A GLIMPSE INTO SCHRAMM-LOEWNER EVOLUTION

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ABSTRACT. The Schramm-Loewner evolution (SLE) is a family of random fractal curves living in a simply connected domain $D \subset \mathbb{C}$. In virtue of the conformal Markov property it features, SLE is believed to be the scaling limits of various discrete models arising in statistical physics. In this paper we give a first glance on the theory of SLE starting from basic definitions. We also talk about properties of SLE curves including the topology and dimension. Finally, we introduce the natural paramterization and give the full version of conformal invariance of SLE.

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1. INTRODUCTION

The Schramm-Loewner evolution (SLE) is a one-parameter family of random fractal curves living in some simply connected domain $D \subset \mathbb{C}$. The distribution of SLE curves features the conformal Markov property in any fixed domain D, and enjoys the conformal invariance between different domains in $\mathbb{C}$. Further, this family of curves turns out to be best suitable for describing scaling limits of various two-dimensional lattice models in statistic physics.

Since SLE was introduced in 1998 by Oded Schramm, it has been studied extensively in the past 25 years. On the one hand, properties of SLE curves have been established by tools in various mathematical branches, including stochastic calculus, complex analysis and conformal field theory, and etc. On the other hand, convergence to SLE has been rigorously proved for several well-studied

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two dimensional discrete models, while some others are strongly believed to be so. Hence, SLE curves are not only of independent interests, but also serve as a bridge between the discrete world and the continuum.

In this paper, we give a brief overview of the story for SLE. We start by giving some necessary preliminaries in complex analysis, and then state the precise definition of SLE_κ curves with $\kappa \in [0, \infty)$. Next we derive some pathwise properties of SLE curves such as topology and dimension, where we take Bessel process and stochastic calculus as the main tool. Finally, we introduce the natural parameterization of SLE_κ curves and give the full statement for the conformal invariance of SLE_κ.

1.1. **Some motivations.** One of the most fascinating topics in modern probability is the universality, which states that the existence of an universal limiting distribution is somewhat independent of microscopic components. The starting example of universality is given by the Central Limit Theorem (CLT): For any i.i.d. random variables $X_1, \ldots, X_n$ with $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_1^2 = 1$, it always holds that as n goes to infinity,

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}} \Rightarrow_{\mathrm{d}} \mathcal{N}(0,1)\,,$$ [1]

no matter what X_i's exactly are. A more advanced example is provided by Donsker's invariance principle, which states for the convergence of normalized random walks to the Brownian motion under mild conditions.

A challenging problem is the universality of various stochastic models on two-dimensional lattices arising form statistical physics. It is widely believed that these models, such as the percolation model, Ising model and Gaussian free field, share certain universal properties (at least at the criticality). Even further, it is conjectured by theoretical physicists that the limiting distributions of such models are conformally invariant, which is a strong sense of symmetry. Most of such problems remained open for decades, and there was even no reasonable guess for what the scaling limits should be. Finally, SLE came up onto the stage and things started to turn clearer.

SLE curves successfully capture many desired features of the scaling limits for some essential curves in the discrete model, such as interfaces and cluster boundaries. Inspired by this, mathematicians started to realize the correct way to formulate the scaling limits of these discrete models. This also set the first step on the long journey of proving universality. Up to now, the connection to SLE has been established in several models rigorously. In order to gain more concrete intuition, we present several discrete models and related questions here, which turn out to be deeply connected with SLE.

- *Loop-erased random walk (LERW).* Consider the simple random walk on $\mathbb{Z}^2$ starting at the origin and ending at the first hitting of $\partial[-n,n]^2$. For such a path ω^n, we define its loop erasure $\mathrm{LE}(\omega^n)$ as the path obtained by erasing loops appeared in ω^n chronologically. More precisely, for each ω^n, we start from its starting point and walk along the path, but stop at the first appearance of a loop. Then we remove this loop in ω^n to get ω_1^n, and do the same thing for ω_1^n; repeat this procedure until we reach $\partial[-n,n]^2$ and end up with $\mathrm{LE}(\omega^n)$.

  Is there an analogue of Donsker's invariance principle? Namely, can we find some appropriate normalization constant β such that as $n \to \infty$, $n^{-\beta}\,\mathrm{LE}(\omega^n)$ converges to some deterministic distribution? If so, what is the limiting distribution?

[1] Here $\mathcal{N}(0,1)$ stands for the standard normal distribution.

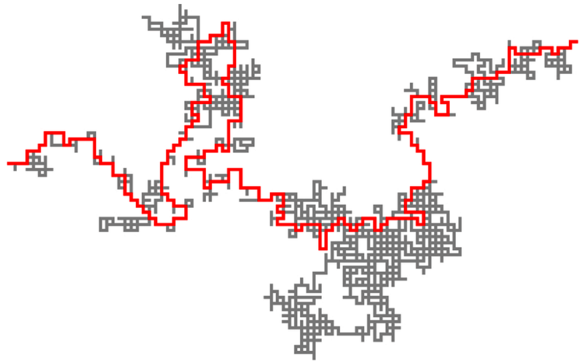

FIGURE 1. A simulation for the LERW, where the gray line is the underline simple random walk, and the red line is the loop erasure. Pictured by ALex Karrila, Kalle Kytőlǎ and Eveliina Peltola.

- *Percolation interface.* Consider a simply connected domain $D_\eta \subset \mathbb{C}$ composed by hexagons with mesh size η. Fix two points $a_\eta, b_\eta \in \partial D_\eta$ and declare all the hexagons intersecting the left (resp. right) boundary to be open (resp. closed), where left and right states the relative location with respect to a_η, b_η. The rest of hexagons in D_η are declared to be either open or closed with equal probability independently. For such a given configuration, there exists a unique interface γ_η starting from a_η and ending at b_η, so that γ_η always have open hexagons on its left, and closed hexagons on its right.

  Is there a scaling limit for γ_η as $\eta \to 0$ and (D_η, a_η, b_η) tends to some fixed triple (D, a, b) in the Carathéodory sense? If so, how can we describe it?

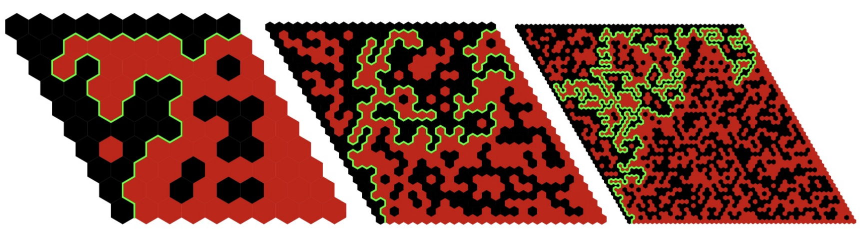

FIGURE 2. An illustration for the interface as (D_η, a_η, b_η) converges to a diamond with its two vertices, where black (red) means open (closed), and the interface is highlighted with green. Pictured by Jason Miller.

- *Uniform spanning tree (UST) and the Peano curve.* Let G_n be the subgraph induced by $\mathbb{Z}^2$ on $\Lambda_n = [-n, n]^2$ with all vertices in $\partial\Lambda_n \cap \big(\{x = n\} \cup \{y = -n\}\big)$ glued a singleton. Consider the uniform spanning tree T_n of G_n (i.e. a subtree of G_n with vertex set same as G_n, chosen with equal probability for any such tree). Define the dual tree T_n^* on the dual graph of Λ_n as follow: an edge appears in T_n^* if and only if its dual edge does not appear in T_n. Then there is a curve P_n from $a = (-n, -n)$ to $b = (n, n)$ winding between T_n and

T_n^*. We call it the Peano curve associated with T_n since it almost covers the whole region of Λ_n. Is there a scaling limit of P_n as $n \to \infty$? If so, what is it?

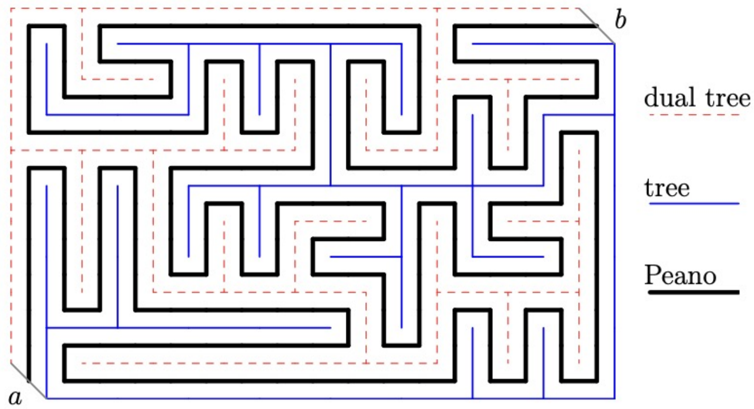

FIGURE 3. An example of the tree T_n, the dual tree T_n^* and the Peano curve P_n. Pictured by Gregory F. Lawler, Oded Schramm and Wendelin Werner.

The questions above can be answered as follow: the LERW with normalization constant 5/4 converges to radial SLE_2, the percolation interface converges to chordal SLE_6 and the Peano curve of UST converges to chordal SLE_8 under suitable normalization. The first and third results are proved by Lawler, Schramm and Werner in [6], and the second result is due to Smirnov in his celebrated work [14]. Similar questions can be raised in settings beyond hexagonal lattice and square lattice, or even on random lattice enviroment. The answers are predicated to be similar based on the belief of universality, and some partial results have been obtained by the communities see e.g. [5], [3]. However, complete result for universality still have a long way to go.

1.2. **Comments on this paper.** Since this paper is dealing with an advanced topic in modern probability theory, we assume the readers to be familiar with most of usual terminologies and results in probability. Specifically, the readers should know about basic facts of Brownian motion and stochastic calculus. For these preliminaries, we refer [11] for a wonderful book about Brownian Motion, and [4] for a concise lecture notes on stochastic calculus, or [10, Section 8] for a quick review of what we will need in this paper. The establishment of SLE also strongly depends on deep results in complex analysis, but this is not the thread of our paper, so we just state the results we shall use and refer the proofs for interested readers.

We also point out that the goal of this paper is to give the readers some heuristics about SLE, but not all down to earth. In other words, we seek a good conceptual understanding for the theory, rather than a detailed one. Although the author tried hard to make most things self-contained, there are numerous technical details which are impossible to be made fully clear given the limitation of spaces. Throughout the presentation, we may leave some gaps in proofs but we will always remark them explicitly. We want to convince the readers that all the “proofs” in this paper work conceptually, and the remaining technical gaps can be found in references. The readers should also fell free to skip technical proofs, this will not affect the overall understanding. We aim to outline the general picture for the grand topic of SLE, and that’s all.

2. Definition of the Schramm-Lowener evolution

In this section we formally define the Schramm-Loewner evolution. We will first focus on the chordal SLE curve from 0 to ∞ in the upper-half plane $\mathbb{H}$ and then talk about other generalizations.

Before plunging into math symbols and formulas, we sketch the main idea behind the construction of SLE. We are seeking a family of canonical probability measures on curves in $\overline{\mathbb{H}}$ starting from 0 and tending to infinity. On the one hand, since we expect this distribution to be the scaling limits for self-avoiding curves in aforementioned models, it is natural to guess the measures should support on simple curves, or at least curves without self-crossing. In addition, it is natural to assume these curves should not glue on a segment with its previous part. We call curves satisfying these conditions **non-crossing**. On the other hand, for a non-crossing curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ from 0 to ∞, there is a theory in complex analysis called *Loewner differential equation* saying that γ can be encoded by a family of exact conformal mappings (see Proposition 2.2 below)

$$\{g_t : \mathbb{H} \setminus \gamma([0, t]) \to \mathbb{H}\}_{t\geq 0}\,,$$

which satisfies a set of ordinary differential equations involving some real valued continuous function, called the driving function. Conversely, given the driving function, we are also able to reconstruct the curve γ by somehow reversing the procedure. Thus we may think (though not precisely true) that there is a correspondence between non-crossing curves in $\overline{\mathbb{H}}$ and continuous functions in $\mathbb{R}$, so any probability measure on $\mathrm{C}([0, \infty), \mathbb{R})$ can be pulled back to become a probability measure on such curves. A natural choice is taking the Wiener measure on $C([0, \infty), \mathbb{R})$ and generate a measure on curves in $\overline{\mathbb{H}}$, and this indeed leads to the definition of SLE.

2.1. Preliminaries in complex analysis.

We first provide some results in complex analysis which are necessary for defining SLE. We will present only the results, and refer to [10] for detailed proofs. We start by some definitions.

Definition 2.1. For a **compact hull**, we mean a compact set $D \subset \mathbb{H}$ such that $\mathbb{H} \setminus D$ is simply connected. Denote the set of compact hulls as $\mathcal{Q}$ and for $D \in \mathcal{Q}$. Let $\mathrm{diam}(D)$ and $\mathrm{rad}(D)$ stand for the diameter of D and the maximal distance to 0 of points in D, respectively.

For compact hulls we have the following slight generalization of Rieman mapping theorem:

Proposition 2.2. *For any $D \in \mathcal{Q}$, there exists a unique conformal transformation g_D which maps $\mathbb{H} \setminus D$ to $\mathbb{H}$, so that $|g_D(z) - z| \to 0$ as $|z| \to \infty$. Further, for some constant $c \geq 0$, it holds*

$$g_D(z) = z + \frac{c}{z} + O\left(\frac{1}{|z|^2}\right), \text{ as } |z| \to \infty\,. \tag{2.3}$$

Definition 2.4. For each $D \in \mathcal{Q}$, the unique transformation $g_D : \mathbb{H} \setminus D \to \mathbb{H}$ in Proposition 2.2 is called the **exact mapping** of $\mathbb{H} \setminus D$, and the constant c in (2.3) is called the **half-plane capacity** of D, denoted as $\mathrm{hcap}(D)$.

As suggested by its name, half-plane capacity is a measurement for the "size" of a set in $\mathcal{Q}$, and we do have $\mathrm{hcap}(D) = 0$ if and only if $D = \emptyset$. Further, for any $D \in \mathcal{Q}$ and any $r \in \mathbb{R}^+, x \in \mathbb{R}$, we have the following scaling property and translation invariance:

$$\mathrm{hcap}(rD) = r^2\,\mathrm{hcap}(D)\,, \quad \mathrm{hcap}(x + D) = \mathrm{hcap}(D)\,. \tag{2.5}$$

Indeed, from the definition of half-plane capacity, (2.5) follows the observations $g_{rD}(\cdot) = rg_D(\cdot/r)$, $g_{x+D}(\cdot) = x + g_D(\cdot - x)$ together with some simple computations. In addition, for $D' \in \mathcal{Q}$ with

$D' \supset D$, note that $g_{D'} = g_{g_D(D'\setminus D)} \circ g_D$, we see as $|z| \to \infty$,

$$g_{D'}(z) = g_{g_D(D'\setminus D)}\left(z + \frac{\mathrm{hcap(D)}}{z} + O(|z|^{-2})\right) = z + \frac{\mathrm{hcap}(D) + \mathrm{hcap}(g_D(D'\setminus D))}{z} + O\left(\frac{1}{|z|^2}\right),$$

which implies the relation

$$\mathrm{hcap}(D') = \mathrm{hcap}(D) + \mathrm{hcap}(g_D(D' \setminus D)) \geq \mathrm{hcap}(D)\,. \tag{2.6}$$

We will need a finer estimation for the exact mappings as the following.

Proposition 2.7. *There exists some constant C, such that for any $D \in \mathcal{Q}$ and any $z \in \mathbb{H}$ with $\mathrm{Im}(z) \geq 2\,\mathrm{rad}(D)$, it holds that*

$$\left| g_D(z) - z - \frac{\mathrm{hcap}(D)}{z} \right| \leq C\, \frac{\mathrm{hcap}(D)\,\mathrm{rad}(D)}{|z|^2}\,. \tag{2.8}$$

The proofs of Proposition 2.2 and 2.7 can be founded in [10, Section 5], we omit the details here. Next we introduce a concept which will help us to describe the SLE curves.

Definition 2.9. For a family of compact hulls $\{A_t\}_{t\geq 0}$, we call it to be
(i) Non-decreasing, if $A_s \subset A_t$ for any $0 \leq s \leq t$;
(ii) Locally growing, if for any $T < \infty$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $0 \leq t \leq s \leq t + \delta \leq T$, it holds $\mathrm{diam}\left(g_{A_t}(A_s \setminus A_t)\right) < \varepsilon$;
(iii) Parameterized by capacity, if $\mathrm{hcap}(A_t) = 2t$ for any $t \geq 0$.

Let $\mathcal{A}$ be the set of families $\{A_t\}_{t\geq 0}$ satisfying the above three conditions.

Remark 2.10. A particular type of elements in $\mathcal{A}$ can be constructed as follow: take a non-crossing curve γ in $\overline{\mathbb{H}}$ from 0 to ∞, define K_t to be the unbounded component of $\mathbb{H} \setminus \gamma\big([0,t]\big)$ and $A_t = \mathbb{H} \setminus K_t$ for each $t \geq 0$. Then the family $\{A_t\}_{t\geq 0}$ is non-decreasing and locally growing. (2.5) and (2.6) suggest that $t \mapsto \mathrm{hcap}(A_t)$ is continuous. Further, non-crossing implies $\mathrm{hcap}(A_t)$ is strictly increasing, and thus we may take a time change $t \mapsto \sigma(t)$ such that $\mathrm{hcap}(A_{\sigma(t)}) = 2t, \forall t \geq 0$. Defining $\tilde{\gamma}(t) = \gamma(\sigma(t))$ and $\tilde{A}_t = A_{\sigma(t)}$, we obtain a family $\{\tilde{A}_t\}_{t\geq 0}$ in $\mathcal{A}$ generated by γ, and we call $\tilde{\gamma}$ as the capacity-reparameterization of γ.

Actually we will only consider elements in $\mathcal{A}$ of such type in this paper.

FIGURE 4. An illustration of $\{A_t\}_{t\geq 0}$ generated by a curve. The results are clear for simple curves, but may seem to be problematic for self-touching curves. For example, one may wonder why the locally growing condition still holds near self-touching points like $\gamma(t_0)$ in the figure, since $\mathrm{diam}(A_s \setminus A_t)$ can always be large no matter how closed s, t are. The secret lies in that the conformal mapping g_t indeed "swallows" the large hole, and thus $\mathrm{diam}\left(g_t(A_s \setminus A_t)\right)$ is still small.

Now we are ready to give the core for the theory of Loewner differential equation, which serves as a fundamental role in the establishment of SLE.

Theorem 2.11. *For any family $\{A_t\}_{t\geq 0}$ in $\mathcal{A}$, denote $K_t = \mathbb{H}\setminus A_t$ and g_t for the exact mapping $g_{A_t} : K_t \to \mathbb{H}$. Then there exists a real-valued continuous function U_t for $t \geq 0$, such that*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}\,, \forall t \geq 0\,, z \in K_t\,. \tag{2.12}$$

Proof. For each $t \geq 0$, since $\overline{g_t(A_s \setminus A_t)}$ is compact for any $s \geq t$, and with diameter shrinking to 0 as $s \downarrow t$ by locally growing, there exists a unique point in $\bigcap_{s>t} \overline{g_t(A_s \setminus A_t)}$, denoted as U_t. It's also clear that for any $0 \leq t \leq s$, $\overline{g_t(A_s \setminus A_t)}$ intersects the real line and $\|g_s - g_t\|_\infty \to 0$as $s \downarrow t$, which implies U_t is a real valued continuous function on $[0, \infty)$.

Now we fix some $t \geq 0, z \in K_t$ and some $\varepsilon > 0$ small enough such that diam $\big(g_t(A_{t+\varepsilon} \setminus A_t)\big)$ is less than half of Im $\big(g_t(z)\big)$. Denote $A_{t,t+\varepsilon} = g_t(A_{t+\varepsilon} \setminus A_t)$ and $g_{t,t+\varepsilon} = g_{A_{t,t+\varepsilon}} = g_{t+\varepsilon} \circ g_t^{-1}$. Since $\{A_t\}_{t\geq 0}$ is parametrized by capacity, we get from (2.6) that

$$\mathrm{hcap}(A_{t,t+\varepsilon}) = \mathrm{hcap}\left(g_t(A_{t+\varepsilon} \setminus A_t)\right) = \mathrm{hcap}(A_{t+\varepsilon}) - \mathrm{hcap}(A_t) = 2\varepsilon\,.$$

Note that $U_t \in A_{t,t+\varepsilon}$ implies $\mathrm{rad}(A_{t,t+\varepsilon} - U_t) \leq \mathrm{diam}(A_{t,t+\varepsilon})$. Hence, applying (2.8) to $A = A_{t,t+\varepsilon} - U_t$ yields that for any z_0 with $\mathrm{Im}(z_0) \geq 2\,\mathrm{diam}(A_{t,t+\varepsilon})$, we have

$$\begin{aligned} g_{A_{t,t+\varepsilon}}(z_0) &= g_A(z_0 - U_t) + U_t = z_0 + \frac{\mathrm{hcap}(A)}{z_0 - U_t} + O\left(\frac{\mathrm{hcap}(A)\,\mathrm{rad}(A)}{|z_0 - U_t|^2}\right) \\ &= z_0 + \frac{2\varepsilon}{z_0 - U_t} + 2\varepsilon\,\mathrm{diam}(A_{t,t+\varepsilon}) O\left(\frac{1}{|z_0 - U_t|^2}\right)\,. \end{aligned}$$

Due to the choice of ε, we may take z_0 as $g_t(z)$ in the above estimation and obtain

$$g_{t+\varepsilon}(z) - g_t(z) = g_{t,t+\varepsilon}\big(g_t(z)\big) - g_t(z) = \frac{2\varepsilon}{g_t(z) - U_t} + 2\varepsilon\,\mathrm{diam}(A_{t,t+\varepsilon}) O\left(\frac{1}{|g_t(z) - U_t|^2}\right)\,,$$

dividing ε from both sides and then sending $\varepsilon \to 0$, we get

$$\lim_{\varepsilon\downarrow 0} \frac{g_{t+\varepsilon}(z) - g_t(z)}{\varepsilon} = \frac{2}{g_t(z) - U_t}\,,$$

and the other direction can be proved similarly. Since this is true for any $t \geq 0$ and $z \in K_t$, we complete the proof. ☐

The function U_t derived as above is called the driving function of the family $\{A_t\}_{t\geq 0}$. Conversely, one can verify that given a real valued continuous function U_t, if we consider the family of ODE's indexed by $z \in \mathbb{H}$ given as

$$\begin{cases} \frac{\mathrm{d}g_t(z)}{\mathrm{d}t} = \frac{2}{g_t(z) - U_t}\,, \\ g_0(z) = z\,. \end{cases}$$

Then for each $t \geq 0$, g_t is a conformal transformation from $K_t = \{z : g_s(z) - U_s \neq 0, \forall s \in [0, t]\}$ to $\mathbb{H}$, and the family $\{A_t\}_{t\geq 0} = \{\mathbb{H} \setminus K_t\}_{t\geq 0}$ is an element in $\mathcal{A}$. Further, for driving functions U_t with some nice property (as in cases we will concerning)[2], the limit

$$\gamma(t) = \lim_{y\downarrow 0} g_t^{-1}(U_t + iy) \tag{2.13}$$

exists for any t and the family $\{A_t\}_{t\geq 0}$ is generated by the non-crossing curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ in the sense of Remark 2.10.

[2]For example, the condition $|U_s - U_t| \leq c_0|s - t|^{1/2}, \forall\, 0 \leq s \leq t$ for some constant c_0 would suffice, see e.g. [9, Section 1, Theorem 1.8.23]. However, since we will take U_t as a Brownian motion path, things will be much trickier.

2.2. **Chordal** SLE_κ **on** $(\mathbb{H}, 0, \infty)$**.** We now turn to the definition of SLE. As mentioned before, this is a family of measures on the set of non-crossing curves from 0 to ∞ in $\overline{\mathbb{H}}$, or equivalently, a family of distribution on the subset of $\mathcal{A}$ with $A_0 = \{0\}$. Let $\mathcal{F}_t = \sigma(A_s : s \le t)$ be the filtration generated by information before time t. To gain a better intuition, we start with some properties that we expect this family of measures to have.

Definition 2.14. We call a measure $\mathbb{P}$ on the set $\mathcal{A}$ to have the **conformal Markov property**, if for any $s \ge 0$, conditioned on $\mathcal{F}_s$, $\{g_s(A_{s+t} \setminus A_s) - U_s\}_{t\ge 0}$ is independent with $\mathcal{F}_s$, and

$$\{g_s(A_{s+t} \setminus A_s) - U_s\}_{t\ge 0} \overset{\mathrm{d}}{=} \{A_t\}_{t\ge 0}\,. \tag{2.15}$$

And we call $\mathbb{P}$ to have the **scaling invariance**, if for any $r > 0$, it holds that

$$\{rA_t\}_{t\ge 0} \overset{\mathrm{d}}{=} \{A_{r^2 t}\}_{t\ge 0}\,. \tag{2.16}$$

From the Loewner differential equation theory, we see that the information for the left hand side of (2.15) can be fully encoded by its driving function $\hat{U}_t = U_{s+t} - U_s$, and *vise versa*. As a result, the conformal Markov property is equivalent to say that conditioning on $\mathcal{F}_s$, $\{U_{s+t} - U_s\}_{t\ge 0}$ is independent with $\mathcal{F}_s$, and has the same distribution with $\{U_t\}_{t\ge 0}$. This suggests $\{U_t\}_{t\ge 0}$ is a stationary process with independent increments. In addition, we know $\{U_t\}_{t\ge 0}$ is pathwise continuous, and thus the only possible measure is the measure for drifted Brownian motion. That is to say, there exists $\kappa > 0$ and $a \in \mathbb{R}$ so that there is a standard Brownian motion $\{B_t\}_{t\ge 0}$ which satisfies $U_t = \sqrt{\kappa} B_t + at$, almost surely. If we further assume that the measure is scaling invariant (which must be true for any scaling limit), then since the driving function of $\{rA_t\}_{t\ge 0}$ is rU_t, (2.16) is equivalent to

$$\{r(\sqrt{\kappa} B_t + at)\}_{t\ge 0} \overset{\mathrm{d}}{=} \{\sqrt{\kappa} B_{r^2 t} + ar^2 t\}_{t\ge 0}\,,$$

which is true if and only if $a = 0$. This leads to the following definition.

Definition 2.17. For each $\kappa \in [0, \infty)$, define $\mathbb{P}_\kappa$ by the probability measure on $\mathcal{A}$ with the following property: for $\{A_t\}_{t\ge 0}$ sampled from $\mathbb{P}_\kappa$, the corresponding driving function has the same distribution with $\{\sqrt{\kappa} B_t\}_{t\ge 0}$, where $\{B_t\}_{t\ge 0}$ is a standard Brownian motion.

As we argued above, $\mathbb{P}_\kappa\}$ with $\kappa \ge 0$ are all of the measures which satisfy both conformal Markov property and scaling invariance. $\mathbb{P}_0$ is a point mass on a deterministic family[3] while $\mathbb{P}_\kappa$ for $\kappa > 0$ support on an uncountable set. Further, it is shown that for each $\kappa \ge 0$ and $\{A_t\}_{t\ge 0}$ sampled from $\mathbb{P}_\kappa$, a.s. $\{A_t\}_{t\ge 0}$ is generated by some non-crossing curve γ from 0 to ∞ in $\overline{\mathbb{H}}$[4]. Given this fact, we may think of $\mathbb{P}_\kappa$ as a probability measure on such curves parameterized by capacity, and we will call the curve sampled from $\mathbb{P}_\kappa$ as chordal SLE_κ curves on $(\mathbb{H}, 0, \infty)$.

2.3. **Other SLE curves.** We have defined for each $\kappa \in [0, \infty)$ the probability measure $\mathbb{P}_\kappa$ on non-crossing curves in $\overline{\mathbb{H}}$ joining 0 with ∞, and the corresponding chordal SLE_κ curves on $(\mathbb{H}, 0, \infty)$. Now we introduce more SLE curves.

We first generalize the definition of chordal SLE_κ curves on any triple (D, a, b) with $D \subsetneq \mathbb{C}$ simply connected and $a, b \in \partial D$ ($a \ne b$). The idea is fairly simple: we just take a conformal transformation $\psi : \mathbb{H} \to D$ with $\psi(0) = a$ and $\psi(\infty) = b$, and then pull everything back to the upper half-plane. To be precise, we make the following definition.

[3] One can verify by simple computations that $\mathbb{P}_0$ is the Dirac-delta measure on $\{[0, 2it]\}_{t\ge 0}$.

[4] The case $\kappa \ne 8$ was first shown in [13], where the authors proved (2.13) exists by methods in complex analysis. The remaining case was shown in [6] via the convergence of Peano curves of UST to SLE_8.

Definition 2.18. For any simply connected domain $D \subsetneq \mathbb{C}$ with a conformal transformation $\psi : \mathbb{H} \to D$ and any curve γ in $\overline{D}$, define the curve $f^{-1} \circ \gamma$ in $\overline{\mathbb{H}}$ by $(f^{-1} \circ \gamma)(t) = f^{-1}\big(\gamma(t)\big)$[5] for any t in the life span of γ. Furthermore, for a probability measure $\mathbb{P}$ on the set of curves in $\overline{D}$, define $\psi^{-1} \circ \mathbb{P}$ as the probability measure on curves in $\mathbb{H}$, with the distribution $f^{-1} \circ \gamma, \gamma \sim \mathbb{P}$.

With these notations and the aforementioned ψ, we may define the probability measure $\mathbb{P}_\kappa^{D,a,b}$ for chordal SLE$_\kappa$ curves on (D, a, b) so that $\psi^{-1} \circ \mathbb{P}_\kappa^{(D,a,b)} = \mathbb{P}_\kappa$. However, there is a bit subtlety in this definition: since $\mathbb{P}_\kappa$ supports on curves parameterized by capacity, $\mathbb{P}_\kappa^{D,a,b}$ supports only on those γ with $\psi^{-1} \circ \gamma$ parameterized by capacity; but ψ is not unique, and thus $\mathbb{P}_\kappa^{D,a,b}$ is not well-defined. The solution is that we take $\mathbb{P}_\kappa^{D,a,b}$ as a probability on curves modulo time reparameterization; in other words, we only concern about the trace of curves. In this way $\mathbb{P}_\kappa^{D,a,b}$ is well-posed, and we are not losing too much information. Also, we will go back to this problem in Section 4, where we consider another parameterization, which is more natural than parameterizing by capacity.

We can also consider SLE curves ending at some interior point of the domain. Let $\mathbb{D}$ be the unit disk, there is an analogue of Theorem 2.11 for non-crossing curves in $\overline{\mathbb{D}}$ from 1 to 0. For such a curve γ, let K_t be the component of $\mathbb{D} \setminus \gamma\big([0,t]\big)$ containing 0, and g_t be the unique conformal transformation from K_t to $\mathbb{D}$ with $g_t(0) = 0$ and $g_t'(0) > 0$. As before, we may describe $\{g_t\}_{t\geq 0}$ by a set of differential equations: under suitable parameterization of γ, there exists a driving function U_t such that $\{g_t\}_{t\geq 0}$ satisfies

$$\partial_t g_t(z) = 2 g_t(z) \frac{e^{iU_t} + g_t(z)}{e^{iU_t} - g_t(z)}\,,\ \forall t \geq 0, z \in K_t\,. \tag{2.19}$$

Taking $U_t = \sqrt{\kappa} B_t$ (where $\kappa \geq 0$ and B_t is a standard Brownian motion) and then reversing the procedure leads to the definition of **radial** SLE$_\kappa$ **curves on** $(\mathbb{D}, 1, 0)$. Via conformal mappings, we can also define radial SLE$_\kappa$ curves (modulo time reparameterization) on any simply connected domain $D \subsetneq \mathbb{C}$ from any boundary point $a \in \partial D$ to any interior point $b \in D$.

It can be shown that for any $\kappa > 0$, chordal SLE$_\kappa$ curves and radial SLE$_\kappa$ curves are absolutely continuous with respect to each other in any domain away from the ends. That is to say, these curves behave similarly, at least in domains away from the ends. In the rest of the paper, we will focus on chordal SLE, and similar results hold for the radial cases.

3. Basic properties of SLE$_\kappa$ curves

In this section, we derive some basic properties of SLE$_\kappa$ curves. The unusual thing is that, one can only study a SLE$_\kappa$ curve by studying points **not** on this curve, since it is defined by a set of differential eqution of $\{g_t : K_t \to \mathbb{H}\}_{t\geq 0}$. We will focus on the case of chordal SLE$_\kappa$ on $(\mathbb{H}, 0, \infty)$, and concern about properties depending only on trace. Hence, all the results can be passed to chordal SLE$_\kappa$ curves on any other domain via conformal mapping. Similar results are also true in the radial case by the absolute continuity, but we omit the details in this paper.

3.1. Bessel process review.

We first briefly discuss some basics of the Bessel process and the radial Bessel process, which will play an essential role in this section.

[5] ψ can always be continuously extended to the boundary of $\mathbb{H}$ since it is a Jordan curve, see e.g. [12]. There is a bit subtlety that ψ^{-1} is not necessary well-defined for points on ∂D, but there is always a natural choice to make $\psi^{-1} \circ \gamma$ a curve in $\overline{\mathbb{H}}$.

For $d > 0$, we mean a d**-dimensional Bessel process** starting at $x \in \mathbb{R}^+$ by a stochastic process $\{X_t\}_{t\geq 0}$ defined as

$$\mathrm{d}X_t = \frac{d-1}{2X_t}\,\mathrm{d}t + \mathrm{d}B_t,\ X_0 = x\,, \tag{3.1}$$

where B_t is a standard Brownian motion, and the stochastic differential equation holds for $t < T = \inf\{t : X_t = 0\}$. We set $X_t = 0$ for any $t > T$ if $T < \infty$. When d is an integer, let $\mathbf{B}_t$ be a d-dimensional Brownian motion starting at $(x, 0, \ldots, 0)$. Then X_t can be viewed as $|\mathbf{B}_t|$ before the first hitting of origin, where $|x|$ denotes the Euclidean distance from x to the origin.

The main result for Bessel process we will need is the following.

Proposition 3.2. *For a d-dimensional Bessel process, it holds that $T < \infty$ a.s. for $0 < d < 2$ and $T = \infty$ a.s. for $d \geq 2$.*

Proof. Let

$$S(x) = \begin{cases} |x|^{2-d}, & d \neq 2\,, \\ \log|x|, & d = 2\,. \end{cases}$$

It is straightforward to check that $S(X_t)$ is a local martingale. For any $c \geq 0$, define $T_c = \inf\{t \geq 0 : X_t = c\}$, and for any $0 \leq a < x < b$, define $T_{a,b} = T_a \wedge T_b$. Then $S(X_{t\wedge T_{a,b}})$ is a bounded martingale and it is easy to see $ET_{a,b} < \infty$[6]. Applying the optimal stopping theorem, we get

$$S(x) = S(X_0) = \mathbb{E}\left[S(X_{T_{a,b}})\right] = S(a)\mathbb{P}[T_a < T_b] + S(b)\mathbb{P}[T_b < T_a]\,.$$

Solving with the relation that $\mathbb{P}[T_a < T_b] + \mathbb{P}[T_b < T_a] = 1$ yields that

$$\mathbb{P}[T_a < T_b] = \frac{S(x) - S(b)}{S(a) - S(b)}\,. \tag{3.3}$$

For $d < 2$, taking $a = 0$ in (3.3), we see $\mathbb{P}[T < T_b] = (x/b)^{2-d}, \forall b > x$. Sending b to infinity we conclude $\mathbb{P}[T_0 < \infty] = 1$. For $d \geq 2$, note that $\{T < \infty\} \subset \liminf_{n\to\infty}\{T_{e^{-n}} < T_n\}$. Thus by Fatou's lemma and (3.3),

$$\mathbb{P}[T < \infty] \leq \mathbb{P}\big[\liminf_{n\to\infty}\{T_{e^{-n}} < T_n\}\big] \leq \liminf_{n\to\infty}\mathbb{P}[T_{e^{-n}} < T_n] = 0\,. \qquad \square$$

Next we turn to the radial Bessel process. For any $a \in \mathbb{R}$, we mean a **radial Bessel process with parameter** a starting at $\theta \in (0, \pi)$ by a stochastic process $\{Y_t\}_{t\geq 0}$ defined as

$$\mathrm{d}Y_t = a\cot Y_t\,\mathrm{d}t + \mathrm{d}B_t\,,\ Y_0 = \theta\,, \tag{3.4}$$

where B_t is a standard Brownian motion and the equation holds until $T = \inf\{t \geq 0 : Y_t \in \{0, \pi\}\}$. We define $Y_t = Y_T$ for $t \geq T$ if $T < \infty$. We will need the following result for radial Bessel process:

Proposition 3.5. *For a radial Bessel process with parameter a, it holds that $T < \infty$ a.s. for $a < 1/2$ and $T = \infty$ a.s. for $a \geq 1/2$. Moreover, when $a \geq 1/2$, the process admits an invariant distribution μ_a on $(0, \pi)$ given by $\mu(x) \propto (\sin x)^{2a}, \forall x \in (0, \pi)$.*

Proof. Let

$$F(x) = \int_{\pi/2}^{x} \frac{\mathrm{d}y}{(\sin y)^{2a}}\,,\ x \in (0, \pi)\,.$$

Then F serves as the role of S in the previous proof, and the first statement follows similarly.

[6]For example, we can compare the process X_t before time $T_{a,b}$ with a drifted Brownian motion.

For the second statement, we first note that $a \geq 1/2$ implies μ_a can indeed be normalized to a probability measure. Then according to the theory of one-dimensional diffusion process, the result follows immediately once we check $f(x) = (\sin x)^{2a}$ satisfies the differential equation

$$\mathcal{L}^* f(x) = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) - \frac{\mathrm{d}}{\mathrm{d}x}\big(a \cot x f(x)\big) = 0\,.$$

Here $\mathcal{L}^*$ is the joint operator of the operator for the radial process with parameter a. We refer to [9, Section 2.4] for more details and further discussions of this proposition. □

3.2. **Topology.** We have mentioned SLE_κ curves are non-crossing curves, but are they simple or self-intersecting? And for the latter, what is the intersecting pattern of such curves? These questions can be answered as in the following theorem.

Theorem 3.6. *Suppose γ is an SLE_κ curve with $\kappa \geq 0$. Then almost surely it hods that*
(i) *γ is a simple curve in $\mathbb{H}$ (except the starting point 0) if $\kappa \in [0, 4]$;*
(ii) *γ is a self-intersecting curve in $\overline{\mathbb{H}}$ but not space-filling if $\kappa \in (4, 8)$;*
(iii) *γ is a space-filling curve in $\overline{\mathbb{H}}$ if $\kappa \geq 8$.*

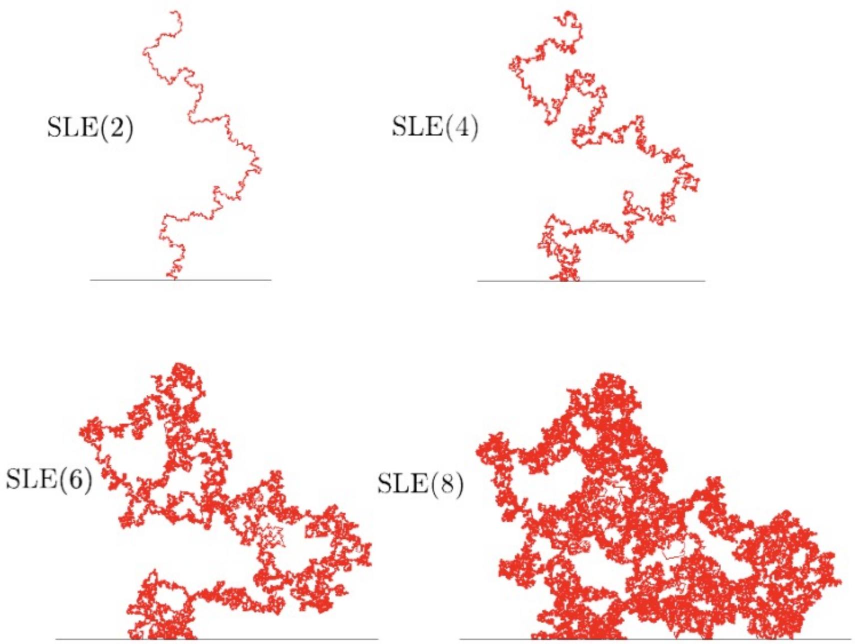

FIGURE 5. A simulation of SLE_κ curves for $\kappa = 2, 4, 6, 8$. Pictured by Tom Kennedy and Hao Wu.

Before proving Theorem 3.6, we sketch the main idea. As suggested at the beginning of this section, the main tool for analyzing SLE_κ curves is the Loewner differential equation (2.12) with $U_t = \sqrt{\kappa} B_t$ for points z not on the curves. Consider a point $x \in \mathbb{R}^+$, define

$$\tau_x = \inf\{t \geq 0 : g_t(x) - \sqrt{\kappa} B_t = 0\}\,. \tag{3.7}$$

Then τ_x is a stopping time, and before such random time, the differential equation

$$\frac{\mathrm{d}g_t(x)}{\mathrm{d}t} = \frac{2}{g_t(x) - \sqrt{\kappa}B_t}\,,\ g_0(x) = x \tag{3.8}$$

always holds. If we further denote $U_t^x = g_t(x) - \sqrt{\kappa}B_t$, then (3.8) becomes

$$\mathrm{d}U_t^x = \frac{2}{U_t^x}\,\mathrm{d}t - \sqrt{\kappa}\,\mathrm{d}B_t\,,\ U_t^x = x\,. \tag{3.9}$$

Take a transformation $\tilde{U}_t^x = U_t^x/\sqrt{\kappa}$, (3.8) turns to

$$\mathrm{d}\tilde{U}_t^x = \frac{2/\kappa}{\tilde{U}_t^x}\,\mathrm{d}t + \mathrm{d}\tilde{B}_t\,,\ \tilde{U}_0^x = x/\sqrt{\kappa}\,, \tag{3.10}$$

where $\tilde{B}_t = -B_t$ is also a standard Brownian motion. As a result, $\tilde{U}_t^x$ is a $(1 + 4/\kappa)$-dimensional Bessel process starting at $x/\sqrt{\kappa}$. Since this is true for any $x \in \mathbb{R}^+$, we can view $\{\tau_x\}_{x>0}$ as hitting times of 0 for a set of Bessel processes driven by a *common* Brownian motion but with different starting points. It is also clear from the SLE view that $\tau_x \le \tau_y, \forall\ 0 < x < y$. Similar things apply for $x < 0$, where τ_x is the hitting time of 0 from a Bessel process starting at $-x$.

Now we examine how behaviors of $\{\tau_x\}_{x\neq 0}$ reflect properties of the curve γ as below:

- If $\tau_x = \infty$ for all $x \neq 0$, then γ never hit the real line. This is because once γ hits some $x_0 \in \mathbb{R}$ at time t_0, it holds $\tau_{x_0/2} \le t_0 < \infty$, a contradiction. Further, γ never hit itself from the conformal Markov property. This means γ is a simple curve.
- If $\tau_x < \tau_y < \infty$ for all $0 < x < y$ and $y < x < 0$, then there is no "holes" surrounding by γ and the real line. Indeed, if such hole exists, then for any two real points x, y on the boundary of the hole, it holds $\tau_x = \tau_y$, a contradiction. Further, there is no holes surrounding by γ itself from the conformal Markov property. Once we can further show that any point in $\mathbb{H}$ is disconnected from ∞ by γ, it implies γ is space-filling.
- If $\tau_x = \tau_y$ for some $0 < x < y$ or $y < x < 0$, then there does exist holes surrounding by γ and the real line. Actually, for such two points x, y, they must lie in a common hole. Furthermore, there are holes surrounding by γ itself from the conformal Markov property. This means γ is self-intersecting but not space-filling.

In light of the geometric arguments given above, Theorem 3.6 reduces to analysis of the hitting time for Bessel process.

Proof of Theorem 3.6. Recall that the d-dimensional Bessel process almost surely never hit 0 for $d \ge 2$. Hence if $1 + 4/\kappa \ge 2$, then $\tau_x = \infty$ for any $x \neq 0$, almost surely.[7] Combining with the first item, it shows that for $\kappa \le 4$, SLE_κ curves are a.s. simple.

For $\kappa > 4$, we have a.s. $\tau_x < \infty$ for all $x \in \mathbb{R}$. For $0 < x < y$, let $g(x, y)$ denote the probability that $\tau_x = \tau_y$. Note that $g(x, y)$ also stands for the probability that x, y are lying in a common hole surrounding by the SLE_κ curve and the real line. Then it follows from the scaling invariance of SLE_κ that $g(x, y) = g(x/y, 1), \forall\ 0 < x < y$. We may assume $y = 1$ and prove that for any $0 < x < 1$, $g(x, 1) = 0$ when $k \ge 8$ and $g(x, 1) > 0$ when $\kappa \in (4, 8)$. For $\kappa > 4$, it can be shown that a.s. any point in $\mathbb{H}$ is disconnected from ∞ by γ[8], and we just take this as an assumption. Then comparing with the last two items, the aforementioned result will complete the proof.

[7]this is a.s. true for all rationals, and the rest follows from monotonicity

[8]Intuitively, from the conformal Markov property, there is an "infinite try" for a SLE_κ curve to surround a given point in $\mathbb{H}$, so it must disconnected it from ∞ eventually. More precisely, this is equivalent to show that a.s. for any $z \in \mathbb{H}$, $\tau_z = \inf\{t \ge 0 : g_t(z) - \sqrt{\kappa}B_t = 0\} < \infty$, and one can find a proof in [8, Section 3, Theorem 3.2.4].

We start by showing that $\lim_{x\downarrow 0} g(x,1)=0$. By the scaling invariance of SLE_κ curve, we see for any $x>0$, $\tau_x \overset{\mathrm{d}}{=} x^2\tau_1$. Thus for any $\varepsilon>0$, it holds

$$\begin{aligned}\liminf_{x\downarrow 0}\big(1-g(x,1)\big) \geq \ &\liminf_{x\downarrow 0}\big(\mathbb{P}[\tau_x<\varepsilon,\tau_1>\varepsilon]\big) \geq \liminf_{x\downarrow 0}\big(\mathbb{P}[\tau_1>\varepsilon]-\mathbb{P}[\tau_x>\varepsilon]\big)\\ = \ &\liminf_{x\downarrow 0}\big(\mathbb{P}[\tau_1>\varepsilon]-\mathbb{P}[\tau_1>x^{-2}\varepsilon]\big) = \mathbb{P}[\tau_1>\varepsilon]\,.\end{aligned}$$

Sending ε to 0, we get $\lim_{x\downarrow 0} g(x,1)=0$.

Now fix $0<x<1$. Denote $U_t=\tilde{U}_t^x$ and $V_t=\tilde{U}_t^1-\tilde{U}_t^x$. We claim that $E_0=\{\tau_x=\tau_1\}$ is equivalent to the event $E_1=\{\sup_{t<\tau_x} V_t/U_t<\infty\}$ (here equivalent means differ by a set of measure zero). On the one hand, for a sample path $\omega\in E_1$, there exists $C=C(\omega)>0$ such that $V_t(\omega)\leq CU_t(\omega), \forall\ t<\tau_x$. Hence $V_{\tau_x}=0$ and thus $\tau_x=\tau_1$. This implies $E_1\subset E_0$. On the other hand, denote a stopping time $T_M=\inf\{t\geq 0: V_t/U_t\geq M\}$ for any $M>0$. Then for any $\omega\in E_0\cap E_1^c$ and $M>x^{-1}$, we have $T_M(\omega)<\tau_x$ and $V_{T_M(\omega)}(\omega)/U_{T_M(\omega)}(\omega)=M$. From the strong Markov property and the scaling invariance, we conclude that

$$\mathbb{P}[E_0\cap E_1^c]\leq \mathbb{P}\big[\tau_{U_{T_M}}=\tau_{U_{T_M}+V_{T_M}}\big]=g\big(U_{T_M},U_{T_M}+V_{T_M}\big)=g\left((M+1)^{-1},1\right).$$

Letting $M\to\infty$ yields $\mathbb{P}[E_0\cap E_1^c]=0$. Hence we verify the claim.

We turn to analyze the event E_1 by tools in stochastic calculus. We have

$$\mathrm{d}U_t=\frac{2/\kappa}{U_t}\,\mathrm{d}t+\mathrm{d}\tilde{B}_t\,,\quad \mathrm{d}V_t=\frac{2/\kappa}{U_t+V_t}\,\mathrm{d}t-\frac{2/\kappa}{U_t}\,\mathrm{d}t=\frac{2V_t\,\mathrm{d}t}{\kappa U_t(U_t+V_t)}\,. \tag{3.11}$$

Denoting $L_t=V_t/U_t$, applying Itő's formula to $\log L_t$ yields

$$\begin{aligned}\mathrm{d}\log L_t = \ &\mathrm{d}\log V_t-\mathrm{d}\log U_t=\frac{\mathrm{d}V_t}{V_t}-\frac{\mathrm{d}U_t}{U_t}+\frac{\mathrm{d}[U]_t}{2U_t^2}\\ = \ &\frac{1}{U_t^2}\left(\frac{1}{2}-\frac{2}{\kappa}-\frac{2U_t}{\kappa(U_t+V_t)}\right)\mathrm{d}t-\frac{\mathrm{d}\tilde{B}_t}{U_t}\\ = \ &\frac{1}{U_t^2}\left(\frac{1}{2}-\frac{2}{\kappa}-\frac{2}{\kappa(1+e^{L_t})}\right)\mathrm{d}t-\frac{\mathrm{d}\tilde{B}_t}{U_t}\,.\end{aligned} \tag{3.12}$$

Taking the random time change $t\mapsto\sigma(t)$ such that

$$\int_0^{\sigma(t)}\frac{\mathrm{d}s}{U_s^2}=t\,,\ \forall t\geq 0\,,$$

and let $\hat{L}_t=L_{\sigma(t)}, \hat{B}_t=\int_0^{\sigma(t)}U_s^{-1}\,\mathrm{d}s$, then

$$[\hat{B}]_t=\int_0^{\sigma(t)}\frac{\mathrm{d}s}{U_s^2}=t\,,\ \forall\ t\geq 0.$$

Therefore from Lévy characterization, $\hat{B}_t$ is a standard Brownian motion, and (3.12) gives

$$\mathrm{d}\log\hat{L}_t=\left(\frac{1}{2}-\frac{2}{\kappa}-\frac{2}{\kappa(1+e^{\hat{L}_t})}\right)\mathrm{d}t+\mathrm{d}\hat{B}_t\,, \tag{3.13}$$

with $\log\hat{L}_0=\log(x^{-1})$. Integrating both sides then yields

$$\log\hat{L}_t=\log(x^{-1})+\int_0^t\left(\frac{1}{2}-\frac{2}{\kappa}-\frac{2}{\kappa(1+e^{\hat{L}_s})}\right)\mathrm{d}s+\hat{B}_t\,. \tag{3.14}$$

Note that E_1 is equivalent to that $\limsup_{t\to\infty} \hat{L}_t < \infty$.

If $\kappa \geq 8$, then the integration in (3.14) is always non-negative, so $\log \hat{L}(t) \geq \log(x^{-1}) + \hat{B}_t$ for any $t \geq 0$. This implies that almost surely,

$$\limsup_{t\to\infty} \hat{L}_t \geq \limsup_{t\to\infty} x^{-1} e^{\hat{B}_t} = \infty,$$

and thus proves $g(x,1) = \mathbb{P}[E_1] = 0, \forall\ 0 < x < 1$.

If $\kappa < 8$, then there is some $N > 0$ such that

$$\frac{1}{2} - \frac{2}{\kappa} - \frac{2}{\kappa(1+e^{-N})} < 0\,.$$

With a positive probability, $\hat{L}$ goes below the level $-N-1$ at some time t_0. Then since t_0, $\log \hat{L}$ is stochastic dominated by a Brownian motion with negative drift until $\hat{L}$ goes above the level $-N$. As a result, $\hat{L}_t$ will always stay below the level $-N$ with positive probability. From the strong Markov property, this shows $\limsup_{t\to\infty} \log \hat{L}_t \leq -N$ with positive probability, and thus proves $g(x,1) = \mathbb{P}[E_1] > 0$ for all $0 < x < 1$. □

Remark 3.15. With additional effort, one can compute the value of $g(x,y)$ explicitly, see e.g. [9, Section 2, Proposition 2.2.12].

3.3. **Dimension.** Now we turn to the Hausdorff dimension of SLE curves. Heuristically speaking, for the Hausdorff dimension of a set $C \subset \mathbb{R}^2$, we mean an appropriate constant d such that as $\varepsilon \downarrow 0$, one needs around ε^{-d} disks with radius ε to cover C. More precisely, for each $d \geq 0$ we let

$$\mathcal{H}^d_\varepsilon(C) = \inf\Big\{\sum_{i=1}^\infty \operatorname{diam}(E_i)^d :\ E_1, E_2, \dots \text{ cover } C, \text{ and } \operatorname{diam}(E_i) \leq \varepsilon \text{ for all i}\Big\}, \tag{3.16}$$

and define the d-dimensional Hausdorff measure of C by

$$\mathcal{H}^d(C) = \sup_{\varepsilon>0} \mathcal{H}^d_\varepsilon(C) = \lim_{\varepsilon\downarrow 0} \mathcal{H}^d_\varepsilon(C)\,. \tag{3.17}$$

Then the Hausdorff dimension of C is given as

$$\dim(C) = \sup\{d : \mathcal{H}^d(C) = \infty\} = \inf\{d : \mathcal{H}^d(C) = 0\}\,. \tag{3.18}$$

We also refer to [11, Chapter 4.1] for a concise and enlightening review. The main result of this subsection is the following, which is first proved by Vincent Beffara in [2].

Theorem 3.19. *For each $\kappa \in [0,\infty)$, sample a SLE_κ curve γ in $\overline{\mathbb{H}}$ from $\mathbb{P}_\kappa$, then almost surely*

$$\dim\big(\gamma[0,\infty)\big) = \Big(1 + \frac{\kappa}{8}\Big) \wedge 2\,. \tag{3.20}$$

We would like to first make some remarks about the proof. In the theory of random geometry, there are several standard approaches to determine the Hausdorff dimension of a random fractal set. For the SLE setting, we will apply a classical method called the **first and second moments estimation**, as we now briefly discuss. For convenience, assume C is a random fractal set lying in $[0,1]^2$, and we fix some $\varepsilon > 0$ together with around ε^{-2} disks $D_1, \dots, D_N$ which cover $[0,1]^2$. The first moment stands for the expected number of disks intersecting C:

$$E(d) \overset{\text{def}}{=} \mathbb{E}\Big[\sum_{i=1}^N \mathbf{1}_{C\cap D_i \neq \emptyset}\Big] = \sum_{i=1}^N \mathbb{P}[C \cap D_i \neq \emptyset]\,. \tag{3.21}$$

Now assume for some $s > 0$, and some constants c_1, c_2 not depending on ε, the one-point estimation

$$c_1\varepsilon^s \leq \mathbb{P}[C \cap D_i \neq \emptyset] \leq c_2\varepsilon^s \tag{3.22}$$

holds for each $1 \leq i \leq N$. We pick the cover of C as the union of disks among D_i which intersect with C. Then it gives an upper bound of $\mathcal{H}^d_\varepsilon(C)$. Taking expectation we see

$$\mathbb{E}[\mathcal{H}^d_\varepsilon(C)] \leq (2\varepsilon)^d E(d) \asymp \varepsilon^{d+s-2}\,.$$

Therefore for any $d > 2 - s$, by Fatou's lemma we see

$$\mathbb{E}[\mathcal{H}^d(C)] \leq \liminf_{\varepsilon\downarrow 0} \mathbb{E}[\mathcal{H}^d_\varepsilon(C)] \lesssim \liminf_{\varepsilon\downarrow 0} \varepsilon^{d+s-2} = 0\,,$$

which implies $\mathcal{H}^d(C) = 0$ almost surely. As a result, $\dim(C) \leq 2 - s$, almost surely.

For a lower bound, we need to show that the expectation $E(d)$ reflects the typical number of disks intersecting with C. To this end, we compute the variance of this quantity, and that's where the second moment appears. For any $1 \leq i < j \leq n$, denote $d(i, j)$ as the maximum of ε and the distance between centers of D_i and D_j. Indeed, it suffice to show that for some constant c_3, the two-points estimation

$$\mathbb{P}[D_i \cap C \neq \emptyset, D_j \cap C \neq \emptyset] \leq C\varepsilon^{2s} d(i, j)^{-s} \tag{3.23}$$

holds for all $1 \leq i < j \leq N$. More precisely, (3.22) together with (3.23) will imply that $\dim(C) \leq 2 - s$ almost surely and $\dim(C) = 2 - s$ with positive probability. Usually with some additional $0 - 1$ law argument, one can improve the result to $\dim(C) = 2 - s$, almost surely.

Remark 3.24. The arguments presented here are certainly incomplete, since for a lower bound of Hausdorff dimension, in principal one needs to exhaust all possible covers, but here we only deal with a particular one. Fortunately, this issue can be tackled by standard approaches in random geometry. We refer to [1, Proposition 1] for rigorous statement and proof of the first and second moments method.

We now apply this framework to prove Theorem 3.19 as the author of [2] did. The cases when $\kappa \geq 8$ follows immediately from the fact that γ is almost surely space-filling. Thus we may assume $\kappa < 8$ is fixed in the rest of this subsection. We focus on the one-point estimation, as given in the following proposition.

Proposition 3.25. *For each point $z_0 \in \mathbb{H}$, there exist two constants c_1, c_2 depending only on z_0, such that for any $\varepsilon > 0$, it holds*

$$c_1\varepsilon^{1-\kappa/8} \leq \mathbb{P}\left[\gamma[0, \infty) \cap \mathrm{B}(z_0, \varepsilon) \neq \emptyset\right] \leq c_2\varepsilon^{1-\kappa/8}\,, \tag{3.26}$$

where $\mathrm{B}(z_0, \varepsilon)$ stands for the disk centered at z_0 with radius ε.

We will only provide an outline of the proof here. For the complete proof, see e.g. [9, Section 3.3]. We will decompose the proof into three main parts to explain the main idea behind it, but just present key steps for each part.

Proof. Denote K_t for the unbounded component of $\mathbb{H} \setminus \gamma\big([0, t]\big)$. Then ∂K_t is formed by a part of γ and a part of the real line. Fix $z_0 \in \mathbb{H}$ and $0 < \varepsilon < \mathrm{Im}(z_0)$, and let

$$T = \inf\{t \geq 0 : g_t(z_0) - \sqrt{\kappa}B_t = 0\} = \inf\{t \geq 0 : z_0 \notin K_t\}\,.$$

Then the event $\gamma[0, \infty) \cap \mathrm{B}(z_0, \varepsilon) \neq \emptyset$ is equivalent to $\inf_{t<T} \mathrm{dist}(z_0, \partial K_t) < \varepsilon$.

The first part of the proof is to replace $\mathrm{dist}(z_0, \partial K_t)$ by some quantity not so rigid. A good choice is the **conformal radius** defined as follow. For a simply connected domain D with a point

$z \in D$, let f be the unique conformal mapping from D to the unit disk with $f(z) = 0$ and $f'(z) > 0$. Then the value of $f'(z)^{-1}$ is called the conformal radius of z in D, denoted as $\mathrm{crad}(z, D)$. There are two virtues of the conformal radius. On the one hand, Koebe 1/4 theorem (see e.g. [10, Section 4]) states that for any simply connected domain D and $z \in D$, it holds

$$\mathrm{dist}(z, \partial D) \le \mathrm{crad}(z, D) \le 4\,\mathrm{dist}(z, \partial D)\,.$$

On the other hand, conformal radius behaves nicely under conformal mappings: for any conformal transformation f on D and $z \in D$, it holds

$$\mathrm{crad}\big(f(z), f(D)\big) = |f'(z)|\,\mathrm{crad}(z, D)\,.$$

A straightforward computation yields that for $z \in \mathbb{H}$, $\mathrm{crad}(z, \mathbb{H}) = 2\,\mathrm{Im}(z)$. Hence, if we define

$$\mathsf{Y}_t = \frac{\mathrm{Im}(g_t(z_0))}{|g_t'(z_0)|} = \frac{\mathrm{crad}(z, K_t)}{2}\,,$$

then $\mathrm{dist}(z_0, \partial K_t)/2 \le \mathsf{Y}_t \le 2\,\mathrm{dist}(z_0, \partial K_t)$. As a result, it suffice to analyze behavior of the probability that Y_t goes below the level ε before time T as $\varepsilon \downarrow 0$.

The second part is to deal with Y_t. Let $X_t, Y_t, u_t, v_t \in \mathbb{R}$ be defined by $g_t(z_0) - \sqrt{\kappa}B_t = X_t + iY_t$ and $\log g_t'(z_0) = u_t + iv_t$. From Loewner equation (2.12) we obtain

$$\partial_t u_t + i\partial_t v_t = \partial\big(\log g_t(z_0)'\big) = \frac{\partial g_t'(z_0)}{g_t'(z_0)} = \frac{1}{g_t'(z_0)}\left(\frac{2}{g_t(z) - \sqrt{\kappa}B_t}\right)' = -\frac{2}{(X_t + iY_t)^2}\,.$$

and thus

$$\partial_t u_t = -\frac{2(X_t^2 - Y_t^2)}{(X_t^2 + Y_t^2)^2}\,, \quad \partial_t v_t = -\frac{4X_tY_t}{(X_t^2 + Y_t^2)^2}\,. \tag{3.27}$$

We further denote $\Theta_t = \arg\big(g_t(z_0) - \sqrt{\kappa}B_t\big) \in (0, \pi)$. Note that $\log|g_t'(z)| = u_t$ and $\Theta_t = \mathrm{Im}\big(\log(g_t(z_0) - \sqrt{\kappa}B_t)\big)$, from (3.27) and Itő's formula we get

$$\partial_t \mathsf{Y}_t = -\mathsf{Y}_t\frac{4Y_t^2}{(X_t^2 + Y_t^2)^2}\,, \quad \mathrm{d}\Theta_t = -\frac{(4-\kappa)X_tY_t}{(X_t^2 + Y_t^2)^2}\,\mathrm{d}t - \frac{\sqrt{\kappa}Y_t}{X_t^2 + Y_t^2}\,\mathrm{d}B_t\,. \tag{3.28}$$

Now we take the random time change $t \mapsto \sigma(t)$ such that $\mathsf{Y}_{\sigma(t)} = \mathsf{Y}_0\exp(-4t/\kappa)$, and denote $\hat{\mathsf{Y}}_t = \mathsf{Y}_{\sigma(t)}, \hat{\Theta}_t = \Theta_{\sigma(t)}, \hat{X}_t = X_{\sigma(t)}, \hat{Y}_t = Y_{\sigma(t)}$ respectively. Then

$$-4\hat{\mathsf{Y}}_t/\kappa = \frac{\mathrm{d}\hat{\mathsf{Y}}_t}{\mathrm{d}t} = -\sigma'(t)\hat{\mathsf{Y}}_t\frac{4\hat{Y}_t^2}{(\hat{X}^2 + \hat{Y}^2)^2} \quad \Rightarrow \quad \sigma'(t) = \frac{(\hat{X}_t^2 + \hat{Y}_t^2)^2}{\kappa\hat{Y}_t^2}\,.$$

This shows for some standard Brownian motion $\hat{B}_t$, $\hat{\Theta}_t$ satisfies

$$\mathrm{d}\hat{\Theta}_t = \frac{(\kappa - 4)\hat{X}_t}{\kappa\hat{Y}_t}\,\mathrm{d}t + \mathrm{d}\hat{B}_t = (1 - 4/\kappa)\cot\hat{\Theta}_t + \mathrm{d}\hat{B}_t\,. \tag{3.29}$$

Hence, $\hat{\Theta}_t$ is a radial Bessel process with parameter $1 - 4/\kappa$, up to the random time $T_0 = \sigma^{-1}(T-)$, at when $\hat{\Theta}_{T_0} \in \{0, \pi\}$[9]. Thus $\inf_{t<T}\mathsf{Y}_t < \varepsilon$ is equivalent to the process $\hat{\Theta}_t$ does not leave $(0.\pi)$ before time $\frac{\kappa}{4}\log(1/\varepsilon)$.

The third part is to estimate the large deviation probability for a radial Bessel process to stay in $(0, \pi)$ for a long time. This is a classical topic in one-dimensional diffusion process, and there are standard techniques to handle such problem. Here we present the tilting method used in [9, Proposition 3.2.9]. Denote $a = 2/\kappa, S_t = \sin\hat{\Theta}_t$ and $M_t = e^{t(2a-\frac{1}{2})}S_t^{4a-1}$. It is straightforward to

[9]Recall Proposition 3.5, the assumption $\kappa < 8$ implies $\hat{\Theta}_t$ almost surely hits $\{0, \pi\}$.

check that $M_{t\wedge T_0}$ is a continuous martingale. Denote $\mathbb{P}$ for the law of radial Bessel process with parameter $1-2a$, and $\mathbb{Q}$ for the law of $\mathbb{P}$ tilted by $M_{t\wedge T}/M_0$. Use the Girsanov theorem one can show $\hat{\Theta}_t$ is a Bessel process with parameter $2a$ under $\mathbb{Q}$. On the other hand, for any t we have

$$\begin{aligned}\mathbb{P}[T_0>t] &= \mathbb{E}_{\mathbb{P}}\left[M_t M_t^{-1}\mathbf{1}_{T_0>t}\right] = e^{t(\frac{1}{2}-2a)}\mathbb{E}_{\mathbb{P}}\left[M_t S_t^{1-4a}\mathbf{1}_{T_0>t}\right] = e^{t(\frac{1}{2}-2a)}\mathbb{E}_{\mathbb{P}}\left[M_{t\wedge T_0} S_t^{1-4a}\mathbf{1}_{T_0>t}\right]\\ &= e^{t(\frac{1}{2}-2a)}S_0^{4a-1}\mathbb{E}_{\mathbb{Q}}[S_t^{1-4a}\mathbf{1}_{T_0>t}] = e^{t(\frac{1}{2}-2a)}S_0^{4a-1}\mathbb{E}_{\mathbb{Q}}[S_t^{1-4a}]\,,\end{aligned}$$

where the last equity follows from $\mathbb{Q}[T_0>t]=1$ by the definition of the tilting. Now take $t=\frac{\kappa}{4}\log(1/\varepsilon)$ in the above equation, and note that $\mathbb{E}[S_t^{1-4a}]$ tends to some fixed positive constant[10] as $\varepsilon\downarrow 0$. Thus we get

$$\mathbb{P}\left[T_0>\frac{\kappa}{4}\log(1/\varepsilon)\right]\asymp \varepsilon^{-\frac{\kappa}{4}\left(\frac{1}{2}-2a\right)}=\varepsilon^{1-\kappa/8}$$

as desired, which completes the proof. □

We have shown (3.22) for SLE_κ curves with $s=1-\kappa/8$. With more effort one can further show a finer estimation as below (see [2, Proposition 4]):

$$\mathbb{P}\left[\gamma\big([0,\infty)\big)\cap \mathrm{B}(z_0,\varepsilon)\right]\asymp\left(\frac{\varepsilon}{\mathrm{Im}(z_0)}\right)^{1-\kappa/8}\big(\sin\arg(z_0)\big)^{8/\kappa-1}. \tag{3.30}$$

This suggests that how we can prove (3.23) for SLE_κ curves: fix two points $z_1,z_2\in\mathbb{H}$ and $\varepsilon>0$ small enough, we want to estimate the probability that an SLE_κ curve γ hits both $\mathrm{B}(z_i,\varepsilon), i=1,2$. Assume γ first hits $\mathrm{B}(z_1,\varepsilon)$ at time t_0. Then (3.30) together with the conformal Markov property implies that unless the set A_{t_0} behaves strange (e.g. $\mathrm{dist}(z_2,A_{t_0})$ is unusually small), the conditional probability $\mathbb{P}[\gamma[t_0,\infty)\cap\mathrm{B}(z_2,\varepsilon)\neq\emptyset\mid\mathcal{F}_{t_0}]$ is still around $\varepsilon^{1-\kappa/8}$. Making these arguments rigorously as in [2], one can show (3.23). Details are omitted here.

Up to now, we have both (3.22) and (3.23) for a SLE_κ curve γ with $s=1-\kappa/8$, so $\gamma[0,\infty)$ has Hausdorff dimension $1+\kappa/8$ with positive probability. Note that conformal mapping preserves Hausdorff dimension, and thus by conformal Markov property, the same is true for $\gamma[t,\infty)$ with any $t>0$. Sending $t\to\infty$ and applying Kolmogrov's $0-1$ law, we get almost surely, $\dim\big(\gamma[0,\infty)\big)=1+\kappa/8$. This completes the proof of Theorem 3.19.

4. Natural parameterization and conformal invariance

Now we turn back to the problem for parameterization. We have discussed about parameterized by capacity, but this is not an intrinsic parameterization for a curve. We now introduce a natural parameterization for SLE curves which does not depend on the domain they living in.

First, let us invest some familiar curves and gain some intuition. Consider a smooth curve $\gamma:[0,T]\to\mathbb{C}$, a natural choice is to parameterize γ by length. In other word, we reparameterize γ by $\tilde{\gamma}:[0,T_\gamma]\to\mathbb{C}$, such that for any $0\le t\le T_\gamma$, it holds

$$\lim_{\Delta\to 0}\sum_{i=1}^{n}|\tilde{\gamma}(t_i)-\tilde{\gamma}(t_{i-1})|=\int_0^t|\tilde{\gamma}'(s)|\,\mathrm{d}s=t\iff|\gamma'(t)|=1\,, \tag{4.1}$$

where the first limit is take over all partitions $0=t_0<t_1<\cdots<t_n=t$ and Δ is defined as $\max_{1\le i\le n}(t_i-t_{i-1})$. Clearly this is an intrinsic parameterization of γ. Further, assume γ lives in the domain D. Then for any conformal mapping f on D, $f\circ\gamma$ is a curve living in $f(D)$. Note that

[10] The constant is given by $\int_0^\pi \sin\theta\,\mathrm{d}\mu_{2a}(\theta)$, where μ_{2a} is the invariant distribution of radial Bessel process with parameter $2a$ given in Proposition 3.5.

even if γ satisfies (4.1), $f \circ \gamma$ is not necessarily parameterized by length. Indeed, to make it so, we need to do a further time change $t \mapsto \sigma(t)$ such that for any $0 \le t \le \sigma(T_\gamma)$,

$$\lim_{\Delta \to 0} \sum_{i=1}^{n} \left| f\big(\gamma(\sigma(t_i)\big) - f\big(\gamma(\sigma(t_{i-1}))\big) \right| \overset{(4.1)}{=} \int_0^{\sigma^{-1}(t)} \left| f'\big(\gamma(s)\big) \right| \mathrm{d}s = t \,, \tag{4.2}$$

and then $f \circ \gamma \circ \sigma$ is parameterized by length. Intuitively, we may think f locally transforms a segment at z with length $\mathrm{d}s$ to a segment at $f(z)$ with length $|f'(z)|\,\mathrm{d}s$, this makes (4.2) clear.

Now we move a step further to remove the smooth condition. Let us see what can we say for fractal curves like a Brownian motion path. Suppose $\{B_t\}_{t\ge 0}$ is sampled from a two-dimensional Brownian motion. We still want to give it some natural parameterization, but the length no longer makes sense, i.e. the limit in (4.1) does not exist anymore. Perhaps the reason for this is that Brownian motion paths are a.s. two-dimensional objects (see e.g. [11, Section 4, Theorem 4.33]), so measuring it by a one-dimensional scaling is unreasonable. From this view, we turn to the "two-dimensional length" defined as

$$\lim_{\Delta \to 0} \sum_{i=1}^{n} |B_{t_i} - B_{t_{i-1}}|^2$$

where limit is taken over a sequence of nested partitions $0 = t_0 < t_1 < \cdots < t_n = t$. Indeed, the above limit does exists and equals to t for any $t \ge 0$ (see e.g. [11, Section 1, Theorem 1.35]), almost surely. Hence we may think Brownian paths are natural parameterized by the two-dimensional length, almost surely.

Moreover, let us see what happens for two-dimensional Brownian motion path after taking conformal mappings. Take a domain $D \subset \mathbb{C}$, let $\{B_t\}_{0\le t\le \tau}$ be a Brownian motion starting at $x \in D$ and τ is the first exit time of D for B_t. For any conformal mapping f on D, $f \circ B_t, 0 \le t \le \tau$ is a curve on D. Motivated by (4.2), we reparameterize $f \circ B_t$ as follow: take a time change $t \mapsto \sigma(t)$ so that for any $0 \le t \le \sigma(\tau)$, it holds

$$\int_0^{\sigma^{-1}(t)} |f'(B_s)|^2 \,\mathrm{d}s = t \,,$$

and then $f(B_{\sigma(t)})$ seems to be parameterized by the two-dimensional length. This is indeed true, and in fact we have

Theorem 4.3 (Conformal invariance of 2d Brownian motion)**.** *With aforementioned notations, let $\tilde{B}_t$ be a Brownian motion starting at $f(x)$, and $\tilde{\tau}$ be the first exit time of $f(D)$ for $\tilde{B}_t$. Then it holds that*

$$\{f(B_{\sigma(t)})\}_{0\le t\le \tau} \overset{\mathrm{d}}{=} \{\tilde{B}_t\}_{0\le \tilde{\tau}} \,.$$

The proof can be founded in [11, Section 7, Theorem 7.20]. Theorem 4.3 states for a stronger version of the conformal invariance of the trace of 2d Brownian motion. Recall that for SLE curves, conformal invariance of trace still holds by definition. One cannot help asking whether there is some analogue of Theorem 4.3 for SLE curves? The answer is yes, at least for cases when $\kappa < 8$, as we explain below.

Denote $d_\kappa = 1 + \kappa/8$ for $\kappa \in [0, 8)$, recall that a SLE_κ curve γ is a.s. with dimension d_κ. To begin with, we need to define some d-dimensional length parameterization for γ. The analogue of (4.1) turns out to be hard to make rigorous, so instead of this, we turn to

Definition 4.4. For a curve $\gamma : [a,b] \to \mathbb{C}$ and $1 \le d < 2$, define its d**-Minkowisky content** by

$$\mathrm{Cont}_d\left(\gamma[a,b]\right) = \lim_{\varepsilon \downarrow 0} \varepsilon^{2-d}\, \mathrm{Area}\left\{z : \mathrm{dist}(z, \gamma[a,b]) \le \varepsilon\right\}, \tag{4.5}$$

provided the limit exists.

For a curve $\gamma \subset \mathbb{C}$, there is at most one d such that $\mathrm{Cont}_d(\gamma[a,b]) \in (0,\infty)$, and in general such d does not exist. However, for SLE_κ curves with $\kappa < 8$, the Minkowisky content does nontrivially exist for $d = d_\kappa$, as in the following theorem.

Theorem 4.6. *For any $\kappa < 8$ and a SLE_κ curve γ, a.s. it holds for any $t > 0$ that $\mathrm{Cont}_{d_\kappa}\left(\gamma[0,t]\right)$ exists in $(0,\infty)$. Further, the function $\mathrm{Cont}_{d_\kappa}\left(\gamma[0,t]\right)$ is a.s. continuous and strictly increasing on $\mathbb{R}^+$, and satisfies for any $0 < s < t$,*

$$\mathrm{Cont}_{d_\kappa}\left(\gamma[0,t]\right) = \mathrm{Cont}_{d_\kappa}\left(\gamma[0,s]\right) + \mathrm{Cont}_{d_\kappa}\left(\gamma[s,t]\right). \tag{4.7}$$

With this result, we may reparameterize γ by $\tilde{\gamma}$ so that $\mathrm{Cont}_{d_\kappa}\left(\gamma[0,t]\right) = t$ for any $t \ge 0$. This is called the **natural parameterization** for SLE_κ curves.

Remark 4.8. Theorem 4.6 is first due to Lawler and Sheffield in [7]. While the authors initially applied an abstract approach to derive a natural parameterization, they realized later it can be described equivalently by the Minkowisky content. The additivity (4.7) follows naturally for $\kappa \le 4$ since γ is a simple curve then. For $4 < \kappa < 8$, this is true essentially because the set of multiple points (i.e. points visited by γ more than once) has Hausdorff dimension strictly less than d_κ.

For $\kappa \ge 8$, the Minkowisky content does not exist anymore, but we can also naturally parameterize a SLE_κ curve γ so that $\mathrm{Area}\left\{\gamma[0,t]\right\} = t$.

Define $\mathbb{P}_\kappa^{\#,(\mathbb{H},0,\infty)}$ for the probability measure on non-crossing curves in $\mathbb{H}$ with natural parameterization, obtained from reparameterize curves sampled from $\mathbb{P}_\kappa$ by d_κ-Minkowisky content. Similarly, we define $\mathbb{P}_\kappa^{\#,(D,a,b)}$ for any simply connected domain D with $a, b \in \partial D$ from the measure $\mathbb{P}_\kappa^{(D,a,b)}$. Note that this probability measure is intrinsic for (D,a,b).

For a triple (D,a,b) as above, and a conformal mapping f on D, sample a chordal SLE_κ curve γ on (D,a,b). Then $f \circ \gamma$ is a curve in $\overline{f(D)}$. Take a random time change $t \mapsto \sigma(t)$ such that

$$\int_0^{\sigma^{-1}(t)} \left|f'\left(\gamma(s)\right)\right|^{d_\kappa} \mathrm{d}s = t, \forall t \ge 0,$$

and then $f \circ \gamma \circ \sigma$ is natural parameterized[11]. We are now ready to state the final theorem of this paper: the full conformal invariance of SLE_κ curves for $\kappa < 8$.

Theorem 4.9. *For any $\kappa \in [0,8)$, with aforementioned notations, sample γ and $\tilde{\gamma}$ from $\mathbb{P}_\kappa^{\#,(D,a,b)}$ and $\mathbb{P}_\kappa^{\#,(f(D),f(a),f(b))}$, respectively. Then it holds*

$$\left\{f\left(\gamma(\sigma(t))\right)\right\}_{t \ge 0} \overset{\mathrm{d}}{=} \left\{\tilde{\gamma}(t)\right\}_{t \ge 0}. \tag{4.10}$$

Give the conformal invariance of Brownian motion (Theorem 4.3), the readers may not find the above theorem too surprising. We refer to [9, Section 3.3] for details.

[11] At least conceptually it is so. Actually, this can be rigorously proved as in [9, Section 3, Proposition 3.3.2]

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