THE GAUSS-BONNET THEOREM

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Abstract. In this paper we will start from preliminary definitions and theorems in differential geometry and gradually work our way first to the proof of the Local Gauss-Bonnet Theorem, and then to the Global Gauss-Bonnet Theorem. Then, we will discuss a number of elementary mathematical facts hidden in the theorem before we present the Hairy Ball Theorem. Familiarity with multivariable calculus will be assumed.

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1. Introduction

The Gauss-Bonnet Theorem is an important theorem that relates differential geometry to topology. It also generalizes a number of mathematical facts in elementary geometry, such as the sum of exterior angles for a polygon and the surface area of a sphere. However, before we can state the Gauss-Bonnet Theorem in full, we have to first define many of the concepts that it involves and prove several important lemmas. The purpose of this section is to introduce regular surfaces and the first fundamental form.

Definition 1.1. A regular surface is defined as a subset $S \subset \mathbb{R}^3$ such that for each element $p \in S$ there exists a neighborhood $V \in \mathbb{R}^3$ of $p$, meaning an open set $V$ containing $p$, and some differentiable map $x: U \to V \cap S$ with $x(u, v) = (x(u, v), y(u, v), z(u, v))$, called a parametrization of $S$, for an open set $U \in \mathbb{R}^2$ that has a continuous inverse and an injective differential matrix.

Example 1.2. An example of a regular surface is the sphere $S^2$, the set of points $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. First we can show that there exists a parametrization around any point $(a, b, c)$ on the sphere, noting that at least one of $a, b, c$ must be non-zero. Without loss of generality, we let $c > 0$. We know

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\[a^2 + b^2 + c^2 = 1\] which implies \(c = \sqrt{1 - a^2 - b^2}\). Then, we can let our map be \(x(u, v) = (u, v, \sqrt{1 - u^2 - v^2})\), with \(U = \{u, v \in \mathbb{R} \mid u^2 + v^2 < 1\}\).

We check that \(x\) is differentiable, since each component function has continuous partial derivatives. We have \(x(u, v) = u\) and therefore \(x_u = 1\) and \(x_v = 0\). Thus \(x(u, v)\) is differentiable. Similarly, \(y(u, v) = v\) can be verified to be differentiable. Lastly, \(z(u, v) = \sqrt{1 - u^2 - v^2}\) has the partial derivative \(z_u = \frac{-u}{\sqrt{1 - u^2 - v^2}}\), which is continuous as long as the radicand in the denominator is positive. By the definition of \(U\) we know that \(u^2 + v^2 < 1\), so we algebraically conclude that \(1 - u^2 - v^2 > 0\). Therefore, \(z(u, v)\) is differentiable, and thus \(x\) is differentiable.

Next, we verify that \(x\) is a homeomorphism by directly finding the inverse \(x^{-1}: V \cap S^2 \to U\). Since the two variables \(u, v \in U\) are found in the expression of \(x\), the inverse can be found to be \(x^{-1}(x, y, z) = (x, y)\). This must be continuous, since the functions \(x\) and \(y\) are continuous in terms of \(x\) and \(y\).

Lastly, we need to check that the map has a one-to-one differential matrix, which we can do by showing that the two columns are linearly independent. The differential is given by

\[
dx_p = \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ z_u & z_v \end{bmatrix}.
\]

We see the two columns are linearly independent, as

\[
\alpha \begin{bmatrix} 1 \\ 0 \\ z_u \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ z_v \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \alpha z_u + \beta z_v \end{bmatrix} = 0
\]

requires both \(\alpha = 0\) and \(\beta = 0\).

**Definition 1.4.** We define the tangent space \(T_p(S)\) at a point \(p\) on a surface \(S\) as the space of vectors tangent to \(S\) at \(p\), represented by the tangent plane at \(p\) and given by the span of \(x_u\) and \(x_v\) at \(p\).

Now, we move on to the inner product on regular surfaces.

**Definition 1.5.** We define an inner product on \(\mathbb{R}^3\) as an operation between two vectors \(v, w \in \mathbb{R}^3\) notated by \(\langle v, w \rangle_p\) satisfying that:

1. The operation is symmetric, meaning \(\langle v, w \rangle = \langle w, v \rangle\)
2. The operation is linear in each argument, meaning \(\langle \alpha v_1 + \beta v_2, w \rangle = \alpha \langle v_1, w \rangle + \beta \langle v_2, w \rangle\)
3. For any \(v \in \mathbb{R}^3\), it must hold that \(\langle v, v \rangle = v^2 \geq 0\)

We can note that we have a natural inner product defined in \(\mathbb{R}^3\) as the dot product, where \(\langle (x_1, y_1, z_1), (x_2, y_2, z_2) \rangle = x_1x_2 + y_1y_2 + z_1z_2\). Since each tangent space has its own basis \(\{x_u, x_v\}\), we will want to define an inner product on \(T_p(S)\).

**Definition 1.6.** The first fundamental form \(I_p\) is defined as \(I_p(v) = \langle v, v \rangle_p\) for a vector \(v \in T_p(S)\).

Since \(v\) is the tangent vector to some parametrized curve on \(S\) at \(t = 0\), denoted by \(v(t) = x(u(t), v(t))\), we can apply the chain rule and find

\[
v = \alpha'(0) = x_u u' + x_v v'.
\]
Plugging (1.7) into \( I_p \) and using the properties of linearity and symmetry, we can distribute and find

\[
I_p(v) = \langle x_u, x_u \rangle (u')^2 + 2 \langle x_u, v \rangle (u'v') + \langle x_v, x_v \rangle (v')^2.
\]

**Definition 1.9.** We define the coefficients of the first fundamental form as functions \( E, F, G \)

\[
(1.8) \\
E = \langle x_u, x_u \rangle \\
F = \langle x_u, x_v \rangle \\
G = \langle x_v, x_v \rangle.
\]

2. THE GAUSS MAP

**Definition 2.1.** We define the Gauss Map for a given parametrization \( x: U \subset \mathbb{R}^2 \rightarrow S \) of a surface \( S \) as the map \( N: x(U) \rightarrow \mathbb{R}^3 \) of unit normal vectors on the surface \( S \), given by

\[
N(p) = \frac{x_u \times x_v}{|x_u \times x_v|}.
\]

**Definition 2.2.** A surface is defined as orientable if there exists a differentiable Gauss map defined for the entire surface.

**Example 2.3.** We can find the Gauss Map for the sphere by considering that, for any parametrized curve \( \alpha(t) = (x(t), y(t), z(t)) \) contained on the sphere, \( x^2(t) + y^2(t) + z^2(t) \) is constant and equals 1. Differentiating with respect to \( t \), we get \( 2x(x') + 2y(y') + 2z(z') = 0 \), which implies then that the vector \( (2x, 2y, 2z) \) is normal to any parametrized curve on the sphere. Since \( x^2 + y^2 + z^2 = 1 \), we can find unit normal vectors to be \( N = (x, y, z) \) or \( N = (-x, -y, -z) \). By convention, we choose \( N(x, y, z) = (x, -y, -z) \).

**Example 2.4.** As a more arbitrary example, we can consider the xy-plane given by the parametrization \( x(u, v) = (u, v, 0) \). Then, we can use \( x_u = (1, 0, 0) \) and \( x_v = (0, 1, 0) \) to find \( x_u \times x_v = (0, 0, 1) \), which is already a unit vector. Therefore, \( N(x, y, z) = (0, 0, 1) \) is the Gauss Map for the xy-plane. More generally, the Gauss Map for any plane will always send every point on the plane to a single point on the sphere.

It will be important to consider the differential of the Gauss Map, \( dN_p: T_p(S) \rightarrow T_{N(p)}(S^2) \) (and we can note here that \( T_p(S) = T_{N(p)}(S^2) \), since \( N(p) \) is the normal vector of \( S \) at \( p \), so the vector from 0 to \( N(p) \) will be normal to the same plane).

**Definition 2.5.** We define the second fundamental form \( II_p \) as

\[
II_p(v) = -\langle dN_p(v), v \rangle.
\]

Next, we want to find coefficient functions \( e, f, g \) for the second fundamental form as done previously with the first fundamental form. As before, we consider a regular surface \( S \), a parametrization \( x: U \rightarrow S \), and a parametrized curve \( \alpha(t) = x(u(t), v(t)) \) lying on the surface with a tangent vector \( v \) at \( \alpha(0) = p \). Then, the second fundamental form is

\[
(2.6) \\
-\langle dN_p(\alpha'(0)), \alpha'(0) \rangle = -\langle N_u u' + N_v v', x_u u' + x_v v' \rangle
\]
Distributing, we get

\[ H_p(v) = -\langle Nu, xu \rangle (u')^2 - 2\langle Nu, xv \rangle (u'v') - \langle Nv, xv \rangle (v')^2. \]

**Definition 2.8.** We define the coefficients of the second fundamental form as functions \( e, f, g \) of \( u \) and \( v \) satisfying

\[ H_p(v) = e (u')^2 + 2f (u'v') + g (v')^2. \]

More explicitly, the functions are given by

\[
\begin{align*}
e &= -\langle Nu, xu \rangle \\
f &= -\langle Nu, xv \rangle = -\langle Nv, xu \rangle \\
g &= -\langle Nv, xv \rangle.
\end{align*}
\]

Next, we introduce the concept of Gaussian curvature and approach the goal of this section, which is to prove a useful lemma for this proof of the Local Gauss-Bonnet Theorem.

**Definition 2.9.** The Gaussian curvature, \( K \), at a point is equal to the determinant of the differential of the Gauss Map \( \det(dN_p) \).

**Example 2.10.** For the xy-plane, the Gauss Map is constant, so its differential is the zero matrix, and we get \( K = 0 \).

**Example 2.11.** For the sphere, the Gauss map is \( N(x, y, z) = (-x, -y, -z) \), or just \( N(v) = -v \). Then, for some parametrized curve \( \alpha(t) = x(u(t), v(t)) \) with \( \alpha(0) = p \), we have \( \alpha'(0) = x_u u'(0) + x_v v'(0) \). Additionally, we have \( dN_p(\alpha'(0)) = N_u u'(0) + N_v v'(0) = -x_u u'(0) - x_v v'(0) \). This produces the matrix equation,

\[
dN_p \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -u' \\ -v' \end{bmatrix}
\]

in the basis \( B = \{x_u, x_v\} \). Thus,

\[
dN_p = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

From here, we calculate that for any point \( p \) on the sphere, \( K = \det(dN_p) = 1 \).

Alternatively, we could have shown this fact using the negative orientation \( N \) which sends the sphere to itself. Then, \( dN_p \) is the identity matrix for any point \( p \), so \( K = 1 \).

Next, we prove the following lemma.

**Lemma 2.12.** At any point \( p \), the Gaussian curvature can be expressed in terms of the coefficients of the first and second fundamental forms as

\[ K = \frac{eg - f^2}{EG - F^2}. \]

**Proof.** First, it is useful to write the differential of the Gauss Map as

\[
dN_p = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.
\]

We can write this out in matrix form as

\[
dN_p(\alpha') = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} (a_{11}u' + a_{12}v')x_u + (a_{21}u' + a_{22}v')x_v. \end{bmatrix}
\]
Since $dN_p$ sends vectors in the tangent space $T_p(S)$ to vectors in the same tangent space, with both the domain and image in the basis $B = \{x_u, x_v\}$, the vector $dN(\alpha')$ can be expressed as a unique linear combination of $x_u$ and $x_v$. Using the fact that $dN_p(\alpha') = N_u u' + N_v v'$, we write

$$N_u u' + N_v v' = (a_{11} u' + a_{12} v') x_u + (a_{21} u' + a_{22} v') x_v,$$

which can then be rearranged as

$$N_u u' + N_v v' = (a_{11} x_u + a_{21} x_v) u' + (a_{12} x_u + a_{22} x_v) v'.$$

This tells us

$$N_u = a_{11} x_u + a_{21} x_v,$$

$$N_v = a_{12} x_u + a_{22} x_v.$$

Now, we use the second fundamental form here, since we know

$$-e = \langle N_u, x_u \rangle = -\langle a_{11} x_u + a_{12} x_v, x_u \rangle = a_{11} E + a_{21} F.$$

By the same process as above, we get the following equations

$$-f = a_{11} F + a_{21} G$$

$$-f = a_{12} E + a_{22} F$$

$$-g = a_{12} F + a_{22} G.$$

These four equations can then be combined into the matrix equation

$$\begin{bmatrix} e & f \\ f & g \end{bmatrix} = dN_p \cdot \begin{bmatrix} E & F \\ F & G \end{bmatrix}.$$

And the matrix $dN_p$ can be found by multiplying by the inverse of the matrix on the right, to get

$$dN_p = -\begin{bmatrix} e & f \\ f & g \end{bmatrix} \cdot \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1}.$$

Since our goal is to find $\det(dN_p)$ and not necessarily find the explicit formula for each $a_{ij}$, from here we can easily calculate $\det(dN_p)$ by multiplying the determinants of the two matrices, which gives

$$K = \det(\begin{bmatrix} -e & -f \\ -f & -g \end{bmatrix}) \cdot \det(\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1}) = \frac{eg - f^2}{EG - F^2}.$$

\[\square\]

Before we prove the final lemma in this section, we must first introduce the notion of the Christoffel symbols. First, we note that, by the definition of $N$, $\{\mathbf{x}_u, \mathbf{x}_v, N\}$ constitutes a basis of $\mathbb{R}^3$. Therefore, we can express any vector in $\mathbb{R}^3$ as a linear combination of the vectors $\mathbf{x}_u, \mathbf{x}_v, N$ at any point $p$.

**Definition 2.13.** The Christoffel symbols are the coefficients $\Gamma_{jk}^i$ that satisfy the following equations

$$\mathbf{x}_{uu} = \Gamma_{11}^1 \mathbf{x}_u + \Gamma_{12}^2 \mathbf{x}_v + e \mathbf{N},$$

$$\mathbf{x}_{uv} = \Gamma_{12}^1 \mathbf{x}_u + \Gamma_{22}^2 \mathbf{x}_v + f \mathbf{N},$$

$$\mathbf{x}_{vv} = \Gamma_{22}^1 \mathbf{x}_u + \Gamma_{22}^2 \mathbf{x}_v + g \mathbf{N}.$$

The fact that the coefficients for $N$ in each equation are the coefficients of the second fundamental form follow directly from the coefficients’ definitions as inner products of $N$ and $\mathbf{x}_{uu}, \mathbf{x}_{uv}, \mathbf{x}_{vv}$, and the fact that $\langle N, \mathbf{x}_u \rangle = \langle N, \mathbf{x}_v \rangle = 0$.  

\[\square\]
Before proving the final lemma of this section, we must first prove one proposition.

**Proposition 2.14.** Given an isothermal parametrization \( \mathbf{x} \) (that is, a parametrization where \( E = G \) and \( F = 0 \)) of a regular surface \( S \), the following equations hold.

\[
\begin{align*}
\Gamma_{11}^1 &= \frac{E_u}{2E} \\
\Gamma_{11}^2 &= -\frac{E_v}{2G} \\
\Gamma_{12}^1 &= \frac{E_v}{2E} \\
\Gamma_{12}^2 &= \frac{G_u}{2G} \\
\Gamma_{22}^1 &= -\frac{G_u}{2E} \\
\Gamma_{22}^2 &= \frac{G_u}{2G}
\end{align*}
\]

**Proof.** The proof for each equation involves the same process, so it suffices to prove the first equation. If we take the inner product \( \langle x_{uu}, x_u \rangle \) we can substitute our equations from Definition 2.13 and calculate \( \langle x_{uu}, x_u \rangle = \langle \Gamma_{11}^1 x_u + \Gamma_{11}^2 x_v + eN, x_u \rangle = \Gamma_{11}^1 E + \Gamma_{11}^2 F + 0 \). Since we are given that \( x \) is an isothermal parametrization, we know \( F = 0 \), so we get \( \langle x_{uu}, x_u \rangle = \Gamma_{11}^1 E \). However, if we consider \( E_u = \langle x_{uu}, x_u \rangle + \langle x_u, x_{uu} \rangle = 2\langle x_{uu}, x_u \rangle \). Therefore, we can combine these two results to get \( \frac{1}{2} E_u = \langle x_{uu}, x_u \rangle = \Gamma_{11}^1 E \), which then gives \( \Gamma_{11}^1 = \frac{E_u}{2E} \). \( \square \)

The following lemma will be useful in the proof of the Local Gauss-Bonnet Theorem.

**Lemma 2.15.** For an isothermal parametrization \( \mathbf{x} \) of a regular surface \( S \), the Gaussian curvature can be expressed as

\[
K = -\frac{1}{2\sqrt{EG}} \left( \left[ \frac{E_v}{\sqrt{EG}} \right]_v + \left[ \frac{G_u}{\sqrt{EG}} \right]_u \right).
\]

**Proof.** First, we find an explicit expression for \( K \) in terms of the Christoffel symbols by noting that \( \langle x_{uu}, x_u \rangle = \langle x_{uv}, x_v \rangle = \langle x_{uv}, x_v \rangle = 2\langle x_{uu}, x_u \rangle \). When we substitute the equations from Definition 2.13, this becomes,

\[
\langle \Gamma_{11}^1 x_u + \Gamma_{11}^2 x_v + eN, x_u \rangle = \langle \Gamma_{12}^1 x_u + \Gamma_{12}^2 x_v + fN, x_u \rangle.
\]

Using the product rule, we rewrite this as

\[
\Gamma_{11}^1 x_{uv} + \Gamma_{11}^2 x_{vu} + eN_v + (\Gamma_{11}^1)_v x_u + (\Gamma_{11}^2)_v x_v + e_v N = \Gamma_{12}^1 x_{uv} + \Gamma_{12}^2 x_{vu} + fN_u + (\Gamma_{12}^1)_u x_u + (\Gamma_{12}^2)_u x_v + f_u N_v + f_v N_u.
\]

Since \( \{x_u, x_v, N\} \) is a basis of \( \mathbb{R}^3 \), we can substitute the equations from Definition 2.13 and look at the coefficients of one of the basis vectors. Choosing \( x_v \), and using our expression for \( N_v \) from the proof of Lemma 2.12, we get,

\[
\Gamma_{11}^1 (\Gamma_{12}^1) + \Gamma_{11}^2 (\Gamma_{22}^1) + (\Gamma_{11}^1)_v + e a_{22} = \Gamma_{12}^1 (\Gamma_{11}^2) + \Gamma_{12}^2 (\Gamma_{22}^2) + (\Gamma_{12}^1)_u + f a_{21}.
\]
Looking back at the proof for Lemma 2.12, we can easily calculate $a_{22}$. Since $a_{22}$ is the bottom right element of the matrix $dN_p$, we just need to use the explicit formula. Substituting $F = 0$ since we’re given $x$ is isothermal, we get

\begin{equation}
(2.19) \quad dN_p = -\begin{bmatrix} e & f \\ f & g \end{bmatrix} \cdot \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix}^{-1} = -\frac{1}{EG} \begin{bmatrix} e & f \\ f & g \end{bmatrix} \cdot \begin{bmatrix} G & 0 \\ 0 & E \end{bmatrix}.
\end{equation}

The bottom right element is calculated to be

\begin{equation}
(2.20) \quad a_{22} = -\frac{1}{EG} (f(0) + g(E)) = \frac{-g}{G}.
\end{equation}

Similarly, we find $a_{21} = -\frac{1}{EG} (fG) = \frac{-f}{E}$. Substituting (2.20) and the equations from Proposition 2.14 into (2.18) gives us

\begin{equation}
(2.21) \quad \frac{E_u G_u}{4EG} - \frac{E_v G_v}{4G^2} - \left(\frac{E_u}{2G}\right)_v - \frac{eg}{G} = -\frac{E_v^2}{4EG} + \frac{G_u^2}{4G^2} + \frac{G_u}{2G}_u - \frac{f^2}{E}.
\end{equation}

Knowing from Lemma 2.12 that $K = \frac{eg}{EG}$, we can rearrange the terms in (2.21) and divide by $E$ to get

\begin{equation}
(2.22) \quad \left(\frac{E_u G_u}{4EG} - \frac{E_v G_v}{4G^2} - \left(\frac{E_u}{2G}\right)_v + \frac{E_v^2}{4EG} - \frac{G_u^2}{4G^2} - \left(\frac{G_u}{2G}\right)_u\right) \frac{1}{E} = \frac{eg - f^2}{EG} = K.
\end{equation}

Here, we have an explicit formula for $K$ in terms of the coefficients of the first fundamental form, which can be algebraically cleaned up into

\begin{align*}
K &= \frac{E_u G_u + E_v^2}{4E^2G} - \frac{E_v G_v + G_u^2}{4EG^2} + \frac{2E_v G_v - 2E_{vv} G}{4EG^2} + \frac{2G_u^2 - 2GG_{uu}}{4EG^2} \\
&= -\frac{1}{2\sqrt{EG}} \left(\frac{-E_u GG_u - E_v^2 G + EE_u G_v + E_G^2 - 2EE_v G_v - 2E_G^2 + 2EE_{vv} G + 2EGG_{uu}}{2EG\sqrt{EG}}\right) \\
&= -\frac{1}{2\sqrt{EG}} \left(\frac{-E_G^2 G - EE_u G_v + 2EE_{vv} G}{2EG\sqrt{EG}} + \frac{-E_u GG_u - EG^2_u + 2EGG_{uu}}{2EG\sqrt{EG}}\right) \\
&= -\frac{1}{2\sqrt{EG}} \left(\frac{E_{vv} \sqrt{EG} - \frac{1}{2\sqrt{EG}} (E_G^2 G + EE_u G_v)}{EG} + \frac{G_{uu} \sqrt{EG} - \frac{1}{2\sqrt{EG}} (E_u GG_u + EG^2_u)}{EG}\right) \\
&= -\frac{1}{2\sqrt{EG}} \left(\frac{(E_v)_v \sqrt{EG} - (\sqrt{EG})_v E_v}{(\sqrt{EG})^2} + \frac{(G_u)_u \sqrt{EG} - (\sqrt{EG})_u G_u}{(\sqrt{EG})^2}\right) \\
&= -\frac{1}{2\sqrt{EG}} \left(\frac{E_v}{\sqrt{EG}}}_v + \frac{G_u}{\sqrt{EG}}_u\right) \\
&= -\frac{1}{2\sqrt{EG}} \left(\begin{bmatrix} E_v \\ G_u \end{bmatrix}_v + \begin{bmatrix} G_v \\ E_u \end{bmatrix}_u\right).
\end{align*}

\[\square\]

3. Geodesic Curvature

The goal of this section is to introduce the notion of geodesic curvature and prove an important lemma for the Local Gauss-Bonnet Theorem. We will start by defining the covariant derivative.

**Definition 3.1.** We define a vector field $w$ along a parametrized curve $\alpha : I \to S$ as an expression $w(t)$ that assigns to each $t \in I$ a vector in the tangent space $T_{\alpha(t)}(S)$. 
**Definition 3.2.** For a differentiable vector field $w$ restricted to a parametrized curve $\alpha(I)$ on a surface $S$ with $\alpha(0) = p$ and $\alpha'(0) = y \in T_p(S)$, the covariant derivative $\left[\frac{Dw}{dt}\right]$ at $p$ in the direction $y$ is defined to be the projection of the vector $w'(0)$ onto the plane $T_p(S)$.

**Definition 3.3.** The algebraic value of the covariant derivative of a differentiable field of unit vectors $w$ at $p$ in the direction $y$ is defined to be the function $\lambda(t) = [Dw/dt]$ that satisfies

\[
\frac{Dw}{dt} = \lambda(t)(N \times w(t)).
\]

To show that such a function $\lambda(t)$ exists, it suffices to show that the covariant derivative $\left[\frac{Dw}{dt}\right]$ is orthogonal to both $N$ and $w(t)$. Since the covariant derivative is the projection of $w'$ onto the plane $T_p(S)$, it lies in $T_p(S)$ and is therefore orthogonal to $N$. Additionally, since $\left[\frac{Dw}{dt}\right]$ is the projection of $w'$ onto $T_p(S)$, $w'$ must equal some linear combination of $\left[\frac{Dw}{dt}\right]$ and $N$, which implies $\left[\frac{Dw}{dt}\right]$ is a linear combination of $w'$ and $N$, both of which are orthogonal to $w(t)$, so $\left[\frac{Dw}{dt}\right]$ is orthogonal to $w(t)$ as well.

Now, we are ready to define geodesic curvature.

**Definition 3.4.** For an oriented regular curve $C$ on an open surface $S$ parametrized by arc length $\alpha(s)$, the geodesic curvature of $C$ at a point $p$ is defined to be the algebraic value $k_g(s) = [D\alpha'(s)/ds]$ of the covariant derivative of the vector field $\alpha'$ at the point $p$.

Now, we are ready to prove two lemmas, the latter of which will be incredibly helpful for proving the Local Gauss-Bonnet Theorem.

**Lemma 3.5.** For two differentiable unit vector fields $w$ and $v$ along a curve $\alpha: I \to S$, the following holds

\[
\left[\frac{Dw}{dt}\right] - \left[\frac{Dv}{dt}\right] = \frac{d\phi}{dt}
\]

where $\phi(t)$ is the angle from $v(t)$ to $w(t)$.

**Proof.** First, we let $\nabla = N \times v$ and $\overline{w} = N \times w$. Then, since $\phi$ is the angle from $v$ to $w$, we know

\[
(3.6) \quad w = (\cos \phi)v + (\sin \phi)\nabla
\]

and if we take the cross product of this with $N$, we get

\[
(3.7) \quad \overline{w} = N \times w = (\cos \phi)\overline{v} - (\sin \phi)v.
\]

Then, differentiating (3.6) with respect to $t$, we get

\[
(3.8) \quad w' = (-\sin \phi)(\phi')v + (\cos \phi)v' + (\cos \phi)(\phi')\nabla + (\sin \phi)\nabla'.
\]

Then, combining (3.7) and (3.8), we calculate $\langle w', \overline{w} \rangle$ to be

\[
(3.9) \quad \langle w', \overline{w} \rangle = \sin^2 \phi(\phi') + \cos^2 \phi(v', \nabla) + \cos^2 \phi(\phi') - \sin^2 \phi(v, \nabla).
\]

Since we know $\langle v, \nabla \rangle = 0$, we know $\langle v', \nabla \rangle + \langle v, \nabla' \rangle = 0$. Using this fact and the pythagorean identity simplifies (3.7) to

\[
(3.10) \quad \langle w', \overline{w} \rangle = \phi'(t) + \langle v', \nabla \rangle.
\]
Lastly, since \( \frac{Dw}{dt} \) represents the scalar function such that \( \frac{Dw}{dt} (N \times w) = \frac{Dw}{dt} \), we know

\[
\langle w', \overline{w} \rangle = \left[ \frac{Dw}{dt} \right] \langle \overline{w}, w \rangle = \left[ \frac{Dw}{dt} \right].
\]

Plugging (3.11) into (3.10) then gives us the result of

\[
\frac{Dw}{dt} - \frac{Dv}{dt} = \frac{d\phi}{dt}.
\]

□

Finally, we can move on to the last lemma for this section which is the basis for the proof of the Local Gauss-Bonnet Theorem.

**Lemma 3.13.** For a differentiable field of unit vectors \( w \) restricted to a curve \( x(u(t), v(t)) \) and an isothermal parametrization \( x \),

\[
\left[ \frac{Dw}{dt} \right] = \frac{1}{2\sqrt{EG}} \left( \frac{G_u dv}{dt} - \frac{E_v dv}{dt} \right) + \frac{d\phi}{dt},
\]

where \( \phi \) represents the angle from \( x_u \) to \( w(t) \).

**Proof.** First, we note that the unit tangent vectors are \( e_1 = \frac{x_u}{\sqrt{E}} \) and \( e_2 = \frac{x_v}{\sqrt{G}} \). Applying Lemma 3.5, we know

\[
\left[ \frac{De_1}{dt} \right] = \langle \frac{de_1}{dt}, N \times e_1 \rangle = \langle \frac{de_1}{dt}, e_2 \rangle = \langle (e_1)_u, e_2 \rangle + \langle (e_1)_v, e_2 \rangle.
\]

Since \( F = 0 \), we have \( \langle x_u u, x_v v \rangle = -\frac{1}{2} E_v \). Thus,

\[
\langle (e_1)_u, e_2 \rangle = \langle \frac{x_u}{\sqrt{E}} u, \frac{x_v}{\sqrt{G}} e_2 \rangle.
\]

We can calculate the partial derivative \( \langle \frac{x_u}{\sqrt{E}} \rangle_u \) here, and knowing that \( F = 0 \) tells us that

\[
\langle \frac{x_u}{\sqrt{E}} \rangle_u = -\frac{E_v}{2\sqrt{EG}}.
\]

And by the same reasoning we can find

\[
\langle (e_1)_v, e_2 \rangle = \frac{G_u}{2\sqrt{EG}}.
\]

Putting (3.18) and (3.17) into (3.15) gives us

\[
\left[ \frac{De_1}{dt} \right] = \frac{1}{2\sqrt{EG}} (G_u v' - E_v u'),
\]

which, plugged back into (3.14) gives us our original goal of

\[
\frac{Dw}{dt} = \frac{1}{2\sqrt{EG}} \left( \frac{G_u dv}{dt} - \frac{E_v dv}{dt} \right) + \frac{d\phi}{dt}.
\]

□
Importantly, if we choose \( w = \alpha' \), we can use this lemma to calculate geodesic curvature.

4. The Local Gauss-Bonnet Theorem

Before the Local Gauss-Bonnet Theorem, we must state the Theorem of Turning Tangents, which will not be proved in this paper. A proof can be found in do Carmo’s *Differential Geometry of Curves and Surfaces* (pp. 396–397).

**Theorem 4.1.** Let \( \alpha \) be a positively oriented, simple, closed, piecewise regular plane curve on an interval \( I = [0, l] \) such that \( \alpha \) does not intersect itself other than at \( \alpha(0) = \alpha(l) \), and there exists a finite subdivision of \( I \) of points \( t_0, t_1, \ldots, t_k+1 \) such that \( \alpha \) is differentiable and regular on each interval \([t_i, t_{i+1}]\) on a surface \( x(U) \subset S \) where \( U \) is homeomorphic to an open disk. If \( \phi_i(t) \) is a function defined on the interval \([t_i, t_{i+1}]\) that measures the signed angle from \( x_u \) to \( \alpha'(t) \), then

\[
\sum_{i=0}^{k} (\phi_i(t_{i+1}) - \phi_i(t_i)) + \sum_{i=0}^{k} \theta_i = 2\pi
\]

where \( \theta_i \) are the external angles at each vertex.

Now, we can state and prove the Local Gauss-Bonnet Theorem.

**Theorem 4.2.** Let \( x: U \to S \) be an isothermal parametrization of an orientable surface \( S \), where \( U \) is homeomorphic to an open disk, and let \( R \subset x(U) \) be a region also homeomorphic to an open disk, with the boundary \( \partial R = \alpha(I) \) where \( \alpha: I \to S \) is a positively oriented, simple, closed curve parametrized by arc length with vertices \( \alpha(s_0), \alpha(s_1), \ldots, \alpha(s_k) \) and corresponding external angles \( \theta_0, \theta_1, \ldots, \theta_k \). Then, the following equation holds

\[
\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_y(s) ds + \iint_R K d\sigma + \sum_{i=0}^{k} \theta_i = 2\pi
\]

**Proof.** We start from Lemma 3.13, which tells us

\[
k_y(s) = \frac{1}{2\sqrt{EG}} \left( G_u \frac{dv}{ds} - E_v \frac{du}{ds} \right) + \frac{d\psi_i}{ds}.
\]

If we integrate the geodesic curvature along the boundary \( \alpha(I) \), we get

\[
\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_y(s) ds = \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{1}{2\sqrt{EG}} \left( G_u \frac{dv}{ds} - E_v \frac{du}{ds} \right) ds + \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \left( \frac{d\psi_i}{ds} \right) ds.
\]

Since \( \alpha(I) \) is the boundary of a simple region, by Green’s Theorem we know

\[
\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \left( \frac{G_u}{2\sqrt{EG}} \frac{dv}{ds} - \frac{E_v}{2\sqrt{EG}} \frac{du}{ds} \right) ds = \iint_{x^{-1}(R)} \frac{1}{2} \left( \left( \frac{E_v}{\sqrt{EG}} \right)_v + \left( \frac{G_u}{\sqrt{EG}} \right)_u \right) du dv
\]

And then we can algebraically rework this into

\[
\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \left( \frac{G_u}{2\sqrt{EG}} \frac{dv}{ds} - \frac{E_v}{2\sqrt{EG}} \frac{du}{ds} \right) ds = \iint_{x^{-1}(R)} \frac{1}{2\sqrt{EG}} \left( \left( \frac{E_v}{\sqrt{EG}} \right)_v + \left( \frac{G_u}{\sqrt{EG}} \right)_u \right) \sqrt{EG} du dv
\]
which, using Lemma 2.15, becomes

\[ \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \left( \frac{G_u}{2\sqrt{EG}} \frac{dv}{ds} - \frac{E_v}{2\sqrt{EG}} \frac{du}{ds} \right) ds = - \int_{A} K(\sqrt{EG})dudv. \]

We also note that, since \( x \) is an isothermal parametrization,

\[ |x_u \times x_v| = |x_u| \cdot |x_v| = \sqrt{I_p(x_u) \cdot I_p(x_v)} = \sqrt{EG}, \]

so we can rewrite the integral as

\[ \int \int_{x^{-1}(R)} K(\sqrt{EG})dudv = \int_{R} Kd\sigma. \]

Plugging this into (4.4) gives us

\[ \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g(s)ds = - \int_{R} Kd\sigma + \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \left( \frac{d\phi_i}{ds} \right) ds. \]

Evaluating the integral on the right, we get

\[ \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \left( \frac{d\phi_i}{ds} \right) ds = \sum_{i=0}^{k} (\phi_i(s_{i+1}) - \phi_i(s_i)), \]

which, by the Theorem of Turning Tangents becomes

\[ \sum_{i=0}^{k} (\phi_i(s_{i+1}) - \phi_i(s_i)) = 2\pi - \sum_{i=0}^{k} \theta_i, \]

since \( \alpha \) is positively oriented. Plugging (4.11) into (4.9) then gives us

\[ \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g(s)ds = - \int_{R} Kd\sigma + 2\pi - \sum_{i=0}^{k} \theta_i. \]

Moving everything over to one side, we arrive at the Local Gauss-Bonnet Theorem

\[ \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g(s)ds + \int_{R} Kd\sigma + \sum_{i=0}^{k} \theta_i = 2\pi. \]

\[ \square \]

5. The Gauss-Bonnet Theorem

First, we must briefly define the Euler characteristic \( \chi(R) \) for a region \( R \).

**Definition 5.1.** The Euler characteristic \( \chi(R) \) of a region \( R \) is equal to the constant value of \( F - E + V \) for any triangulation of \( R \), where \( F \) is the number of faces, \( E \) is the number of edges, and \( V \) is the number of vertices.

The fact that the Euler characteristic of a region \( R \) is constant for the region and not dependent on the specific triangulation will be assumed without proof in this paper.

**Example 5.2.** The Euler characteristic of a sphere is 2. We can see this by imagining a basic triangulation of two triangles covering the entire sphere, with edges and vertices all lying on the same equator. Then, there are 2 faces, 3 edges, and 3 vertices, so the Euler characteristic is \( \chi(S^2) = 2 - 3 + 3 = 2. \)
Example 5.3. The Euler characteristic of a disk is 1.

Example 5.4. The Euler characteristic of a torus is 0.

Finally, we are ready to state and prove the Global Gauss-Bonnet Theorem.

Theorem 5.5. Let $R \subset S$ be a regular region of an oriented surface $S$. Let $\partial R$ comprise the closed, simple, piecewise regular and positively oriented curves $C_1, C_2, \ldots, C_n$. If $\theta_1, \theta_2, \ldots, \theta_p$ are the external angles of each $C_i$, then the following equation holds

$$\sum_{i=1}^{n} \int_{C_i} k_g(s) \, ds + \int_{R} K \, d\sigma + \sum_{i=1}^{p} \theta_i = 2\pi \chi(R).$$

Proof. To prove the Global Gauss-Bonnet Theorem, we need to consider a triangulation $T$ of $R$ with each triangle having a positively oriented boundary and being contained in a single coordinate neighborhood with an isothermal parametrization, apply the Local Gauss-Bonnet Theorem to each triangle, and add up the results.

We will assume without proof that such a triangulation exists. Doing this gives us (5.6)

$$\sum_{j=1}^{F} \left( \sum_{i=0}^{k} \int_{s_{i}^{s_{i+1}}} k_g(s) \, ds \right) + \sum_{j=1}^{F} \int_{T_j} K \, d\sigma + \sum_{j=1}^{F} \sum_{i=1}^{3} \theta_{ji} = 2F \pi$$

Since each triangle in $T$ is positively oriented, every triangle edge on the interior of $R$ will appear in the sum $\sum_{j=1}^{F} \left( \sum_{i=0}^{k} \int_{s_{i}^{s_{i+1}}} k_g(s) \, ds \right)$ two times, first going in one direction and then going in the other. Since $\int_{s_{i}^{s_{i+1}}} k_g(s) \, ds + \int_{s_{i}}^{s_{i+1}} k_g(s) \, ds = 0$, all of the interior triangle edges introduced by $T$ cancel out and we are left with (5.7)

$$\sum_{j=1}^{F} \left( \sum_{i=0}^{k} \int_{s_{i}^{s_{i+1}}} k_g(s) \, ds \right) = \sum_{i=1}^{n} \int_{C_i} k_g(s) \, ds.$$

Plugging (5.7) into (5.6) gives us (5.8)

$$\sum_{i=1}^{n} \int_{C_i} k_g(s) \, ds + \sum_{j=1}^{F} \int_{T_j} K \, d\sigma + \sum_{j=1}^{F} \sum_{i=1}^{3} \theta_{ji} = 2F \pi.$$

Since the next sum is just adding the same integral over the entire triangulation $T$, we know (5.9)

$$\sum_{j=1}^{F} \int_{T_j} K \, d\sigma = \int_{R} K \, d\sigma,$$

and substituting (5.9) into (5.8) gives us (5.10)

$$\sum_{i=1}^{n} \int_{C_i} k_g(s) \, ds + \int_{R} K \, d\sigma + \sum_{j=1}^{F} \sum_{i=1}^{3} \theta_{ji} = 2F \pi.$$

Next, we introduce the corresponding internal angles $\phi_i = \pi - \theta_i$. Then, we get (5.11)

$$\sum_{j=1}^{F} \sum_{i=1}^{3} \theta_{ji} = \sum_{j=1}^{F} \sum_{i=1}^{3} (\pi - \phi_{ji}) = \sum_{j=1}^{F} \sum_{i=1}^{3} \pi - \sum_{j=1}^{F} \sum_{i=1}^{3} \phi_{ji} = 3F \pi - \sum_{j=1}^{F} \sum_{i=1}^{3} \phi_{ji}.$$

Next, we need to distinguish between interior and exterior edges and vertices. Since each interior edge is an edge for two different faces, if we use the subscript $e$ to
represent exterior and the subscript $i$ to represent interior, we know $3F = 2E_i + E_e$. Substituting this into (5.11) gives us

\[(5.12) \sum_{j=1}^{F} \sum_{i=1}^{3} \theta_{ji} = 2E_i \pi + E_e \pi - \sum_{j=1}^{F} \sum_{i=1}^{3} \phi_{ji}.\]

Since $\sum \sum \phi_{ij}$ is the sum of interior angles for each vertex in $T$, we need to further split $V$ into $V = V_i + V_{ec} + V_{et}$, where $V_i$ are interior vertices, $V_{ec}$ are exterior vertices introduced by the curves $C_i$, and $V_{et}$ are the exterior vertices introduced by the triangulation. Then, we can split the sum into

\[(5.13) \sum_{j=1}^{F} \sum_{i=1}^{3} \phi_{ji} = \sum_{V_i} \phi_{ji} + \sum_{V_{ec}} \phi_{ji} + \sum_{V_{et}} \phi_{ji}.\]

Since each interior vertex will have interior angles that add up to $2\pi$, we know

\[(5.14) \sum_{V_i} \phi_{ij} = 2V_i \pi.\]

Additionally, interior angles of exterior vertices lying on the curves $C_i$ which are not vertices of the curves themselves should add up to $\pi$, so we similarly conclude

\[(5.15) \sum_{V_{et}} \phi_{ij} = V_{et} \pi.\]

Plugging (5.15) and (5.14) into (5.13) as well as replacing $\phi_{ji}$ with $\pi - \theta_i$ for the external vertices of the curves $C_i$, we get

\[(5.16) \sum_{j=1}^{F} \sum_{i=1}^{3} \phi_{ji} = 2V_i \pi + V_{et} \pi + V_{ec} \pi - \sum_{i=1}^{p} \theta_i.\]

Plugging (5.16) into (5.12), we get

\[(5.17) \sum_{j=1}^{F} \sum_{i=1}^{k} \theta_i = 2E_i \pi + E_e \pi - 2V_i \pi - V_{et} \pi - V_{ec} \pi + \sum_{i=1}^{p} \theta_i.\]

Since $E_e = V_e$, we can add $E_e \pi$ and subtract $V_e \pi$ from the right side of the equation to get

\[(5.18) \sum_{j=1}^{F} \sum_{i=1}^{k} \theta_i = 2E_i \pi + 2E_e \pi - 2V_i \pi - 2V_{et} \pi - 2V_{ec} \pi + \sum_{i=1}^{p} \theta_i.\]

Cleaning this up, we get

\[(5.19) \sum_{j=1}^{F} \sum_{i=1}^{k} \theta_i = 2E \pi - 2V \pi + \sum_{i=1}^{p} \theta_i.\]

Substituting this back into (5.10), we get

\[(5.20) \sum_{i=1}^{n} \int_{C_i} k_g(s) ds + \int_R K d\sigma + 2E \pi - 2V \pi + \sum_{i=1}^{p} \theta_i = 2F \pi,\]

which we can rearrange to get

\[(5.21) \sum_{i=1}^{n} \int_{C_i} k_g(s) ds + \int_R K d\sigma + \sum_{i=1}^{p} \theta_i = 2\pi(F - E + V) = 2\pi \chi(R).\]
Hidden inside the Gauss-Bonnet Theorem are many basic facts which can help contextualize the end result.

**Example 5.22.** If we apply the Gauss-Bonnet Theorem to a triangle on a flat plane, the first two integrals will equal 0, because straight lines on a plane are geodesics and because the Gaussian curvature of a plane is always \( K = 0 \), we get

\[
\sum_{i=1}^{3} \theta_i = 2\pi \chi(R) = 2\pi. 
\]

Replacing the exterior angles \( \theta_i \) with expressions in terms of interior angles \( \alpha, \beta, \gamma \), we get

\[
3\pi - \alpha - \beta - \gamma = 2\pi. 
\]

or

\[
\alpha + \beta + \gamma = \pi, 
\]

which is the basic fact that the sum of the interior angles of a triangle is 180°. Of course, since the Gauss-Bonnet Theorem is dependent on more advanced results, this does not function as a proof. We could also do this for any n-gon and find that the sum of the interior angles is \((n - 2)\pi\).

**Example 5.23.** Additionally, if we consider geodesic triangles on a sphere of radius 1, meaning triangles whose edges are geodesics, which, on a sphere, are segments of great circles, we can find the following relationship

\[
\int_R Kd\sigma + \sum_{i=1}^{p} \theta_i = 2\pi \chi(R). 
\]

Knowing \( K = 1 \) and \( \chi(R) = 1 \), and replacing exterior angles with corresponding interior angles, we get

\[
\int_R d\sigma + 3\pi - \alpha - \beta - \gamma = 2\pi 
\]

or

\[
\alpha + \beta + \gamma - \pi = A 
\]

where \( A \) is the area of the triangle.

**Example 5.24.** In the reverse direction, if we didn’t know the Euler characteristic of a cylinder, we could remember it by imagining two parallel circles bounding a section of the cylinder. If the axis of the cylinder is normal to the planes in which the circles lie, then the circles will have zero geodesic curvature and exterior angles of 0, so for the sectioned off region \( R \) we get

\[
\int_R Kd\sigma = 2\pi \chi(R). 
\]

Since \( K = 0 \) everywhere on the cylinder, we quickly remember that \( \chi(R) = 0 \).

**Corollary 5.25.** Another result emerges if we apply the Global Gauss-Bonnet Theorem to the entirety of a surface without any parametrized curve serving as a border. Then, we get the much simpler result that

\[
\int_S Kd\sigma = 2\pi \chi(S). 
\]
Example 5.26. Applying Corollary 5.25 to any torus, we get the result that
\[ \iint_S K d\sigma = 0. \]

Example 5.27. Additionally, for a sphere of radius \( r \), the Gaussian curvature at any point can be found to be \( K = \frac{1}{r^2} \). Applying Corollary 5.25, we get
\[ \iint_{S^2} \frac{1}{r^2} d\sigma = 2\pi \chi(S^2). \]
Since we know \( \chi(S^2) = 2 \), and because \( r \) is constant, we can conclude
\[ \iint_{S^2} d\sigma = 4\pi r^2, \]
or that the surface area of a sphere is \( 4\pi r^2 \).

6. The Hairy Ball Theorem

As one final interesting result of the Gauss-Bonnet Theorem, we can state and prove the Hairy Ball Theorem, which intuitively tells us that it is impossible to comb a hairy ball without parting the hair or creating a tuft or whorl. The proof requires a few definitions which, for brevity, we will assume without proof are well defined.

Definition 6.1. A singular point \( p \) of a vector field \( v \) on a surface \( S \) is a point such that \( v(p) = 0 \).

Definition 6.2. The index of a vector field \( v \) at a point \( p \) on a surface \( x(U) \) is the integer \( I \) such that for a simple closed piecewise parametrized curve \( \alpha([0, l]) \) enclosing a simple region \( R \) containing \( p \) and a differentiable function \( \phi(t) \) restricted to the curve \( \alpha \) that measures the angle from \( x_u \) to \( v(t) \), \( I \) satisfies that
\[ 2\pi I = \phi(l) - \phi(0) = \int_0^l \phi'(t) dt. \]

The following statement will also be stated without proof.

Proposition 6.3. For any point \( p \) that is not a singular point, the index \( I \) is 0.

Now, we can prove the Hairy Ball Theorem for the 2-sphere.

Theorem 6.4. A continuous vector field of tangent vectors \( v \) on a sphere \( S^2 \) must have at least one singular point.

Proof. We start by taking some triangulation of the surface of the sphere such that each triangle is positively oriented and contains at most one singular point inside it, with no singular points on any edge. Similar to the proof for the Global Gauss-Bonnet Theorem, we apply the Local Gauss-Bonnet Theorem to each triangle and add up the results for the entire sphere to get
\[ \sum \sum k_g(s) ds + \sum \iint_T K d\sigma + \sum \sum \theta_i = 2\pi F. \]  

Once again, all of the integrals of geodesic curvature cancel each other out, since each edge is counted twice in opposite directions, and we can add the surface integrals of Gaussian curvature together to get
\[ \iint_{S^2} K d\sigma + \sum \sum \theta_i = 2\pi F. \]
From Theorem 4.1 we know in each triangle, $\sum \theta_i = 2\pi \ - \ (\phi(l) \ - \ \phi(0)) = 2\pi \ - \ 2\pi I$ by Definition 6.2. Plugging this into (6.6), we get

\[ \int \int_{S^2} K d\sigma + \sum_{i=1}^{F} (2\pi - 2\pi I) = 2\pi F, \]

which we can simplify to

\[ \int \int_{S^2} K d\sigma + 2\pi F - 2\pi \sum I = 2\pi F, \]

and then to

\[ \int \int_{S^2} K d\sigma - 2\pi \sum I = 0, \]

where we use $\sum I$ to represent the total sum of the indices of points across the entire sphere. However, we can also use Corollary 5.25 to conclude that

\[ 2\pi (\chi(S^2) - \sum I) = 0. \]

Therefore, the sum of indices across the entire sphere must equal 2. This means the vector field must have at least one singular point. □

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