CONNECTIVE PERIODIC SPECTRA REVISITED

YUQIN KEWANG

Abstract. In this paper, we will define three categories: periodic E_∞ ring spaces, periodic E_∞ ring spectra, and connective periodic E_∞ ring spectra. We will further prove that there is an equivalence on the homotopy categories of these three categories using localization and the multiplicative infinite loop space machine. This result improves the classical ideas in [12, II.3].

Contents

1. Introduction

Periodicity plays an important role in algebraic topology in the sense that many spectra of interest in this area are periodic. For example, the K-theory spectra, the periodic complex cobordism spectra, Morava K-theory spectra and Morava E-theory spectra, elliptic spectra, etc. Consequently, it is interesting to think about the general nice structures of these periodic spectra. In [12, II.3], Peter May introduced a notion called *connective periodic spectra* which is inspired by the connective K-spectra ku and ko, and studied the relationships between periodic spaces, periodic spectra and connective periodic spectra. Such relationships can be illustrated by the following diagram

$$\begin{tikzcd} \text{Periodic spectra} \arrow[rr, "\theta"] \arrow[dr, shift right, "\Omega^\infty"'] & & \text{Connective periodic spectra} \arrow[dl, "\Omega^\infty"'] \\ & \text{Periodic spaces} \arrow[ul, shift right, "K"'] & \end{tikzcd}$$

where θ is the functor which takes any spectrum to its connective cover [12, II.2.11] and K is a functor which constructs a periodic spectrum level wise from the data of a periodic space [12, II.3.6]. May further generalized the result to ring spaces and ring spectra classically in the sense of Whitehead [16]. Although the functor K which is constructed level wise works fine on objects, it fails to take a map between periodic spaces to a strict map between prespectra: the resulting map

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between prespectra on each level only commutes with the structure maps up to homotopy equivalences. Such a map between prespectra is called a *weak map* in [12]. When we try to apply the spectrification functor to promote the weak maps between periodic prespectra to strict maps between periodic spectra, there is a non-vanishing $\lim^1$ term studied by J. McClure [3, VII] which obstructs our desired promotion. Such $\lim^1$ terms vanish in some special cases, for example, the K-theory spectra, but do not vanish in general. This prevents us from reaching the category of periodic spectra with K, instead landing us in an intermediary stage: the weak periodic spectra, where all maps between spectra are only weak maps. Hence, the conclusion in [12, II.3] is that the homotopy categories of the following three categories are equivalent: the category of periodic spaces, the category of weak periodic spectra, and the category of weak connective periodic spectra. A problem of such result is that the homotopy category of weak spectra is not the same as our stable homotopy category. Therefore, there is a desire to improve the result by getting rid of the weak maps.

The problem of weak maps comes from the level wise construction of the functor K. However, in the E_∞ ring world, the functor K can be replaced by two well-behaved functors: the multiplicative infinite loop space machine and the localization functor. These two functors have nice functorial properties and successfully solve the issues of the level wise construction in the classical case. Our goal for this paper is to prove that the following commutative diagram of three categories

$$\begin{tikzcd}
\text{Periodic } E_\infty \text{ ring spectra} \arrow[dr, "\Omega^\infty"] & & \text{Connective periodic } E_\infty \text{ ring spectra} \arrow[ll, "{[\beta^{-1}]}"'] \arrow[dl, shift right, "\Omega^\infty"'] \\
& \text{Periodic } E_\infty \text{ ring spaces} \arrow[ur, shift right, "\mathbb{E}"'] &
\end{tikzcd}$$

induces an equivalence of categories between their homotopy categories. We will first give an explicit definition of the three categories appearing in the diagram: the category of periodic E_∞ ring spaces, the category of periodic E_∞ ring spectra, and the category of connective periodic E_∞ ring spectra. Then we will briefly introduce the two functors in the diagram: the localization $[\beta^{-1}]$ and the multiplicative infinite loop space machine $\mathbb{E}$. Finally we will prove how these functors induce an equivalence on the homotopy categories.

2. Connective periodic E_∞ spectra

2.1. A brief review of E_∞ ring spaces and E_∞ ring spectra. We choose our canonical additive E_∞ operad $\mathscr{C}$ to be the the Steiner operad $\mathscr{K}_U$, where U is a chosen universe (i.e. a countably infinite dimensional inner product space), and the canonical multiplicative E_∞ operad to be the linear isometries operad $\mathscr{L}$. Let $\mathscr{L}_+$ be the augmented operad of the linear isometries operad $\mathscr{L}$.

First let's recall the definition of E_∞ ring spaces. Let X be a based space with basepoint denoted as 0 and let 1 denote a distinct point in X, which will turn out to be the identity of the multiplication structure on X. Recall that we say X is an E_∞ *ring space* if X is a $(\mathscr{C}, \mathscr{L}_+)$-space, i.e. there are actions

$$\mu : \mathscr{C}(k) \times X^k \to X$$

for each degree k such that $\mu(*) = 0$, which parametrize the additions on X, and actions

$$\xi : \mathscr{L}_+(k) \wedge X^{\wedge k} \to X \tag{2.1}$$

for each degree k such that when $k = 0$, ξ maps the unique point $*$ in $\mathscr{L}(0)$ to 1 and maps the added point to 0, which parametrize the multiplications on X; both actions need to satisfy certain axioms and these two actions are compatible in the sense that there are actions of $\mathscr{L}$ on $\mathscr{C}$ given by

$$\zeta : \mathscr{L}(k) \times \mathscr{C}(j_1) \times \cdots \times \mathscr{C}(j_k) \to \mathscr{C}(j_1 \cdots j_k)$$

satisfying some axioms and such that the following diagram

$$\begin{tikzcd}
\mathscr{L}(k) \times \mathscr{C}(j_1) \times X^{j_1} \times \cdots \times \mathscr{C}(j_k) \times X^{j_k} \arrow[r, "\mu^k"] \arrow[d, "\zeta"'] & \mathscr{L}(k) \times X^k \arrow[d, "\xi"] \\
\mathscr{C}(j_1 \cdots j_k) \times X^{j_1 \cdots j_k} \arrow[r, "\mu"'] & X
\end{tikzcd}$$

commutes, where the action $\xi : \mathscr{L}(k) \times X^k \to X$ comes from restricting the augmented action (2.1) to $\mathscr{L}(k) \subseteq \mathscr{L}_+(k)$; notice that for any $l \in \mathscr{L}(k)$, this requires $\xi(l; x_1, \cdots, x_k) = 0$ if any x_i is the basepoint 0.

If X is an E_∞ ring space, then $\pi_*(X)$ naturally inherits an additive structure from the action of μ on X, which coincides with the original additive structure of homotopy groups by the Eckmann-Hilton argument. $\pi_*(X)$ further inherits a graded multiplicative structure from the action of ξ on X in the following way: for any two classes $[\alpha] \in \pi_i(X)$ and $[\beta] \in \pi_j(X)$, we consider

$$S^{i+j} \cong S^i \wedge S^j \xrightarrow{\alpha\wedge\beta} X \wedge X \xrightarrow{\xi(f;-,-)} X$$

where we choose an arbitrary element f in $\mathscr{L}(2)$. This is well-defined up to homotopy: since $\mathscr{L}(2)$ is contractible, different choices of $f \in \mathscr{L}(2)$ give homotopy equivalent products. Hence, we get a well-defined class in $\pi_{i+j}(X)$ which we denote as their product $[\alpha][\beta]$. This multiplication is also commutative when we pass to homotopy groups.

In this sense, $\pi_*(X)$ is almost like a commutative graded ring except that $\pi_0(X)$ might not have additive inverses. We say that X is *ringlike* if $\pi_0(X)$ has additive inverses; in this case $\pi_*(X)$ is truly a commutative graded ring.

On the spectra level, we say that a Lewis-May spectrum E is an E_∞ *ring spectrum* if there is an action of the linear isometries operad $\mathscr{L}$ on E

$$\xi : \mathscr{L}(k) \ltimes E^{\wedge k} \to E \tag{2.2}$$

satisfying certain axioms. Similar as E_∞ ring spaces, for an E_∞ ring spectrum E, the linear isometries operad $\mathscr{L}$ parametrizes the multiplications on E and $\pi_*(E)$ becomes a commutative graded ring.

2.2. **Periodic spaces, periodic spectra and connective periodic spectra.** Now we are ready to introduce our three objects in the E_∞ ring theories:

Definitions 2.3.

- An E_∞ ring space X is called *periodic* if there exists a class $[\beta] \in \pi_d(X)$ $(d > 0)$ such that multiplying by $[\beta]$
$$- \cdot [\beta] : \pi_*(X) \to \pi_{*+d}(X)$$
induces an isomorphism as graded modules, where $* \geq 0$.
- An E_∞ ring spectrum E is called *periodic* if there exists a class $[\beta] \in \pi_d(E)$ $(d > 0)$ such that multiplying by $[\beta]$ induces an isomorphism
$$- \cdot [\beta] : \pi_*(E) \to \pi_{*+d}(E)$$
as graded modules.
- An E_∞ ring spectrum E is called *connective periodic* if E is first connective and there exists a class $[\beta] \in \pi_d(E)$ $(d > 0)$ such that multiplying by $[\beta]$ induces an isomorphism
$$- \cdot [\beta] : \pi_*(E) \to \pi_{*+d}(E)$$

as graded modules, where $* \geq 0$.

Although we didn't require anything particularly for the degree d of the class $[\beta]$, when d is odd , we have

$$[\beta]^2 = (-1)^{d^2}[\beta]^2 = -[\beta]^2$$

implying that $2[\beta]^2 = 0$ in, for example, $\pi_{2d}(X)$ for a periodic E_∞ ring space X. If $\pi_{2d}(X)$ is not 2-torsion, then this forces $[\beta]^2 = 0$ which contradicts to our assumption that X is periodic. Since in general requiring $\pi_{nd}(X)$ to be 2-torsion for all $n > 0$ is too strong, we may expect the degree d to be even in most cases. We call such $[\beta]$ which induces the periodicity in homotopy groups the *Bott element*.

Also, notice that a periodic E_∞ ring space X is naturally ringlike because $\pi_0(X) \cong \pi_d(X)$.

Now we specify the morphisms between the three new objects.

Definitions 2.4. Let X, Y be periodic E_∞ ring spaces. We say that $f : X \to Y$ is a *map between periodic E_∞ ring spaces* if f is a map between E_∞ ring spaces and $f([\beta_X]) = [\beta_Y]$, where $[\beta_X], [\beta_Y]$ are the Bott elements of X, Y respectively.

Similarly, we may define maps between periodic E_∞ ring spectra (resp. connective periodic E_∞ ring spectra) as maps between E_∞ ring spectra (resp. connective E_∞ ring spectra) which send the Bott element to the Bott element.

Now, we have given full specification of the categories of periodic E_∞ ring spaces, periodic E_∞ ring spectra, and connective periodic E_∞ ring spectra. We say that a morphism in any such category is a *weak equivalence* if it induces isomorphisms on homotopy groups. Then, their *homotopy categories* will be specified as the localization of these three categories with respect to their weak equivalences [8, III.2.2]. Other constructions of homotopy categories such as the simplicial localization [5] or the hammock localization [6] might lead to better homotopy behaviors.

3. The machineries

3.1. **Localizations of spectra.** With nice model category structures, localizations of spectra can be formalized as Bousfield localizations [7, VIII]. But as our main theorem only requires the homotopical level properties, we introduce a more straightforward construction of localizations which does not require model structures.

Let E be an E_∞ ring spectrum and $[\beta] \in \pi_d(E)$. We may write the representative β as a map

$$\beta : \mathbb{S}^0 \to \mathbb{S}^{-d} \wedge E$$

Consider the following composition of maps

$$\begin{aligned} \mathbb{S}^{-nd} \wedge E \cong \mathbb{S}^{-nd} \wedge \mathbb{S}^0 \wedge E &\xrightarrow{\mathrm{id} \wedge \beta \wedge \mathrm{id}} \mathbb{S}^{-nd} \wedge \mathbb{S}^{-d} \wedge E \wedge E \\ &\cong \mathbb{S}^{-(n+1)d} \wedge E \wedge E \xrightarrow{\mathrm{id} \wedge \xi(f;-,-)} \mathbb{S}^{-(n+1)d} \wedge E \end{aligned}$$

for every integer $n \geq 0$, where ξ in the last arrow is the multiplicative structure on E (2.2) and f is an arbitrary element in $\mathscr{L}(2)$. For simplicity we still denote this map as β. Then the localization of E at β is constructed as the telescope

$$E[\beta^{-1}] = \mathrm{tel}(E \xrightarrow{\beta} \mathbb{S}^{-d} \wedge E \xrightarrow{\beta} \mathbb{S}^{-2d} \wedge E \xrightarrow{\beta} \cdots).$$

We may regard the inclusion of the initial stage E of the telescope as a natural map $E \to E[\beta^{-1}]$. The nice properties of localization of spectra are summarized in the following proposition.

Proposition 3.1 ([7, V.1.13, V.1.14, V.2.3]). *Let E be an E_∞ ring spectrum and $[\beta] \in \pi_d(E)$. Then there exists an E_∞ ring spectrum $E[\beta^{-1}]$ together with a natural map of E_∞ ring spectra $E \to E[\beta^{-1}]$ satisfying the universal property of localization, and*

$$\pi_*(E[\beta^{-1}]) \cong \pi_*(E)([\beta]^{-1})$$

an isomorphism of E_∞ ring spectra.

3.2. **Additive and multiplicative infinite loop space machines.** The multiplicative infinite loop space machine is a specialization of the additive one by defining

$$\mathbb{E}X = B(\Sigma^\infty, C, X)$$

where C is the associated monad of the Steiner operad $\mathscr{C}$. It turns out that $\mathbb{E}X$ is a connective spectrum if X is an E_∞ space, where we call $\mathbb{E}$ the *additive infinite loop space machine*, and $\mathbb{E}X$ is a connective E_∞ ring spectrum if X is an E_∞ ring space, where we call $\mathbb{E}$ the *multiplicative infinite loop space machine*. Further, we have the following result.

Proposition 3.2 ([14, 9.3, 9.12]). *Let X be a ringlike E_∞ ring space. There is natural diagram of maps of E_∞ ring spaces*

$$X \xleftarrow{\varepsilon} B(C, C, X) \xrightarrow{B(\alpha,\mathrm{id},\mathrm{id})} B(Q, C, X) \xrightarrow{\zeta} \Omega^\infty\mathbb{E}X$$

where $Q = \Omega^\infty\Sigma^\infty$, ε is a homotopy equivalence with natural homotopy inverse η, and both $B(\alpha, \mathrm{id}, \mathrm{id})$ and ζ are weak equivalences. Therefore we have a composition

$$\eta : X \to \Omega^\infty\mathbb{E}X$$

which is a weak equivalence as a map between E_∞ ring spaces.

For any E_∞ ring spectrum E, there is a composition of maps between E_∞ ring spectra

$$\varepsilon : \mathbb{E}\Omega^\infty E \xrightarrow{B(\mathrm{id},\alpha,\mathrm{id})} B(\Sigma^\infty, Q, \Omega^\infty E) \xrightarrow{\varepsilon} E$$

which is a weak equivalence if E is connective.

Therefore, $\mathbb{E}$ and Ω^∞ induces an adjoint equivalence between the homotopy category of ringlike E_∞ ring spaces and the homotopy category of connective E_∞ ring spectra.

4. The main theorem

Theorem 4.1. *The following commutative diagram of three categories*

$$\begin{tikzcd}
\textit{Periodic } E_\infty \textit{ ring spectra} \arrow[dr, "\Omega^\infty"'] & & \textit{Connective periodic } E_\infty \textit{ ring spectra} \arrow[ll, "{[\beta^{-1}]}"'] \arrow[dl, shift right, "\Omega^\infty"'] \\
& \textit{Periodic } E_\infty \textit{ ring spaces} \arrow[ur, shift right, "\mathbb{E}"'] &
\end{tikzcd}$$

induces an equivalence on homotopy categories.

Proof. We prove that the homotopy category of periodic E_∞ ring spaces is equivalent to the homotopy category of connective periodic E_∞ ring spectra.

Notice that the functors Ω^∞ and $\mathbb{E}$ are originally defined between E_∞ ring spaces and connective E_∞ ring spectra, so we need to first show that they induce functors between the periodic categories. This is straightforward because

$$\pi_*(X) \cong \pi_*(\Omega^\infty\mathbb{E}X) \cong \pi_*(\mathbb{E}X)$$

for $* \geq 0$, where the first isomorphism is induced by the weak equivalence $\eta : X \to \Omega^\infty\mathbb{E}X$. Hence, from the result in Proposition 3.2, we know that Ω^∞ and $\mathbb{E}$ induce an equivalence between the homotopy categories of periodic E_∞ ring spaces and connective periodic E_∞ ring spectra. It follows similarly that Ω^∞ also induces a functor from periodic E_∞ ring spectra to periodic E_∞ ring spaces.

Now we show that there is an equivalence between the homotopy categories of periodic E_∞ ring spectra and periodic E_∞ ring spaces

$$\text{Periodic } E_\infty \text{ ring spectra} \underset{\Omega^\infty}{\overset{\mathbb{E}(-)[\beta^{-1}]}{\leftrightarrows}} \text{Periodic } E_\infty \text{ ring spaces.}$$

We need to construct natural isomorphisms $\eta : X \to \Omega^\infty(\mathbb{E}X[\beta^{-1}])$ for any periodic E_∞ ring space X and $\varepsilon : \mathbb{E}\Omega^\infty E[\beta^{-1}] \to E$ for any periodic E_∞ ring spectrum E.

The unit η is defined as the following composition

$$\eta : X \xrightarrow{\ \eta\ } \Omega^\infty\mathbb{E}X \longrightarrow \Omega^\infty(\mathbb{E}X[\beta^{-1}])$$

where the first arrow is the unit in Proposition 3.2 which is a weak equivalence and the second arrow is induced by the natural map of localization $\mathbb{E}X \to \mathbb{E}X[\beta^{-1}]$. Since $\mathbb{E}X$ is a connective periodic spectrum, by definition we know that multiplying by its Bott element $[\beta]$ induces an isomorphism on homotopy groups in positive degrees. Further, since $\pi_*(\mathbb{E}X[\beta^{-1}]) \cong \pi_*(\mathbb{E}X)[\beta^{-1}]$ (Proposition 3.1), we have

$$\pi_i(\mathbb{E}X[\beta^{-1}]) \cong \pi_i(\mathbb{E}X)$$

for $i \geq 0$. Hence,

$$\pi_i(\Omega^\infty\mathbb{E}X) \cong \pi_i(\mathbb{E}X) \cong \pi_i(\mathbb{E}X[\beta^{-1}])$$

for any $i \geq 0$, implying that the second arrow is also a weak equivalence. This shows that $\eta : X \to \Omega^\infty(\mathbb{E}X[\beta^{-1}])$ is a weak equivalence, and it is natural because $X \to \Omega^\infty\mathbb{E}X$ and $\Omega^\infty\mathbb{E}X \to \Omega^\infty(\mathbb{E}X[\beta^{-1}])$ are both natural.

Now we construct the counit $\varepsilon : \mathbb{E}\Omega^\infty E[\beta^{-1}] \to E$. Notice that in $\varepsilon : \mathbb{E}\Omega^\infty E \to E$ (Proposition 3.2), the Bott element $[\beta]$ of $\mathbb{E}\Omega^\infty E$ is sent to the Bott element of E which is a unit in $\pi_*(E)$, so by the universal property of localization (Proposition 3.1),

$$\begin{tikzcd} \mathbb{E}\Omega^\infty E \arrow[r] \arrow[dr, "\varepsilon"'] & \mathbb{E}\Omega^\infty E[\beta^{-1}] \arrow[d, dashed] \\ & E \end{tikzcd}$$

we have a unique map of E_∞ ring spectra $\mathbb{E}\Omega^\infty E[\beta^{-1}] \to E$, which is our counit so we still denote it as ε. The counit ε follows from the construction being natural. To show that it is a weak equivalence, it suffices to show that

$$\pi_*(E) \cong \pi_*(\mathbb{E}\Omega^\infty E[\beta^{-1}]) \cong \pi_*(\mathbb{E}\Omega^\infty E)[\beta^{-1}].$$

For any ring S and a ring homomorphism $f : \pi_*(\mathbb{E}\Omega^\infty E) \to S$ such that the Bott element $[\beta]$ is sent to a unit in S, we may construct a graded ring homomorphism $\tilde{f} : \pi_*(E) \to S$ in the following way

$$\begin{aligned} \tilde{f} : \pi_*(E) &\to S \\ [x] &\mapsto \begin{cases} f([x]), & \text{if } [x] \in \pi_i(E) \text{ for } i \geq 0; \\ f([\beta]^t[x]), & \text{if } [x] \in \pi_i(E) \text{ for } i < 0 \end{cases} \end{aligned}$$

where $[\beta]$ is the corresponding Bott element in E, and t is chosen to be the smallest positive integer such that $t+di \geq 0$. This construction of $\tilde{f}$ is completely determined by f, so it is the unique choice that fits into the commutative diagram

$$\begin{tikzcd} \pi_*(\mathbb{E}\Omega^\infty E) \arrow[r] \arrow[dr, "f"'] & \pi_*(E) \arrow[d, dashed, "\tilde{f}"] \\ & S \end{tikzcd}$$

By the universal property of localizations of rings, we know that $\pi_*(E) \cong \pi_*(\mathbb{E}\Omega^\infty E)[\beta^{-1}]$. This shows that the counit $\varepsilon : \mathbb{E}\Omega^\infty E[\beta^{-1}] \to E$ is a weak equivalence. $\square$

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