NON-ASSOCIATIVITY OF THE SMASH PRODUCT

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Abstract. We show that the smash product is not associative in the category of based topological spaces, and discuss in what "convenient" categories the smash product is associative.

Contents

1. Introduction

The main counterexample that shows that the smash product is not associative has some history to it. It is originally due to Dieter Puppe, but he did not provide a proof that it is a counterexample. Kathleen Lewis figured out a proof which was then included as an appendix to Peter May and J. Sigurdsson's book *Parametrized Homotopy Theory*. Here we flesh out some of the details of this proof to make it more accessible to people who, like myself, are new to algebraic topology and to clear up some confusion that seems to have found its way onto the StackExchange forums.

2. Definitions

First, the most basic definition we will need in this paper:

Definition 2.1. A *based* (or *pointed*) topological space is a pair (X, x_0) where X is a topological space and x_0 is some element of X. The point x_0 is called the *basepoint* of X.

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When there is a chance of ambiguity (e.g. if we are considering the same topological space with two different basepoints), we denote a based topological space by (X, x_0) where X is the topological space and x_0 is the basepoint.

Next we define some important operations on based topological spaces.

Definition 2.2. Given a topological space X (not necessarily based) and an equivalence relation $\sim$, we can define the *quotient space* $X/\sim$ by the set of equivalence classes of X under $\sim$. This comes with a canonical map $q : X \to X/\sim$ defined by $q(x) = [x]$ for all $x \in X$. There is a natural topology on $X/\sim$ where the open sets are exactly the subsets $U \subseteq X/\sim$ whose preimage under the canonical map q is open in X.

When X is in fact based (say with basepoint x_0), we just get a new based space out of the quotient with basepoint $[x_0]$.

One useful thing to do is to take the quotient of a topological by a subspace. (Recall that every subset of a topological space can be thought of as a sub*space* under the subspace topology). Let X be a topological space and let Y be a subspace of X. Then if we were to take the quotient X/Y, what we would be doing is taking the quotient as above where the equivalence relation $\sim$ is defined by $y \sim y'$ if and only if $y \in Y$ and $y' \in Y$.

One nice way to think about this (at least for "nice" topological spaces!) is to imagine "collapsing" all of the points of the subspace to a single point.

Example 2.3. Consider $[0, 1]$ with the usual topology. We can think of taking the quotient by $\{0, 1\}$ as gluing the ends of the interval together. This way we can see that $[0, 1]/\{0, 1\} = S^1$, the one-dimensional circle. Note that the equals sign denotes homeomorphism of topological spaces.

It will become necessary to think about when certain functions are quotient maps. To that end, we need the following definition:

Definition 2.4. A map $f : X \to Y$ between topological spaces X and Y is a *quotient map* if it is surjective and if subsets $U \subseteq Y$ are open in Y if and only if $f^{-1}(U)$ is open in X.

However, we can also characterize quotient maps with the following universal property:

Proposition 2.5. *Consider the quotient space $X/\sim$ together with its quotient map $q : X \to X/\sim$. Then if $g : X \to Z$ is any continuous map with $a \sim b \implies g(a) = g(b)$, then there exists a unique continuous map $f : X/\sim \to Z$ such that $g = f \circ q$. Equivalently, the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "q"'] & Z \\ X/\sim \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

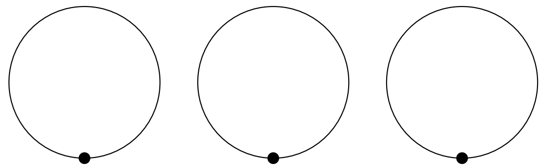
The above diagram will be incredibly useful in proving our counterexample.

Definition 2.6. Let (X, x_0) and (Y, y_0) be two based topological spaces. The *wedge sum* of X and Y, denoted $X \vee Y$, is the quotient $X \amalg Y/\sim$ where $\sim$ is the equivalence relation in which x_0 and y_0 are the only distinct points that are equivalent to each other.

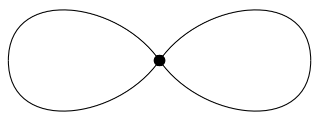
As defined before with quotient spaces, the basepoint of the wedge sum of two based spaces is the equivalence class of the basepoints of the original spaces. Continuing the collapsing analogy from before, we can think of taking the wedge sum of topological spaces as just sticking them together at their basepoints.

It is easy to see that the wedge sum is associative. Given three based topological spaces X, Y, and Z, it does not matter in what order we attach the basepoints as long as we are consistent with those basepoints. Here is a simple example:

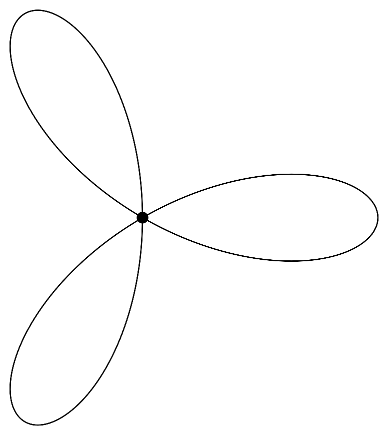
Example 2.7. Consider three based circles S^1.



The first step is to find the sum $S^1 \vee S^1$. This results in a figure-8 shape.



Then once we add a second S^1, either pre-fixing or post-fixing, we have the following "bouquet" of circles:



Applying the logic of this example to an arbitrary collection of topological spaces, we see that the wedge sum is both commutative and associative.

Definition 2.8. Let I be some nonempty set and for each $i \in I$, let X_i be a topological space. We will define the *product topology* on the Cartesian product

$$X = \prod_{i \in I} X_i.$$

First, for each $i \in I$, define $p_i : X \to X_i$ by $p((x_j)_{j \in I}) = x_i$. That is, each p_i is the projection from X to X_i. The product topology on X is generated by the sets $p_i^{-1}(U_i)$ where U_i is open in X_i.

When the X_i are based, say with basepoints x_{i0}, the basepoint of X is the element $(x_{i0})_{i \in I}$.

When the set I is finite, this topology is exactly what you'd hope it is: it is generated by the products of open sets in the X_i. However, this is not the case when I is infinite. Even though we will not be taking any infinite products in this paper, it is important to keep track of our topologies.

Definition 2.9. The *smash product* of two based topological spaces (X, x_0) and (Y, y_0) is the quotient space $X \wedge Y = (X \times Y)/(X \vee Y)$.

The wedge sum $X \vee Y$ can be considered a subspace of the Cartesian product $X \times Y$ as

$$\{(x_0, y) \mid y \in Y\} \cup \{(x, y_0) \mid x \in X\} \subset \{(x, y) \mid x \in X, y \in Y\}.$$

Generally, the smash product is significantly more difficult to visualize than the wedge sum is, but here is a particularly tractable example:

Example 2.10. The smash product of two circles is homeomorphic to the 2-sphere. That is,

$$S^1 \wedge S^1 = S^2.$$

Going step by step, recall that $S^1 \times S^1 = T^2$, or the 2-torus. Now recall that $S^1 \vee S^1$ is a figure-eight-like shape. On the torus in particular, we can think of this space as two intersecting circles on the torus, one going around the hole and one going through the hole. Imagining this space collapsing to a single point, say the intersection point, we end up with a space that is homeomorphic to the 2-sphere.

This example does in fact generalize, so $S^n \wedge S^m = S^{n+m}$ for positive integers n and m. Sadly, though, we cannot visualize higher-dimensional spheres very well.

3. The Smash Product is not Associative

We will think of $\mathbb{Q}$ and $\mathbb{N}$ as subspaces of $\mathbb{R}$ and with the basepoint 0. In particular, this gives $\mathbb{N}$ what amounts to the discrete topology. There are two plausible ways to go about computing the three-fold smash product $\mathbb{Q} \wedge \mathbb{Q} \wedge \mathbb{N}$. One could either smash the two $\mathbb{Q}$s together first, or smash the second $\mathbb{Q}$ and $\mathbb{N}$ together first. These two operations result in (based) topological spaces that are not homeomorphic to each other, meaning that **this operation is not associative**. We illustrate the process in a diagram:

$$\begin{tikzcd}
 & \mathbb{Q} \times \mathbb{Q} \times \mathbb{N} \arrow[dl, "\mathrm{id} \times p'"'] \arrow[dd, "q"] \arrow[dr, "p \times \mathrm{id}"] & \\
\mathbb{Q} \times (\mathbb{Q} \wedge \mathbb{N}) \arrow[d, "r"'] & & (\mathbb{Q} \wedge \mathbb{Q}) \times \mathbb{N} \arrow[d, "s"] \\
\mathbb{Q} \wedge (\mathbb{Q} \wedge \mathbb{N}) & \mathbb{Q} \wedge \mathbb{Q} \wedge \mathbb{N} \arrow[l, "t"'] \arrow[r, "t'"] & (\mathbb{Q} \wedge \mathbb{Q}) \wedge \mathbb{N}
\end{tikzcd}$$

where $\mathbb{Q} \wedge \mathbb{Q} \wedge \mathbb{N}$ is defined naively as the quotient space $(\mathbb{Q} \times \mathbb{Q} \times \mathbb{N})/(\mathbb{Q} \vee \mathbb{Q} \vee \mathbb{N})$ with q the associated quotient map. The other quotient maps in the diagram are p', p, r, s, and, trivially, id. The maps t and t' are given by the universal property of q. The key to this counterexample is that one of $\mathrm{id} \times p'$ and $p \times \mathrm{id}$ is a quotient map while the other is not.

Definition 3.1. A topological space X is *locally compact* if every point of X has a compact neighborhood. That is, if for all $x \in X$, there exists an open set $U \subseteq X$ containing x and a compact set $K \subseteq X$ such that $U \subseteq K$.

Theorem 3.2. *Let X be a Hausdorff space. Then X is locally compact if and only if $\mathrm{id}_X \times g$ is a quotient map for every quotient map g.*

The proof of this theorem is a combination of work done in [8] and [5].

Example 3.3. We can easily see that $\mathbb{N}$ is locally compact: it has the discrete topology, and therefore every subset of $\mathbb{N}$ is both open and closed. Moreover, $\mathbb{N}$ is a subspace of $\mathbb{R}$, meaning that the Heine-Borel theorem applies and every closed and bounded subset (i.e. every finite subset) of $\mathbb{N}$ is compact. Thus any finite subset of $\mathbb{N}$ containing a given point serves as a compact neighborhood for that point.

Applying the theorem to this example, we see that $p \times \mathrm{id}$ is a quotient map. In particular, the right-hand side of the diagram commutes! By the universal property of the quotient map and the fact that the composition of quotient maps is a quotient map, we know that t' must be a homeomorphism. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathbb{Q} \times \mathbb{Q} \times \mathbb{N} \arrow[d, "q"'] \arrow[dr, "s\circ(p\times\mathrm{id})"] & \\
\mathbb{Q} \wedge \mathbb{Q} \wedge \mathbb{N} \arrow[r, shift left, "t'"] & (\mathbb{Q} \wedge \mathbb{Q}) \wedge \mathbb{N} \arrow[l, shift left, "t''"]
\end{tikzcd}$$

The map t'' comes from the universal property for the quotient map $s \circ (p \times \mathrm{id})$. It is the unique continuous map such that $q = t'' \circ s \circ (p \times \mathrm{id})$. From the universal property for q, t' is the unique continuous map such that $s \circ (p \times \mathrm{id}) = t' \circ q$. Then we have

$$q = t'' \circ s \circ (p \times \mathrm{id})$$
$$t' \circ q = t' \circ t'' \circ s \circ (p \times \mathrm{id}).$$

But the universal property for q implies that

$$t' \circ t'' \circ s \circ (p \times \mathrm{id}) = s \circ (p \times \mathrm{id}),$$

meaning that $t' \circ t''$ is the identity map, or that $t'' = (t')^{-1}$. This invertibility implies that t' is in fact a homeomorphism.

Example 3.4. Now note that $\mathbb{Q}$ is *not* locally compact. To see this, first recall that open sets in $\mathbb{Q}$ are of the form $U \cap \mathbb{R}$ for open subsets U of $\mathbb{R}$, that open sets in $\mathbb{R}$ can all be written as a disjoint union of open intervals, and that a neighborhood of a point is a set that contains an open set containing the point in question.

Now let $x \in \mathbb{Q}$ and consider a neighborhood $N \subseteq \mathbb{Q}$ of x. Then by definition, N must contain an open subset U. This U may be written as $V \cap \mathbb{Q}$ for V open in $\mathbb{R}$, and V may in turn be written as a countable union of disjoint open intervals. Without loss of generality, let us focus on the interval that contains x (there is only one). Denote this interval (a, b), and note that its intersection with U is entirely contained within U. There is a countable number of rationals contained in this interval, and we can choose an increasing sequence of them indexed by $\mathbb{N}$. Denote this sequence by $\{r_i\}_{i \in \mathbb{N}}$. Then the open cover

$$\left(\bigcup_{i \in \mathbb{N}} (r_{2i}, r_{2i+2}) \right) \cup \left(\bigcup_{i \in \mathbb{N}} (r_{2i+1}, r_{2i+3}) \right)$$

has no finite subcover. Thus U has a non-compact subset, which precludes N from being compact, which means that $\mathbb{Q}$ is not locally compact.

Because $\mathbb{Q}$ is not locally compact, $\mathrm{id} \times p'$ cannot be a quotient map!

The proof presented in May and Sigurdsson's appendix does not explicitly show that there exists no homeomorphism between $\mathbb{Q} \wedge (\mathbb{Q} \wedge \mathbb{N})$ and $(\mathbb{Q} \wedge \mathbb{Q}) \wedge \mathbb{N}$. Instead, it shows that the map t from the diagram is not a homeomorphism. But because of the way t is defined, it turns out that this is in fact sufficient to show that the two spaces are not homeomorphic. If they were, the map t would be forced to be a homeomorphism.

Suppose that $\mathbb{Q} \wedge (\mathbb{Q} \wedge \mathbb{N})$ and $(\mathbb{Q} \wedge \mathbb{Q}) \wedge \mathbb{N}$ are homeomorphic. That is, suppose that there exists a homeomorphism $f : \mathbb{Q} \wedge (\mathbb{Q} \wedge \mathbb{N}) \to (\mathbb{Q} \wedge \mathbb{Q}) \wedge \mathbb{N}$. Then the bottom part of the following diagram commutes:

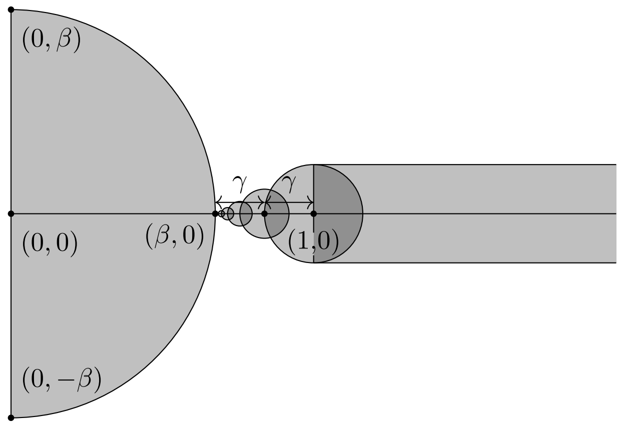
$$\begin{tikzcd}
 & \mathbb{Q} \times \mathbb{Q} \times \mathbb{N} \arrow[dl] \arrow[d, "q"] \arrow[dr] & \\
\mathbb{Q} \wedge (\mathbb{Q} \wedge \mathbb{N}) \arrow[rr, bend right, "f"'] & \mathbb{Q} \wedge \mathbb{Q} \wedge \mathbb{N} \arrow[l, "t"'] \arrow[r, "t'"] & (\mathbb{Q} \wedge \mathbb{Q}) \wedge \mathbb{N}
\end{tikzcd}$$

where the two blank arrows are compositions of arrows given in the first diagram. Then we may write $f \circ t = t'$. Since f is a homeomorphism, we can invert it and write $t = f^{-1} \circ t'$. Both of the maps in that composition are homeomorphisms, and therefore their composition is a homeomorphism. That is, if the two spaces were homeomorphic, the map t in particular would be a homeomorphism.

3.1. **The Counterexample.** All that remains is to find an open subset of $\mathbb{Q} \wedge \mathbb{Q} \wedge \mathbb{N}$ whose image under t is not open, i.e. to show that t is not a homeomorphism. Note that this is sufficient because while t is continuous and a bijection, we show here that its *inverse* is not continuous, and therefore t is not a homeomorphism. Following the notation of May and Sigurdsson, choose an irrational number β between 0 and 1 and define $\gamma = \frac{1-\beta}{2}$. Then define $V'(\beta) \subset \mathbb{R} \times \mathbb{R}$ as the union of:

- the open ball of radius β, centered at $(0, 0)$.
- the "open tubes" of radius γ around the x- and y-axes starting from 1. The rightmost "tube" is $[1, \infty) \times (-\gamma, \gamma)$, for example.
- the open balls of radius γ centered at $(\pm 1, 0)$ and $(0, \pm 1)$.
- $\left(\bigcup\limits_{n \in \mathbb{N}} B_{\gamma/2^n}(\pm\gamma_n, 0) \right) \cup \left(\bigcup\limits_{n \in \mathbb{N}} B_{\gamma/2^n}(0, \pm\gamma_n) \right)$ for $\gamma_n = 1 - \sum\limits_{k=0}^{n-1} \gamma/2^k$.

The following diagram is a picture of the right axis of $V'(\beta)$. It is symmetric with respect to rotation by $\pi/2$, so one can take the union of the rotations of the depicted set to find the entirety of $V'(\beta)$. Note that this set is open as the union of open sets. This means that while the solid black lines bordering whitespace make up the boundary of this part of $V'(\beta)$, none but the one that makes up the diameter of the leftmost semicircle is actually completely contained in $V'(\beta)$. In particular, neither $(\pm\beta, 0)$ nor $(0, \pm\beta)$ is in $V'(\beta)$. In fact, they are the only axis-points not contained in $V'(\beta)$.



Now let $V(\beta) = V'(\beta) \cap (\mathbb{Q} \times \mathbb{Q})$. Note that every rational axis-point is contained in $V(\beta)$. But given $\varepsilon > \beta$, there does not exist any $\delta > 0$ such that $((-\varepsilon, \varepsilon) \times (-\delta, \delta)) \cap (\mathbb{Q} \times \mathbb{Q})$ is contained in $V(\beta)$.

The set we ultimately land on will be obtained from $\mathbb{Q} \times \mathbb{Q} \times \mathbb{N}$ via q. Choose another irrational number α between 0 and 1 and define $\bullet$ to be the basepoint of $\mathbb{Q} \wedge \mathbb{N}$ and $*$ the basepoint of $\mathbb{Q} \wedge \mathbb{Q} \wedge \mathbb{N}$. Define $U \subset \mathbb{Q} \wedge \mathbb{Q} \wedge \mathbb{N}$ by

$$U = \{*\} \cup q \left(\bigcup_{n \in \mathbb{N}} V(\alpha/n) \times \{n\} \right)$$

and note that this is in fact open in $\mathbb{Q} \wedge \mathbb{Q} \wedge \mathbb{N}$ as its preimage under q is open in $\mathbb{Q} \times \mathbb{Q} \times \mathbb{N}$ and because q is a quotient map it is continuous. In particular

$$q^{-1}(U) = (\mathbb{Q} \times \mathbb{Q} \times \{0\}) \cup \left(\bigcup_{n \in \mathbb{N}} V(\alpha/n) \times \{n\} \right)$$

is the union of products of open subspaces (recall that $\mathbb{N}$ is endowed with the discrete topology), so in the product topology it is open.

Now suppose that $t(U)$ is open in $\mathbb{Q} \wedge (\mathbb{Q} \wedge \mathbb{N})$. Then $r^{-1}(t(U))$ is open (quotient maps are continuous) which means that it contains an open neighborhood V of $(0, \bullet)$. "Undoing" the product topology it follows that V contains a set of the form $((-\varepsilon, \varepsilon) \cap \mathbb{Q}) \times W$ for $\varepsilon > 0$ and W an open neighborhood of $\bullet$ in $\mathbb{Q} \wedge \mathbb{N}$.

Note that $\mathbb{Q} \wedge \mathbb{N}$ is homeomorphic to the wedge sum of all $\mathbb{Q} \times \{n\}$ for $n \in \mathbb{N}$. We can see this informally by noting that the latter sets have basepoints $(0, n)$, and wedging them together results in a "pinwheel" of copies of $\mathbb{Q}$, all intersecting at 0. The smash of these two spaces can be visualized just as easily: the cartesian product is like $|\mathbb{N}|$ copies of $\mathbb{Q}$ stacked on top of each other, spaced apart equally. The quotient by $\mathbb{Q} \vee \mathbb{N}$ amounts to collapsing all points $(0, n)$ and $(q, 0)$ to a single point, and we can think of this as a new origin. There are still $|\mathbb{N}|$ copies of $\mathbb{Q}$ "left over," and we can think of the pinwheel again.

Thus $\mathbb{Q} \wedge \mathbb{N} \cong \bigvee_{n \in \mathbb{N}} (\mathbb{Q} \times \{n\})$ implies that for each $n \in \mathbb{N}$, there exists $\delta_n > 0$ such that

$$\bigvee_{n \in \mathbb{N}} (((-\delta_n, \delta_n) \cap \mathbb{Q}) \times \{n\}) \subset W.$$

Now by definition of U, it follows that

$$((-\varepsilon, \varepsilon) \times (-\delta_n, \delta_n)) \cap (\mathbb{Q} \times \mathbb{Q}) \subset V(\alpha/n).$$

But we just noted that if $\varepsilon > \alpha/n$, there are no $\delta_n > 0$ that satisfy the above inclusion. We have reached a contradiction, so $t(U)$ cannot be open.

4. Why Do We Care?

Even the Cartesian product is *technically* not associative, but it is up to a very natural bijection, so we say (even though this is usually omitted) that it is associative *up to isomorphism*. However, there are even plenty of important operations that are just not associative at all. Some examples include the commutator bracket $[x, y] = xy - yx$, the cross product of vectors in $\mathbb{R}^3$, and set difference.

So why is it so bad that the smash product is not associative up to isomorphism? Because, according to algebraic topologists, it is incredibly *inconvenient* if this is not the case. Computations become long and arduous, and it is difficult to prove that properties hold "across" the smash product—something very desirable to do. As an analogy, consider how useful Tychonoff's theorem is:

Theorem 4.1 (Tychonoff's theorem)**.** *Any product of compact topological spaces is compact with respect to the product topology.*

Moreover, if the smash product *were* associative up to isomorphism, it would make the category of based topological spaces into a *symmetric monoidal* category. Such a thing is very useful, but in order to talk about it we need some more definitions:

4.1. Category theory primer.

First we will say what a category actually is.

Definition 4.2. A **category** $\mathscr{C}$ is a collection obj$(\mathscr{C})$ of **objects** together with for every (ordered!) pair of objects A and B of $\mathscr{C}$ a set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ of **morphisms** such that:

- for each object A of C, there is at least one element of $\mathrm{Hom}_{\mathscr{C}}(A, A)$ denoted 1_A and called the **identity** morphism of A;
- morphisms can be composed with one another: given three objects A, B, C of the category $\mathscr{C}$, for any two morphisms $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathscr{C}}(B, C)$ there exists a morphism $g \circ f \in \mathrm{Hom}_{\mathscr{C}}(A, C)$;
- this composition of morphisms is associative: given objects A, B, C, and D of $\mathscr{C}$ and homomorphisms $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathscr{C}}(B, C)$, and $h \in \mathrm{Hom}_{\mathscr{C}}(C, D)$, then the morphism $h \circ (g \circ f) \in \mathrm{Hom}_{\mathscr{C}}(A, D)$ is the same morphism as $(h \circ g) \circ f$;
- the identity morphisms defined above are actually identities: given any two obejcts A, B of $\mathscr{C}$, for any morphism $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ we have $f \circ 1_A = f$ and $1_B \circ f = f$.

From one point of view, categories are a way to talk about things that are "too big" to be sets. We can't, as Russell famously showed, consider things like the set of all sets. But for simplicity, all of the categories we talk about today will be **locally small**, i.e. given any pair of objects, the collection of morphisms between them is actually a set. We can, however, think of the category of sets.

Example 4.3. The *category of sets*, denoted **Set**, has as objects sets and as morphisms functions between sets.

This is valid because we never said that the objects of a category can be thought of as belonging to a set—they usually can't! It is a simple exercise to verify that **Set** is in fact a category. It is often useful to think categorically when dealing with mathematical objects of the form "a set with some extra structure." For instance, we have the category **Top** of topological spaces whose morphisms are continuous functions. We can also consider $\mathbf{Top}_*$, the category of based topological spaces whose morphisms are basepoint-preserving homeomorphisms.

Certain morphisms are also *isomorphisms*. The categorical definition of an isomorphism is very similar to the definition of a bijection (not least because bijections are the isomorphisms in **Set**)!

Definition 4.4. A morphism $f : X \to Y$ is an **isomorphism** if it has an inverse morphism, i.e. if there exists a morphism $g : Y \to X$ such that $f \circ g = 1_Y$ and $g \circ f = 1_X$.

Even though categories are not sets, we can think of maps in between them.

Definition 4.5. A **functor** F between two categories $\mathscr{C}$ and $\mathscr{D}$ is a mapping such that:

- for each object X of $\mathscr{C}$, there is an associated object $F(X)$ of $\mathscr{D}$;
- for each morphism $f : X \to Y$ in $\mathscr{C}$, there is an associated morphism $F(f) : F(X) \to F(Y)$ in $\mathscr{D}$ where the source and target of the morphisms are determined by the previous bullet;
- $F(1_X) = 1_{F(X)}$ for each object X in $\mathscr{C}$ (i.e. identities map to identities);
- $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathscr{C}$ (i.e. functors preserve composition).

Example 4.6. A useful type of functor is the *forgetful* functor, where we "forget" about certain structures. For example, there exists a forgetful functor $F : \mathbf{Top} \to \mathbf{Set}$. For every topological space X, $F(X)$ is the underlying set, and for every continuous function $f : X \to Y$ in **Top** (recall that these are the morphisms in **Top**), $F(f)$ is the function $X \to Y$ such that $F(f)(x) = f(x)$. We just ignore the topology and consider the spaces as sets.

We can define a product of categories, which behaves exactly as you would expect: given two categories $\mathscr{C}$ and $\mathscr{D}$, objects are pairs of objects (C, D), morphisms are pairs of morphisms, etc. The advantage of this is, for us, the ability to define functors with multiple "arguments."

Example 4.7. In the category of vector spaces, denoted **Vect**, the *tensor product* of vector spaces can be thought of as a functor $\otimes : \mathbf{Vect} \times \mathbf{Vect} \to \mathbf{Vect}$.

We can also talk about maps between functors that respect the structure of the categories, and this is incredibly useful.

Definition 4.8. Given two categories $\mathscr{C}$ and $\mathscr{D}$ and two functors $F, G : \mathscr{C} \to \mathscr{D}$, a **natural transformation** η between F and G is a collection of morphisms such that for every object X in $\mathscr{C}$, there is a morphism $\eta_X : F(X) \to G(X)$ in $\mathscr{D}$. This morphism is called the **component** of η at X. Moreover, we require that for every morphism $f : X \to Y$ in $\mathscr{C}$ that $\eta_Y \circ F(f) = G(f) \circ \eta_X$. Equivalently, this is the statement that the diagram

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"] & G(Y) \end{tikzcd}$$

commutes.

A natural transformation η is a **natural isomorphism** when each component is an isomorphism.

Definition 4.9. Let $\mathscr{C}$ be a category. The **opposite category** of $\mathscr{C}$, denoted $\mathscr{C}^{\mathrm{op}}$, has as objects the objects of $\mathscr{C}$ and as morphisms the morphisms of $\mathscr{C}$ but with sources and targets reversed. Essentially, given a morphism $f : X \to Y$ in $\mathscr{C}$, this corresponds to a morphisms $f : Y \to X$ in $\mathscr{C}^{\mathrm{op}}$.

4.2. **So why the smash product anyway?** In short, it's the thing that is left adjoint to the Hom functor, but that's a bit of a mouthful.

Recall how we defined the quotient in terms of a universal property. We can do the same for the Cartesian product! It actually serves as a *generalization* of the usual definition of Cartesian product we know and love.

Definition 4.10. Let $\mathscr{C}$ be a category and let X and Y be two objects of C. Then the **product** of X and Y (which may or may not exist) is another object of $\mathscr{C}$ denoted $X \times Y$ together with a pair of morphisms $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ such that for every object Z of $\mathscr{C}$ and every pair of morphisms $p_X : Z \to X$ and $p_Y : Z \to Y$, there exists a unique morphism $f : Z \to X \times Y$ such that the following diagram commutes:

$$\begin{tikzcd} & Z \arrow[dl, "p_X"'] \arrow[d, dashed, "\exists! f"] \arrow[dr, "p_Y"] & \\ X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y \end{tikzcd}$$

The Cartesian product of sets satisfies a very nice property, namely the *tensor-hom adjunction*. What this means is that for sets X, Y, and Z, there exists a natrual isomorphism between $\mathrm{Hom}_{\mathbf{Set}}(X \times Y, Z)$ and $\mathrm{Hom}_{\mathbf{Set}}(X, \mathrm{Hom}_{\mathbf{Set}}(Y, Z))$ where $\mathrm{Hom}_{\mathbf{Set}}(X, Y)$ denotes the set of morphisms $f : X \to Y$. Note that here we are thinking of $\mathrm{Hom}_{\mathscr{C}}(-,-)$ as a functor from $\mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$!

More details can be found in [4], but also note that this holds in **Top**, the category of topological spaces and that it does *not* hold in $\mathbf{Top}_*$, the category of based topological spaces. What would it take to make it hold?

Let's consider based spaces (X, x_0), (Y, y_0), and (Z, z_0). Suppose that we do have an adjunction $\mathrm{Hom}_{\mathbf{Top}_*}(X \otimes Y, Z) \cong \mathrm{Hom}_{\mathbf{Top}_*}(X, \mathrm{Hom}_{\mathbf{Top}_*}(Y, Z))$. The $\otimes$ just indicates a generic tensor product (which as we saw is also a functor); we will see that this tensor product should actually correspond to the smash product. Since as in multilinear algebra the "underlying set" of such a tensor product should be a Cartesian product, consider a morphism $f : X \times Y \to Z$ and the corresponding map $g : X \to \mathrm{Hom}_{\mathbf{Top}_*}(Y, Z)$ and let's think about what we would need for this correspondence to hold for based maps. If we wanted our morphisms to be based, the only thing x_0 could map to is the morphism that sends every element of Y to z_0. That is, for all $y \in Y$, $g(x_0)(y) = z_0$. Similarly, we would have for all $x \in X$ that $g(x)(y_0) = z_0$.

But recall from our discussion of the definition of the smash product that the wedge sum can be considered as a subspace of the Cartesian product. Because we are expecting a natural transformation, the properties we just discovered g satisfies means that the wedge sum as a subspace of $X \times Y$ would, under every such morphism f, be mapped to the basepoint of Z. Thus we are essentially restricting to morphisms on the smash product.

The above is not a completely rigorous proof that the smash product is the tensor product in the category of based topological spaces, but it gives a good indication that the smash product is a thing we should be looking at.

4.3. **Symmetric monoidal categories.** A symmetric monoidal category is one of the "nicest" kind of category there is. The definition is rather involved, but we keep in mind that we are building the machinery that gets us something that works a lot like the usual tensor product of vector spaces.

Definition 4.11. A **monoidal category** is a category $\mathscr{C}$ together with a functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ called the **tensor product**, a distinguished object I called the **unit**, and three natural isomorphisms that say that the tensor product is **associative** and **unital**. The first natural isomorphism, called α for associativity, is natural in three arguments and it says that for objects X, Y, and Z of $\mathscr{C}$,

$$\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z.$$

The other two natural isomorphisms are left (λ) and right (ρ) identity natural isomorphisms, and they have components

$$\lambda_X : I \otimes X \cong X \quad \text{and} \quad \rho_X : X \times I \cong X.$$

These natural isomorphisms must also satisfy some **coherence conditions**, namely that the pentagon diagram:

$$\begin{tikzcd}
 & (W \otimes X) \otimes (Y \otimes Z) \arrow[dr, "\alpha_{W\otimes X,Y,Z}"] & \\
W \otimes (X \otimes (Y \otimes Z)) \arrow[ur, "\alpha_{W,X,Y\otimes Z}"] \arrow[d, "1_W \otimes \alpha_{X,Y,Z}"] & & ((W \otimes X) \otimes Y) \otimes Z \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "\alpha_{W,X\otimes Y,Z}"'] & & (W \otimes (X \otimes Y)) \otimes Z \arrow[u, "\alpha_{W,X,Y} \otimes 1_Z"']
\end{tikzcd}$$

and the triangle diagram:

$$\begin{tikzcd}
X \otimes (I \otimes Y) \arrow[rr, "\alpha_{X,I,B}"] \arrow[dr, "1_X \otimes \lambda_Y"'] & & (X \otimes I) \otimes Y \arrow[dl, "\rho_X \otimes 1_Y"] \\
 & X \otimes Y &
\end{tikzcd}$$

commute.

This just gives us associativity of the tensor product. We've seen that the smash product is like a tensor product, but that we don't get associativity in the boring category of based topological spaces. If we are a little pickier, we can actually get this associativity and more. Keeping in mind the tensor product of vector spaces, we can also get symmetry, like how for vector spaces V and W, $V \otimes W$ is isomorphic to $W \otimes V$:

Definition 4.12. A **symmetric monoidal category** $\mathscr{C}$ is a monoidal category with (yet another) natural isomorphism s (for symmetry) such that for every pair of objects X and Y, the component $s_{X,Y}$ has source $X \otimes Y$ and target $Y \otimes X$ in $\mathscr{C}$ and such that the following diagrams commute:

$$\begin{tikzcd}
X \otimes I \arrow[rr, "s_{X,I}"] \arrow[dr, "\rho_X"'] & & I \otimes X \arrow[dl, "\lambda_X"] \\
 & X &
\end{tikzcd}$$

$$\begin{tikzcd}
& (X \otimes Y) \otimes Z \arrow[r, "s_{X,Y} \otimes 1_Z"] \arrow[dl, "\alpha_{X,Y,Z}"'] & (Y \otimes X) \otimes Z \arrow[dr, "\alpha_{Y,X,Z}"] & \\
X \otimes (Y \otimes Z) \arrow[dr, "s_{X,Y \otimes Z}"'] & & & Y \otimes (X \otimes Z) \arrow[dl, "1_Y \otimes s_{X,Z}"] \\
& (Y \otimes Z) \otimes X \arrow[r, "\alpha_{Y,Z,X}"'] & Y \otimes (Z \otimes X) &
\end{tikzcd}$$

$$\begin{tikzcd}
& Y \otimes X \arrow[dr, "s_{Y,X}"] & \\
X \otimes Y \arrow[ur, "s_{X,Y}"] \arrow[rr, equal] & & X \otimes Y
\end{tikzcd}$$

4.4. **Convenient categories of topological spaces.** There are a few different notions of "convenience" when doing algebraic topology, but they largely coincide for the most important notions. According to May and Sigurdsson, a convenient category of (based) topological spaces should be complete, cocomplete, and closed cartesian monoidal. We won't go into what "complete" and "cocomplete" mean here, but note that **Top** itself is complete and cocomplete.

Essentially, we want our category of topological spaces to be closed under finite products, and for the exponential A^B of objects A and B (which consists of continuous functions from $A \to B$ and whose topology depends on uniform convergence) to also be an object of this category. Monoidal corresponds to the notion of symmetric monoidal category discussed above.

Norman Steenrod, who was the first to talk about "convenient" categories of topological spaces, devised a few "test propositions" which one can verify to see whether a category is convenient or not, some of which involve concepts we will not discuss in this paper. The test propositions are as follows for objects X, Y, and Z:

- $(Y \times Z)^X = Y^X \times Y^Z$
- $Z^{Y \times X} = (Z^Y)^X$
- A product of decomposition spaces is a decomposition space of the product
- A product of unions is a union of products
- A decomposition space of a union is a union of decomposition spaces.

The nLab gives a full-blown definition of a convenient category:

Definition 4.13. A **convenient** category of topological spaces is a "nice" subcategory of **Top** such that

- every CW complex is an object of the subcategory
- it is cartesian closed
- it is complete and cocomplete.

CW complexes are a type of topological space that is of great use in algebraic topology. All of these concepts are very similar, and they all reflect the needs of algebraic topologists. Here we focus on the symmetric monoidal requirement The question is now: does such a category exist?

5. How Do We Fix This?

As it turns out, there are in fact convenient categories of topological spaces in which the smash product is associative (up to isomorphism) in the way that we want. In the based world, we just replace all mentions of the Cartesian product in our stipulations for convenience with the smash product and go from there.

There are two convenient categories mentioned by May and Sigurdsson. Before we can define them, we need some other definitions:

Definition 5.1. A topological space X is **weak Hausdorff** if given a compact space K and any continuous function $f : K \to X$, $g(K)$ is closed in X.

Definition 5.2. A topological space X is a k**-space** if given a compact space K and any continuous function $g : K \to X$, every subspace $A \subset X$ such that $g^{-1}(A)$ is closed in K is closed in X.

Definition 5.3. A topological space is **compactly generated** if it is a weak Hausdorff k-space.

Both the category of based k-spaces and the category of based compactly generated spaces are convenient categories of topological spaces. That is, they are both closed and symmetric monoidal with respect to the smash product.

In particular, k-spaces share with locally compact spaces the property that the product of quotient maps is a quotient map [12]. In fact, every locally compact space is compactly generated! Then for k-spaces and consequently compactly generated spaces s, we avoid the problem we encountered with $\mathbb{Q}$ and can think of both $(X \wedge Y) \wedge Z$ and $X \wedge (Y \wedge Z)$ as $(X \times Y \times Z)/(X \vee Y \vee Z)$.

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