GROUPS OF EXOTIC SPHERES

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Abstract. The exotic sphere is defined as a topological sphere with a smooth structure that is different from the standard one. In a fixed dimension, all exotic spheres with the standard sphere form a group under connected sum. In this paper, we will discuss these groups and prove their finiteness in dimensions $\geqslant 5$. Along the way, some explicit computation of order of these groups can be obtained.

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1. Introduction

It is a long-standing problem in topology to determine the number of smooth structures on a topological sphere. It was proven by Kervaire and Milnor in 1963 that the number of non-isomorphic smooth structures on a topological sphere is finite in dimensions $\geqslant 5$. In this paper, we will follow Kervaire and Milnor's work [1] to present the proof.

Shown later by Kirby and Siebenmann, more generally, every closed topological manifold in dimensions $\geqslant 5$ has at most finitely many smooth structures, but the proof depends on the result for spheres in the same dimension.

It is worth mentioning the story in lower dimensions: every topological manifold has a unique smooth structure in dimensions $\leqslant 3$, and the case of dimension 3 was proven earlier in 1952 by Moise. However, this is not true in dimension 4, where there are homeomorphic manifolds with non-isomorphic smooth structures, and topological manifolds without a smooth structure. Moreover, whether there is a unique smooth structure on a topological 4-sphere and whether there are infinitely many remain unknown.

Now let us get to the point by starting with an important definition. Throughout this paper, all manifolds (with or without boundary) are assumed compact, oriented and smooth. Denote by $\overline{M}$ the manifold M with the orientation reversed.

Definition 1.1. Manifolds M_1 and M_2 are said to be h**-cobordant** if the disjoint union $M_1 \sqcup \overline{M}_2$ is the boundary of some manifold W and both M_1 and $\overline{M}_2$ are deformation retracts of W. W is called a h**-cobordism** between M_1 and M_2.

Since smooth manifolds have CW-structures by the Morse theory, we only need the inclusions $M_1 \to W$, $\overline{M}_2 \to W$ to be homotopy equivalences in place of deformation retracts. Then it is easy to see that h-cobordance is an equivalence relation on manifolds in a fixed dimension.

One of the reasons why this definition is important comes from the h-cobordism theorem by Smale in 1962. In this paper, we denote diffeomorphisms by $\cong$ and homotopy equivalences by $\simeq$.

Theorem 1.2 (Smale)**.** *Let $n \geqslant 5$ and M_1, M_2 be simply-connected n-manifolds. If M_1 and M_2 are h-cobordant by W, then $W \cong M_1 \times [0,1]$ and hence $M_1 \cong M_2$.*

The proof using Morse functions is presented nicely in Milnor's lecture notes [7]. The theorem was proven to fail for $n = 4$ by Donaldson in 1983.

It will be shown in the next section that h-cobordism classes of homotopy n-spheres form a group under connected sum (which is called the h**-cobordism group of homotopy spheres**). By the topological Poincaré conjecture, a homotopy sphere is a topological sphere for all n; by the h-cobordism theorem, h-cobordant spheres are the same thing as diffeomorphic spheres for $n \geqslant 5$. Therefore, the h-cobordism group of homotopy spheres is none other than the group of exotic spheres (including the standard sphere) under connected sum.

In this paper, we will only consider the h-cobordism group of homotopy spheres without resorting to the topological Poincaré conjecture and the h-cobordism theorem. Thus the main result can be stated as follows.

Theorem 1.3. *The h-cobordism classes of homotopy n-spheres form a finite abelian group under connected sum.*

This group will be denoted by Θ_n. To prove its finiteness, we prove that bP_{n+1} and Θ_n/bP_{n+1} are both finite, where bP_{n+1} is a subgroup of Θ_n, consisting of homotopy spheres that bound parallelizable manifolds (Definition 4.1). Moreover, we shall show that bP_{2k+1} is trivial and that bP_{2k} is cyclic, without going into details of computation of the order $|bP_{2k}|$.

2. Preliminaries

2.1. **Connected Sum.** Let M_1, M_2 be closed n-manifolds, $i_1 : D^n \to M_1$ and $i_2 : D^n \to M_2$ be (smooth) embeddings from n-disks to them, such that i_1 preserves orientation and i_2 reverses orientation.

Definition 2.1. The **connected sum** $M_1 \sharp M_2$ is the quotient space of

$$(M_1 - i_1(0)) \sqcup (M_2 - i_2(0))$$

identifying $i_1(tu)$ with $i_2((1-t)u)$ for each $t \in (0,1)$ and $u \in S^{n-1} = \partial D^n$.

Proposition 2.2. *(1) $M_1 \sharp M_2$ is a well-defined closed manifold.*
(2) The connected sum operation is associative and commutative (up to orientation diffeomorphism) with an identity element S^n, the standard sphere.

Proof. (1) Clearly, $M_1 \sharp M_2$ is a closed smooth manifold and has an orientation compatible with that of M_1 and M_2 since the attaching is orientation-preserving.

The non-trivial part is that $M_1 \sharp M_2$ is independent of the embeddings i_1, i_2. This follows from the "disc theorem" of Palais [9] that any two orientation preserving embeddings $i_1, i_1' : D^n \to M_1$ are isotopic.

(2) These follow from the definition. □

For manifolds with boundary, we also need a definition of connected sum.

Let H^{n+1} denotes the closed upper half-disk of D^{n+1}, containing points with the last coordinate $\geqslant 0$. D^n denotes the subset of D^{n+1} with the last coordinate $= 0$.

Let W_1, W_2 be $(n+1)$-manifolds with boundary,

$$i_q : (H^{n+1}, D^n) \to (W_q, \partial W_q),\ q = 1, 2$$

be embeddings such that one preserves orientation and the other reverses it.

Definition 2.3. The **connected sum along boundary** $W_1 \natural W_2$ is the quotient space of

$$(W_1 - i_1(0)) \sqcup (W_2 - i_2(0))$$

identifying $i_1(tu)$ with $i_2((1-t)u)$ for each $t \in (0,1)$ and $u \in H^{n+1} \cap S^n$.

Similarly, one can prove the well-definedness. It is easy to see the following facts about the resulting manifold $W_1 \natural W_2$.

- (boundary) $\partial(W_1 \natural W_2) = \partial W_1 \sharp \partial W_2$.
- (homotopy type) $W_1 \natural W_2 \simeq W_1 \vee W_2$.

2.2. **Stable Parallelizability.** Let M be an n-manifold with tangent bundle TM. If it is immersed in some larger manifold, we will denote the normal bundle by NM. Denote the trivial r-bundle on M by ϵ^r.

Definition 2.4. M is said to be **stably parallelizable** if TM is stably trivial, i.e., $TM \oplus \epsilon^r$ is trivial for some $r \geqslant 0$.

The following lemma is useful for judging whether a bundle on an n-manifold (with or without boundary) is trivial.

Lemma 2.5. *Let ξ be an r-bundle on an n-manifold M, where $r > n$. If ξ is stably trivial, then ξ is trivial.*

Proof. By induction, assume that $\xi \oplus \epsilon^1$ is trivial. An isomorphism $\xi \oplus \epsilon^1 \cong \epsilon^{r+1}$ gives rise to a bundle map from ξ to tangent bundle TS^r as follows.

Identify all fibers of ϵ^{r+1} with a fixed Euclidean space $\mathbb{R}^{r+1}$. Each fiber of ξ corresponds to a r-subspace of $\mathbb{R}^{r+1}$. Then we define $M \to S^r$ by sending each point to the normal vector of its fiber. This defines a bundle map $\xi \to TS^r$.

Since $\dim M < \dim S^r$, the map $M \to S^r$ between bases is null-homotopic by cellular approximation. Thus, ξ is trivial. □

The following two corollaries about stable parallelizability follow immediately from Lemma 2.5. They will be used in the future.

Corollary 2.6. *M is stably parallelizable iff $TM \oplus \epsilon^1$ is trivial.*

Corollary 2.7. *Let M be an n-submanifold of S^{n+k} where $n < k$. Then M is stably parallelizable iff NM is trivial.*

For the former, apply Lemma 2.5 to $TM \oplus \epsilon^1$; for the latter, apply it to NM. (Note that the tangent bundle of S^{n+k} minus a point is trivial.)

Proposition 2.8. *A connected manifold M with boundary is parallelizable iff it is stably parallelizable.*

Proof. Claim: for any n-manifold M,

$$H^n(M) \cong [M, S^n],$$

where $[\cdot,\cdot]$ denotes the homotopic classes of maps.

In fact, the Eilenberg-MacLane space $K(\mathbb{Z}, n)$ can be constructed by gluing cells of dimension $\geqslant n+2$, with $(n+1)$-skeleton $K(\mathbb{Z}, n)^{(n+1)} = S^n$. Then the claim follows since for any n-manifold M, we have

$$H^n(M) \cong [M, K(\mathbb{Z}, n)] \cong [M, K(\mathbb{Z}, n)^{(n+1)}] \cong [M, S^n].$$

Now suppose $TM \oplus \epsilon^1 \cong \epsilon^{n+1}$. Similar to the proof of Lemma 2.5, we have a bundle map $TM \to TS^n$. Since M has boundary here, $H^n(M) = 0$. Thus, every map $M \to S^n$ is null-homotopic by the claim, and hence TM is trivial. □

Finally, let us state a theorem that will be used when studying homotopy spheres. The proof is an application of the obstruction theory. See [1, §3].

Theorem 2.9. *Every homotopy sphere is stably parallelizable.*

2.3. **Framing and Cobordism.** Recall that a **framing** of a vector bundle ξ is a trivialization of ξ. Existence of a framing implies being trivial, but different trivializations may not be homotopic.

Definition 2.10. Let M be a n-manifold.

- A **stable framing** of M is a framing Φ of the stable tangent bundle $TM \oplus \epsilon^1$ (if one exists). (M, Φ) is then called a **framed manifold**.
- If M is a submanifold of W, a **normal framing** of M is a framing Ψ of the normal bundle NM (if one exists). (M, Ψ) is then called a **normally framed submanifold** of W.

Next we relate the concept of framing to cobordism.

Definition 2.11. Let (M_1, Φ_1) and (M_2, Φ_2) be framed n-manifolds. A **framed cobordism** between them is a parallelizable $(n+1)$-manifold W together with a framing Ψ of TW, such that $\partial W = M_1 \sqcup \overline{M}_2$ and $\Psi|_{M_q} = \Phi_q,\ q = 1, 2$.

Remark 2.12. Note that $TW|_{\partial W}$ can be identified with $T(\partial W) \oplus \epsilon^1$ by assigning an outward pointing vector field.

Definition 2.13. Let (M_1, Φ_1) and (M_2, Φ_2) be normally framed n-submanifolds of W. A **normally framed cobordism** between them is a normally framed $(n+1)$-submanifold (V, Ψ) of $W \times [0, 1]$, such that $\partial V = M_1 \sqcup \overline{M}_2$ where $M_q \subset W \times \{q\}$ and $\Psi|_{M_q} = \Phi_q,\ q = 1, 2$.

Closely related to the normal framing is the Pontryagin(-Thom) construction, which relates normally framed cobordism classes of n-submanifolds of W to homotopy classes of maps $[W, S^k]$, where W is a closed $(n+k)$-manifold.

- Let $f : W \to S^k$ be a smooth map and $p \in S^k$ be a regular value. Consider the normally framed submanifold $(f^{-1}(p), \Phi)$ of W, where Φ is the pullback of standard trivialization of $TS^k|_p$.
- Let (M, Φ) be a normally framed manifold, where $\Phi : NM \to M \times \mathbb{R}^k$. Regard NM as a tubular neighborhood of M in W. Consider the map $f : W \to S^k$ defined by setting $f|_{NM} : NM \overset{\Phi}{\to} M \times \mathbb{R}^k \overset{\mathrm{pr}}{\to} \mathbb{R}^k \hookrightarrow S^k$ and $f|_{W-NM}$ collapses $W - NM$ to the point $S^k - \mathbb{R}^k$. (Regard S^k as the one-point compactification of $\mathbb{R}^k$.)

Theorem 2.14. *Let W be a closed $(n+k)$-manifold. The above construction gives a one-to-one correspondence between normally framed cobordism classes of n-submanifolds of W and homotopy classes of maps $[W, S^k]$.*

One needs to check that the correspondences are well-defined (independent of various choices), and are inverses of each other. See [8, §7].

2.4. **Homotopy Groups of Orthogonal Groups.** Consider the exact sequence of homotopy groups associated to the fibration

$$SO_k \overset{i_k}{\to} SO_{k+1} \overset{j_k}{\to} S^k, \quad \text{where } j_k : \rho \mapsto \rho \cdot x_0,\ x_0 = (0, ..., 0, 1).$$

One defines the stable homotopy group $\pi_k(SO) := \pi_k(SO_{k+2}) = \pi_k(SO_{k+3}) = \cdots$ and $\pi_k(O)$ similarly. They are equal when $k > 0$.

Theorem 2.15 (Bott's Periodicity)**.** *$\pi_k(O)$ is* mod 8 *periodic with repect to k and the actual homotopy groups are*

k mod 8	0	1	2	3	4	5	6	7
$\pi_k(O)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0	0	$\mathbb{Z}$

We will encounter unstable homotopy groups of orthogonal groups, so we state the needed results here. Consider the diagram with the row and column exact.

$$\begin{array}{ccccccc}
 & & \downarrow & & & & \\
 & & \pi_k(S^k) & \overset{f_k}{\dashrightarrow} & & & \\
 & & \downarrow \partial_k & & & & \\
\longrightarrow & \pi_{k-1}(SO_{k-1}) & \xrightarrow{(i_{k-1})_*} \pi_{k-1}(SO_k) & \xrightarrow{(j_{k-1})_*} & \pi_{k-1}(S^{k-1}) & \longrightarrow & . \\
 & & \downarrow (i_k)_* & & & & \\
 & & \pi_{k-1}(SO_{k+1}) & & & & \\
 & & \downarrow & & & &
\end{array}$$

Proposition 2.16. *The map $f_k : \mathbb{Z} = \pi_k(S^k) \to \pi_{k-1}(S^{k-1}) = \mathbb{Z}$ is $\cdot 2$ if k is even and $\cdot 0$ if k is odd.*

Proof. By clutching functions, elements of $\pi_{k-1}(SO_k)$ are in one-to-one correspondence with the isomorphic classes of k-bundles on S^k. We identify them.

It can be computed that ∂_k maps the standard generator of $\pi_k(S^k)$ to the tangent bundle TS^k, and that $(j_{k-1})_*$ maps a k-bundle to its obstruction class to finding a section ([2, §1.B]). Therefore, $(j_{k-1})_*(\partial_k(1)) = e(TS^k)[S^k] = \chi(S^k)$, the Euler characteristic. The proposition then follows. □

Proposition 2.17. *(1) $(i_k)_* : \pi_{k-1}(SO_k) \to \pi_{k-1}(SO_{k+1})$ is surjective. Its kernel is a cyclic group with generator TS^k.*

(2)

$$\mathrm{Coker}(\pi_k(SO_k) \to \pi_k(SO)) = \begin{cases} \mathbb{Z}_2, & k = 1, 3, 7; \\ 0, & k \neq 1, 3, 7. \end{cases}$$

Proof. (1) This is clear by the long exact sequence. Further, $\mathrm{Ker}(i_k)_* = 0$ iff $k = 1, 3, 7$, using the fact that TS^k is trivial iff $k = 1, 3, 7$.

(2) First let k be even. Since

$$\pi_k(SO_k) \overset{(i_k)_*}{\to} \pi_k(SO_{k+1}) \overset{(j_k)_*}{\to} \pi_k(S^k) \overset{\partial_k}{\to} \pi_{k-1}(SO_k)$$

is exact and ∂_k is injective (Proposition 2.16), we see that $(i_k)_*$ is surjective. The other cases are equally easy. □

Combining (1) above and $\pi_2(O) = \pi_6(O) = 0$, we have $\pi_2(SO_3) = \pi_6(SO_7) = 0$.

3. Construction of Θ_n

Now we turn to the precise definition of the h-cobordism group of homotopy n-spheres. We will show in this section that the connected sum is well-defined over h-cobordism classes of homotopy n-spheres and makes it a abelian group. This abelian group will be denoted by Θ_n.

First, we need to note the following.

Lemma 3.1. *The connected sum of two homotopy n-spheres is a homotopy n-sphere.*

Proof. By van Kampen's theorem and the Mayer-Vietoris sequence, this connected sum is a simply-connected homology sphere, and hence has homotopy groups of a sphere in dim $\leqslant n$. Then, it has to be homotopic to a sphere if we take a generator of its n^{th} homotopy group and apply Whitehead's theorem. □

Next, we show that the connected sum is well-defined over h-cobordism classes. Denote the equivalence relation of h-cobordance by $\overset{h}{\sim}$.

Lemma 3.2. *Let M_1, M_1', M_2 be simply-connected closed n-manifolds. If $M_1 \overset{h}{\sim} M_1'$, then $M_1 \sharp M_2 \overset{h}{\sim} M_1' \sharp M_2$.*

The proof is to construct a h-cobordism $M_1 \sharp M_2 \overset{h}{\sim} M_1' \sharp M_2$ by somehow gluing the h-cobordism $W : M_1 \overset{h}{\sim} M_1'$ with $M_2 \times [0, 1]$, and is not so relevant. See [1, §2].

We now turn to the group structure of h-cobordism classes of homotopy n-spheres. Since we have shown the associativity, commutativity and that S^n represents the identity by Proposition 2.2, we only need to show that inverse exists.

Before that, we introduce the following useful criterion for judging whether a manifold represents the identity. Note that a manifold h-cobordant to S^n has to be a homotopy sphere by definition.

Lemma 3.3. *Let M be a simply-connected n-manifold. $M \overset{h}{\sim} S^n$ iff M bounds a contractible manifold.*

Proof. (1) If W is the h-cobordism with $\partial W = M \sqcup \overline{S^n}$, then attaching a disk along S^n yields a manifold W' with boundary M. Since $S^n \simeq W$, W' is contractible by a Mayer-Vietoris computation.

(2) If $M = \partial W'$ with W' contractible, removing the interior of an embedded disk yields a simply-connected W with $W = M \sqcup \overline{S^n}$. Considering homology long exact sequences of pairs $(D^{n+1}, S^n) \hookrightarrow (W', W)$, we have isomorphisms $H_k(S^n) \to H_k(W)$ and hence a homotopy equivalence $S^n \hookrightarrow W$. Now since

$$H_k(W, M) \cong H^{n+1-k}(W, S^n) = 0$$

by the Poincaré duality, $M \hookrightarrow W$ is a homotopy equivalence as well since both spaces are simply-connected. □

Theorem 3.4. *The h-cobordism classes of homotopy n-spheres form an abelian group Θ_n under connected sum.*

Proof. By Lemma 3.2 and Proposition 2.2, all that is left is to show that every element $[M]$ has an inverse. The inverse is $[\overline{M}]$ by the claim below.

Claim: if M is a homotopy sphere, then $M \sharp \overline{M}$ bounds a contractible manifold.

In fact, let W be the cylinder $(M - D^n) \times [0, 1]$, where $D^n \hookrightarrow M$ is an embedded disk. Clearly $W \simeq M - D^n$ is contractible. After suitably smoothing out the corners, W becomes a manifold with boundary $M \sharp \overline{M}$ ([1, §2]). □

We will finally prove that Θ_n is a finite group. Here is a table of order of Θ_n for small n (from [4]). The subgroup bP_{n+1} of Θ_n will be defined later.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\lvert\Theta_n\rvert$	1	1	1	1	1	1	28	2	8	6	992	1	3	2	16256	2
$\lvert bP_{n+1}\rvert$	1	1	1	1	1	1	28	1	2	1	992	1	1	1	8128	1
$\lvert\Theta_n/bP_{n+1}\rvert$	1	1	1	1	1	1	1	2	4	6	1	1	3	2	2	2

When $n \leqslant 3$, every topological n-manifold admits a unique smooth structure, therefore Θ_n is trivial by the Poincaré conjecture.

When $n \geqslant 4$, proving the finiteness of Θ_n will be the goal of this paper. When $n \geqslant 5$, Θ_n is the "group of exotic spheres" by the h-cobordism theorem. When $n = 4$, we will show that Θ_4 is trivial, but still, we can say nothing about smooth structure on a topological 4-sphere.

4. Finiteness of Θ_n/bP_{n+1}

Definition 4.1. Define $bP_{n+1} \subset \Theta_n$ as a subset whose elements are classes that can be represented by a homotopy sphere M that bounds a parallelizable manifold.

We will see that the condition of being the boundary of a parallelizable manifold depends only on the h-cobordism class $[M]$, and that bP_{n+1} is actually a subgroup of Θ_n. In order to prove the finiteness of Θ_n, we prove that bP_{n+1} and Θ_n/bP_{n+1} are both finite.

Denote n^{th} stable homotopy group of spheres by $\pi_n^S (= \pi_{k+n}(S^k), k \geqslant n+2)$. Let M be a stably parallelizable closed n-manifold (e.g., a homotopy sphere, by Theorem 2.9). We associate a specific subset $p(M)$ of π_n^S to M as follows.

By Whitney's theorem, there exists an embedding (unique up to isotopy)

$$i : M \hookrightarrow S^{k+n}$$

for $k \geqslant n+2$. The normal bundle NM is trivial by Corollary 2.7. Each of its framing Φ yields by the Pontryagin construction a map

$$f_{M,\Phi} : S^{k+n} \to S^k,$$

whose homotopy class is a well-defined element in $\pi_{k+n}(S^k)$. We collect these elements as a subset $p(M)$ as the normal framing varies

$$p(M) := \{[f_{M,\Phi}]\} \subset \pi_{k+n}(S^k) \cong \pi_n^S.$$

(Here we need to note that the Pontryagin construction is compatible with suspension so that $p(M)$ is well-defined in π_n^S.) It turns out that $p(M)$ can be defined over h-cobordism classes:

Lemma 4.2. *If $M_1 \overset{h}{\sim} M_2$, then $p(M_1) = p(M_2)$.*

Proof. Let W be the h-cobordism with $\partial W = M_1 \sqcup \overline{M}_2$. Choose an embedding $W \hookrightarrow S^{k+n} \times [0,1]$ such that $M_q \hookrightarrow S^{k+n} \times \{q\}, q = 0, 1$. Now a normal framing Φ_1 of M_1 extends to a normal framing Ψ of W by pullback along $W \to M_1$, which restricts to a normal framing Φ_2 of M_2. Then (M_1, Φ_1) is normally framed cobordant to (M_2, Φ_2) and consequently $f_{(M_1,\Phi_1)} \simeq f_{(M_2,\Phi_2)}$ by nature of the Pontryagin construction.

This shows that $p(M_1) \subset p(M_2)$. Similarly, $p(M_2) \subset p(M_1)$. □

Next consider $[M] \in \Theta_n$. We investigate $p(M) \subset \pi_n^S$ further and see the following group structure.

Proposition 4.3. *Let M be a homotopy n-sphere.*

(1) $p(S^n) \subset \pi_n^S$ is a subgroup. $p(M) \subset \pi_n^S$ is a coset of $p(S^n)$.

(2) The map

$$\Theta_n \to \pi_n^S / p(S^n)$$
$$[M] \mapsto [p(M)]$$

is a group homomorphism.

(3) M bounds a parallelizable manifold iff $[p(M)] = 0$.

Proof. Key claim:

$$p(M_1) + p(M_2) \subset p(M_1 \sharp M_2).$$

In fact, consider the connected sum of $M_1 \times [0,1]$ and $M_2 \times [0,1]$ along boundary components $M_1 \times \{1\}$ and $M_2 \times \{1\}$ (Definition 2.3). The resulting manifold W has boundary $(M_1 \sharp M_2) \sqcup (\overline{M}_1) \sqcup (\overline{M}_2)$.

Now let $W \hookrightarrow S^{k+n} \times [0,1]$ be an embedding such that $(\overline{M}_1) \sqcup (\overline{M}_2) \hookrightarrow S^{k+n} \times \{0\}$ and $(M_1 \sharp M_2) \hookrightarrow S^{k+n} \times \{1\}$. Any normal framings Φ_1, Φ_2 of M_1, M_2 can extend to W and restrict to a normal framing Φ_0 of $M_1 \sharp M_2$. Therefore, $(M_1, \Phi_1) \sqcup (M_2, \Phi_2)$ is normally framed cobordant to $(M_1 \sharp M_2, \Phi_0)$ and $[f_{M_1,\Phi_1}] + [f_{M_2,\Phi_2}] = [f_{M_1 \sharp M_2, \Phi_0}]$, which implies the claim.

(1) Apply the claim to the following identities:

$$S^n \sharp S^n \cong S^n, \quad M \sharp S^n \cong M, \quad M \sharp \overline{M} \overset{h}{\sim} M.$$

We obtain relations

$$p(S^n) + p(S^n) \subset p(S^n), \quad p(M) + p(S^n) \subset p(M), \quad p(M) + p(\overline{M}) \subset p(S^n).$$

Since π_n^S is finite ([6, §4.2.4]), the first relation implies that $p(S^n)$ is a subgroup. The second implies that $p(M)$ is a union of $p(S^n)$-cosets, and hence the third implies that $p(M)$ is a single coset.

(2) This follows immediately from the claim.

(3) $0 \in p(M)$ iff $f_{M,\Phi} \simeq 0$ for some Φ, which is in turn equivalent to that (M, Φ) is normally framed null-cobordant for some Φ.

If this holds, M bounds a W and Φ extends to a normal framing of W. Then by Corollary 2.7 and Proposition 2.8, W is parallelizable.

Conversely, if $\partial W = M$ with W parallelizable, choose an embedding $W \hookrightarrow D^{k+n+1}$ with $M \hookrightarrow S^{k+n}$. By Corollary 2.7 again, NW is trivial. □

By (3) above and Lemma 4.2, the condition of bounding a parallelizable manifold depends only on the h-cobordism class $[M]$.

Moreover, elements of $\mathrm{Ker}(\Theta_n \to \pi_n^S/p(S^n))$ are precisely classes of homotopy spheres that bound parallelizable manifolds, i.e., elements of bP_{n+1}. Thus, we have proved that $bP_{n+1} \subset \Theta_n$ is a subgroup.

Theorem 4.4. *Θ_n/bP_{n+1} is a finite group.*

Proof. We have an injection $\Theta_n/bP_{n+1} \hookrightarrow \pi_n^S/p(S^n)$, and the target is finite. □

Remark 4.5. One can see that $p(S^n)$, the range of the Pontryagin construction of S^n, is exactly the image of the J-homomorphism $J : \pi_n(SO) \to \pi_n^S$ by checking definitions. Thus we actually have an injection

$$\Theta_n/bP_{n+1} \hookrightarrow \mathrm{Coker} J,$$

whose target is more well-studied.

In the next few sections, we will study the groups bP_{n+1} and prove that they are always finite cyclic. Here are some results.

$$bP_{n+1} = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 2, \\ 0 \text{ or } \mathbb{Z}_2, & \text{if } n \equiv 1 \pmod 4, \\ \mathbb{Z}_{k_n}, & \text{if } n \equiv 3 \pmod 4, \end{cases}$$

where $\{k_n\}$ is a sequence growing faster than exponential (see Table 3).

5. Surgery and Framed Surgery

The key technique of studying the group bP_{n+1} is called "surgery". This section is an introduction to surgery and framed surgery. The main reference is Milnor's paper [3] on surgery. Let M be an n-manifold.

Definition 5.1. Let $\phi : S^p \times D^{n-p} \to M$ be an embedding. Define $\chi(M, \phi)$ to be the quotient of

$$(M - \phi(S^p \times \{0\})) \sqcup (D^{p+1} \times S^{n-p-1})$$

identifying $\phi(u, tv)$ and (tu, v) for each $u \in S^p, v \in S^{n-p-1}, t \in (0, 1]$. We say that $\chi(M, \phi)$ is obtained from M by the **surgery** $\chi(\phi)$.

Remark 5.2. Topologically, $\chi(M,\phi)$ can be obtained in the following neater way: Remove the interior of the tube $\phi(S^p \times D^{n-p})$ from M so it adds a boundary component, the torus $\phi(S^p \times S^{n-p-1})$. Then gluing a new tube $D^{p+1} \times S^{n-p-1}$ along this torus yields a space homeomorphic to $\chi(M,\phi)$. Thus, by smoothing out the corners where it glues, we have the previous definition of surgery.

M and $\chi(M,\phi)$ actually play symmetric roles, in the sense that M can be obtained back from $\chi(M,\phi)$ by surgery $\chi(\varphi)$ where $\varphi : D^{p+1} \times S^{n-p-1} \to \chi(M,\phi)$ is the natural embedding. The following theorem allows us to see a surgery as an elementary cobordism.

Theorem 5.3. *If M, M' are closed n-manifolds, M' can be obtained from M by a sequence of surgeries iff M and M' are cobordant.*

Proof. Suppose $M' = \chi(M,\phi)$. Let W be the quotient of

$$(M \times [0,1]) \sqcup (D^{p+1} \times D^{n-p})$$

identifying $\phi(S^p \times D^{n-p}) \subset M \times \{1\}$ and $S^p \times D^{n-p} \subset \partial(D^{p+1} \times D^{n-p})$. Then $\partial W = M \sqcup \overline{\chi(M,\phi)}$. In fact, one needs to smooth out corners of W (see [3, §2]). The resulting manifold is called the **trace** of the surgery $\chi(\phi)$.

The converse can be proven by a standard argument of Morse functions but we do not need it here. □

To study the group bP_{n+1}, we consider parallelizable manifolds bounded by a homotopy sphere. By Lemma 3.3, we aim to make this parallelizable manifold homotopically simpler by surgery (clearly surgery does not change the boundary). If it can always be modified into a contractible one, then by Lemma 3.3, $bP_{n+1} = 0$ (which is true for even n).

The reason why surgery works is partly the following proposition, which shows that surgery can kill homotopy groups below the middle dimension, like what attaching cells does (but in the manifold category).

Now let M be an n-manifold (with or without boundary) and $\phi : S^p \times D^{n-p} \to M$ be an embedding. Denote $\chi(M,\phi)$ by M' and the homotopy class of $\phi|_{S^p\times\{0\}}$ in $\pi_p(M)$ by λ.

Proposition 5.4. *Homotopy groups of M and M' below the middle dimension are related by*

$$\begin{aligned} \pi_k(M') &\cong \pi_k(M), && \text{if } k < \min(p, n-p-1), \\ \pi_p(M') &\cong \pi_p(M)/\Lambda, && \text{if } p \leqslant n/2 - 1, \end{aligned}$$

where Λ is some subgroup containing λ.

Proof. Let V be the attaching space $M \cup_\phi (D^{p+1} \times D^{n-p})$ containing both M and M' as subspaces (as in Remark 5.2). The subspace $V_0 = M \cup_\phi (D^{p+1} \times \{0\})$ is a deformation retract of V, and is also M with a $(p+1)$-cell attached. Thus, $M \hookrightarrow V$ induces isomorphisms on π_k for $k < p$ and a surjection on π_p with kernel Λ containing λ.

Similarly, $M' \hookrightarrow V$ induces isomorphisms on π_k for $k < n-p-1$, hence the first relation follows. Since $p \leqslant n/2 - 1 \Leftrightarrow p < n-p-1$, the second follows as well. □

However, not every class of $\pi_p(M)$ arises from an embedding $\phi : S^p \times D^{n-p} \to M$ for general M. First, this class needs to be representable by an embedding $f : S^p \to$

M. On top of that, the image $f(S^p)$ needs to have a product neighborhood, or equivalently, to have a trivial normal bundle by the tubular neighborhood theorem. We will shortly see that the stable parallelizability guarantees conditions above.

Still, stable parallelizability of M may be destroyed after surgery. For this reason, we introduce the framed surgery first.

Recall that a manifold can be framed iff it is stably parallelizable. Let (M, Φ) be a framed manifold. If $\chi(\phi)$ is a surgery on M, its trace W satisfies that $\partial W = M \sqcup \overline{M'}$ where $M' = \chi(M, \phi)$. Again we identify $TW|_M$ with $TM \oplus \epsilon^1$.

Definition 5.5. A surgery $\chi(\phi)$ is called a **framed surgery** if there is a framing Ψ of TW such that $\Psi|_M = \Phi$.

If $\chi(\phi)$ is framed, M' automatically acquires a stable framing $\Phi' := \Psi|_{M'}$, i.e., the stable parallelizability is preserved by framed surgery.

Lemma 5.6. *Let (M, Φ) be a stably parallelizable n-manifold and $\lambda \in \pi_p(M)$ where $p < n/2$. Then there exists an embedding $\phi : S^p \times D^{n-p} \to M$ representing λ, such that $\chi(\phi)$ is a framed surgery.*

Proof. Since $2p + 1 \leqslant n$, Whitney's theorem implies that λ can be represented by an embedding $f : S^p \to M$. The normal bundle $N(f(S^p))$ is stably trivial since both TM and $T(f(S^p))$ are stably trivial. Then by Lemma 2.5, $N(f(S^p))$ is trivial. Hence there exists an embedding $\phi : S^p \times D^{n-p} \to M$ representing λ.

Let W be the trace of $\chi(\phi)$. The obstructions to extending the framing Φ (of $TW|_M$) to TW live in cohomology groups $H^{k+1}(W, M; \pi_k(SO_{n+1}))$. Note that there is a homotopy equivalence $(W, M) \simeq (W_0, M \times [0,1])$, where $W_0 = (M \times [0,1]) \cup (D^{p+1} \times \{0\})$, i.e., $(M \times [0,1])$ with a $(p+1)$-cell attached. Hence,

$$H^{k+1}(W, M; \pi_k(SO_{n+1})) = \begin{cases} \pi_p(SO_{n+1}), & k = p; \\ 0, & k \neq p. \end{cases}$$

Therefore, the only obstruction to extending Φ is a well-defined class in $\gamma(\phi) \in \pi_p(SO_{n+1})$, which may be non-vanishing. However, we can "twist" the tubular embedding ϕ by a map $\alpha : S^p \to SO_{n-p}$ by setting

$$\begin{aligned} \phi_\alpha : S^p \times D^{n-p} &\to M, \\ (u, v) &\mapsto \phi(u, \alpha(u)v). \end{aligned}$$

Claim: If $s_* : \pi_p(SO_{n-p}) \to \pi_p(SO_{n+1})$ is the map induced by inclusion, then

$$\gamma(\phi_\alpha) = \gamma(\phi) + s_*(\alpha).$$

Note that $s_* : \pi_p(SO_{n-p}) \to \pi_p(SO_{n+1})$ is surjective when $2p < n$. Thus we can choose an α so that $\gamma(\phi_\alpha) = 0$ by the claim. Then the framing Φ extends and makes $\chi(\phi_\alpha)$ a framed surgery.

For a proof of the claim, see [3, §6]. ☐

With these preparations, we are now able to show that homotopy groups below middle dimension can be all killed by framed surgery, provided that the manifold is stably parallelizable.

Proposition 5.7. *Let M be a stably parallelizable connected n-manifold. By a sequence of framed surgeries on M, one can obtain a stably parallelizable manifold which is $([n/2] - 1)$-connected.*

Proof. By Lemma 5.6, choose an embedding $\phi : S^1 \times D^{n-1} \to M$ representing some $\lambda \neq 0 \in \pi_1(M)$, such that $\chi(\phi)$ is a framed surgery. Then we obtain a stably parallelizable manifold $M' = \chi(M, \phi)$ with $\pi_1(M')$ generated by fewer elements than $\pi_1(M)$ by Proposition 5.4.

Here we use the fact that $\pi_1(M)$ is finitely generated. This is because M is a finite CW-complex, so its π_1 is mapped onto by π_1 of the 1-skeleton.

Now assume that M is 1-connected. Since $H_2(M) \cong \pi_2(M)$ then, $\pi_2(M)$ is finitely generated. Similarly by a finite number of framed surgeries, we kill $\pi_2(M)$. This process goes on and the proposition follows. □

Note that the $([n/2]-1)$-connected manifold will be contractible if we could kill the middle homotopy group as well, and this is what we want as explained before Proposition 5.4.

Lemma 5.8. *Let M be a compact n-manifold bounded by a homotopy sphere. If M is $[n/2]$-connected, then M is contractible.*

Proof. By Hurewicz's theorem, $H_i(M, \partial M) = 0$ for $i \leqslant [n/2]$. By the Poincaré duality, M has homology as a point, hence homotopy type as a point. □

For this reason, we now focus on the effect of surgery on the middle dimension, which is more delicate than the case of dimensions below the middle.

First consider the case where $\dim M$ is odd. Let M be a $(k-1)$-connected $(2k+1)$-manifold where $k > 1$, so that $\pi_k(M) \cong H_k(M)$. Let $\phi : S^k \times D^{k+1} \to M$ be an embedding. $M' := \chi(M, \phi)$ is also $(k-1)$-connected and has a corresponding embedding $\phi' : D^{k+1} \times S^k \to M$.

Let $\lambda \in H_k(M)$ and $\lambda' \in H_k(M')$ be the classes corresponding to $[\phi|_{S^k \times \{0\}}] \in \pi_k(M)$ and $[\phi'|_{\{0\} \times S^k}] \in \pi_k(M')$. Consider maps

$$\cdot\lambda : H_{k+1}(M) \to \mathbb{Z}, \quad \cdot\lambda' : H_{k+1}(M') \to \mathbb{Z}$$

defined by taking the intersection pairing with λ, λ'.

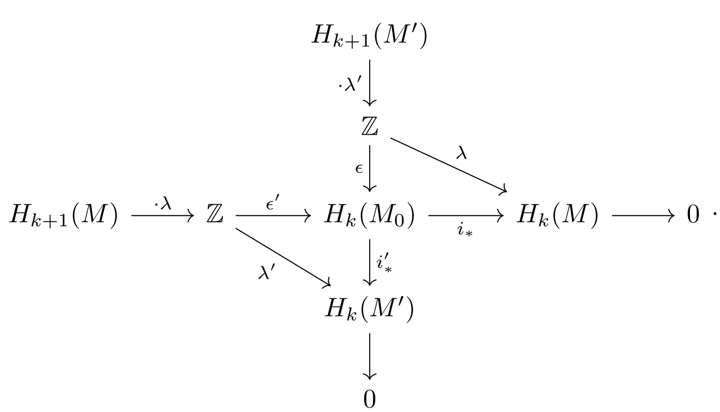
Lemma 5.9. *With settings above, there is an isomorphism*

$$H_k(M)/\langle\lambda\rangle \cong H_k(M')/\langle\lambda'\rangle\,.$$

Proof. Removing the interior of $\phi(S^k \times D^{k+1})$ from M yields a space M_0. We have

$$M_0 = M - (\phi(S^k \times D^{k+1}))^\circ = M' - (\phi'(D^{k+1} \times S^k))^\circ.$$

Consider a commutative diagram



Its row and column are exact sequences of pair (M, M_0) and (M', M_0). To see this, by excision,

$$H_j(M, M_0) = H_j(S^k \times D^{k+1}, S^k \times S^k) = \begin{cases} \mathbb{Z}, & j = k+1; \\ 0, & j \leqslant k. \end{cases}$$

Since a generator of $H_{k+1}(M, M_0)$ clearly has intersection number ± 1 with λ which is represented by $\phi(S^k \times \{0\})$, the map $H_{k+1}(M) \to \mathbb{Z}$ can be described as $\mu \mapsto \mu \cdot \lambda$. Also, it is easy to see that the element $\epsilon' := \epsilon'(1) \in H_k(M_0)$ can be represented by the meridian $\phi(\{x_0\} \times S^k)$ of the torus $\phi(S^k \times S^k)$. Since $\lambda' \in H_k(M')$ is also represented by $\phi(\{x_0\} \times S^k)$, we have $i'_*(\epsilon') = \lambda'$.

Analogous descriptions hold for the column. The lemma then follows since

$$H_k(M)/\langle \lambda \rangle \cong H_k(M_0)/(\langle \epsilon \rangle + \langle \epsilon' \rangle) \cong H_k(M')/\langle \lambda' \rangle .$$

□

Applying this lemma, we can reduce $H_k(M)$ to its torsion subgroup, provided that M is stably parallelizable.

Corollary 5.10. *Let M be a stably parallelizable $(k-1)$-connected $(2k+1)$-manifold where $k > 1$. By a sequence of surgeries, one can obtain a stably parallelizable $(k-1)$-connected manifold M', such that $H_k(M') = (H_k(M))^{\mathrm{tor}}$.*

Proof. Suppose $H_k(M) \cong \mathbb{Z}^{\oplus r} \oplus T$ where $T = (H_k(M))^{\mathrm{tor}}$. Let λ be a generator of a $\mathbb{Z}$-summand. By the Poincaré duality, we have $\mu \cdot \lambda = 1$ for some $\mu \in H_{k+1}(M)$ (similar to Definition 7.1).

Now choose an embedding ϕ by Lemma 5.6, so that $\chi(\phi)$ is a framed surgery. By the row's exactness in the diagram in Lemma 5.9 and that $\mu \cdot \lambda = 1$, we have $\epsilon' = 0$ and hence $\lambda' = 0$. Therefore, $H_k(M') \cong H_k(M)/\langle \lambda \rangle$.

Repeating this process, we obtain the desired manifold, whose middle homology H_k is reduced to torsion. □

For the case of even dimensions, we have a similar diagram. Let M be a $(k-1)$-connected $2k$-manifold where $k > 1$. Let $\chi(\phi)$ be a surgery with $\phi : S^k \times D^k \to M$, $M' = \chi(M, \phi)$. Define λ, λ' and M_0 similarly.

Lemma 5.11. *With settings above, there is a commutative diagram*

$$\begin{array}{ccccccccc} & & \mathbb{Z} & \overset{\lambda}{\searrow} & & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & H_k(M_0) & \longrightarrow & H_k(M) & \xrightarrow{\cdot\lambda} & \mathbb{Z} & \longrightarrow & H_{k-1}(M_0) \longrightarrow 0 \\ & & \downarrow & & & & & & \\ & & H_k(M') & & & & & & \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array},$$

with the row and the column exact.

The proof is analogous. The diagram changes since the non-zero $H_j(M, M_0) = H_j(S^k \times D^k, S^k \times S^{k-1})$ shifts by one dimension. For this reason, we cannot derive an explicit isomorphism like that in Lemma 5.9.

6. The Group bP_{2k+1}

In this section, we are going to show that

$$bP_{n+1} = 0, \text{ for } n \text{ even}.$$

That is to say, every even dimensional homotopy sphere that bounds a parallelizable manifold also bounds a contractible manifold, by Lemma 3.3.

Denote $n+1$ by $2k+1$. By Proposition 5.7, we have been able to make a parallelizable $(n+1)$-manifold $(k-1)$-connected without changing its boundary by framed surgeries. Then by Lemma 5.8, all we need to do now is to kill its k^{th} homotopy group.

Assumption 6.1. Throughout this section, M is a stably parallelizable and $(k-1)$-connected $(2k+1)$-manifold. The boundary ∂M is a homotopy sphere.

Using Corollary 5.10, we have been able to reduce the middle homology group $H_k(M)(\cong \pi_k(M))$ to $(H_k(M))^{\text{tor}}$. Now in order to kill this torsion group, we have to specialize further to cases where k is even or odd.

6.1. For k Even.

Let M satisfy Assumption 6.1.

Lemma 6.2. *If k is even, then any surgery $\chi(\phi)$ with some $\phi : S^k \times D^{k+1} \to M$ necessarily changes* $\operatorname{rank}(H_k(M))$.

The proof will be delayed until after proving the main result below.

Theorem 6.3. *If a homotopy sphere of dimension $2k$ (with k even) bounds a parallelizable manifold M, then it bounds a contractible manifold M'.*

Proof. First assume that M is $(k-1)$-connected and $H_k(M)$ is a finite group by Corollary 5.10. Take a $\lambda \neq 0 \in H_k(M)$ and an embedding $\phi : S^k \times D^{k+1} \to M$ as in Lemma 5.6. By Lemma 5.9, we have

$$H_k(M)/\langle\lambda\rangle \cong H_k(M')/\langle\lambda'\rangle .$$

Since $\operatorname{rank}(H_k(M')) \neq 0$ by Lemma 6.2, $\langle\lambda'\rangle$ is infinite, so $(H_k(M'))^{\text{tor}}$ is mapped injectively into $H_k(M)/\langle\lambda\rangle$, i.e., the torsion subgroup of H_k shrinks after surgery. Then apply Corollary 5.10 again to kill the free part of $H_k(M')$.

Repeating the process, H_k can be killed completely by a finite number of framed surgeries. Then the theorem follows by Lemma 5.8. □

Hence we have proved that $bP_{2k+1} = 0$ for k even.

Now we turn to the proof of Lemma 6.2.

Definition 6.4. Let W be a $2r$-manifold. Its **semi-characteristic** $\chi^*(\partial W)$ is the mod 2 residue class

$$\chi^*(\partial W) \equiv \sum_{i=0}^{r-1} \operatorname{rank}(H_i(\partial W)) \pmod 2.$$

We prove Lemma 6.2 by considering the semi-characteristic of the surgery trace:

Proof. Let $M' := \chi(M, \phi)$ and W be the trace with dimension $2k+2$.

First suppose M is closed. Then $\partial W = M \sqcup \overline{M'}$. Consider the exact sequence

$$H_{k+1}(W) \xrightarrow{h} H_{k+1}(W, \partial W) \to H_k(\partial W) \to \cdots \to H_0(W, \partial W) \to 0.$$

Since $H_p(W, \partial W) \cong H^{2k+2-p}(W)$, we see that

$$\chi^*(\partial W) \equiv \mathrm{rank}(\mathrm{Im} h) + \chi(W) \pmod 2 \qquad (\star)$$

by a rank counting. (Here $\chi(W)$ denotes the Euler characteristic.)

Noting that W has homotopy type of M with a $(k+1)$-cell attached, and M has odd dimension $2k+1$, we have

$$\chi(W) = \chi(M) + (-1)^{k+1} = -1.$$

(k is even.) Moreover, note that $\mathrm{rank}(\mathrm{Im} h)$ is none other than the rank of the intersection form (Definition 7.1) of W, since

$$h \otimes \mathbb{Q} : H_{k+1}(W; \mathbb{Q}) \to H_{k+1}(W, \partial W; \mathbb{Q}) \cong H^{k+1}(W; \mathbb{Q}) \cong \mathrm{Hom}(H_{k+1}(W; \mathbb{Q}), \mathbb{Q}),$$

and $\mathrm{rank}(\mathrm{Im} h) = \dim(\mathrm{Im}(h \otimes \mathbb{Q}))$. Therefore, this rank is even since the intersection form is skew-symmetric ($k+1$ is odd).

Then by ($\star$), we have $\chi^*(\partial W) \equiv \chi^*(M) + \chi^*(M') \equiv 1$. Finally, note that $H_i(M) = H_i(M') = 0$ for $0 < i < k$, we obtain by definition of χ^* that

$$\mathrm{rank}(H_k(M)) \not\equiv \mathrm{rank}(H_k(M')).$$

For the case where M is bounded by a homotopy $2k$-sphere, we can adjoin a cone over ∂M to obtain a closed manifold M_* without changing homology groups in dimensions $\leqslant 2k-1$. Then the result of M_* implies that of M. $\square$

6.2. **For k Odd.** Let k be odd, M satisfy Assumption 6.1 and $\chi(\phi)$ be a framed surgery. Keeping the notations in the proof of Theorem 6.3, we have

$$H_k(M)/\langle\lambda\rangle \cong H_k(M')/\langle\lambda'\rangle.$$

Since we do not have Lemma 6.2 this time, $H_k(M)$ does not necessarily shrink if λ' has finite order. More precisely, $H_k(M)$ shrinks iff $\mathrm{ord}(\lambda') < \mathrm{ord}(\lambda)$.

$\mathrm{ord}(\lambda')$ does not need to be smaller, but we can twist the tubular embedding ϕ to ϕ_α by a map $\alpha : S^k \to SO_{k+1}$ (see the proof of Lemma 5.6) and consider $\lambda'_\alpha := [\phi'_\alpha|_{\{0\}\times S^k}] \in \pi_k(M'_\alpha)$, where $M'_\alpha = \chi(M, \phi_\alpha)$ with the embedding $\phi'_\alpha : D^{k+1} \times S^k \to M'_\alpha$.

Now suppose λ, λ' have finite order l, l'. We aim to represent $\mathrm{ord}(\lambda'_\alpha)$ in terms of l, l' and $\alpha \in \pi_k(SO_{k+1})$.

Lemma 6.5.

$$\mathrm{ord}(\lambda'_\alpha) = |l' - j_*(\alpha)l|,$$

where $j_* : \pi_k(SO_{k+1}) \to \pi_k(S^k) = \mathbb{Z}$ is induced by the standard map $j : \rho \mapsto \rho \cdot x_0$.

Proof. As in the proof of Lemma 5.9, Let $M_0 = M - (\phi(S^k \times D^{k+1}))^\circ$ and define

- $\epsilon := [\phi(S^k \times \{x_0\})] \in H_k(M_0)$, the parallel of $\phi(S^k \times S^k)$.
- $\epsilon_\alpha := [\phi_\alpha(S^k \times \{x_0\})] \in H_k(M_0)$, the parallel of $\phi_\alpha(S^k \times S^k)$.
- $\epsilon' := [\phi(\{x_0\} \times S^k)] (= [\phi_\alpha(\{x_0\} \times S^k)]) \in H_k(M_0)$, the meridian of $\phi(S^k \times S^k) = \phi_\alpha(S^k \times S^k)$.

It is clear that

$$\epsilon_\alpha = \epsilon + j_*(\alpha)\epsilon'.$$

Now consider the relation between ϵ and ϵ' in $H_k(M_0)$ by investigating the diagram in Lemma 5.9. $l\epsilon$ must be a multiple of ϵ' since $i_*(l\epsilon) = l\lambda = 0$. This multiple m must be divisible by l' since $i'_*(m\epsilon') = m\lambda' = 0$. Further, $m = \pm l$ since we can similarly consider $l'\epsilon'$ as a multiple of ϵ. (Note that ϵ and ϵ' are not torsion and hence can only satisfy one linear relation.)

Assume that $l\epsilon + l'\epsilon' = 0$. Since $\epsilon_\alpha = \epsilon + j_*(\alpha)\epsilon'$, we have

$$l\epsilon_\alpha + (l' - j_*(\alpha)l)\epsilon' = 0.$$

Then by a similar argument as the preceding paragraph, we then have $\mathrm{ord}(\lambda'_\alpha) = |l' - j_*(\alpha)l|$ since $\mathrm{ord}(\lambda) = l$.

One checks that when $l' - j_*(\alpha)l = 0$, λ'_α will be an infinte-order element, rather than the zero element, but we denote its order by 0 still. □

However, $j_*(\alpha)$ may not be able to take value of every integer, because there is one more restriction on α: $\chi(\phi_\alpha)$ needs to be framed. By the claim in Lemma 5.6, $\chi(\phi_\alpha)$ is framed iff the obstruction class $\gamma(\phi_\alpha) = \gamma(\phi) + s_*(\alpha)$ vanishes, where $\gamma(\phi)$ vanishes already since $\chi(\phi)$ is framed.

In summary, we want to choose $\alpha \in \pi_k(SO_{k+1})$ such that

- $s_*(\alpha) = 0$, where $s_* : \pi_k(SO_{k+1}) \to \pi_k(SO_{2k+1}) = \pi_k(SO)$,
- $0 < |l' - j_*(\alpha)l| < l$, i.e., $\mathrm{ord}(\lambda'_\alpha) < \mathrm{ord}(\lambda)$,

Lemma 6.6. *If there exists a framed surgery $\chi(\phi)$ such that $l \nmid l'$, then there is an α such that M'_α still satisfies Assumption 6.1 but $H_k(M'_\alpha)$ is smaller than $H_k(M)$.*

Proof. First we examine the kernel of $s_* : \pi_k(SO_{k+1}) \to \pi_k(SO_{k+2})$. Since

$$\pi_{k+1}(S^{k+1}) \xrightarrow{\partial} \pi_k(SO_{k+1}) \xrightarrow{s_*} \pi_k(SO_{k+2})$$

is exact, $\mathrm{Ker} s_* = \mathrm{Im}\partial$. Then consider the composition

$$\mathbb{Z} = \pi_{k+1}(S^{k+1}) \xrightarrow{\partial} \pi_k(SO_{k+1}) \xrightarrow{j_*} \pi_k(S^k) = \mathbb{Z},$$

which is $\cdot 2$ provided that k is odd (by Proposition 2.16). Therefore, the range of $j_*(\alpha)$ is $j_*(\mathrm{Ker} s_*) = j_*(\partial(\mathbb{Z})) = 2\mathbb{Z}$. Thus one can always choose $j_*(\alpha)$ such that

$$-l < l' - j_*(\alpha)l \leqslant l,$$

and $0 < |l' - j_*(\alpha)l| < l$ unless $l|l'$. The lemma then follows by the argument before it. □

All that is left is the case where $l|l'$ for any framed surgery $\chi(\phi)$, and the method for this (interpreting $l|l'$ as a linking number) gets even more technical. So we refer to [1, §6] so as not to make this paper too lengthy.

Finally, one can prove the following and hence that $bP_{2k+1} = 0$ for k odd.

Theorem 6.7. *If a homotopy sphere of dimension $2k$ (with k odd) bounds a parallelizable manifold M, then it bounds a contractible manifold M'.*

7. The Group bP_{2k}

In this section, we are going to show that

$$bP_{n+1} \text{ is finite cyclic, for } n \text{ odd.}$$

Denote $n+1$ by $2k$. The cases $k = 1, 2$ are trivial since Θ_1, Θ_3 are trivial. Thus we may always assume that $k > 2$ in this section.

Similar to the previous section, we will consider the conditions under which a parallelizable $2k$-manifold M bounded by a homotopy sphere can be surgically turned into a contractible one.

By Proposition 5.7, we have been able to make M $(k-1)$-connected by framed surgeries. A suitable condition for killing the k^{th} homotopy group is given in Lemma 7.2 below, which is the basis of this section.

Before stating it, let us recall the intersection form first for its importance to the topology of an even-dimensional manifold.

Definition 7.1. The **intersection form** of M is the bilinear form

$$\cdot : H_k(M)^{\text{free}} \otimes H_k(M)^{\text{free}} \to \mathbb{Z},$$

i.e., the middle-dimensional intersection pairing restricted to the free part of $H_k(M)$.

The intersection form is symmetric when k is even and skew-symmetric when k is odd. If M is closed or bounded by a homotopy sphere, by the Poincaré duality, the intersection form is unimodular, i.e., its matrix under a $\mathbb{Z}$-basis of $H_k(M)^{\text{free}}$ has determinat ± 1. See [2, §3].

If in addition M is $(k-1)$-connected, then $H_k(M) \cong H^k(M, \partial M) \cong H^k(M) \cong \text{Hom}(H_k(M), \mathbb{Z})$ is free abelian.

Lemma 7.2. *Let M be a $(k-1)$-connected $2k$-manifold with $k \geqslant 3$. If $H_k(M)$ has a basis $\{\lambda_1, ..., \lambda_r, \mu_1, ..., \mu_r\}$ where*

$$\lambda_i \cdot \lambda_j = 0, \qquad \lambda_i \cdot \mu_j = \delta_{ij},$$

and every embedded k-sphere representing a class in the subgroup $\langle \lambda_1, ..., \lambda_r \rangle$ has a trivial normal bundle in M, then $H_k(M)$ can be killed by surgeries.

Remark 7.3. Such a basis $\{\lambda_1, ..., \lambda_r, \mu_1, ..., \mu_r\}$ with the above condition on intersection numbers is called **weakly symplectic**. If $\mu_i \cdot \mu_j = 0$ in addition, the basis is called **symplectic**. Note that we then have $\mu_i \cdot \lambda_j = \delta_{ij}$ or $-\delta_{ij}$, when the pairing is symmetric or skew-symmetric.

Proof. Any homology class in $H_k(M)$ can be represented by an embedded sphere according to [3, §6]. Let $\phi_0 : S^k \to M$ be an embedding representing λ_r. It extends to a tubular embedding $\phi : S^k \times D^k \to M$ by the condition on normal bundle. Let $M' := \chi(M, \phi)$ and

$$M_0 := M - (\phi(S^k \times D^k))^\circ = M' - (\phi'(D^{k+1} \times S^{k-1}))^\circ.$$

We investigate the diagram in Lemma 5.11. First, $H_{k-1}(M_0) = 0$ since $\cdot\lambda_r$ is onto $\mathbb{Z}$. Then M_0 and M' are both $(k-1)$-connected by homology exact sequences of pairs. Next, since in the diagram in Lemma 5.11

$$0 \to H_k(M_0) \to H_k(M) \xrightarrow{\cdot\lambda_r} \mathbb{Z} \to 0$$

is exact, $H_k(M_0)$ is the subgroup $\langle \lambda_1, ..., \lambda_r, \mu_1, ..., \mu_{r-1} \rangle$ of $H_k(M)$. Then, since

$$\mathbb{Z} \xrightarrow{\lambda_r} H_k(M_0) \to H_k(M') \to 0$$

is exact, $H_k(M')$ is the quotient $H_k(M_0)/\langle \lambda_r \rangle = \langle \overline{\lambda}_1, ..., \overline{\lambda}_{r-1}, \overline{\mu}_1, ..., \overline{\mu}_{r-1} \rangle$. The rank of $H_k(M')$ shrinks, but we still need to check that its basis satisfies the condition in the lemma.

Since $\overline{\lambda}_i, \overline{\mu}_j$ come from classes in $H_k(M_0)$, they are weakly symplectic. If an embedding $\psi : S^k \to M'$ represents some class in $\langle \overline{\lambda}_1, ..., \overline{\lambda}_{r-1} \rangle$. $\psi(S^k)$ can always be deformed to be disjoint with $\phi'(\{0\} \times S^{k-1})$ so represent a class in $\langle \lambda_1, ..., \lambda_r \rangle$ in $H_k(M_0)$. Thus it has a trivial normal bundle. □

There are two conditions needed to apply this lemma:

- $H_k(M)$ admits a weakly symplectic basis.
- Embeddings representing $\sum n_i \lambda_i$ have trivial normal bundles.

For k even, there is one obstruction to finding a (weakly) symplectic basis. If one obtains the desired basis, the normal bundles are automatically trivial.

For k odd, the symplectic basis always exists due to the skew-symmetry. Triviality of normal bundles requires one more condition.

7.1. **For k Even.** Let k be even. Since the intersection form on $H_k(M)$ is symmetric, an important invariant arises: the signature $\sigma(M)$. We are going to prove that $\sigma(M)$ is the only obstruction to surgically making M contractible.

We sometimes denote k by $2m$ in this subsection.

Proposition 7.4. *The signature of a $4m$-manifold, which is closed or bounded by a homotopy sphere, is invariant under surgery.*

Proof. For closed manifolds, it follows from Hirzebruch's signature theorem that the signature is a cobordism invariant (since it is a Pontryagin number). Then Theorem 5.3 implies the proposition.

For manifolds bounded by a homotopy sphere, one adjoins a cone over the boundary to obtain a closed manifold, without changing the intersection form and hence the signature. □

To apply Lemma 7.2, we need one more lemma which relates the self-intersection with normal bundle, in the $4m$-dimensional case.

Lemma 7.5. *Let M be a stably parallelizable $2k$-manifold with k even. Let $f : S^k \to M$ be an embedding which represents a class $\lambda \in H_k(M)$. Then the normal bundle of $f(S^k)$ is trivial iff its self-intersection number $\lambda \cdot \lambda$ is 0.*

Proof. Since $N(f(S^k))$ is stably trivial, $N(f(S^k)) \oplus \epsilon^1$ is trivial by Lemma 2.5. Denote $f(S^k)$ by S^k. We have $NS^k \in \pi_{k-1}(SO_k)$ lies in $\mathrm{Ker} i_*$, where

$$\begin{array}{ccccc} \pi_k(S^k) & \xrightarrow{\partial} & \pi_{k-1}(SO_k) & \xrightarrow{i_*} & \pi_{k-1}(SO_{k+1}) \\ & \overset{2}{\searrow} & \big\downarrow j_* & & \\ & & \pi_{k-1}(S^{k-1}) & & \end{array}.$$

By Proposition 2.16, $j_* \circ \partial = 2$. Thus j_* is injective when restricted to $\mathrm{Im}\partial$, and hence NS^k is trivial iff $j_*(NS^k) = 0$. Further, as mentioned in Proposition 2.16, $j_*(NS^k) = e(NS^k)[S^k]$, where $e(NS^k)$ is the Euler class. So in summary,

$$Nf(S^k) \text{ is trivial iff } e(Nf(S^k))[f(S^k)] = 0.$$

However, one sees that $e(Nf(S^k))[f(S^k)]$ is equal to the self-intersection number $[f(S^k)] \cdot [f(S^k)]$, using the fact that $e(Nf(S^k))$ is the Poincaré dual of $[f(S^k)]$ restricted to $f(S^k)$. The lemma then follows. □

From the proof, we see additionally that for any class $\lambda \in H_k(M)$, the self-intersection number $\lambda \cdot \lambda$ is always even. This leads to the following.

Lemma 7.6. *Let M be a stably parallelizable $2k$-manifold with k even. If the signature $\sigma(M) = 0$, then there exists a symplectic basis of $H_k(M)^{\mathrm{free}}$ with respect to the intersection form.*

Proof. Algebraic fact: on $\mathbb{Z}^{\oplus r}$, every unimodular even (i.e., every self-pairing is even) symmetric bilinear form with signature 0 admits a symplectic basis.

The proof is essentially a Schmidt orthogonalization. See [3, §6]. □

With all ingredients for Lemma 7.2 prepared, we are now ready to prove that the signature is the only obstruction to killing the k^{th} homotopy group.

Proposition 7.7. *Let M be a parallelizable $2k$-manifold bounded by a homotopy sphere, where $k \geqslant 4$ and is even. M can be modified into a contractible manifold by a sequence of framed surgeries iff $\sigma(M) = 0$.*

Proof. By Proposition 7.4, we only need to suppose that $\sigma(M) = 0$ and prove that M can be made k-connected by a sequence of surgeries.

Assume that M is $(k-1)$-connected so that $H_k(M)$ is free abelian. Take a symplectic basis. Lemma 7.5 allows us to apply Lemma 7.2 to make M k-connected and consequently contractible by surgeries.

By the claim in Lemma 5.6, we can require these surgeries to be framed since $s_* : \pi_k(SO_k) \to \pi_k(SO)$ is surjective when k is even. □

All the discussion above focuses on the zero element of bP_{4m}, which bounds a contractible manifold. Now we turn to the whole group bP_{4m}.

Note that the connected sum of framed manifolds can also be framed as in the proof of the claim in Proposition 4.3 (combine Corollary 2.7). Thus the range of signatures of parallelizable $4m$-manifolds bounded by a homotopy sphere, denoted by $\sigma(P_{4m})$, is a subgroup of $\mathbb{Z}$, by connected sum along boundary.

Assume that $m \geqslant 2$. A topological construction in [1, §7] shows that $\sigma(P_{4m}) \neq 0$:

Lemma 7.8. *There exists a parallelizable $4m$-manifold M_0 whose boundary is the standard S^{4m-1}, such that $\sigma(M_0)$ is non-zero.*

Let $\sigma_m > 0$ denote the generator of the range group $\sigma(P_{4m}) \subset \mathbb{Z}$. Define a map

$$\begin{aligned} q : \mathbb{Z} &\to bP_{4m}, \\ t &\mapsto [\partial M], \end{aligned}$$

where M is a parallelizable $4m$-manifold bounded by a homotpy sphere, which has signature $t\sigma_m$. $[\cdot]$ denotes the h-cobordism class.

Lemma 7.9. *Let q be the map defined above.*

1. *(1) q is well-defined.*
2. *(2) q is a group homomorphism.*
3. *(3) q is surjective with non-trivial kernel.*

Proof. (1) If M_1, M_2 are parallelizable $4m$-manifolds with the same signature $t\sigma_m$, whose boundaries are homotopy spheres, then the connected sum along boundary $M_1 \natural \overline{M}_2$ has boundary $V := \partial M_1 \sharp \overline{\partial M}_2$, a homotopy sphere.

Since the intersection form, and hence the signature are additive with respect to connected sum, we have $\sigma(M_1 \natural \overline{M}_2) = 0$. Thus by Proposition 7.7, $M_1 \natural \overline{M}_2$ can be modified into a contractible manifold with boundary V. Thus V is h-cobordant to the standard S^{4m-1} and then $[\partial M_1] = [\partial M_2]$.

(2) This follows from $\sigma(M_1) + \sigma(M_2) = \sigma(M_1 \natural M_2)$.

(3) The surjectivity is trivial. The kernel is non-trivial by Lemma 7.8. □

Thus by the preceding lemma, we have proved the following.

Theorem 7.10. *bP_{4m} is a finite cyclic group.*

7.2. **For k Odd.** Let k be odd. Since the intersection form on $H_k(M)$ is now skew-symmetric, the symplectic basis always exists:

Proposition 7.11. *Let M be a $2k$-manifold with k odd. Then there always exists a symplectic basis of $H_k(M)^{\text{free}}$ with respect to the intersection form.*

Proof. This is similar to Lemma 7.6 but easier: on $\mathbb{Z}^{\oplus r}$, every unimodular skew-symmetric bilinear form admits a symplectic basis. □

Corollary 7.12. *The groups bP_2, bP_6, bP_{14} are zero.*

Proof. We only consider bP_6 and bP_{14}. For a parallelizable manifold M bounded by a homotopy sphere with dimension $2k = 6$ or 14, first assume that M is $(k-1)$-connected.

To apply Lemma 7.2, the symplectic basis exists already. Since $\pi_2(SO_3) = \pi_6(SO_7) = 0$ by Proposition 2.17, any 3-bundle over S^3 or 7-bundle over S^7 is trivial. Hence the triviality of normal bundles is also satisfied. Thus M can be modified into a contractible one by Lemma 7.2. □

For general k odd, the normal bundles are not necessarily trivial. The Kervaire invariant comes up in determining the triviality.

Let M be a parallelizable $(4m+2)$-manifold bounded by a homotopy sphere. The **Kervaire invariant** $c(M)$ is an element in $\mathbb{Z}_2$ satisfying

- c is additive with respect to connected sum.
- M can be made contractible by framed surgeries iff $c(M) = 0$.

I am not able to give details due to space reasons. See [2, §4]. Finally, one can use the above properties to prove the following.

Theorem 7.13. *bP_{4m+2} is either 0 or $\mathbb{Z}_2$.*

8. Examples and Further Results

Since a homotopy sphere is a topological sphere and h-cobordant spheres are the same as diffeomorphic spheres for $n \geqslant 5$, the group $\Theta_n (n \geqslant 5)$ is just the group of smooth structures on a topological sphere under connected sum. Therefore we have proved the following.

Theorem 8.1. *For $n \neq 4$, the number of smooth structures on the topological n-spheres is $|\Theta_n|$, which is finite.*

Here we list some examples of Θ_n. Recall that we have an injection

$$\Theta_n / bP_{n+1} \hookrightarrow \text{Coker}J.$$

The study of the J-homomorphism gives us the order of Coker$J$ in low dimensions as follows: (from [4])

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\lvert\pi_n^S/\text{Im}J\rvert$	1	2	1	1	1	2	1	2	4	6	1	1	3	4	2	2

The triviality of Coker$J$ in some dimensions can be deduced from the triviality of the stable homotopy groups of spheres π_n^S, e.g., $n = 4, 5, 12$.

- $\mathbf{\Theta_1, \Theta_2, \Theta_3 = 0}$.
  As we have mentioned at the end of Section 3, these are trivial group 0.
- $\mathbf{\Theta_4, \Theta_{12} = 0}$.
  $\Theta_4 / bP_5 = 0$ since Coker$J = 0$; $bP_5 = 0$ since 5 is odd. Θ_{12} is similar.

- $\mathbf{\Theta_5 = 0}$.
  $\Theta_5/bP_6 = 0$ since Coker$J = 0$; $bP_6 = 0$ by Corollary 7.12.

[2] contains some further material on explicit computation of Θ_n, which its author Levine believes would have appeared in the sequel of [1]. The remaider of this section is some further results from it and more recent papers ([10]).

Proposition 8.2. *The group bP_{4m} ($m \geqslant 2$) has order*

$$2^{2m-2}(2^{2m-1}-1)B,$$

where B is the numerator of $4B_{2m}/m$, and B_{2m} is a Bernoulli number.

- $\mathbf{\Theta_7 = 28}$, $\mathbf{\Theta_{11} = 992}$.
  $\Theta_7/bP_8 = 0$ since Coker$J = 0$; $bP_8 = 28$ by the proposition above. Θ_{11} is similar.

Proposition 8.3. *The group bP_{4m-2} has order*

$$|bP_{4m-2}| = \begin{cases} 0, & m = 1, 2, 4, 8, 16; \\ 0 \text{ or } 2, & m = 32; \\ 2, & \text{otherwise.} \end{cases}$$

The next result says that Coker$J \hookrightarrow \Theta_n/bP_{n+1}$ is not far from an isomorphism.

Proposition 8.4. *The index of the injection is*

$$[\mathrm{Coker}J : \Theta_n/bP_{n+1}] = \begin{cases} 2, & n = 2, 6, 14, 30, 62; \\ 1 \text{ or } 2, & n = 126; \\ 1, & \text{otherwise.} \end{cases}$$

- $\mathbf{\Theta_6 = 1}$, $\mathbf{\Theta_8 = 2}$, $\mathbf{\Theta_9 = 8}$, $\mathbf{\Theta_{10} = 6}$.

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