PROOF OF RIEMANN-ROCH THEOREM BY WAY OF ČECH COHOMOLOGY

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Abstract. In this paper, we will present a proof of Riemann-Roch Theorem for compact Riemann surfaces. The proof roughly follows Forster in [3]. The main reference for the part about Čech Cohomology is Mumford and Oda's notes [4].

Contents

1. Introduction

The *Riemann surface* is an interesting object, for it allows study from both geometric and algebraic perspectives. For geometry, we study its manifold structure, and for algebra, its vector space of functions. The *Riemann-Roch Theorem* demonstrated this dual aspect of the Riemann surfaces. For a compact Riemann surface X, the theorem gives a relation between the "genus" (geometric information) of X, and the vector space of meromorphic functions (algebraic information) on X, as in the following equation.

$$dim(L(D)) - dim(L(K-D)) = 1 - g + deg(D). \tag{1.1}$$

In this equation, g is the genus of X, which refers roughly to the number of "holes" of a surface, and $L(D)$ and $L(K-D)$ both refer to certain subspaces of the $\mathbb{C}$-vector space of meromorphic functions on X. In this sense, the Riemann-Roch Theorem is a bridge between algebra and geometry.

Also, D and $K-D$ are both *divisors*, which are finite formal sums of points of X. This means that $L(D)$ and $L(K-D)$ can be viewed as 'local" information of X, because D and $K-D$ concern only finitely many points of X, which means that $L(D)$ and $L(K-D)$ can be studied in an open set containing the finitely many points. On the other hand, the genus is a kind of "global" information of X. From

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this perspective, the Riemann-Roch Theorem is a bridge in another sense: between local and global information of a Riemann surface.

While (1.1) is the normal statement of Riemann-Roch Theorem (as in [7]), we will cover only the following version of the theorem in this paper. This version of the theorem is stated in [3].

Theorem 1.2. *For a compact Riemann surface X, we have the following equation.*

$$dim(\check{H}^0(X,\mathcal{O}_D)) - dim(\check{H}^1(X,\mathcal{O}_D)) = 1 - g + deg(D)$$

The meaning of $\mathcal{O}_D$ and $\check{H}^n$ will be revealed in this paper. We will first take the formal perspective: we introduce the concept of sheaves and cohomology of sheaves. Then, we take the concrete perspective: we introduce the concept of Riemann surfaces, divisors and the sheaf associated to a divisor. Finally, we will present a proof combining the two strands of thoughts.

2. Sheaves

Definition 2.1 (Presheaves)**.** A *presheaf of abelian groups* over a topological space X is a contravariant functor $\mathcal{F} : C(X) \to \mathbf{Ab}$. The category $C(X)$ refers to the category of open sets of X and inclusion maps between them. $\mathbf{Ab}$ is the category of abelian groups. For inclusion map $i_{VU} : V \to U$ between two open sets, we refer to $\mathcal{F}(i_{VU}) : \mathcal{F}(U) \to \mathcal{F}(V)$ as ρ_{UV}. These homomorphisms of the form ρ_{UV} are called the *restriction morphisms*. Sometimes given $s \in \mathcal{F}(V)$, we would refer to $\rho_{UV}(s) \in \mathcal{F}(U)$ as $s|_U$.

A *morphism of presheaves* between the two presheaves $\mathcal{F}, \mathcal{G} : C(X) \to \mathbf{Ab}$ is a natural transformation $f : \mathcal{F} \to \mathcal{G}$. In other words, a morphism of presheaves is a set of group homomorphisms $f(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ such that these homomorphisms commute with the restriction morphisms.

Definition 2.2 (Sheaves)**.** A *sheaf of abelian groups* over X is a presheaf $\mathcal{F}$ satisfying:

(1) (locality) given an open set $V \subseteq X$ and an open cover $\{U_i\}$ of V, then for $s, t \in \mathcal{F}(V)$, if $s_{|U_i} = t_{|U_i}$ for each i, then $s = t$;
(2) (gluing) if there exists $s_i \in \mathcal{F}(U_i)$ for each i, such that $s_i|U_i \cap U_j = s_j|_{U_i \cap U_j}$, then there exists $s \in V$ such that $s|_{U_i} = s_i$. s constructed by gluing is unique by (a).

A *morphism of sheaves* is the same as a morphism of presheaves.

The category of sheaves over X is denoted as $\mathbf{Sh}(X)$.[1]

Example 2.3 (Skyscraper sheaf)**.** For X a topological space and $p \in X$, define $\mathbb{C}_p$, the *skyscraper sheaf with value* $\mathbb{C}$ *centered at* p as follows.

$$\mathbb{C}_p(U) = \begin{cases} \mathbb{C} & p \in U \\ 0 & p \notin U \end{cases}$$

For ρ_{UV}, when $p \in U \subseteq V$, $\rho_{UV} = id_{\mathbb{C}}$. In the other case, the restriction morphism is the zero morphism.

[1] To be sure, while we have defined only sheaves and presheaves of abelian groups, there are also sheaves and presheaves of other types, such as sheaves of rings or sets. To define them, we simply replace **Ab** in the definition with the category of rings or the category of sets. In this paper, when we use the word "sheaf" or "presheaf," we refer only to a sheaf or presheaf of abelian groups.

On a literal level, our definition of sheaves states the concept as a way of associating algebraic data with the open sets, such that "local" data can be glued into global data. In comparison, the idea of "point" is absent in the definition of sheaf. The concept of *stalk* provides a way to organize algebraic data associated with a single point.

Definition 2.4 (Stalks)**.** The *stalk* of $\mathcal{F}$ at $p \in X$ is the direct limit of all $\mathcal{F}(U)$ $(p \in U)$ and morphisms ρ_{UV} $(p \in V \subseteq U)$.[2] It is denoted as

$$\mathcal{F}p := \varinjlim_{p \in U} \mathcal{F}(U)$$

The stalk at $p \in X$ encodes the "local" properties at p, with "local" meaning that the property can be studied in an arbitrarily small neighborhood of p.

Example 2.5. The stalk of $\mathbb{C}_p$ at point x is as follows:

$$(\mathbb{C}_p)_x = \begin{cases} \mathbb{C} & x = p \\ 0 & x \neq p \end{cases}.$$

Also, a sheaf morphism $f : \mathcal{F} \to \mathcal{G}$ would induce a homomorphism on stalks $f_p : \mathcal{F}_p \to \mathcal{G}_p$, in the following way.

Definition 2.6. Given a sheaf morphism $f : \mathcal{F} \to \mathcal{G}$, there is an *induced homomorphism on stalks* $f_p : \mathcal{F}_p \to \mathcal{G}_p$, which is defined as follows. For each open neighborhood U of p, we have the following diagram.

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f(U)} & \mathcal{G}(U) \\ \downarrow{\scriptstyle c_U} & & \downarrow{\scriptstyle d_U} \\ \mathcal{F}_p & \overset{f_p}{\dashrightarrow} & \mathcal{G}_p \end{array}$$

In the diagram, $c_U : \mathcal{F}(U) \to \mathcal{F}_p$ and $d_U : \mathcal{G}(U) \to \mathcal{G}_p$ for each U $(p \in U)$ are the morphisms composing the colimit cones, respectively. $\{d_U \circ f(U) : \mathcal{F}(U) \to \mathcal{G}_U\}_{p \in U}$ would also constitute a cone around $\mathcal{G}_p$, so by the universal property of colimit, there is the dotted morphism f_p making the above diagram commute for each U.

Now, temporarily moving away from the stalks, we try to make sense of an exact sequence in the category of sheaves. We know that an exact sequence of abelian groups refers to a sequence of abelian groups and group morphisms

$$... \to A_{i-1} \xrightarrow{f_i} A_i \xrightarrow{f_{i+1}} A_{i+1} \to 0$$

such that for each i, $im(f_i) = ker(f_{i+1})$. Therefore, to define an exact sequence of sheaves, we first define kernel and image for a morphism of sheaves (or presheaves).

Definition 2.7. Given a presheaf morphism $f : \mathcal{F} \to \mathcal{G}$, the *presheaf kernel* of ϕ is defined as the presheaf $ker(\phi)$ given by $U \mapsto ker(f(U))$, and the *presheaf image* is defined as the presheaf $im(f)$ defined as $U \mapsto im(f(U))$.

It is easy to verify that $ker(f)$ and $im(f)$ are presheaves. Using this definition, we can define the exact sequences of presheaves.

[2]There is a more concrete definition of stalk, as the set of equivalent classes over ordered pairs of open set U and elements $x \in F(U)$, but since we adopt the formal viewpoint on the section of sheaves, we should refrain from such concreteness.

Definition 2.8 (Exact Sequence of Presheaves)**.** A sequence of presheaves and presheaf morphism

$$... \to \mathcal{F}_{i-1} \xrightarrow{f_i} \mathcal{F}_i \xrightarrow{f_{i+1}} \mathcal{F}_{i+1} \to ...$$

is an *exact sequence of presheaves* when $im(f_i) = ker(f_{i+1})$ for each i.

Equivalently, the sequence is an exact sequence of presheaves, when the following sequence of abelian groups is exact for each U.

$$... \to \mathcal{F}_{i-1}(U) \xrightarrow{f_i(U)} \mathcal{F}_i(U) \xrightarrow{f_{i+1}(U)} \mathcal{F}_{i+1}(U) \to ...$$

Now, since we have defined exact sequences of presheaves, this definition would also apply to sequences of sheaves. The problem with applying this definition to sequences of sheaves is that, according to [1], when $f : \mathcal{F} \to \mathcal{G}$ is a morphism between two sheaves, while $ker(f)$ is a sheaf, $im(f)$ is not necessarily a sheaf. This would mean that we would not be able to make sense of the exact sequence within the category of sheaves.

The best way to solve this problem is to "make" $im(f)$ into a sheaf. For this purpose, we introduce the *sheafification of a presheaf.*

Proposition 2.9. *When $\mathcal{F}$ is a presheaf, there is an associated sheaf $\mathcal{F}^+$ and an associated morphism $h : \mathcal{F} \to \mathcal{F}^+$ such that for every sheaf $\mathcal{G}$, there is a bijection $Hom(\mathcal{F}^+, \mathcal{G}) \cong Hom(\mathcal{F}, \mathcal{G})$ given by composition with a unique morphism $h : \mathcal{F} \to \mathcal{F}^+$. $\mathcal{F}^+$ is called the sheafification of $\mathcal{F}$.*

(The construction of sheafification is not related to our discussion, so we omit it. One can refer to [1, II, Proposition 1.2] for an explicit construction.)

Thus, we can define the *sheaf image* of a sheaf morphism (compared to the presheaf image), and *exact sequence of sheaves* (compared to exact sequence of presheaves).

Definition 2.10 (Exact Sequence of Sheaves)**.** Given a sheaf morphism $f : \mathcal{F} \to \mathcal{G}$, we define the *sheaf image* as the sheafification of the presheaf image. The sheaf image is denoted as $Im(f)$.

The following sequence is an *exact sequence of sheaves*

$$... \to \mathcal{F}_{i-1} \xrightarrow{f_i} \mathcal{F}_i \xrightarrow{f_{i+1}} \mathcal{F}_{i+1} \to ...$$

when for every i, $ker(f_{i+1}) = Im(f_i)$.

We can see that the concept of an exact sequence of sheaves is *weaker* than an exact sequence of presheaves. This difference has to be noted in our discussions of Čech Cohomology. This is because while exact sequence of presheaves has better properties, most exact sequences of sheaves that we care about are not exact sequences of presheaves.

Since we want to avoid the construction of sheafification, we look a simpler criterion for exactness. Preferably, we wish for a criterion that can be computed with the information we have at hand. For example, when we work with manifolds, generally we carry out our computation on the charts, which form a basis on the manifold. Thus, we conjecture that if a sequence of sheaves is exact on the basis, then the sequence is an exact sequence of sheaves. This turns out to be true, but we would need two more lemmas to prove this.

Lemma 2.11. *For a sequence of sheaves,*

$$... \to \mathcal{F}_{i-1} \xrightarrow{f_i} \mathcal{F}_i \xrightarrow{f_{i+1}} \mathcal{F}_{i+1} \to ...$$

the sequence is an exact sequence of sheaves, iff for each point $p \in X$, the following sequence of stalks at p is an exact sequence of abelian groups.

$$... \to (\mathcal{F}_{i-1})_p \xrightarrow{f_i} (\mathcal{F}_i)_p \xrightarrow{f_{i+1}} (\mathcal{F}_{i+1})_p \to ...$$

Proof. See [1, II, Exercise 1.2]. □

Lemma 2.12. *The direct limit preserves short exact sequences. In other words, Let $\mathcal{D}$ be a directed set viewed as a category. (A directed set is a set with a preorder such that every pair of elements have an upper bound.) Let there be functors $\mathcal{F}, \mathcal{G}, \mathcal{H} : \mathcal{D} \to \mathbf{Ab}$, and natural transformations $f : \mathcal{F} \to \mathcal{G}$ and $g : \mathcal{G} \to \mathcal{H}$. If the following sequence is an SES for each $U \in Obj(\mathcal{D})$,*

$$0 \to \mathcal{F}(U) \xrightarrow{f(U)} \mathcal{G}(U) \xrightarrow{g(U)} \mathcal{H}(U) \to 0$$

then,

$$0 \to \varinjlim \mathcal{F} \xrightarrow{f'} \varinjlim \mathcal{G} \xrightarrow{g'} \varinjlim \mathcal{H} \to 0$$

is also an SES, with the morphisms f' and g' naturally induced from f and g, as in Definition 2.6.

Proof. [6, Proposition 5.33] □

Proposition 2.13 (Criterion for SES of Sheaves to be Used in Section 5)**.** *Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be sheaves over a topological space X, and $f : \mathcal{F} \to \mathcal{G}$ and $g : \mathcal{G} \to \mathcal{H}$ be sheaf morphisms. Let $\mathfrak{B}$ be a basis of X. If for every open set $U \in \mathfrak{B}$,*

$$0 \to \mathcal{F}(U) \xrightarrow{f(U)} \mathcal{G}(U) \xrightarrow{g(U)} \mathcal{H}(U) \to 0$$

is an SES, then for every $p \in X$

$$0 \to \mathcal{F}_p \xrightarrow{f_p} \mathcal{G}_p \xrightarrow{g_p} \mathcal{H}_p \to 0$$

is also an SES. Consequently, by Lemma 2.11,

$$0 \to \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \to 0$$

would be an SES of sheaves.

Proof. Fix $p \in X$. Then, because the local basis around p, $\{U \ open | p \in U \in \mathfrak{B}\}$ is a directed set (ordered by reversed inclusion), by Lemma 2.12, we have an SES

$$0 \to \varinjlim_{p \in U \in \mathfrak{B}} \mathcal{F}(U) \xrightarrow{f'} \varinjlim_{p \in U \in \mathfrak{B}} \mathcal{G}(U) \xrightarrow{g'} \varinjlim_{p \in U \in \mathfrak{B}} \mathcal{H}(U) \to 0.$$

with induced morphisms f' and g'. We try to prove that this short exact sequence of abelian groups is identical with our desired sequence of abelian groups,

$$0 \to \mathcal{F}_p \xrightarrow{f_p} \mathcal{G}_p \xrightarrow{g_p} \mathcal{H}_p \to 0$$

so that this sequence of abelian groups is an SES.

We first prove that $\varinjlim_{p \in U \in \mathfrak{B}} \mathcal{F}(U) = \mathcal{F}_p$. Let $L = \varinjlim_{p \in U \in \mathfrak{B}} \mathcal{F}(U)$. We shall prove that L satisfies universal property of $\mathcal{F}_p$. Let $\{d_U : \mathcal{F}(U) \to L\}_{p \in U \in \mathfrak{B}}$ be the universal cone around L.

For every cone $\{c_U : \mathcal{F}(U) \to A\}_{p \in U}$ around an abelian group A, by the universal property of L, there is a unique map $u : L \to A$ such that we have the following identity for each $p \in U \in \mathfrak{B}$:

$$c_U = u \circ d_U$$

Now, define another cone around L: for each open set U such that $p \in U$, select open subset $B(U) \subseteq U$ such that $p \in B(U) \subseteq \mathfrak{B}$. Then we have a set of map

$$\{d'_U : \mathcal{F}(U) \to L : d'_U = d'_{B(U)} \circ \rho_{U,B(U)}\}_{p \in U}.$$

We can verify easily that (1) the morphisms $\{d'_U\}$ constitute a cone, i.e. these morphisms commute with the restriction morphisms; (2) $d_U = d'_U$ when $U \in \mathfrak{U}$; and (3) the cone $\{d'_U\}$ does not depend on the selection of $B(U)$. Then, for every U such that $p \in U$,

$$c_U = c_{B(U)} \circ \rho_{U,B(U)} = u \circ d_{B(U)} \circ \rho_{U,B(U)} = u \circ d'_U$$

Thus, $\{d'_U\}_{p \in U}$ is a universal cone around $\mathcal{F}_p$, so $L = \mathcal{F}_p$, i.e. $\varinjlim_{p \in U \in \mathfrak{B}} \mathcal{F}(U) = \mathcal{F}_p$. The same applies to $\mathcal{G}$ and $\mathcal{H}$.

Finally, we prove that $f' = f_p$ and $g' = g_p$. By the construction of f', we know that f' is the unique morphism making the following diagram commutes for every $U \in \mathfrak{U}$.

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f(U)} & \mathcal{G}(U) \\ \downarrow{\scriptstyle c_U} & & \downarrow{\scriptstyle d_U} \\ \varinjlim_{U \in \mathfrak{U}} \mathcal{F}(U) & \dashrightarrow_{f'} & \varinjlim_{U \in \mathfrak{U}} \mathcal{G}(U) \end{array}$$

By the construction of f_p, if we put f_p in the place of the dotted arrow, then the diagram would commute as well. Then, because f' is unique, we have $f' = f_p$. Similarly we have $g' = g_p$. Thus, we have the following SES:

$$0 \to \mathcal{F}_p \xrightarrow{f_p} \mathcal{G}_p \xrightarrow{g_p} \mathcal{H}_p \to 0.$$

□

As a side note, the approach on the above proposition, i.e. understanding the entire sheaf through the partial data on a basis, is called the "sheaf on a basis," which is described more thoroughly in the Section 2.5 of [**?**, vaki]

3. Čech Cohomology

Definition 3.1 (Čech complex)**.** Given a sheaf $\mathcal{F}$ over X and a fixed open cover $\mathfrak{U} = \{U_i\}_{i \in I}$ (I being a totally ordered set of indices) of X, define the *Čech complex* as a cochain complex of abelian group: for each $n \geq 0$,

$$C^n(\mathfrak{U}, \mathcal{F}) = \prod_{i_0 < \ldots < i_n} \mathcal{F}(U_{i_0} \cap \ldots \cap U_{i_n})$$

For each $n \geq 0$, the boundary map $d_n : C^n(\mathfrak{U}, \mathcal{F}) \to C^{n+1}(\mathfrak{U}, \mathcal{F})$ is defined as

$$d_n(s)_{i_0,\ldots,i_{n+1}} = \sum_{j=0}^{n+1} (-1)^j s_{i_0,\ldots,i_{j-1},i_{j+1},\ldots,i_{n+1}}|_{\cap_{k=0}^{n+1} U_{i_k}}.$$

It is easy to verify that for each n, $Im(d_n) \subseteq ker(d_{n+1})$.

Define the *n-th Čech cohomology group with respect to* $\mathfrak{U}$ as:

$$\check{H}^n(\mathfrak{U}, \mathcal{F}) = ker(d_n)/Im(d_{n-1}).$$

Specially, $\check{H}^0(\mathfrak{U}, \mathcal{F}) = ker(d_0)$.

Now, we define the "standard" Čech cohomology, which does not require fixing an open cover, as the colimit of all $\check{H}^n(\mathfrak{U}, \mathcal{F})$ for all open covers of X. In order for the colimit to make sense, we need to make the open covers of X a category, and make $\check{H}^n(\cdot, \mathcal{F})$ a functor.

Let the open covers of X be partially ordered by refinement (i.e. $\mathfrak{U} \leq \mathfrak{V}$ when $\mathfrak{U}$ is a refinement of $\mathfrak{V}$, i.e. each element of $\mathfrak{U}$ is an open subset of an element of $\mathfrak{V}$). Consider the poset of open covers as a category. By [4], for each pair of open covers $\mathfrak{U} \leq \mathfrak{V}$, there is a canonical map $\check{H}^n(\mathfrak{V}, \mathcal{F}) \to \check{H}^n(\mathfrak{U}, \mathcal{F})$. Thus, $\check{H}^n(\cdot, \mathcal{F})$ becomes a contravariant functor, so it make sense to take the colimit of all $\check{H}^n(\mathfrak{U}, \mathcal{F})$. We can define the Čech cohomology groups as follows.

Definition 3.2. For a sheaf $\mathcal{F}$ over a topological space X, we define the *n-th Čech cohomology groups* $\check{H}^n(X, \mathcal{F})$ as follows:

$$\check{H}^n(X, \mathcal{F}) := \varinjlim_{\mathfrak{U}} \check{H}^n(\mathfrak{U}, \mathcal{F}).$$

Now, to better understand the concept, we compute some examples of Čech cohomology groups. We first have a general fact about $\check{H}^0$.

Lemma 3.3. *For any sheaf $\mathcal{O}$ over a topological space X, $\mathcal{O}(X) \cong \check{H}^0(X, \mathcal{O})$.*

Proof. Given an open cover $\mathfrak{U} = \{U_i\}_{i \in I}$, define $f : \mathcal{O}(X) \to C_0(\mathfrak{U}, \mathcal{O})$ as $f(s) = (s|_{U_i})_{i \in I}$. Then, $f(s) \in ker(d_0)$. Thus, f can be seen as a group homomorphism $f : \mathcal{O}(X) \to \check{H}^0(X, \mathcal{O})$. Given $(s_i)_{i \in I} \in ker(d_0)$, then by gluing there is $f(t) = (s_i)_{i \in I}$, so f is surjective. Also, assume that $f(s) = 0$ for $s \in \mathcal{O}(X)$, then $s|_{U_i} = 0$, so by the definition of sheaf, we have $s = 0$, so f injective. Thus we have an isomorphism $\mathcal{O}(X) \cong H_0(\mathfrak{U}, \mathcal{O})$ for every open cover $\mathfrak{U}$. Then, by the definition of colimits, $H_0(X, \mathcal{F}) = \mathcal{O}(X)$. □

In Example 2.3, we defined the skyscraper sheaf $\mathbb{C}_p$. Here we shall compute its Čech cohomology groups in the following lemma, which is going to be extremely useful in the proof of Riemann-Roch Theorem.

Lemma 3.4. *$\check{H}^0(X, \mathbb{C}_p) = \mathbb{C}$; $\check{H}^1(X, \mathbb{C}_p) = 0$.*

Proof. First, we already know $\check{H}^0(X, \mathbb{C}_p) = \mathbb{C}_p(X) = \mathbb{C}$ by the above lemma.

Then we deal with $\check{H}^1(X, \mathbb{C}_p)$. Let $\mathfrak{U}$ be an open cover of X. Then, there is $\mathfrak{B}$ a refinement of $\mathfrak{U}$ such that only one open set in $\mathfrak{B}$ contains p. ($\mathfrak{B}$ can be constructed by removing p from every element of $\mathfrak{U}$ and then adding a neighborhood of p.) Then, let $s \in ker(d_1) \subseteq C_1(\mathfrak{B}, \mathbb{C}_p)$. Then, $s = 0$, because every $\mathbb{C}_p(\mathfrak{B}_i \cap \mathfrak{B}_j) = 0$ for $i < j$. Thus, $C_1(\mathfrak{B}, \mathbb{C}_p) = 0 = \check{H}^1(\mathfrak{B}, \mathbb{C}_p)$. Now because every $\mathfrak{U}$ has a refinement $\mathfrak{B}$ such that $C_1(\mathfrak{B}, \mathbb{C}_p) = \check{H}^1(\mathfrak{B}, \mathbb{C}_p) = 0$, using the definition of colimits, we have that$\check{H}^1(X, \mathbb{C}_p) = 0$. □

Shifting away from examples, we now consider some key properties of the Čech complex and cohomology groups.

Lemma 3.5. *$C_n(\mathfrak{U}, \cdot)$ defines a functor $\mathbf{Sh}(X) \to \mathbf{Ab}$, and the functor maps short exact sequences of presheaves to short exact sequences of abelian groups.*

Proof. For a sheaf morphism $f : \mathcal{F} \to \mathcal{G}$, we can construct $C_n(\mathfrak{U}, f) : C_n(\mathfrak{U}, \mathcal{F}) \to C_n(\mathfrak{U}, \mathcal{G})$ canonically from the natural morphisms $f(U) : \mathcal{F}(U) \to \mathcal{G}(U)$. $C_n(\mathfrak{U}, f)$ is constructed as the dotted arrow that makes the following diagram commute for each $n+1$-tuple of indices $i_0, ..., i_n$. This arrow exists by the universal property of product.

$$\begin{array}{ccc} C_n(\mathfrak{U}, \mathcal{F}) & \overset{C_n(\mathfrak{U}, f)}{\dashrightarrow} & C_n(\mathfrak{U}, \mathcal{G}) \\ \downarrow \pi & & \downarrow \pi \\ \mathcal{F}(\bigcap_{k=0}^n U_{i_k}) & \xrightarrow{f(\bigcap_{k=0}^n U_{i_k})} & \mathcal{G}(\bigcap_{k=0}^n U_{i_k}) \end{array}$$

For preservation of SES, assume $0 \to \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \to 0$ to be an SES of presheaves. Since the sequence is exact on each open set, we have an SES for each $(n+1)$-tuple of indices $i_0, ..., i_n$:

$$0 \to \mathcal{F}(U_{i_0} \cap ... \cap U_{i_n}) \xrightarrow{f} \mathcal{G}(U_{i_0} \cap ... \cap U_{i_n}) \xrightarrow{g} \mathcal{H}(U_{i_0} \cap ... \cap U_{i_n}) \to 0.$$

Thus, we have the following SES:

$$\begin{aligned} 0 \to \prod_{i_0, \ldots, i_n \in I} \mathcal{F}(U_{i_0} \cap ... \cap U_{i_n}) \xrightarrow{f} \prod_{i_0, \ldots, i_n \in I} \mathcal{G}(U_{i_0} \cap ... \cap U_{i_n}) \xrightarrow{g} \\ \xrightarrow{g} \prod_{i_0, \ldots, i_n \in I} \mathcal{H}(U_{i_0} \cap ... \cap U_{i_n}) \to 0 \end{aligned}.$$

□

Now, given that for each n, we have the SES $0 \to C_n(\mathfrak{U}, \mathcal{F}) \xrightarrow{f} C_n(\mathfrak{U}, \mathcal{G}) \xrightarrow{g} C_n(\mathfrak{U}, \mathcal{H}) \to 0$, our instinct leads us to question whether these SESs commute with the boundary map d. The answer is yes, because $d : C_n(\mathfrak{U}, \cdot) \to C_{n+1}(\mathfrak{U}, \cdot)$ as defined above is a natural transformation.

Lemma 3.6. *$d : C_n(\mathfrak{U}, \cdot) \to C_{n+1}(\mathfrak{U}, \cdot)$ is a natural transformation, so the SES $0 \to C_n(\mathfrak{U}, \mathcal{F}) \xrightarrow{f} C_n(\mathfrak{U}, \mathcal{G}) \xrightarrow{g} C_n(\mathfrak{U}, \mathcal{H}) \to 0$ commutes with d.*

Proof. Given a sheaf morphism $f : \mathcal{F} \to \mathcal{G}$, it can be checked easily that the following diagram commutes:

$$\begin{array}{ccc} C_n(\mathfrak{U}, \mathcal{F}) & \xrightarrow{C_n(\mathfrak{U}, f)} & C_n(\mathfrak{U}, \mathcal{G}) \\ \downarrow d_{\mathcal{F}} & & \downarrow d_{\mathcal{G}} \\ C_{n+1}(\mathfrak{U}, \mathcal{F}) & \xrightarrow[C_{n+1}(\mathfrak{U}, f)]{} & C_{n+1}(\mathfrak{U}, \mathcal{G}) \end{array}$$

Thus, d is a natural transformation. Then, for an SES $0 \to \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \to 0$, we have the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_n(\mathfrak{U}, \mathcal{F}) & \xrightarrow{C_n(\mathfrak{U}, f)} & C_n(\mathfrak{U}, \mathcal{G}) & \xrightarrow{C_n(\mathfrak{U}, g)} & C_n(\mathfrak{U}, \mathcal{H}) & \longrightarrow & 0 \\ & & \downarrow d & & \downarrow d & & \downarrow d & & \\ 0 & \longrightarrow & C_{n+1}(\mathfrak{U}, \mathcal{F}) & \xrightarrow{C_{n+1}(\mathfrak{U}, f)} & C_{n+1}(\mathfrak{U}, \mathcal{G}) & \xrightarrow{C_{n+1}(\mathfrak{U}, g)} & C_{n+1}(\mathfrak{U}, \mathcal{H}) & \longrightarrow & 0 \end{array}$$

□

Having the commutative diagram above, we can apply the zig-zag lemma, so that we have the following long exact sequence (LES).

$$\begin{aligned}0 &\longrightarrow \check{H}^0(\mathfrak{U},\mathcal{F}) \longrightarrow \check{H}^0(\mathfrak{U},\mathcal{G}) \longrightarrow \check{H}^0(\mathfrak{U},\mathcal{H}) \xrightarrow{\delta} \\ &\longrightarrow \check{H}^1(\mathfrak{U},\mathcal{F}) \longrightarrow \check{H}^1(\mathfrak{U},\mathcal{G}) \longrightarrow \check{H}^1(\mathfrak{U},\mathcal{H}) \xrightarrow{\delta} \\ &\longrightarrow \check{H}^2(\mathfrak{U},\mathcal{F}) \longrightarrow \check{H}^2(\mathfrak{U},\mathcal{G}) \longrightarrow \check{H}^2(\mathfrak{U},\mathcal{H}) \xrightarrow{\delta} \dots\end{aligned} \tag{3.7}$$

Now by [4, Ch. VII, Sec. 1], we can pass the LES (3.7) to the colimit, so we have the following theorem.

Theorem 3.8. *For sheaves $\mathcal{F},\mathcal{G},\mathcal{H}$ over X, and an SES of presheaves $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$, we have the following LES of abelian groups.*

$$\begin{aligned}0 &\longrightarrow \check{H}^0(X,\mathcal{F}) \longrightarrow \check{H}^0(X,\mathcal{G}) \longrightarrow \check{H}^0(X,\mathcal{H}) \xrightarrow{\delta} \\ &\longrightarrow \check{H}^1(X,\mathcal{F}) \longrightarrow \check{H}^1(X,\mathcal{G}) \longrightarrow \check{H}^1(X,\mathcal{H}) \xrightarrow{\delta} \\ &\longrightarrow \check{H}^2(X,\mathcal{F}) \longrightarrow \check{H}^2(X,\mathcal{G}) \longrightarrow \check{H}^2(X,\mathcal{H}) \xrightarrow{\delta} \dots\end{aligned}$$

The above LES is key in the proof of Riemann-Roch Theorem. Using this LES, we are able to compute the dimension of cohomology groups $\check{H}^0(X,\mathcal{O}_D)$ and $\check{H}^1(X,\mathcal{O}_D)$ (as $\mathbb{C}$-vector spaces).

However, Theorem 3.8 holds only for SES of presheaves, and we wish to have the above LES also for SES of sheaves. To this purpose, we can add more limitations to the premises of the theorem. According to [4, Ch. VII, Sec. 1], we have the following altered version of Theorem 3.8.

Theorem 3.9. *When X is a paracompact Hausdorff space, $\mathcal{F},\mathcal{G},\mathcal{H}$ are sheaves over X, and when $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an SES of sheaves, we have the LES of abelian groups stated in Theorem 3.8.*

As a side note, [3] states another altered version of Theorem 3.8, which can also satisfy our purpose, even though this theorem provides only part of the original LES.

Theorem 3.10. *When X is any topological space, $\mathcal{F},\mathcal{G},\mathcal{H}$ are sheaves over X, and when $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an SES of sheaves, we have the following LES of abelian groups.*

$$\begin{aligned}0 &\longrightarrow \check{H}^0(X,\mathcal{F}) \longrightarrow \check{H}^0(X,\mathcal{G}) \longrightarrow \check{H}^0(X,\mathcal{H}) \xrightarrow{\delta} \\ &\longrightarrow \check{H}^1(X,\mathcal{F}) \longrightarrow \check{H}^1(X,\mathcal{G}) \longrightarrow \check{H}^1(X,\mathcal{H})\end{aligned}$$

Proof. [3, Theorem 15.12]. □

4. Riemann Surface and Divisors

Having established the basic results of Čech Cohomology on sheaves in general, now we apply these result to the specific sheaves in concern, namely the sheaves of holomorphic or meromorphic functions on a Riemann surface. We first define these geometric concepts. We follow the definitions of [5] and [3].

Definition 4.1. Let U be an open subset of $\mathbb{C}$. A *holomorphic* function $f : U \to \mathbb{C}$ is a complex differentiable function. A holomorphic function is locally equal to its Taylor series, i.e. for each $z_0 \in U$, there is an open disc $D \subseteq U$ centered at z_0 such that for every $z \in D$,

$$f(z) = \sum_{i=0}^{\infty} a_i (z - z_0)^i.$$

A holomorphic function $f : U \to \mathbb{C}$ has a *zero of order* n at z_0, when there exists a holomorphic function $g : U \to \mathbb{C}$ such that $f(z) = g(z)(z - z_0)^n$ on an open neighborhood of z and $g(z_0) \neq 0$. Notice that we do not consider f to have a zero of any finite order, when f is constantly 0 at a neighborhood of z_0.

Suppose $z_0 \in \mathbb{C}$ and D is an open neighborhood of z_0. Suppose there is a holomorphic function f defined on $D - z_0$. We say that z_0 is a *pole of order* n of f when f is locally equal to a Laurent series with finitely many terms of negative power, and the lowest power being $-n$. In other words, for $z \in D - z_0$,

$$f(z) = \sum_{i=-n}^{\infty} a_i (z - z_0)^i,$$

where $a_{-n} \neq 0$.

We call the function $f : U \to \mathbb{C} \cup \{\infty\}$ a *meromorphic* function, if f is holomorphic on $U - f^{-1}(\infty)$, and $f^{-1}(\infty)$ is a set of isolated points, and each $z \in f^{-1}$ is a pole. ($V \subseteq U$ is a *set of isolated points* when for every point $x \in V$, there is an open neighborhood of x disjoint from $V - \{x\}$.)

For a meromorphic function f, we define the following notation.

$$ord_p(f) = \begin{cases} n & f \text{ has a zero of order } n \text{ at } x \\ -n & f \text{ has a pole of order } n \text{ at } x \\ +\infty & f \text{ is constantly 0 at a neighborhood of } x \\ 0 & \text{otherwise} \end{cases}$$

Definition 4.2. A **complex n-manifold** is a topological space X admitting a cover by coordinate charts $\phi : U \to V \subseteq \mathbb{C}^n$ such that ϕ is an homeomorphism, and U, V are open subsets of X and $\mathbb{C}^n$ respectively.

A *complex structure* of X is an equivalent class of atlases on X. An atlas on X is a set of coordinate charts that cover M such that each transition map is biholomorphic (i.e. is holomorphic and bijective, and has a holomorphic inverse). Two atlases are equal if their union is also an atlas.

A *Riemann surface* is a pair $(X, \mathcal{A})$, where X is a connected complex 1-manifold, and $\mathcal{A}$ is a complex structure.

Definition 4.3. For X a Riemann surface and an open subset $V \subseteq X$, a function $f : U \to \mathbb{C}$ is *holomorphic* at a point x, when there is a chart (U, ϕ) $(x \in U)$ such that $f \circ \phi^{-1} : \phi(U) \to \mathbb{C}$ is holomorphic on $\phi(x)$. f is *meromorphic* when $f \circ \phi^{-1}$ is holomorphic except at some isolated points.

The operator $ord_p(f)$ contains key information about meromorphic functions by reflecting the order of poles and zeros of f, so it is natural to extend this operator to meromorphic functions over Riemann surface by coordinate chart. We still need to check that $ord_p(f)$ is independent of the choices of coordinate charts around a point p.

Proposition 4.4. *For two coordinate charts (U,ϕ) and (V,ψ) around p, we have*

$$ord_{\phi(p)}(f\circ\phi^{-1}) = ord_{\psi(p)}(f\circ\psi^{-1}).$$

Proof. Let $t:\phi(U\cap V)\to\psi(U\cap V)$ be the transition map. Let $z_0=\phi(p), a_0=\psi(p)$. Because $t(z_0)=a_0$, t is defined as follows in a neighborhood around z_0.

$$t(z) = a_0 + \sum_{i=1}^{\infty} a_i(z-z_0)^i$$

Let $n = ord_{\psi(p)}(f\circ\psi^{-1})$. Then, in a neighborhood around z_0, we have

$$(f\circ\psi^{-1})(z) = \sum_{i=n}^{\infty} b_i(z-a_0)^i.$$

Thus, we have:

$$\begin{aligned}(f\circ\phi^{-1})(z) &= (f\circ\psi^{-1}\circ t)(z)\\ &= \sum_{i=n}^{\infty} b_i\left(\left(a_0+\sum_{j=1}^{\infty}a_j(z-z_0)^j\right)-a_0\right)^i\\ &= \sum_{i=n}^{\infty} b_i\left(\sum_{j=1}^{\infty}a_j(z-z_0)^j\right)^i\\ &= b_n a_1^n(z-z_0)^n + \ldots.\end{aligned}$$

Thus, we conclude that $ord_{\phi(p)}(f\circ\phi^{-1}) = n = ord_{\psi(p)}(f\circ\psi^{-1})$. □

Thus, we have defined $ord_p(f)$ for meromorphic function on a Riemann surface. $ord_p(f)$ also have the following two properties, according to [7].

Proposition 4.5. *For meromorphic functions f and g on a compact Riemann surface X, we have:*

$$ord_p(fg) = ord_p(f) + ord_p(g)$$
$$ord_p(f+g) \geq min(ord_p(f), ord_p(g)).$$

Proposition 4.6. *For a non-zero meromorphic function f defined on a compact Riemann surface X, the set of zeros and poles is finite.*

Now, to provide a framework to deal with the key information of poles and zeros, we introduce the concept of *divisors*.

Definition 4.7 (Divisors)**.** For a compact Riemann surface X, define *the group of divisors $Div(X)$* as the free abelian group generated by the points of X. An element $D\in Div(X)$ is called a *divisor*. We express D as a formal sum: $D=\sum_{p\in X} D(p)\cdot p$.

Let f be a meromorphic function on X such that f is not constantly zero on any neighborhoods. Define the divisor $(f)=\sum_{p\in X} ord_p(f)\cdot p$, which is called *the divisor of f*. (f) is well-defined by Proposition 4.6.

We say $D\geq 0$ when $D(p)\geq 0$ for each $p\in X$, $D\geq D'$ when $D-D'\geq 0$. Define $deg(D)=\sum_{p\in X} D(p)$.

Now we define the sheaf associated to a divisor.

Definition 4.8. For a Riemann surface X and a divisor D, and for U a open subset of X, let $\mathcal{M}(U)$ be the set of meromorphic functions f on U. define the sheaf associated to D as the sheaf $\mathcal{O}_D$ over X such that for an open set $U \subseteq X$,

$$\mathcal{O}_D(U) = \{f \in \mathcal{M}(U) | \forall p \in X, ord_p(f) \geq -D(p)\}$$

When $D = 0$, $\mathcal{O}_D$ becomes the sheaf of holomorphic functions on X. In this case we use the notation $\mathcal{O} = \mathcal{O}_D$.

Each $\mathcal{O}_D(U)$ is an abelian group, and a $\mathbb{C}$-vector space by Proposition 4.5. Verifying that $\mathcal{O}_D$ is a sheaf is easy, so we omit it here. In addition, $\mathcal{O}_D(U)$ is a $\mathbb{C}$-vector space, so $C^n(X, \mathcal{O}_D)$ is also a $\mathbb{C}$-vector space. Because $d : C^n(X, \mathcal{O}_D) \to C^{n+1}(X, \mathcal{O}_D)$ by definition is also a $\mathbb{C}$-linear transformation, $H^n(X, \mathcal{O}_D)$ is also a $\mathbb{C}$-vector spaces, so we will be able to discuss the $\mathbb{C}$-dimension of $H^n(X, \mathcal{O}_D)$.

5. Riemann-Roch Theorem

Now we have covered all the important components of a proof of Riemann-Roch Theorem. To remind ourselves, we once again state the theorem.

Theorem 5.1 (Riemann-Roch Theorem)**.** *For a compact Riemann surface X, we have the following equation:*

$$dim(\check{H}^0(X, \mathcal{O}_D)) - dim(\check{H}^1(X, \mathcal{O}_D)) = 1 - g + deg(D).$$

In this equation, g refers to the genus of X, which is defined as $dim(\check{H}^1(X, \mathcal{O}))$.

Remark 5.2. Intuitively, genus refers to the number of "holes" in a Riemann surface. (For example, a torus is of genus 1, while a sphere is of genus 0.) The formal definition of genus as $dim(\check{H}^1(X, \mathcal{O}))$ comes from [3, Definition 14.11].

The following SES is the center of our proof, and we will apply Theorem 3.9 to this SES.

Lemma 5.3. *For a divisor D and a point $p \in X$, we have an SES of sheaves:*

$$0 \to \mathcal{O}_D \xrightarrow{i} \mathcal{O}_{D+p} \xrightarrow{F} \mathbb{C}_p \to 0,$$

where i is the inclusion map, and F is defined as follows: for $p \in U$, we define $F(U)(f) = a_{-D(p)-1}$, where $a_{-D(p)-1}$ is the $(-D(p) - 1)$-th term of the Laurent expansion of f in a neighborhood of p. In other words,

$$f(z) = \sum_{i=-D(p)-1}^{\infty} a_i (z - p)^i$$

Otherwise, if we have $p \notin U$, then define $F(U)(f) = 0$.

Proof. Let (U, ϕ) be a coordinate chart. We need only to prove that

$$0 \to \mathcal{O}_D(U) \xrightarrow{i(U)} \mathcal{O}_{D+p}(U) \xrightarrow{F(U)} \mathbb{C}_p(U) \to 0$$

is an SES of abelian groups. Then, because the above SES holds for every coordinate chart U, we can apply Proposition 2.13 and we would have the SES of sheaves stated in the lemma.

We first prove that $i(U)$ is injective, then that $F(U)$ is surjective, and finally that $Im(i(U)) = ker(F(U))$.

First, $i(U)$ being the inclusion map is obviously injective. Then we prove that $F(U)$ is surjective. When $p \notin U$, $\mathbb{C}_p(U) = 0$, so in this case $F(U)$ is again obviously surjective. Now we consider when $p \in U$, and $\mathbb{C}_p(U) = \mathbb{C}$. By the construction we know $F(U)$ is $\mathbb{C}$-linear, so we need only to construct a meromorphic function $f \in \mathcal{O}_{D+p}(U)$ such that $F(U)(f) \neq 0$. Define $f, g \in \mathcal{O}_{D+p}(U)$ as:

$$f(z) = \prod_{z_0 \in U} (z - z_0)^{-(D+p)(z_0)}$$

$$g(z) = \prod_{z_0 \in U, z_0 \neq p} (z - z_0)^{-(D+p)(z_0)}.$$

Thus, we have $f(z) = g(z)(z-p)^{-(D+p)(p)}$. Then, in the Laurent expansion of f at p, the $-(D+p)(p)$-th coefficient would be the 0-th coefficient of g, that is:

$$g(p) = \prod_{z_0 \in U, z_0 \neq p} (p - z_0)^{-(D+p)(z_0)} \neq 0.$$

Thus, we have $F(U)(f) \neq 0$. $F(U)$ is surjective.

We finally prove $Im(i(U)) = ker(F(U))$. When $p \notin U$, $ker(F(U)) = \mathcal{O}_{D+p}(U)$, so we shall prove that $i(U)$ in this case is surjective. Let $f \in \mathcal{O}_{D+p}(U)$. Then, for every $z_0 \in U$, $ord_f(z_0) \geq -(D+p)(z_0) = -D(z_0)$, so $f \in Im(i(U))$. Thus, $i(U)$ is surjective in this case.

When $p \in U$, we first prove that $Im(i(U)) \subseteq ker(F(U))$. Let $f \in Im(i(U))$. Then, we have $ord_f(p) \geq -D(p)$, so the $-(D+p)(p)$-th coefficient in the Laurent expansion of f at p is 0, so $f \in ker(F(U))$. We then prove that $ker(F(U)) \subseteq Im(i(U))$. Let $f \in ker(F(U))$. Then, the $-(D+p)(p)$-th coefficient in the Laurent expansion of f at p is 0. Since $f \in \mathcal{O}_{D+p}(U)$, the lower coefficients would also be 0, so $ord_f(p) \geq -D(p)$, so $f \in Im(i(U))$. Thus, we conclude $Im(i(U)) = ker(F(U))$. □

Then, applying Theorem 3.9, we have the following LES.

$$\begin{aligned} 0 \longrightarrow \check{H}^0(X, \mathcal{O}_D) \longrightarrow \check{H}^0(X, \mathcal{O}_{D+p}) \longrightarrow \check{H}^0(X, \mathbb{C}_p) \xrightarrow{\delta} \\ \longrightarrow \check{H}^1(X, \mathcal{O}_D) \longrightarrow \check{H}^1(X, \mathcal{O}_{D+p}) \longrightarrow \check{H}^1(X, \mathbb{C}_p) \end{aligned} \tag{5.4}$$

Using Lemma 3.4, we now have an LES as follows.

$$\begin{aligned} 0 \longrightarrow \check{H}^0(X, \mathcal{O}_D) \longrightarrow \check{H}^0(X, \mathcal{O}_{D+p}) \longrightarrow \mathbb{C} \xrightarrow{\delta} \\ \longrightarrow \check{H}^1(X, \mathcal{O}_D) \longrightarrow \check{H}^1(X, \mathcal{O}_{D+p}) \longrightarrow 0 \end{aligned} \tag{5.5}$$

Calculating the dimensions of each group in this long exact sequence, we get the following equation.

$$\begin{aligned} &dim(\check{H}^0(X, \mathcal{O}_D)) + 1 + dim(\check{H}^1(X, \mathcal{O}_{D+p})) \\ &= dim(\check{H}^0(X, \mathcal{O}_{D+p})) + dim(\check{H}^1(X, \mathcal{O}_D)) \end{aligned} \tag{5.6}$$

Finally, we need one last lemma to state before formally proving the theorem.

Lemma 5.7. *A holomorphic function on a compact Riemann surface X is constant, so by Lemma 3.3, $dim(\check{H}^0(X, \mathcal{O})) = 1$.*

Proof. [3, Corollary 2.8]. □

Proof of Theorem 5.1. We shall prove the theorem by induction. Because every divisor D is a finite sum of $+p$ or $-p$ for $p \in X$, we can do induction on the number of terms of the finite sum, i.e. we first prove the theorem for $D = 0$, and then prove the theorem for $D + p$ and $D - p$ when the theorem holds for D.

First the theorem for $D = 0$. Because $D = 0$, $deg(D) = 0$, and $\mathcal{O}_D = \mathcal{O}$. Then, by Lemma 5.7

$$\begin{aligned} dim(\check{H}^0(X, \mathcal{O}_D)) - dim(\check{H}^1(X, \mathcal{O}_D)) &= 1 - dim(\check{H}^1(X, \mathcal{O})) \\ &= 1 - g + deg(D) \end{aligned}$$

Then, assume that the theorem holds for D, then we prove the theorem holds for $D + p$. By (5.6),

$$\begin{aligned} &dim(\check{H}^0(X, \mathcal{O}_{D+p})) - dim(\check{H}^1(X, \mathcal{O}_{D+p})) \\ &= 1 + dim(\check{H}^0(X, \mathcal{O}_D)) - dim(\check{H}^1(X, \mathcal{O}_D)) \\ &= 2 - g + deg(D) \\ &= 1 - g + deg(D + p) \end{aligned}$$

For $D - p$, we again apply (5.6),

$$\begin{aligned} &dim(\check{H}^0(X, \mathcal{O}_{D-p})) - dim(\check{H}^1(X, \mathcal{O}_{D-p})) \\ &= dim(\check{H}^0(X, \mathcal{O}_D)) - dim(\check{H}^1(X, \mathcal{O}_D)) - 1 \\ &= g + deg(D) \\ &= 1 - g + deg(D - p) \end{aligned}$$

Thus finishes the proof.

□

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