objects of concern will now be pairs (L, α) on $X \times T$. These are sometimes called *rigidified line bundles.*

Definition 3.2. An isomorphism $(L, \alpha) \to (L', \alpha')$ on $X \times_k T$ is an isomorphism of line bundles $f : L \to L'$ that is compatible with the trivializations:

$$\begin{tikzcd} L|_{e\times T} \arrow[rr, "f|_{e\times T}"] \arrow[dr, "\alpha'"'] & & L'|_{e\times T} \arrow[dl, "\alpha'"] \\ & \mathcal{O}_T & \end{tikzcd}$$

We introduced these definitions in the hope that they can get rid of the automorphisms, and this is indeed the case.

Lemma 3.3. *A rigidified line bundle (L, α) has no non-trivial automorphisms.*

Proof. We first prove that any automorphism of L comes multiplication by units in $\mathcal{O}_T$, i.e. $\mathrm{Aut}(L) = \Gamma(T, \mathcal{O}_T^\times)$. Indeed, an application of the base change theorem on the diagram

$$\begin{tikzcd} X \times T \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d] \\ T \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

shows that

$$\mathcal{O}_T = \mathcal{O}_T \otimes \Gamma(X, \mathcal{O}_X) \cong p_{2*}(\mathcal{O}_{X\times T})$$

Checking on all stalks, this shows that $\mathcal{O}_T^\times \cong p_{2*}(\mathcal{O}_{X\times T}^\times)$. Taking global sections of both sides, we obtain

$$\Gamma(T, \mathcal{O}_T^\times) \cong \Gamma(X \times T, \mathcal{O}_{X\times T}^\times).$$

The right side is equal to $\mathrm{Aut}(L)$ by definition of line bundles as locally free vector bundles of rank 1.

Hence, if f is an automorphism of L that preserves the trivialization α, then $\alpha \circ f|_{e\times T} = \alpha$. Let u be the unit in $\Gamma(T, \mathcal{O}_T^\times)$ that corresponds to f. Then the above equality says $\alpha : L_{e\times T} \to \mathcal{O}_T$ multiplied by u is still α, so $u = 1$ and f is trivial. □

Definition 3.4. Fix X with its rational point e as above. The *Picard functor* is defined to be the following functor from the category of k-schemes to the category of abelian groups

$$T \mapsto \{(L, \alpha) \text{ on } X \times T \text{ identified up to the isomorphism defined above}\}.$$

These are groups whose operation is tensor product and whose identity is $(\mathcal{O}_{X\times T}, \mathrm{id})$.

It is a theorem of Grothendieck (see [1]) that the Picard functor is representable by a k-scheme $\mathrm{Pic}_{X/k}$ that is locally of finite type, and the connected component $\mathrm{Pic}^0_{X/k}$ of the identity is projective if X is smooth projective. Note also that the usual group of isomorphism classes of line bundles on X is $\mathrm{Pic}(X) = \mathrm{Pic}_{X/k}(k)$.

The following alternative description of the Picard functor shows that the choice of the rational point e is irrelevant.

Proposition 3.5. *Let T be a k-scheme and let $p_2 : X \times T \to T$ be the projection onto T. We have an isomorphism of groups*

$$\operatorname{Pic}_{X/k}(T) \to \operatorname{Pic}(X \times T)/p_2^* \operatorname{Pic}(T)$$

by sending (L, α) to L.

Proof. The group structure on both sides are given by tensor products, so the proposed map is a group homomorphism. For injectivity, suppose (L, α) is in the kernel, i.e. $L \cong p_2^* M$ for some line bundle $M \in \operatorname{Pic}(T)$. Then α gives an isomorphism $M = (p_2^* M)|_{e \times T} \cong \mathcal{O}_T$. Therefore $L \cong p_2^* M \cong \mathcal{O}_{X \times T}$. It then follows that α is an automorphism of $(\mathcal{O}_{X \times T}, \mathrm{id})$, and by Lemma 3.3 we have $\alpha = \mathrm{id}$, so $(L, \alpha) = (\mathcal{O}_{X \times T}, \mathrm{id})$ is the identity element.

For surjectivity, for any $L \in \operatorname{Pic}(X \times T)$ we modify it to $L \otimes p_2^*(L|_{e \times T})^{-1}$, which is a still the same element in the quotient. We claim that restricting to $e \times T$ is a canonical trivialization of $L \otimes p_2^*(L|_{e \times T})^{-1}$. To see this, note that restricting to $e \times T$ is the same as pullback by the map $e : T \mapsto X \times T$ given by $t \mapsto (e, t)$. So

$$e^*(L \otimes p_2^*(L|_{e \times T})^{-1}) = e^* L \otimes e^* p_2^*(L_{e \times T}^{-1}).$$

Of course $p_2 \circ e = \mathrm{id}_T$, so we get $L|_{e \times T} \otimes L|_{e \times T}^{-1} = \mathcal{O}_T$. The upshot is that any L can modified so that it has a canonical trivialization, so the surjectivity follows. $\square$

Since $\operatorname{Pic}_{X/k}$ is the representing object of the Picard functor, it comes with a universal line bundle with a universal trivialization (P, α) on $X \times \operatorname{Pic}_{X/k}$. Explicitly, the universal property is the following: given any k-scheme T and a rigidified line bundle (L, θ) on $X \times T$, there exists a unique map $f : T \to \operatorname{Pic}_{X/k}$ such that $f^* P|_{X \times \{f(t)\}} \cong L|_{X \times \{t\}}$.

The dual abelian variety. Now we specialize to the case where $X = A$ is an abelian variety, with the rational point 0. We would like to define the dual abelian variety of A to be $\operatorname{Pic}^0_{A/k}$, but for this to make sense we must prove $\operatorname{Pic}^0_{A/k}$ is smooth. We mention that this is automatic in characteristic 0, since all group schemes in characteristic 0 are smooth.

To prove smoothness, we study the (Zariski) tangent spaces of $\operatorname{Pic}^0_{A/k}$. Since it is a group scheme, it suffices to only study the tangent space $T_0(\operatorname{Pic}^0_{A/k}) = T_0(\operatorname{Pic}_{A/k})$ at 0. Smoothness is then equivalent to the equality of dimensions

$$\dim T_0(\operatorname{Pic}_{A/k}) = \dim \operatorname{Pic}^0_{A/k}.$$

Lemma 3.6. $T_0(\operatorname{Pic}_{A/k}) = H^1(A, \mathcal{O}_A)$.

Proof. By Hartshorne [4] exercise II.2.8 $T_0(\operatorname{Pic}_{A/k})$ is in bijection with the set of scheme morphisms $D := \operatorname{Spec} k[\epsilon]/(\epsilon^2) \to \operatorname{Pic}_{A/k}$ sending the only point of D to 0. Therefore, the tangent space contains rigidified line bundles (L, α) on $A \times D$ such that L is trivial when restricted to A. Now, by Hartshorne [4] exercise III.4.5, the isomorphisms classes of line bundles $\operatorname{Pic}(A \times D)$ is isomorphic to $H^1(A \times D, \mathcal{O}_{A \times D}^\times)$. Let $\mathcal{I}$ be the ideal sheaf $\mathcal{O}_A \otimes (\epsilon) \cong \mathcal{O}_A$, so $\mathcal{I}^2 = 0$. Then exercise III.4.6 in [4] gives a short exact sequence

$$0 \to \mathcal{O}_A \to \mathcal{O}_{A \times D}^\times \to \mathcal{O}_A^\times \to 0$$

where $\mathcal{O}_A \to \mathcal{O}_{A \times D}^\times$ is the truncated exponential. There is an evident map $\mathcal{O}_A \to \mathcal{O}_{A \times D}$ given by send f to $f \otimes 1$, so the short exact sequence splits. Taking the first degree cohomology, we obtain

$$H^1(A \times D, \mathcal{O}_{A\times D}^\times) = H^1(A, \mathcal{O}_A) \oplus H^1(A, \mathcal{O}_A^\times).$$

Recall that the tangent space are those line bundles in $\mathrm{Pic}(A \times D) = H^1(A \times D, \mathcal{O}_{A\times D})$ that is trivial in $\mathrm{Pic}(A) = H^1(A, \mathcal{O}_A^\times)$, so indeed the tangent space is $H^1(A, \mathcal{O}_A)$. □

Now it suffices to show $\dim H^1(A, \mathcal{O}_A) \leq \dim A$. This uses a theorem of Borel on Hopf algebras.

Theorem 3.7 (Borel, see [6])**.** *Let H be a finite-dimensional graded anticommutative Hopf algebra over a perfect field k. Let H^i denote the degree i piece, and let $\mu : H \to H \otimes H$ be the comultiplication. Suppose $H^0 = k$ and $H^r = 0$ for all r greater than some fixed integer n. Suppose further that for all x with* $\deg(x) > 0$ *we have*

$$\mu(x) = 1 \otimes x + x \otimes 1 + \sum_i x_i \otimes y_i$$

where $\deg(x_i), \deg(y_i) > 0$*. Then* $\dim(H^1) \leq n$*, and equality holds if and only if H is isomorphic to the exterior algebra on H^1.*

We want to apply this theorem to the Hopf algebra $H = \bigoplus_i H^i(A, \mathcal{O}_A)$ with $n = \dim A$, so the dimension inequality $\dim H^1(A, \mathcal{O}_A) \leq \dim A$ follows. We must check all conditions.

Proposition 3.8. $\dim H^1(A, \mathcal{O}_A) \leq \dim A$.

Proof. The proof follows the relevant discussion in [6] too. Let $H = \bigoplus_i H^i(A, \mathcal{O}_A)$ and $n = \dim A$. The direct sum gives a grading, and indeed $H^0 = H^0(A, \mathcal{O}_A) = k$ by properness, and $H^r = H^r(A, \mathcal{O}_A) = 0$ for all $r > n$. There is a graded-commutative multiplication given by the cup product: namely, a map $H \otimes H \to H$ defined on pure tensors by $a \otimes b \mapsto a \smile b$.

We next verify the last condition. Let the multiplication map be $m : A \times A \to A$. Then we can factorize the identity map on A in two ways:

$$A \underset{a \mapsto (0,a)}{\overset{a \mapsto (a,0)}{\longrightarrow}} A \times A \xrightarrow{m} A$$

This gives maps of structure sheaves in the reversed direction, and taking the sheaf cohomology we obtain two maps

$$H^\bullet(A, \mathcal{O}_A) \to H^\bullet(A \times A, \mathcal{O}_{A\times A}) \rightrightarrows H^\bullet(A, \mathcal{O}_A)$$

which still compose to the identity map. By the Kunneth formula, we have the isomorphism coming from the cup product

$$H^\bullet(A, \mathcal{O}_A) \times H^\bullet(A, \mathcal{O}_A) \xrightarrow{\sim} H^\bullet(A \times A, \mathcal{O}_{A\times A})$$

Substituting this into the above diagram and taking the direct sum over all degrees, we obtain

$$H \xrightarrow{\mu} H \otimes H \rightrightarrows H$$

where the map μ is a morphism of algebras, so it defines the bialgebra structure of H, and the composition is still the identity map on H. Since $H^0 = k$, projection $H \otimes H \to H \otimes H^0$ sends $\mu(x)$ to $x \otimes 1$, and the projection $H \otimes H \to H^0 \otimes H$ sends $\mu(x)$ to $1 \otimes x$. Hence, indeed $\mu(x) = 1 \otimes x + x \otimes 1 +$ higher order terms. The desired inequality now follows from Theorem 3.7. □

Now we have shown

$$\dim \mathrm{Pic}^0_{A/k} \leq \dim T_0(\mathrm{Pic}^0_{A/k}) \leq \dim A$$

Recall the map $\phi_L : A \to \mathrm{Pic}(A)$. In view of the discussion of the Picard scheme, we may now view this as a map $\phi_L : A \to \mathrm{Pic}^0_{A/k}$ since A is connected and ϕ_L maps 0 to 0.

Lemma 3.9. *If L is an ample line bundle on A, then the map ϕ_L has finite kernel.*

Proof. Suppose first L is ample. If $\ker \phi_L$ is not finite, let Y be the connected component of 0 of $\ker \phi_L$, so that Y is an abelian variety of positive dimension. Let L_Y be the restriction of L to Y, so it is ample since L is. (For all coherent sheaves $\mathcal{F}$ on Y, pushing forward by the inclusion $i : Y \hookrightarrow X$ gives a coherent sheaf $i_*\mathcal{F}$ on X, so for all large n we know $\Gamma(X, i_*\mathcal{F} \otimes L^n)$ generates $i_*\mathcal{F} \otimes L^n$. Restricting to Y shows $\Gamma(Y, \mathcal{F} \otimes L_Y^n)$ generates $\mathcal{F} \otimes L_Y^n$.) Furthermore, for any $y \in Y$, we have $T_y^*(L_Y) \cong L_Y$ by definition of $\ker \phi_L$. Applying Proposition 2.9 with $f = \mathrm{id}_Y$ and $g = -\mathrm{id}_Y$, we get $\mathcal{O}_Y \cong L_Y \otimes (-1_Y)^* L_Y$. Since L_Y is ample, so is $(-1_Y)^* L_Y$ since -1_Y is an automorphism of Y, and thus $L_Y \otimes (-1_Y)^* L_Y$ is ample too. But this shows $\mathcal{O}_Y$ is ample, and it is a fact that trivial line bundle being ample implies the variety is quasi-affine, and therefore has dimension 0. This is a contradiction. $\square$

Hence, if we choose L to an ample line bundle, then the map ϕ_L has finite kernel, so in fact $\dim A \leq \dim \mathrm{Pic}^0_{A/k}$. Combining this with the above inequality, we see that we must have equality everywhere. In particular, we have $\dim \mathrm{Pic}^0_{A/k} = \dim T_0(\mathrm{Pic}^0_{A/k})$, which means $\mathrm{Pic}^0_{A/k}$ is smooth.

Now that we proved the desired smoothness, we finally make the definition:

Definition 3.10. Let A be an abelian variety. The *dual abelian variety* of A is $\widehat{A} \coloneqq \mathrm{Pic}^0_{A/k}$.

Since $\mathrm{Pic}_{A/k}$ is the representing object of the Picard functor, it comes with a universal line bundle with a universal trivialization.

Definition 3.11. The *Poincare bundle* P is the universal line bundle restricted to $A \times \widehat{A}$ together with a universal trivialization

$$\theta : P|_{\{0\} \times \widehat{A}} \cong \mathcal{O}_{\widehat{A}}.$$

Explicitly, the universal property of (P, α) is that given any (rigidified) line bundle (L, α) on $A \times T$ such that $L|_{A \times \{t\}}$ is in $\mathrm{Pic}^0_{A/k}(k(t))$ for all $t \in T$, there exists a unique map $f : T \to \widehat{A}$ such that $L|_{A \times \{t\}}$ is carried to $P|_{A \times \{f(t)\}}$ and L is carried to $(\mathrm{id} \times f)^* P$. Moreover, the definition of $0 \in \widehat{A}$ implies $P|_{A \times \{0\}} = \mathcal{O}_A$.

The Albanese variety. Now let X be a smooth projective variety over an algebraically closed characteristic zero field k. Then $\mathrm{Pic}^0_{X/k}$ is a smooth projective group scheme, so an abelian variety.

Definition 3.12. Let X be a smooth projective variety. The *Albanese variety* of X, denoted by $\mathrm{Alb}(X)$, is the dual abelian variety of $\mathrm{Pic}^0_{X/k}$.

It turns out the Albanese variety is a key construction for the generic vanishing theorem. Our goal in this section is to see that there is a natural map $\mathrm{alb} : X \to \mathrm{Alb}(X)$.

Consider the universal line bundle P_X on $X \times \mathrm{Pic}^0_{X/k} = \mathrm{Pic}^0_{X/k} \times X$. It satisfies the condition that $P_X|_{\{0\}\times X} \cong \mathcal{O}_X$, and that $P_X|_{\mathrm{Pic}^0_{X/k}\times\{x\}}$ is a line bundle in the connected component of 0 in the Picard scheme of $\mathrm{Pic}^0_{X/k}$ by connectedness. Therefore, by the universal property of the dual of $\mathrm{Pic}^0_{X/k}$, there exists a unique map

$$\mathrm{alb} : X \to \mathrm{Alb}(X)$$

such that $(\mathrm{id} \times \mathrm{alb})^* P_{\mathrm{Pic}^0_{X/k}} \cong P_X$. This is the *Albanese mapping* of X.

4. Grothendieck Duality

Recall that for an n-dimensional (smooth projective) variety X, the canonical bundle is $\omega_X = \bigwedge^n \Omega^1_X$. It is a *dualizing sheaf* on X, meaning it represents the contravariant functor on the category of coherent sheaves on X

$$\mathcal{F} \to H^n(X, \mathcal{F})^\vee.$$

So in other words, there exists a universal *trace map* $t : H^n(X, \omega_X) \to k$ such that for any coherent sheaf $\mathcal{F}$ on X, we have

$$\mathrm{Hom}_X(\mathcal{F}, \omega_X) \cong H^n(X, \mathcal{F})^\vee, \text{ via } \phi \mapsto t \circ H^n(\phi).$$

Then Serre duality is the following theorem:

Theorem 4.1 (Serre duality). *For any coherent $\mathcal{F}$, there exists a unique functorial k-linear isomorphism*

$$\mathrm{Ext}^{n-i}_X(\mathcal{F}, \omega_X) \cong H^i(X, \mathcal{F})^\vee.$$

In the language of derived categories, let $D^b(X)$ denote the bounded derived category of the category of coherent sheaves on X. This notation will be used throughout this paper. Using Proposition **??**, we have

$$\mathrm{Ext}^{n-i}_X(\mathcal{F}, \omega_X) \cong \mathrm{Hom}_{D^b(X)}(\mathcal{F}, \omega_X[n-i]) \cong \mathrm{Hom}_{D^b(X)}(\mathcal{F}[i], \omega_X[n])$$

and

$$H^i(X, \mathcal{F}) \cong \mathrm{Ext}^i_X(\mathcal{O}_X, \mathcal{F}) \cong \mathrm{Hom}_{D^b(X)}(\mathcal{O}_X, \mathcal{F}[i])$$

Recall that the first step is nothing fancy: it is just $\mathrm{Hom}(\mathcal{O}_X, \mathcal{F}) \cong \Gamma(X, \mathcal{F})$, a globalization of $\mathrm{Hom}_R(R, M) = M$ for rings and modules. So Serre duality can be rewritten as

$$\mathrm{Hom}_{D^b(X)}(\mathcal{F}[i], \omega_X[n]) \cong \mathrm{Hom}_{D^b(X)}(\mathcal{O}_X, \mathcal{F}[i])^\vee.$$

In more generality, Serre duality also says that for any two complexes $\mathcal{E}^\bullet, \mathcal{F}^\bullet \in D^b(X)$, we have

$$\mathrm{Hom}_{D^b(X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)^\vee \cong \mathrm{Hom}_{D^b(X)}(\mathcal{F}^\bullet, \mathcal{E}^\bullet \otimes \omega_X[n])$$

functorial in $\mathcal{E}^\bullet$ and $\mathcal{F}^\bullet$.

Now we turn to Grothendieck duality. Let $f : X \to Y$ be a proper morphism of varieties X and Y. The Grothendieck duality is a relativized version of Serre duality. The duality is stated in terms of adjoint functors. Specifically, we want to find left and right adjoints to the right derived functor $\mathbf{R}f_*$. In the case where X and Y are smooth projective (which we are interested in), the right adjoint of the right derived functor $\mathbf{R}f_*$ can be constructed easily. We denote the right adjoint by $f^! : D^b(Y) \to D^b(X)$. Then we have the following theorem

Theorem 4.2 (Grothendieck Duality). *If $f : X \to Y$ is a morphism between two smooth projective varieties, then*

$$f^!\mathcal{F}^\bullet = \omega_X[\dim X] \otimes \mathbf{L}f^*(\mathcal{F}^\bullet \otimes \omega_Y^{-1}[-\dim Y])$$

for any complex $\mathcal{E}^\bullet \in D^b(Y)$.

Proof. We want to check the adjointness, i.e.

$$\mathrm{Hom}_{D^b(Y)}(\mathbf{R}f_*\mathcal{E}^\bullet, \mathcal{F}^\bullet) \cong \mathrm{Hom}_{D^b(X)}(\mathcal{E}^\bullet, f^!\mathcal{F}^\bullet)$$

for any complexes $\mathcal{E}^\bullet$ in $D^b(Y)$ and $\mathcal{F}^\bullet$ in $D^b(Y)$. Applying Serre duality to the left side, and using adjointness of $\mathbf{R}f_*$ and $\mathbf{L}f^*$, and finally applying Serre duality on X, we get

$$\begin{aligned}
\mathrm{Hom}_{D^b(Y)}(\mathbf{R}f_*\mathcal{E}^\bullet, \mathcal{F}^\bullet) &= \mathrm{Hom}_{D^b(Y)}(\mathbf{R}f_*\mathcal{E}^\bullet, \mathcal{F}^\bullet \otimes \omega_Y^{-1}[-\dim Y] \otimes \omega_Y[\dim Y]) \\
&\cong \mathrm{Hom}_{D^b(Y)}(\mathcal{F}^\bullet \otimes \omega_Y^{-1}[-\dim Y], \mathbf{R}f_*\mathcal{E}^\bullet)^\vee \\
&\cong \mathrm{Hom}_{D^b(X)}(\mathbf{L}f^*(\mathcal{F}^\bullet \otimes \omega_Y^{-1}[-\dim Y]), \mathcal{E}^\bullet)^\vee \\
&\cong \mathrm{Hom}_{D^b(X)}(\mathcal{E}^\bullet, \omega_X[\dim X] \otimes \mathbf{L}f^*(\mathcal{F}^\bullet \otimes \omega_Y^{-1}[-\dim Y])).
\end{aligned}$$

□

To make this a statement of the same form as Serre duality, we introduce the *relative dualizing bundle* on X

$$\omega_f = \omega_X \otimes f^*\omega_Y^{-1}.$$

Then we have

$$f^! = \omega_f[\dim X - \dim Y] \otimes \mathbf{L}f^*,$$

and the adjointness means

$$\mathrm{Hom}_{D^b(Y)}(\mathbf{R}f_*\mathcal{E}^\bullet, \mathcal{F}^\bullet) \cong \mathrm{Hom}_{D^b(\mathbf{Coh}X)}(\mathcal{E}^\bullet, \omega_f[\dim X - \dim Y] \otimes \mathbf{L}f^*\mathcal{F}^\bullet).$$

5. The Fourier-Mukai Transform

We first introduce the setting. Let A be a (complex) abelian variety, and let $\widehat{A} = \mathrm{Pic}^0(A)$ be its dual abelian variety. Then there is the Poincare bundle P on $A \times \widehat{A}$. Let

$$P_\alpha = P|_{A\times\{\alpha\}} \quad \text{and} \quad \widehat{P_a} = P|_{\{a\}\times\widehat{A}}.$$

Mukai's idea is to use the Poincare bundle P as the "kernel", just like using $e^{2\pi ix}$ as the kernel of Fourier transform. Precisely, for a coherent sheaf $\mathcal{F}$ on A, we can pull it back along the natural projection from $A \times \widehat{A}$ to A, tensor by P, and then push forward to $\widehat{A}$. So if $p_1 : A \times \widehat{A} \to A$ and $p_2 : A \times \widehat{A} \to \widehat{A}$, then we would like to do the transform

$$\mathcal{F} \mapsto p_{2*}(p_1^*\mathcal{F} \otimes P).$$

The resulting sheaf is again coherent on $\widehat{A}$, because p_2 is proper. However, the composition of the functors pullback, tensoring, and pushforward is not exact, and so we use the derived functors instead. Thus, for any object $\mathcal{F}^\bullet \in D^b(A)$, we define

$$\mathcal{F}^\bullet \mapsto \mathbf{R}p_{2*}(p_1^*\mathcal{F}^\bullet \otimes P)$$

Note that in this situation, p_1^* and $\otimes P$ are already exact. We denote this map by

$$\mathbf{R}\Phi_P : D^b(A) \to D^b(\widehat{A})$$

and call it the *Fourier-Mukai transform*.

Well, the Fourier transform is an isomorphism, and it has an explicit inverse transform, so a natural question to ask is whether the Fourier-Mukai transform is an equivalence of categories, and whether it has an explicit inverse. Mukai showed that the answers to both questions are positive.

Lemma 5.1. *Let $n = \dim A$. Then*

$$R^i p_{2*} P \cong \begin{cases} \mathcal{O}_0, & i = n \\ 0, & i \neq n \end{cases}.$$

Equivalently, $\mathbf{R} p_{2*} P \cong \mathcal{O}_0[-n]$ *in* $D^b(\widehat{A})$.

Proof. If L is any element in $\operatorname{Pic}^0(A)$ and $L \neq \mathcal{O}_A$, then $H^i(A, L) = 0$ for all $i \geq 0$ (This is well-known fact, see for example [7] Chapter 2). So for any $\alpha \in \operatorname{Pic}^0(A)$ non-zero, $P_\alpha = P|_{A \times \{\alpha\}}$ is identified with the element α, so

$$H^i(A, P_\alpha) = 0$$

for all $i = 0, 1, \cdots, n$. Now we consider

$$\begin{tikzcd} A \times \widehat{A} \arrow[r, "p_2"] \arrow[d, "p_1"'] & \widehat{A} \arrow[d] \\ A \arrow[r] & k \end{tikzcd}$$

as a base change diagram, where P is a line bundle on $A \times \widehat{A}$ that is flat over $\widehat{A}$, and p_2 is a proper morphism. Then for non-zero α, the base change theorem (Corollary 2.3) then tells us that

$$R^i p_{2*} P \otimes k(\alpha) \cong H^i(A \times \{\alpha\}, P_\alpha)$$

and we saw that this is zero for any non-zero α. (the condition $\alpha \mapsto h^i(A, P_\alpha)$ being constant is obvious, since it's always 0.) So we see that the sheaves $R^i p_{2*} P$ have non-zero stalk only possibly at 0. For $i > n$, $R^i p_{2*} P$ is just zero everywhere for dimension reasons.

We now want to show $R^i p_{2*} P = 0$ also for $i < n$. Upon considering the Leray spectral sequence, one obtains the isomorphism

$$H^i(A \times \widehat{A}, P) \cong H^0(\widehat{A}, R^i p_{2*} P).$$

The dualizing sheaf (canonical bundle) on $A \times \widehat{A}$ is trivial, so Serre duality implies

$$\begin{aligned} H^i(A \times \widehat{A}, P)^\vee &\cong \operatorname{Ext}^{2n-i}(P, \omega_{A \times \widehat{A}}) \\ &\cong \operatorname{Ext}^{2n-i}(P, \mathcal{O}_{A \times \widehat{A}}) \\ &\cong H^{2n-i}(A \times \widehat{A}, P^{-1}) \end{aligned}$$

where the final isomorphism comes from the isomorphism of sheaves (and thus global sections) $\mathscr{H}om(P, \mathcal{O}_{A \times \widehat{A}}) = P^{-1}$. This vanishes when $2n - i > i$, i.e. $i < n$. Hence $H^0(\widehat{A}, R^i p_{2*} P) = 0$ when $i < n$. But since $R^i p_{2*} P$ is supported only at 0, in fact the global section is exactly the stalk at 0. Thus we see that $R^i p_{2*} P$ has zero stalk also at 0 for $i < n$, so it has zero stalks everywhere, so it is zero.

It remains to consider the case $i = n$. The base change theorem (since H^{n+1} is always 0) gives

$$R^n p_{2*} P \otimes k(0) \cong H^n(A \times \{0\}, P_0) = H^n(A, \mathcal{O}_A) \cong k$$

So the fiber at 0 is free of rank 1 over k, and thus the Nakayama's lemma implies that the stalk of $R^n p_{2*}P$ at 0 is generated by one element as a module over the stalk $\mathcal{O}_{\widehat{A},0}$. Thus the natural generating map $\mathcal{O}_{\widehat{A}} \to R^n p_{2*}P$ is surjective, and we denote the kernel by $\mathcal{J}$. Our goal is to show that $\mathcal{J}$ is the ideal sheaf of the closed point 0, which we denote by $\mathcal{I}_0$. Equivalently, letting Z be the closed subscheme defined by $\mathcal{J}$, we want to show $Z = \{0\} \subset \widehat{A}$ as subschemes. Now Z has structure sheaf $\mathcal{O}_Z = \mathcal{O}_{\widehat{A}}/\mathcal{J} = R^n p_{2*}P$, which is supported at 0, so we know that $Z = \{0\}$ as topological spaces. The real question is whether Z as a scheme is reduced.

By Grothendieck duality, for any complex $\mathcal{F}^\bullet \in D^b(\widehat{A})$, we have a functorial isomorphism (again using the fact that the canonical bundles are trivial)

$$\mathrm{Hom}_{D^b(\widehat{A})}(\mathbf{R}p_{2*}P, \mathcal{F}^\bullet) \cong \mathrm{Hom}_{D^b(A\times\widehat{A})}(P, \mathbf{L}p_2{}^*\mathcal{F}^\bullet).$$

We have seen that $\mathbf{R}p_{2*}P$ is concentrated in degree n, so $\mathbf{R}p_{2*}P \cong R^n p_{2*}P[-n]$. We apply the above functorial isomorphism to the quotient map of coherent sheaves $\mathcal{O}_Z = R^n p_{2*}P \to \mathcal{O}_0$ to get a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_Z) & \longrightarrow & \mathrm{Hom}(P, i_*\mathcal{O}_{A\times Z}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(\mathcal{O}_Z, \mathcal{O}_0) & \longrightarrow & \mathrm{Hom}(P, j_*\mathcal{O}_{A\times\{0\}})
\end{array}$$

where the rows are isomorphisms, and $i : A \times Z \hookrightarrow A \times \widehat{A}$ and $j : A \times \{0\} \hookrightarrow A \times \widehat{A}$ are inclusions. The identity map on $\mathcal{O}_Z$ corresponds to a natural map $P \to i_*\mathcal{O}_{A\times Z}$, and the quotient map $\mathcal{O}_Z \to j_*\mathcal{O}_0$ corresponds naturally to a map $P \to \mathcal{O}_{A\times\{0\}}$. Upon using adjointness, we obtain the maps

$$i^*P = P|_{A\times Z} \to \mathcal{O}_{A\times Z} \text{ and } j^*P = P|_{A\times\{0\}} \to \mathcal{O}_{A\times\{0\}}$$

The commutativity of the diagram tells us the second map is a reduction of the first, and the definition of the Poincare bundle gives $P|_{A\times\{0\}} = \mathcal{O}_{A\times\{0\}}$. This means $P|_{A\times Z} \to \mathcal{O}_{A\times Z}$ is an isomorphism too. Then the universal property of the Poincare bundle implies $A \times Z = A \times \{0\}$ as schemes, so $Z = \{0\}$ as schemes.

□

Here are two examples. Let $\mathcal{O}_a$ be the structure of a closed point $a \in A$. What is $\mathbf{R}\Phi_P\mathcal{O}_a$? We have

$$\begin{aligned}
\mathbf{R}\Phi_P\mathcal{O}_a &= \mathbf{R}p_{2*}(p_1^*\mathcal{O}_a \otimes P) \\
&= \mathbf{R}p_{2*}(\mathcal{O}_{\{a\}\times\widehat{A}} \otimes P) \\
&= \mathbf{R}p_{2*}\widehat{P_a} \\
&= \widehat{P_a}.
\end{aligned}$$

This means the Fourier-Mukai transformation takes a point of A to the corresponding bundle $\widehat{P_a}$ on $\widehat{A}$. This is analogous to the classical Fourier transform, where a delta function gets transformed into a member of the kernel (i.e. a function of the form e^{inx}).

The second example is the Fourier-Mukai transform of $\mathcal{O}_A$. We have

$$\begin{aligned}
\mathbf{R}\Phi_P\mathcal{O}_A &= \mathbf{R}p_{2*}(p_1^*\mathcal{O}_A \otimes P)\\
&= \mathbf{R}p_{2*}(\mathcal{O}_{A\times\widehat{A}} \otimes P)\\
&= \mathbf{R}p_{2*}P\\
&= \mathcal{O}_0[-n].
\end{aligned}$$

by Lemma 5.1. Again, this is analogous to the classical Fourier transform, where the constant function gets transformed into the delta function at 0.

The inverse transformation. There is a natural functor that goes in the opposite direction: we just switch the role of A and $\widehat{A}$ (recall that $\hat{\widehat{A}} \cong A$), and in the formula switch p_1 and p_2. So we define

$$\begin{aligned}
\mathbf{R}\Psi : D^b(\widehat{A}) &\to D^b(A)\\
\mathcal{G}^\bullet &\mapsto \mathbf{R}p_{1*}(p_2^*\mathcal{G}^\bullet \otimes P).
\end{aligned}$$

We would like this to be the inverse of $\mathbf{R}\Phi$. It is not, but not far off.

Theorem 5.2. *Denote the inverse on the abelian variety A by ι, so $\iota(a) = -a$. Let $n = \dim A$. Then there are natural isomorphisms of functors*

$$\mathbf{R}\Psi_P \circ \mathbf{P}\Phi_P \cong \iota^*[-n] \quad \textit{and} \quad \mathbf{R}\Phi_P \circ \mathbf{R}\Psi_P \cong \iota^*[-n]$$

In particular, $\mathbf{R}\Phi$ and $\mathbf{R}\Psi$ are equivalences of categories.

Proof. Let $m : A\times A \to A$ be the addition on A. The maps in the following diagram all have obvious definition suggested by their names:

$$\begin{tikzcd}
A \times \widehat{A} & A \times A \times \widehat{A} \arrow[l, "m\times\mathrm{id}"'] \arrow[r, "p_{23}"'] \arrow[d, "p_{13}"] \arrow[rr, bend left, "\pi_2"] \arrow[dd, bend right, "\pi_1"'] & A \times \widehat{A} \arrow[r, "p_1"'] \arrow[d, "p_2"'] & A \\
 & A \times \widehat{A} \arrow[r, "p_2"'] \arrow[d, "p_1"] & \widehat{A} & \\
 & A & &
\end{tikzcd}$$

The names of the projections are duplicated, but there is no better notation without having ten different maps. Moreover, the context should make it clear what is meant, and we would point it out whenever a change in meaning occurs.

Fix an object $\mathcal{F}^\bullet \in D^b(A)$. We want to compute

$$\mathcal{G}^\bullet = (\mathbf{R}\Psi_P \circ \mathbf{P}\Phi_P)(\mathcal{F}^\bullet) = \mathbf{R}p_{1*}(P \otimes p_2^*(\mathbf{R}p_{2*}(p_1^*\mathcal{F}^\bullet \otimes P))).$$

The object $P \otimes p_1^*\mathcal{F}^\bullet$ is a complex of sheaves on $A \times \widehat{A}$. Applying flat base change to the square in the above diagram to this object, we have

$$p_2^*(\mathbf{R}p_{2*}(P \otimes p_1^*\mathcal{F}^\bullet)) \cong \mathbf{R}p_{23*}(p_{13}^*(P \otimes p_1^*\mathcal{F}^\bullet))$$

We rewrite the right side as

$$\mathbf{R}p_{23*}(p_{13}^*P \otimes \pi_1^*\mathcal{F}^\bullet)$$

where we used that pullback commutes with tensoring. Now we need to tensor this with P. The projection formula gives the natural isomorphism

$$P \otimes \mathbf{R}p_{23*}(p_{13}^*P \otimes \pi_1^*\mathcal{F}^\bullet) \cong \mathbf{R}p_{23*}(p_{23}^*P \otimes p_{13}^*P \otimes \pi_1^*\mathcal{F}^\bullet).$$

Finally we apply $\mathbf{R}p_{1*}$ and get

$$\mathbf{R}p_{1*}\mathbf{R}p_{23*}(p_{23}^*P \otimes p_{13}^*P \otimes \pi_1^*\mathcal{F}^\bullet) = \mathbf{R}\pi_{2*}(p_{23}^*P \otimes p_{13}^*P \otimes \pi_1^*\mathcal{F}^\bullet).$$

Recall that the Poincare bundle has the property that $p_{13}^*P \otimes p_{23}^*P \cong (m \times \mathrm{id})^*P$. So we obtain

$$\mathcal{G}^\bullet = \mathbf{R}\pi_{2*}((m \times \mathrm{id})^*P \otimes \pi_1^*\mathcal{F}^\bullet).$$

Now note that the π_1 and π_2 in this formula can be given by composition in another way:

$$\begin{tikzcd}
 & & A \\
A \times A \times \widehat{A} \arrow[r, "p_{12}"] \arrow[urr, bend left, "\pi_2"] \arrow[drr, bend right, "\pi_1"'] & A \times A \arrow[ur, "p_2"] \arrow[dr, "p_1"] & \\
 & & A
\end{tikzcd}$$

So

$$\mathcal{G}^\bullet = \mathbf{R}p_{2*}\mathbf{R}p_{12*}((m \times \mathrm{id})^*P \otimes p_{12}^*(p_1^*\mathcal{F}^\bullet)) \cong \mathbf{R}p_{2*}(\mathbf{R}p_{12*}(m \times \mathrm{id})^*P \otimes p_1^*\mathcal{F}^\bullet).$$

where the first equality is just composition of maps, and the second isomorphism is the projection formula. Now consider the base change diagram

$$\begin{tikzcd}
A \times A \times \widehat{A} \arrow[r, "p_{12}"] \arrow[d, "m\times\mathrm{id}"'] & A \times A \arrow[d, "m"] \\
A \times \widehat{A} \arrow[r, "p_1"'] & A
\end{tikzcd}$$

giving

$$\mathbf{R}p_{12*}(m \times \mathrm{id})^*P \cong m^*\mathbf{R}p_{1*}P \cong m^*\mathcal{O}_0[-n].$$

where the second isomorphism is by Lemma 5.1. So now we arrive at

$$\mathcal{G}^\bullet = \mathbf{R}p_{2*}(m^*\mathcal{O}_0[-n] \otimes p_1^*\mathcal{F}^\bullet).$$

Let $i : A \to A \times A$ be the closed immersion given by $i(a) = (a, -a)$. Then $m^*\mathcal{O}_0 \cong i_*\mathcal{O}_A$ by the base change theorem again. Finally, if we rename $p_1 : A \times A \to A$, then $p_1 \circ i = \iota$ and $p_2 \circ i = \mathrm{id}_A$, and we have

$$\begin{aligned}
\mathcal{G}^\bullet &= \mathbf{R}p_{2*}\mathbf{R}i_*\mathbf{L}i^*(m^*\mathcal{O}_0[-n] \otimes p_1^*\mathcal{F}^\bullet) \\
&\cong \mathbf{R}p_{2*}\mathbf{R}i_*(\mathcal{O}_A[-n] \otimes \mathbf{L}i^*p_1^*\mathcal{F}^\bullet) \\
&\cong \mathcal{O}_A[-n] \otimes \mathbf{L}i^*p_1^*\mathcal{F}^\bullet \\
&\cong \iota^*\mathcal{F}^\bullet[-n]
\end{aligned}.$$

This is exactly what we wanted to show. Note that all isomorphisms are functorial since they are either supplied by the base change theorem, the projection formula, or the property of the Poincare bundle. The other functor isomorphism is proved in the exact same way, using $\widehat{\widehat{A}} \cong A$ to switch the roles of A and $\widehat{A}$. □

The Fourier-Mukai transform and ample line bundles. Using the Fourier-Mukai transform to prove the generic vanishing theorem requires us to understand how ample line bundles behave under the Fourier-Mukai transform. Let L be an ample line bundle on an abelian variety A. We investigate $\mathbf{R}\Phi_P(L)$. By the Kodaira vanishing theorem,

$$H^i(A, L \otimes P_\alpha) \cong H^i(A, \omega_A \otimes L \otimes P_\alpha) = 0$$

for $i > 0$ and $\alpha \in \widehat{A}$. Then the base change theorem tells us $R^i\Phi_P(L)$ has zero fibers at all points, so $R^i\Phi_P(L) = 0$ for all $i > 0$. This means that the complex $\mathbf{R}\Phi_P(L)$ is in fact just one sheaf $p_{2*}(P \otimes p_1^*L)$.

Proposition 5.3. *Let L be an ample line bundle on A and let $\phi_L : A \to \widehat{A}$ be given by $\phi_L(a) = T_a^*L \otimes L^{-1}$. Then $\phi_L^*\mathbf{R}\Phi_P(L) \cong H^0(A, L) \otimes L^{-1}$.*

Proof. Let $\mathcal{E}_L = \mathbf{R}\Phi_P(L)$: it is in fact a vector bundle on $\widehat{A}$ by the base change theorem. Consider the base change diagram

$$\begin{tikzcd} A \times A \arrow[r, "p_2"] \arrow[d, "\mathrm{id} \times \phi_L"'] & A \arrow[d, "\phi_L"] \\ A \times \widehat{A} \arrow[r, "p_2"'] & \widehat{A} \end{tikzcd}$$

So the base change theorem implies

$$\phi_L^*\mathcal{E}_L = \phi_L^*\mathbf{R}p_{2*}(P \otimes p_1^*L) \cong \mathbf{R}p_{2*}((\mathrm{id} \times \phi_L)^*P \otimes p_1^*L).$$

Be careful that the meaning of the p_1 in front of L changed from a map $A \times \widehat{A} \to A$ to $A \times A \to A$. We know $(\mathrm{id} \times \phi_L)^*P \cong m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$, so we get

$$\phi_L^*\mathcal{E}_L = \mathbf{R}p_{2*}(m^*L \otimes p_2^*L^{-1}) \cong L^{-1} \otimes \mathbf{R}p_{2*}m^*L.$$

It remains to show $\mathbf{R}p_{2*}m^*L$ is the sheaf $H^0(A, L) \otimes \mathcal{O}_A$. Let $f : A \times A \to A \times A$ be given by $f(a, b) = (a + b, b)$. Then there is a clever factorization of m:

$$\begin{tikzcd} A \times A \arrow[r, "f"] \arrow[rr, bend left, "p_2"] \arrow[dr, "m"'] & A \times A \arrow[r, "p_2"] \arrow[d, "p_1"] & A \\ & A & \end{tikzcd}$$

So

$$\begin{aligned} \mathbf{R}p_{2*}m^*L &\cong \mathbf{R}p_{2*}\mathbf{R}f_*(f^*p_1^*L) \\ &\cong^{\text{projection formula}} \mathbf{R}p_{2*}(p_1^*L \otimes \mathbf{R}f_*\mathcal{O}_{A \times \widehat{A}}) \\ &\cong \mathbf{R}p_{2*}(p_1^*L). \end{aligned}$$

We then apply the base change theorem to the following diagram

$$\begin{tikzcd} A \times A \arrow[r, "p_2"] \arrow[d, "p_1"'] & A \arrow[d, "p"] \\ A \arrow[r, "p"'] & \operatorname{Spec} k \end{tikzcd}$$

yielding

$$\mathbf{R}p_{2*}(p_1^*L) \cong p^*\mathbf{R}p_*(L) = p^*H^0(A, L) = H^0(A, L) \otimes \mathcal{O}_A.$$

$\square$

6. The Generic Vanishing Theorem

Recall that there is a natural map alb : $X \to \mathrm{Alb}(X)$. The generic vanishing theorem states

Theorem 6.1. *Let X be a smooth projective variety. Then*

$$\operatorname{codim}_{\operatorname{Pic}^0(X)}\{L \in \operatorname{Pic}^0(X) \mid H^i(X, L) \neq 0\} \geq \dim \operatorname{alb}(X) - i$$

for every $i \geq 0$.

By Serre duality, this is equivalent to

$$\operatorname{codim}_{\operatorname{Pic}^0(X)}\{L \in \operatorname{Pic}^0(X) \mid H^i(X, \omega_X \otimes L) \neq 0\} \geq i - (\dim X - \dim \operatorname{alb}(X))$$

for every $i \geq 0$. To simplify notation, let $A = \mathrm{Alb}(X)$ and let $f : X \to A$ be the Albanese mapping. Let $k = \dim X - \dim f(X)$. Let $\widehat{A} = \operatorname{Pic}^0(A)$ and let P be the Poincare bundle on $A \times \widehat{A}$. Then the line bundle on A corresponding to $\alpha \in \widehat{A}$ is denoted by P_α. Since $\operatorname{Pic}^0(X)$ is isomorphic to $\widehat{A}$, the generic vanishing theorem is equivalent to

$$\operatorname{codim}_{\widehat{A}}\{\alpha \in \widehat{A} \mid H^i(X, \omega_X \otimes f^* P_\alpha) \neq 0\} \geq i - k. \tag{6.2}$$

Hacon proved the following general theorem.

Theorem 6.3 (Hacon)**.** *Let $\mathcal{F}$ be a coherent sheaf on an abelian variety over an algebraically closed field k. The following are equivalent:*

(1) $\operatorname{codim}\{\alpha \in \widehat{A} \mid H^i(A, \mathcal{F} \otimes P_\alpha) \neq 0\} \geq i$ *for all $i \in \mathbf{Z}$.*

(2) The Fourier-Mukai transform $\mathbf{R}\Phi_P(\mathcal{F})$ satisfies

$$\operatorname{codim} \operatorname{Supp} R^i\Phi_p(\mathcal{F}) \geq i$$

for all $i \in \mathbf{Z}$.

(3) For every finite etale morphism $\phi : B \to A$ of abelian varieties, and every ample line bundle L on B, we have

$$H^i(B, L \otimes \phi^* \mathcal{F}) = 0$$

for all $i > 0$.

(4) There exists a coherent sheaf $\mathcal{G}$ such that $\mathbf{R}\Phi_P(\mathcal{F}) \cong \mathbf{R}\mathscr{H}om(\mathcal{G}, \mathcal{O}_{\widehat{A}})$.

If $\mathcal{F}$ satisfies these conditions, we say $\mathcal{F}$ is a GV-sheaf.

Proof. We first prove the equivalence between 1 and 2. Fix i_0. Let

$$Z_1 = \bigcup_{i \geq i_0} \{\alpha \in \widehat{A} \mid H^i(A, \mathcal{F} \otimes P_\alpha) \neq 0\}.$$

Since $\mathcal{F} \otimes P_\alpha$ is $p_1^*\mathcal{F} \otimes P$ restricted to the fiber $A \times \{\alpha\} = p_2^{-1}(\alpha)$, we have

$$Z_1 = \bigcup_{i \geq i_0} \{\alpha \in \widehat{A} \mid H^i(A \times \{\alpha\}, (p_1^*\mathcal{F} \otimes P)|_{A \times \{\alpha\}}) \neq 0\}.$$

Let

$$Z_2 = \bigcup_{i \geq i_0} \operatorname{Supp} R^i\Phi_P\mathcal{F} = \bigcup_{i \geq i_0} \operatorname{Supp} R^i p_{2*}(p_1^*\mathcal{F} \otimes P).$$

The equivalence between 1 and 2 would follow if we prove $Z_1 = Z_2$. Note that the map

$$\alpha \mapsto H^i(A \times \{\alpha\}, (p_1^*\mathcal{F} \otimes P)|_{A \times \{\alpha\}})$$

is upper semi-continuous by Corollary 2.2, so Z_1 is a closed subset of $\widehat{A}$. Also, Z_2 is also closed, since the the support of a coherent sheaf is closed, and Z_2 is a finite union of those.

Let U_1, U_2 be the complement of Z_1, Z_2, and we prove $U_1 = U_2$. On U_1, $H^i(A \times \{\alpha\}, (p_1^*\mathcal{F} \otimes P)|_{A\times\{\alpha\}}) = 0$ for $i \geq i_0$, so by Corollary 2.3, we have the isomorphism

$$R^i p_{2*}(p_1^*\mathcal{F} \otimes P) \otimes k(\alpha) \cong H^i(A \times \{\alpha\}, (p_1^*\mathcal{F} \otimes P)|_{A\times\{\alpha\}}) = 0.$$

for all $i \geq i_0$. Therefore $U_1 \subset U_2$.

Conversely, for any $\alpha \in U_2$ we have

$$R^i p_{2*}(p_1^*\mathcal{F} \otimes P) \otimes k(\alpha) = 0$$

for $i \geq i_0$. Since $R^i p_{2*}(p_1^*\mathcal{F} \otimes P)$ is coherent, this means $R^i p_{2*}(p_1^*\mathcal{F} \otimes P) = 0$ on U_2 for $i \geq i_0$. Thus the complex $\mathbf{R}p_{2*}(p_1^*\mathcal{F} \otimes P) = \mathbf{R}\Phi_P(\mathcal{F})$ has vanishing cohomology in degree $i \geq i_0$. We tensor in the derived category by $k(\alpha)$ for $\alpha \in U_2$ to get $\mathbf{R}\Phi_P(\mathcal{F}) \otimes k(\alpha)$ has vanishing cohomology in degree $i \geq i_0$. We may choose an affine open neighborhood around such α, and then Theorem 2.1 implies

$$H^i(A \times \{\alpha\}, (p_1^*\mathcal{F} \otimes P)|_{A\times\{\alpha\}}) \cong H^i(\mathbf{R}\Phi_P\mathcal{F} \otimes k(\alpha)) = 0$$

for $i \geq i_0$. This shows $\alpha \in U_1$, so $U_1 = u_2$ as desired.

Now we show 2 implies 3, so assume $\operatorname{codim} \operatorname{Supp} R^i\Phi_P(\mathcal{F}) \geq i$ for all $i \in \mathbf{Z}$. We only treat the case where ϕ is the identity map on A. So let L be any ample line bundle on A, and we want to show $H^i(A, \mathcal{F} \otimes L) = 0$ for all $i > 0$. Recall that if $\iota : A \to A$ is the inversion map, then letting $\mathcal{M} = \mathbf{R}\Phi_P(\mathcal{F})$ we have the natural isomorphism

$$\mathcal{F} \cong \iota^*\mathbf{R}\Psi_P(\mathcal{M})[n].$$

Thus in the derived category we have

$$H^i(A, \mathcal{F} \otimes L) = \operatorname{Hom}(\mathcal{O}_A, \mathcal{F} \otimes L[i]) \cong \operatorname{Hom}(\mathcal{O}_A, L \otimes \iota^*\mathbf{R}\Psi_P(M)[n+i]).$$

Now note that proving statement 3 for L is the same as proving it for ι^*L, so we can replace L with ι^*L, and now we want to show

$$H^i(A, \mathcal{F} \otimes \iota^*L) = \operatorname{Hom}(\mathcal{O}_A, L \otimes \mathbf{R}\Psi_P(\mathcal{M})[n+i]) = 0$$

We compute

$$\begin{aligned}
\operatorname{Hom}(\mathcal{O}_A, L \otimes \mathbf{R}\Psi_P(\mathcal{G})[n+i]) &= \operatorname{Hom}(\mathcal{O}_A, L \otimes \mathbf{R}p_{1*}(p_2^*\mathcal{M} \otimes P)[n+i]) \\
&\cong^{\text{p.f.}} \operatorname{Hom}(\mathcal{O}_A, \mathbf{R}p_{1*}(p_1^*L \otimes p_2^*\mathcal{M} \otimes P)[n+i]) \\
&\cong^{\text{ad.}} \operatorname{Hom}(p_1^*\mathcal{O}_A, p_1^*L \otimes p_2^*\mathcal{M} \otimes P[n+i]) \\
&\cong \operatorname{Hom}(\mathcal{O}_{A\times\widehat{A}}, p_1^*L \otimes p_2^*\mathcal{M} \otimes P[n+i]) \\
&\cong^{\text{ad.}} \operatorname{Hom}(\mathcal{O}_{\widehat{A}}, \mathbf{R}p_{2*}(p_1^*L \otimes P \otimes p_2^*\mathcal{M}[n+i])) \\
&\cong^{\text{p.f.}} \operatorname{Hom}(\mathcal{O}_{\widehat{A}}, \mathbf{R}p_{2*}(p_1^*L \otimes P) \otimes \mathcal{M}[n+i])) \\
&= \operatorname{Hom}(\mathcal{O}_{\widehat{A}}, \mathbf{R}\Phi_P(L) \otimes \mathcal{M}[n+i]))
\end{aligned}$$

where "p.f." means we used the projection formula, and "ad." means we used the adjointness of pushforward and pullback.

By Proposition 5.3, $\mathbf{R}\Phi_P(L) = \mathcal{E}_L$ is a vector bundle, so

$$H^i(A, \mathcal{F} \otimes \iota^*L) \cong H^{n+i}(\widehat{A}, \mathcal{E}_L \otimes \mathbf{R}\Phi_P(\mathcal{F})).$$

We have the spectral sequence

$$E_2^{p,q} = H^p(\widehat{A}, \mathcal{E}_L \otimes R^q\Phi_p(\mathcal{F})) \implies H^{p+q-n}(A, \mathcal{F} \otimes \iota^* L).$$

Statement 2 says the dimension of $\operatorname{Supp} R^q\Phi_p(\mathcal{F})$ is at most $n-q$. So $E_2^{p,q} = 0$ if $p > n-q$. Thus $H^{p+q-n}(A, \mathcal{F} \otimes \iota^* L) = 0$ if $p+q-n > 0$, which is exactly what we wanted to show.

Now we prove 3 implies 4. This is a formal computation of derived functors. Recall our notation that $\mathcal{M} = \mathbf{R}\Phi_P(\mathcal{F})$. First observe that "double dual" does nothing, i.e.

$$\mathbf{R}\mathscr{H}om(\mathbf{R}\mathscr{H}om(\mathcal{M}, \mathcal{O}_{\widehat{A}}), \mathcal{O}_{\widehat{A}}) \cong \mathcal{M}.$$

So statement 4 will be proved once we prove the object $\mathbf{R}\mathscr{H}om(\mathcal{M}, \mathcal{O}_{\widehat{A}})$, which a priori is a complex, is in fact a single sheaf. Namely, we want show that

$$R^i\mathscr{H}om(\mathcal{M}, \mathcal{O}_{\widehat{A}}) = 0$$

for all $i \neq 0$. We have the following criterion:

Lemma 6.4. *Let X be a smooth projective variety, and $\mathcal{E}^\bullet \in D^b(X)$ an object in the derived category. Then $\mathcal{H}^i(\mathcal{E}^\bullet) = 0$ if and only if $H^i(X, \mathcal{E}^\bullet \otimes L) = 0$ for every sufficiently ample line bundle L.*

Applying this lemma to our situation, and using the functor isomorphism $\mathbf{R}\Gamma(\widehat{A}, -) \circ \mathbf{R}\mathscr{H}om = \mathbf{R}\operatorname{Hom}$ and the tensor-hom-dual isomorphism, we want to show that

$$\mathbf{R}\Gamma(\mathbf{R}\mathscr{H}om(\mathcal{M}, \mathcal{O}_{\widehat{A}}) \otimes L) \cong \mathbf{R}\operatorname{Hom}(\mathcal{G}, L)$$

is zero in non-zero degrees. We have that for $i \neq 0$,

$$R^i\operatorname{Hom}(\mathcal{M}, L) = \operatorname{Hom}_{D^b(\widehat{A})}(\mathcal{M}, L[i])$$

We compute using Grothendieck duality and the fact that canonical bundles are trivial

$$\begin{aligned}
\operatorname{Hom}_{D^b(\widehat{A})}(\mathcal{M}, L[i]) &= \operatorname{Hom}_{D^b(X)}(\mathbf{R}\Phi_P\mathcal{F}, L[i]) \\
&= \operatorname{Hom}_{D^b(\widehat{A})}(\mathbf{R}p_{2*}(P \otimes p_1^*\mathcal{F}), L[i]) \\
&\cong \operatorname{Hom}_{D^b(\widehat{A}\times A)}(P \otimes p_1^*\mathcal{F}, p_2^! L[i]) \\
&\cong \operatorname{Hom}_{D^b(\widehat{A}\times A)}(P \otimes p_1^*\mathcal{F}, \omega_{p_2}[2n-n] \otimes p_2^* L[i]) \\
&\cong \operatorname{Hom}_{D^b(\widehat{A}\times A)}(P \otimes p_1^*\mathcal{F}, p_2^* L[n+i])
\end{aligned}$$

Now we dualize P and use adjointness of p_1^*, p_{1*} to get

$$\begin{aligned}
\operatorname{Hom}_{D^b(A\times\widehat{A})}(P \otimes p_1^*\mathcal{F}, p_2^* L[n+i]) &\cong \operatorname{Hom}_{D^b(A\times\widehat{A})}(p_1^*\mathcal{F}, P^{-1} \otimes p_2^* L[n+i]) \\
&\cong \operatorname{Hom}_{D^b(A)}(\mathcal{F}, \mathbf{R}p_{1*}(P^{-1} \otimes p_2^* L)[n+i]))
\end{aligned}$$

The Poincare bundle satisfies that $P^{-1} \cong (\mathrm{id} \times \iota)^* P$, and using base change we have

$$\mathbf{R}p_{1*}(P^{-1} \otimes p_2^* L) \cong \mathbf{R}p_{1*}((\mathrm{id} \times \iota)^* P \otimes p_2^* L) \cong \iota^* \mathbf{R}p_{1*}(P \otimes (\mathrm{id} \times \iota)^* p_2^* L)$$

Note that $p_2 \circ (\mathrm{id} \times \iota) = \iota \circ p_2$, so we obtain $\iota^* \mathbf{R}p_{1*}(P \otimes p_2^* \iota^* L) = \iota^* \mathbf{R}\Psi_P(\iota^* L)$. Hence, it now suffices to show

$$\operatorname{Hom}_{D^b(A)}(\mathcal{F}, \mathbf{R}\Psi_P(L)[n+i])) = 0$$

for $i \neq 0$ and L any ample line bundle on $\widehat{A}$.

Recall that $\phi_L : \widehat{A} \to A$ is finite etale (where again we identified A with its double dual), and we know $\phi_L^* \mathbf{R}\Psi_P(L) \cong H^0(\widehat{A}, L) \otimes L^{-1}$. Also since ϕ_L is finite etale, the structure sheaf $\mathcal{O}_A$ is a direct summand of $\phi_{L*}\mathcal{O}_{\widehat{A}}$. This implies that

$$\mathrm{Hom}_{D^b(A)}(\mathcal{F}, \mathbf{R}\Psi_P(L)[n+i]))$$

is a direct summand of

$$\mathrm{Hom}_{D^b(\widehat{A})}(\phi_L^*\mathcal{F}, \phi_L^*\mathbf{R}\Psi_P(L)[n+i])) \cong \mathrm{Hom}_{D^b(\widehat{A})}(\phi_L^*\mathcal{F}, L^{-1}[n+i])) \otimes H^0(\widehat{A}, L).$$

Statement 3 tells us

$$H^i(\widehat{A}, L^{-1} \otimes \phi_L^* F) = \mathrm{Hom}_{D^b(\widehat{A})}(\mathcal{O}_{\widehat{A}}, L^{-1} \otimes \phi_L^* F[i]) = 0$$

So by Serre duality,

$$\mathrm{Hom}_{D^b(\widehat{A})}(\phi_L^* F, L^{-1}[n+i]) = 0,$$

as desired.

Finally, we assume 4 and prove 2. Let $\mathcal{G}$ be a coherent sheaf such that $\mathbf{R}\Phi_P\mathcal{F} \cong \mathbf{R}\mathscr{H}om(\mathcal{G}, \mathcal{O}_{\widehat{A}})$. Then what we need to show is that

$$\operatorname{codim} \operatorname{Supp} \mathcal{E}xt^i(\mathcal{G}, \mathcal{O}_{\widehat{A}}) \geq i$$

for all i. Since $\mathcal{G}$ is coherent, we are allowed to prove this at all stalks. So upon choosing an affine cover, the problem is reduced to showing

$$\operatorname{codim} \operatorname{Supp} \mathrm{Ext}^i(M, R) \geq i.$$

for R a regular local ring (because all local rings of $\widehat{A}$ are regular), where the codimension is taken inside of $\operatorname{Spec} R$. By definition, the support consists of prime ideals $\mathfrak{p}$ of R such that the localization of $\mathrm{Ext}^i(M, R)$ at $\mathfrak{p}$ is non-zero. Localization commutes with taking Ext, so this is equivalent to $\mathrm{Ext}^i_{R_\mathfrak{p}}(M_\mathfrak{p}, R_\mathfrak{p}) \neq 0$. Note that for any prime $\mathfrak{p}$ in the support, $V(\mathfrak{p}) \cong \operatorname{Spec} R/\mathfrak{p}$ is also contained in the support because the support of any coherent sheaf is closed. Therefore

$$\{V(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Supp} \mathrm{Ext}^i(M, R)\}$$

is a closed cover of $\operatorname{Supp} \mathrm{Ext}^i(M, R)$, and it suffices to show $\operatorname{codim} V(p) \geq i$ for all $\mathfrak{p} \in \operatorname{Supp} \mathrm{Ext}^i(M, R)$. Since $\dim R = \dim R/\mathfrak{p} + \dim R_\mathfrak{p}$, we have $\operatorname{codim} V(p) = \dim R_\mathfrak{p}$. In effect, we need to show that if $\mathrm{Ext}^i_{R_\mathfrak{p}}(M_\mathfrak{p}, R_\mathfrak{p}) \neq 0$, then $\dim R_\mathfrak{p} \geq i$.

Since $R_\mathfrak{p}$ is still regular local, we know that $\dim R_\mathfrak{p}$ is equal to the sum of the projective dimension of M and the depth of M. In particular, M has a free resolution of length at most $\dim R_\mathfrak{p}$, so by computing on this free resolution, $\mathrm{Ext}^i_{R_\mathfrak{p}}(M_\mathfrak{p}, R_\mathfrak{p}) \neq 0$ implies $\dim R_\mathfrak{p} \geq i$. □

Let's see how Hacon's theorem implies the generic vanishing theorem. Recall that what we wanted to prove is

$$\operatorname{codim}_{\widehat{A}}\{\alpha \in \widehat{A} \mid H^i(X, \omega_X \otimes f^* P_\alpha) \neq 0\} \geq i - k.$$

Using the functor isomorphism $\mathbf{R}\Gamma(X,-) \cong \mathbf{R}\Gamma(A,-) \circ \mathbf{R}f_*$, we have

$$\begin{aligned} H^i(X, \omega_X \otimes f^*P_\alpha) &\cong R^i\Gamma(A, \mathbf{R}f_*(\omega_X \otimes f^*P_\alpha)) \\ &\cong^{\text{p.f.}} R^i\Gamma(A, \mathbf{R}f_*\omega_X \otimes P_\alpha)) \\ &\cong R^i\Gamma(A, \bigoplus_{j=0}^{k} R^j f_*\omega_X[-j] \otimes P_\alpha)) \\ &\cong \bigoplus_{j=0}^{k} H^{i-j}(A, R^j f_*\omega_X \otimes P_\alpha). \end{aligned}$$

So the non-vanishing of $H^i(X, \omega_X \otimes f^*P_\alpha)$ implies at least one of the summands $H^{i-j}(A, R^j f_*\omega_X \otimes P_\alpha)$ is non-zero. Hence (since $j \le k$) it suffices to show

$$\operatorname{codim}_{\widehat{A}}\{\alpha \in \widehat{A} \mid H^{i-j}(A, R^j f_*\omega_X \otimes P_\alpha) \neq 0\} \geq i - j (\geq i - k)$$

i.e.

$$\operatorname{codim}_{\widehat{A}}\{\alpha \in \widehat{A} \mid H^{i}(A, R^j f_*\omega_X \otimes P_\alpha) \neq 0\} \geq i$$

for all $i \geq 0$. Recall Kollar's vanishing theorem:

Theorem 6.5 (Kollar)**.** *Let $g : X \to Y$ be a morphism of smooth projective varieties over a field of characteristic 0. Let L be an ample line bundle on Y. Then*

$$H^i(A, L \otimes R^j g_*\omega_X) = 0$$

for all $i > 0$ and $j \geq 0$.

Proof. This is Theorem 2.1 in [5]. □

So in our situation we have that

$$H^i(A, L \otimes R^j f_*\omega_X) = 0$$

for any ample line bundle L and all $i > 0$. The next lemma shows that this implies statement 3 in Hacon's theorem.

Lemma 6.6. *Let $\phi : B \to A$ be a finite morphism of abelian varieties. Then*

$$H^i(B, L \otimes \phi^* R^j f_*\omega_X) = 0$$

for any ample line bundle L on B and every $i > 0$.

Proof. Do not forget that A is the Albanese variety of X and $f : X \to A$ is the Albanese mapping. Let $Y = B \times_A X$, and we have the base change diagram

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[\phi]{} & A \end{array}$$

So by the base change theorem, we have

$$\phi^* R^j f_*\omega_X \cong R^j g_*\psi^*\omega_X \cong R^j g_*\omega_Y.$$

We know $H^i(Y, R^j g_*\omega_Y) = 0$ for all $j > 0$ and L ample by Kollar's vanishing theorem. □

Thus the coherent sheaf $R^j f_*\omega_X$ satisfies statement 3 in Hacon's theorem, so it satisfies statement 1 by the theorem. This proves the generic vanishing theorem.

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