CRITICAL ORBITS AND THE FILLED JULIA SET

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Abstract. We prove that the filled Julia set of a nonlinear polynomial is connected if and only if each of the polynomial's finite critical points has bounded orbit. Along the way, we construct the Böttcher map, and we show that the Böttcher map extends to the entire immediate basin of a superattracting fixed point if and only if the only critical point contained in the immediate basin is the superattracting fixed point itself.

Contents

1. Introduction

Dynamics is the study of how systems evolve over time. In this paper, we study systems modeled by the iteration of a rational function on the extended complex plane.

Rational maps of the Riemann sphere display a wide variety of dynamical behaviors. Often, a given rational map displays many of these behaviors simultaneously. If this is the case, we can carve up the Riemann sphere into the regions on which these different behaviors are displayed. For example, for every rational map f of degree at least two, there is an invariant subset of the Riemann sphere on which the dynamics of f are "chaotic," in a precise sense [2]. This set, together with its complement, partition the Riemann sphere into the region where f is chaotic and the region where f is not chaotic. Another example—and the central topic of this paper—is the following: for every nonlinear polynomial f, there is an invariant open subset of the Riemann sphere, called the *basin* of ∞ for f, whose points all diverge to ∞ under iteration. The basin of ∞ for f, together with its complement, called the *filled Julia set* of f, partition the Riemann sphere into the regions where all orbits diverge to ∞ and where all points have bounded orbits, respectively.

In this paper, besides showing that every nonlinear polynomial has a filled Julia set, we study how topological properties of a polynomial's filled Julia set are reflected in that polynomial's dynamics. In particular, we characterize the connectedness of the filled Julia set in terms of the dynamics of the polynomial's critical

Date: AUGUST 28, 2022.

points. We prove that the filled Julia set of a nonlinear polynomial f is connected if and only if it contains all finite critical points of f, or equivalently, if and only if the basin of ∞ contains *no* finite critical points. We also show that if the filled Julia set is not connected, then it has uncountably many connected components.

In Section 2, we give the basic definitions of complex dynamics needed to state this theorem, and we illustrate its proof with two simple examples. Section 3 is the technical heart of the paper. In this section we construct the *Böttcher map*, which describes the dynamics of rational maps near "superattracting" fixed points. In particular, the Böttcher map describes the dynamics of nonlinear polynomials near ∞. We then prove an important result which relates analytic continuation of the Böttcher map to the presence of critical points near superattracting fixed points. Finally, in Section 4, we use this result about continuing the Böttcher map to prove that the filled Julia set of a nonlinear polynomial f is connected if and only if it contains every finite critical point of f.

2. Basins of Attraction

In this paper, we study the iterative behavior of rational functions (also called *rational maps*) from the Riemann sphere $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ to itself. A rational function is a map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form $z \mapsto P(z)/Q(z)$, where P and Q are complex polynomials with no common factors. The *degree* of the rational map $z \mapsto P(z)/Q(z)$ is equal to the maximum of the degree of P and the degree of Q.

Given a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and an arbitrary point $z \in \hat{\mathbb{C}}$, we would like to study the sequence of points $z, f(z), f(f(z)), \ldots$, which is called the *orbit*, or *forward orbit*, of z under f. Denoting the n-fold iterate of f by

$$f^n := \underbrace{f \circ f \circ \ldots \circ f}_{n \text{ times}},$$

the orbit of z is the sequence of points $\{f^n(z)\}_{n=0}^{\infty}$. We can often study these orbits all at once by analyzing the sequence of functions $f, f^2, f^3, \ldots$, and so on. This change in perspective is subtle but of great importance in complex dynamics.

The behavior we are most concerned with in this paper is when an orbit $\{f^n(z)\}_{n=0}^{\infty}$ converges to a particular point p in $\hat{\mathbb{C}}$. The point p is necessarily a *fixed point*—that is, $f(p) = p$—since continuity of f implies

$$p = \lim_{n\to\infty} f^n(p) = f\left(\lim_{n\to\infty} f^{n-1}(p)\right) = f(p).$$

Since f maps neighborhoods of p into neighborhoods of p by continuity, we can speak properly of the dynamics of f near p. The local behavior of f near p is determined by the derivative of f at p, as the following definitions suggest:

Definition 2.1. A fixed point p of a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is said to be *attracting* if $|f'(p)| < 1$. In particular, if $0 < |f'(p)| < 1$, then p is called *geometrically attracting*, and if $f'(p) = 0$, then p is called *superattracting*. On the other hand, if $|f'(p)| > 1$, then p is called *repelling*, and if $|f'(p)| = 1$, then p is called *indifferent*.

The qualitative behavior of f near a fixed point is affected dramatically by its classification into one of the above types. For example, near attracting fixed points, f is locally contracting, while near repelling fixed points, f is locally expanding [4]. To give another example, while the Inverse Function Theorem implies that f is a local biholomorphism (meaning f has a local holomorphic inverse) near repelling, indifferent, and geometrically attracting fixed points, f is not injective in any neighborhood of a superattracting fixed point. This property of superattracting fixed points is simply a rephrasing of the fact that superattracting fixed points are critical points, and it can be proved using the Argument Principle (see [5]). This property can also be derived from Theorem 3.1 proved below, which says that the behavior of f near a superattracting fixed point, up to a holomorphic change of coordinates, is the same as the behavior of a power map $z \mapsto z^n$, $n \geq 2$, near 0. Note in particular that $z \mapsto z^n$, $n \geq 2$, is not injective in any neighborhood of 0.

We now describe the most important example of a superattracting fixed point.

Example 2.2. Let $f(z) = a_d z^d + \ldots + a_1 z + a_0$, where each $a_i \in \mathbb{C}$ and $a_d \neq 0$, be a polynomial of degree $d \geq 2$. The point at infinity is a fixed point for f since $f(\infty) = \infty$. To see that ∞ is superattracting, we conjugate f by $z \mapsto 1/z$ and show that $F(z) = 1/f(1/z)$ has 0 as a critical fixed point. Indeed, we have

$$F(z) = \frac{1}{a_d z^{-d} + \ldots + a_1 z^{-1} + a_0} = z^d \left(\frac{1}{a_d + \ldots + a_1 z^{d-1} + a_0 z^d} \right).$$

The rational function $(a_d + \ldots + a_1 z^{d-1} + a_0 z^d)^{-1}$ takes the finite value $1/a_d \neq 0$ at 0 and hence is holomorphic near 0. Thus F has a Taylor expansion at 0 of the form

$$F(z) = a_d^{-1} z^d + o(z^d) \text{ as } z \to 0.$$

It follows that $F'(0) = 0$, since $d \geq 2$. Thus, $f'(\infty) = 0$ by definition, and so ∞ is a superattracting fixed point for f. Note that ∞ is a critical point of multiplicity $d-1$ for f since 0 is a root of F' of multiplicity $d-1$.

For the remainder of this section, we explore properties of attracting fixed points. In particular, we begin our investigation of basins of attraction and their topological properties.

Definition 2.3. If p is an attracting fixed point for a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, the *attracting basin* of p, denoted $\Omega = \Omega(p, f)$, is the set of points z whose forward orbit $\{f^k(z)\}_{k=0}^{\infty}$ converges to p. If Ω has several connected components, then the component of Ω containing p is called the *immediate basin* of p, denoted Ω_0.

Consider, for example, the map $w \mapsto w^n$, $n \geq 2$, which has both 0 and ∞ as superattracting fixed points. The attracting basin of 0 is the open unit disk D, since if $|z| < 1$, then the iterates z^{n^k} of z under $w \mapsto w^n$ converge to 0 as $k \to \infty$. Similarly, the attracting basin of ∞ is $\hat{\mathbb{C}} \setminus \overline{D}$, the complement of $\overline{D}$. Note that both of these superattracting basins are connected and hence coincide with the respective immediate basins.

The closed disk $\overline{D}$ is called the *filled Julia set* of the map $w \mapsto w^n$, $n \geq 2$. In general, the *filled Julia set* of an arbitrary nonlinear polynomial is defined to be the complement of the superattracting basin of ∞. (Recall from Example 2.2 that every nonlinear polynomial has ∞ as a superattracting fixed point.) The basin of ∞ for a nonlinear polynomial is the primary example of an attracting basin studied in this paper.

There are two key properties of attracting basins that we use throughout this paper. The first is that an attracting basin $\Omega(p, f)$ is *totally invariant* under f, meaning that $\Omega = f(\Omega) = f^{-1}(\Omega)$.* Indeed, the orbit $\{f^k(z)\}_{k=0}^{\infty}$ of a point z

*If f is a diffeomorphism, the equations $\Omega = f(\Omega)$ and $\Omega = f^{-1}(\Omega)$ are equivalent.

converges to p if and only if the orbit of $f(z)$ and the orbit of each preimage of z converges to p. Note that since Ω is completely invariant, its complement $\hat{\mathbb{C}} \setminus \Omega$ and boundary $\partial\Omega$ are also totally invariant. In particular, the filled Julia set of a nonlinear polynomial is totally invariant.

The second key property of an attracting basin $\Omega(p, f)$ is that there is an open neighborhood U of p in Ω such that the orbit of every point of U converges to p. Such a neighborhood is called an *attracting neighborhood* of p for f. We now prove that attracting neighborhoods exist.

Proposition 2.4. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map with an attracting fixed point p. There is a neighborhood U of p such that $f(\overline{U}) \subset U$ and $f^k(z) \to p$ as $k \to \infty$ for all z in U.*

Proof. First, for convenience, we make a holomorphic change of coordinates taking p to 0. That is, we consider the conjugate map $g = \phi \circ f \circ \phi^{-1}$, where ϕ is a biholomorhic function with $\phi(p) = 0$. For example, we could take $\phi(z) = z - p$ if $p \neq \infty$, and otherwise we can take $\phi(z) = 1/z$. The map g then has an attracting fixed point at 0, with $g'(0) = f'(0) =: \lambda$. Once we find an attracting neighborhood U' of 0 for g, then its image $U := \phi^{-1}(U')$ is an attracting neighborhood of p for f. We now find the desired neighborhood U' of 0, dealing separately with the cases $\lambda \neq 0$ and $\lambda = 0$.

First suppose $\lambda \neq 0$. Then since $0 < |\lambda| < 1$, there is a constant $c > 0$ such that $0 < |\lambda| < c < 1$. By Taylor's Theorem, we may write

$$g(z) = \lambda z + z \cdot o(1) = z(\lambda + o(1)) \text{ as } z \to 0.$$

Thus there is an open disk B centered at 0 such that

$$|g(z)| < c|z| \text{ for all } z \in B.$$

In particular, $g(\overline{B})$ is contained in B since $c|z| < |z|$. By repeatedly applying the inequality $|g(z)| < c|z|$, we obtain, for all $k > 0$ and all $z \in B$,

$$|g^k(z)| < c^k|z|.$$

Since $0 < c < 1$, this implies $g^k(z) \to 0$ as $k \to \infty$.

Now suppose $\lambda = 0$. By Taylor's Theorem, we may write

$$g(z) = z^2\eta(z),$$

where the function η is holomorphic in the closure of a neighborhood V of 0. In particular, there is $C > 0$ such that

$$|g(z)| \leq C|z|^2 \text{ for all } z \in V.$$

Now choose $0 < r < 1/2C$ such that the open disk D_r of radius r centered at 0 is contained in V. Then for all z in D_r, we have

$$|g(z)| \leq C|z|^2 < Cr|z| < \frac{1}{2}|z| < |z|.$$

Thus g maps $\overline{D}_r$ into D_r, and by repeatedly applying the inequality $|g(z)| < |z|/2$, we obtain

$$|g^k(z)| < 2^{-k}|z| \text{ for all } z \in D_r.$$

Thus for all z in D_r the orbit of z converges to 0. $\square$

An important consequence of Proposition 2.4 is that the attracting basin of an attracting fixed point is an open set. Indeed, first notice that if U is an attracting neighborhood of an attracting fixed point p for f, then the orbit of an arbitrary point $z \in \hat{\mathbb{C}}$ converges to p if and only if $f^k(z)$ belongs to U for some $k > 0$. That is, a point $z \in \hat{\mathbb{C}}$ converges to p under iteration if and only if z is eventually mapped into the neighborhood U. Put another way, all orbits must converge to p *through* the neighborhood U. It follows that the basin of p is precisely equal to the union

$$\Omega = \bigcup_{k \geq 0} f^{-k}(U).$$

The set $f^{-k}(U)$ denotes the preimage of U under f^k and thus represents all points mapped into U by f^k. In particular, Ω is open, since each preimage $f^{-k}(U)$ is open. Note that the sequence of preimages $f^{-k}(U)$ is an ascending sequence, meaning $U \subset f^{-1}(U) \subset f^{-2}(U)$, since $f(U) \subset U$.

Besides implying that Ω is open, this formula expressing Ω as a union of preimages is useful for determining Ω for specific choices of f and p. We illustrate this with two examples. These examples also display the main ideas for how we can characterize the connectedness of filled Julia sets in terms of the dynamics of critical points.

Example 2.5. Consider the quadratic polynomial $f_{-3}(z) = f(z) = z^2 - 3$. The point ∞ is a superattracting fixed point for f. Now we determine the superattracting basin $\Omega = \Omega(-3)$ of ∞ for f. First, we find an attracting neighborhood of ∞. Then, we obtain Ω by taking the union of the preimages of this neighborhood.

We claim that the complement of the closed disk $\overline{D}_3$ of radius 3 about the origin is an attracting neighborhood of ∞. Indeed, if $|z| > 3$, then

$$|f(z)| \geq |z|^2 - 3 > |z|(|z| - 1) > 2|z| > |z|.$$

Thus f maps $\hat{\mathbb{C}} \setminus \overline{D}_3$ into itself, and the orbit of any point in $\hat{\mathbb{C}} \setminus \overline{D}_3$ converges to ∞.

We can now find Ω as the union of the iterated preimages of $\hat{\mathbb{C}} \setminus \overline{D}_3$. Since these neighborhoods are unbounded, it will be easier instead to consider the basin's complement $\hat{\mathbb{C}} \setminus \Omega$, which is given by

$$K = K(-3) = \bigcap_{k \geq 0} f^{-k}(\overline{D}_3).$$

Note that since $\hat{\mathbb{C}} \setminus \overline{D}_3$ is contained in its preimage $f^{-1}(\hat{\mathbb{C}} \setminus \overline{D}_3)$, we have the opposite inclusion $f^{-1}(\overline{D}_3) \subset \overline{D}_3$ for the complement $\overline{D}_3$. Thus K is the intersection of a descending sequence of compact subsets and hence is nonempty. (This same reasoning shows that the complement of an attracting basin is nonempty for all rational maps.) Moreover, to get a general picture of what K looks like, we only need to consider the preimages $f^{-k}(\overline{D}_3)$ one by one, since K is approximated better and better by $f^{-k}(\overline{D}_3)$ as k becomes large. The first several preimeages of $\overline{D}_3$ are pictured in Figure 1.

The first preimage $f^{-1}(\overline{D}_3)$ is the region bounded by horizontal figure "8" passing through the origin. Each "lobe" of $f^{-1}(\overline{D}_3)$ is mapped biholomorphically onto $\overline{D}_3$ and is contained in the *interior* of $\overline{D}_3$. Note that the interior of $f^{-1}(\overline{D}_3)$ is disconnected. Thus $f^{-2}(\overline{D_3})$ is disconnected, since it is contained in the interior of $f^{-1}(\overline{D}_3)$ and meets both of its components. In Figure 1, the components of

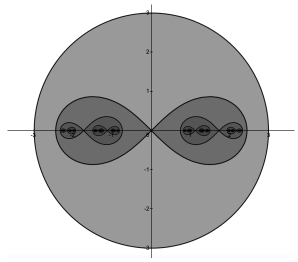

FIGURE 1. Iterated preimages of the disk D_3 under $z \mapsto z^2 - 3$. The intersection of these preimages is a Cantor set contained in the interval $[-3, 3]$.

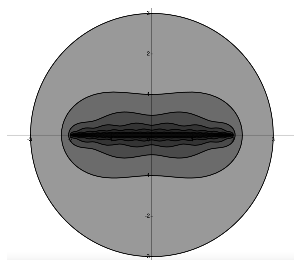

FIGURE 2. Iterated preimages of the disk D_3 under $z \mapsto z^2 - 2$. The intersection of these preimages is the interval $[-2, 2]$.

$f^{-2}(\overline{D}_3)$ are the two smaller figure 8s contained in the two lobes of $f^{-1}(\overline{D}_3)$. Now, since $f^{-2}(\overline{D}_3)$ is disconnected, and since the preimage of a disconnected set under a continuous map is disconnected, it follows that $f^{-k}(\overline{D}_3)$ is connected for all $k \geq 2$. In fact, each preimage $f^{-k}(\overline{D}_3)$ is a union of 2^{k-1} disjoint figure 8s. In particular, $f^{-2}(\overline{D}_3)$ consists of two figures 8s, one contained in the interior of each lobe of $f^{-1}(\overline{D}_3)$. Similarly, $f^{-3}(\overline{D}_3)$ consists of four figure 8s contained in the interiors of the four lobes of $f^{-2}(\overline{D}_3)$, and so on. Thus for each $k > 0$, the set K is contained in the union of 2^{k-1} tiny figure 8s. See Figure 1.

It is clear, at least pictorially, that K is a Cantor set. The total disconnectedness of K follows from the fact that any descending sequence of lobes is a contracting sequence—that is, the diameters of the lobes in the sequence tend to 0. Hence the intersection of a descending sequence of lobes contains a single point. This implies that each point of K belongs to a unique connected component. In addition, for any $k > 0$, each of the 2^{k-1} tiny figure 8s of $f^{-k}(\overline{D}_3)$ contains infinitely many smaller disjoint figure 8s inside of it, each harboring different points of K. Thus K has no isolated points. Since K is compact, perfect, and totally disconnected, K is a Cantor set.

The map in the next example has a completely different filled Julia set.

Example 2.6. Consider the polynomial $f_{-2}(z) = f(z) = z^2 - 2$. To find the basin $\Omega = \Omega(-2)$ of ∞ for f_{-2}, we first choose an attracting neighborhood of ∞. One possible choice is the complement of the closed disk $\overline{D}_3$, as in the previous example. We can find the complement $K(-2) = \hat{\mathbb{C}} \setminus \Omega$ by intersecting the iterated preimages

$f^{-k}(\overline{D}_3)$ of $\overline{D}_3$. These iterated preimages of $\overline{D}_3$ are depicted in Figure 2. They appear all to be complements of open disks, and their intersection appears to be the interval $[-2, 2]$.

To actually prove that $K = [-2, 2]$, however, we will take a different approach. The key observation is that f_{-2} is semiconjugate to the power map $w \mapsto w^2$ on all of $\hat{\mathbb{C}}$. Indeed, the holomorphic map $g(w) = w + w^{-1}$ satisfies the equation

$$f \circ g(w) = g(w^2) \text{ for all } w \in \hat{\mathbb{C}}.$$

The map g sends the superattracting fixed point ∞ of $w \mapsto w^2$ to the superattracting fixed point ∞ of f_{-2}. Thus g sends the basin of ∞ for $w \mapsto w^2$ to the basin of ∞ for f. Since the attracting basin of ∞ for $w \mapsto w^2$ is the complement of the closed unit disk $\overline{D}$, the attracting basin of ∞ for f therefore is $\Omega = g(\hat{\mathbb{C}} \setminus \overline{D})$. In particular, Ω is connected.

We now show that $\Omega = g(\hat{\mathbb{C}} \setminus \overline{D})$ is equal to $\hat{\mathbb{C}} \setminus [-2, 2]$. First note that, for all $\theta \in [0, 2\pi)$, we have

$$g(e^{i\theta}) = e^{i\theta} + e^{-i\theta} = 2\cos\theta.$$

Thus $g(\partial D)$ is precisely equal to $[-2, 2]$. Since g is an open map, g sends boundaries of open sets to boundaries of open sets. Thus the boundary of $\Omega = g(\hat{\mathbb{C}} \setminus \overline{D})$ is equal to $[-2, 2]$. This means $\hat{\mathbb{C}} \setminus [-2, 2]$ is equal to the union of Ω with its exterior $\operatorname{ext}(\Omega)$. But since $\hat{\mathbb{C}} \setminus [-2, 2]$ is connected and intersects Ω (as both contain ∞, for example), it follows that $\hat{\mathbb{C}} \setminus [-2, 2]$ is equal to Ω. Hence $K = [-2, 2]$.

Though we did not find K here by taking intersections of preimages, the semiconjugacy g shows us how, in contrast to the previous example, intersecting the preimages of an attracting neighborhood of ∞ for f_{-2} must give us a connected set—in this case an interval—instead of a Cantor set.

First, note that g maps $\hat{\mathbb{C}} \setminus \overline{D}$ biholomorphically onto $\hat{\mathbb{C}} \setminus [-2, 2]$; that is, g is actually a *conjugacy* between f on Ω and $w \mapsto w^2$ on $\hat{\mathbb{C}} \setminus \overline{D}_1$. To see this, note that g has no poles in $\hat{\mathbb{C}} \setminus [-2, 2]$, and its derivative does not vanish there. (Note that the derivative of g at ∞ is 1, since $1/(zg(1/z)) \to 1$ as $z \to 0$.) Moreover, g is injective on $\hat{\mathbb{C}} \setminus \overline{D}_1$, since for every z in $\hat{\mathbb{C}} \setminus [-2, 2]$, the equation

$$z = w + w^{-1}$$

has exactly one solution in $\hat{\mathbb{C}} \setminus \overline{D}_1$, the other belonging to D_1. Thus g is actually a conformal isomorphism of $\hat{\mathbb{C}} \setminus \overline{D}_1$ onto Ω.

We can use the conjugacy g to choose a convenient attracting neighborhood of ∞ for f_{-2}. Indeed, the advantage of having this conjugacy between f_{-2} and $w \mapsto w^2$ is that the dynamics of $w \mapsto w^2$ are geometrically much simpler. In particular, if we first choose a disk $B_0 \subset \hat{\mathbb{C}} \setminus \overline{D}_1$ as an attracting neighborhood of ∞ for $w \mapsto w^2$, then each of its preimages B_k under $w \mapsto w^{2^k}$ is *also* a disk. Thus, since g is a conformal isomorphism on $\hat{\mathbb{C}} \setminus \overline{D}_1$, the attracting neighborhood $U := g(B_0)$ of ∞ for f_{-2} is a topological disk, and so are all of its iterated preimages $f^{-k}(U) = g(B_k)$. In particular, it follows that the complement of each preimage $f^{-k}(U)$ is connected, and hence that the intersection $K(-2)$ of these complements is also connected (see [1, §5.1] and [5]). This is the key difference between the current example and the previous one. In the current example, each complement $\hat{\mathbb{C}} \setminus f_{-2}^{-k}(U)$ is connected, and so we obtained "in the limit" the connected set $K(-2)$. By contrast, in the last example, we obtained a disconnected set at the second iteration of taking preimages. The number of components of these preimages then doubled with each iteration, and we obtained "in the limit" the Cantor set $K(-3)$ with uncountably many components.

The attracting basins in Examples 2.5 and 2.6 are strikingly different from a topological point of view. In particular, the basin $\Omega(-2)$ in Example 2.6 is simply connected, while the attracting basin $\Omega(-3)$ in Example 2.5 is *uncountably connected*, meaning its complement has uncountably many components. We saw moreover that the basin $\Omega(-2)$ is simply connected precisely because each preimage $f^{-k}(U)$ of the attracting neighborhood U is simply connected. Equivalently, the filled Julia set $K(-2)$ is connected precisely because each preimage complement $\hat{\mathbb{C}} \setminus f^{-k}(U)$ is connected. On the other hand, the disconnectedness of $K(-3)$ in Example 2.5 became evident when we found that the preimage $f_{-3}^{-2}(\overline{D}_3)$ is disconnected.

Examples 2.5 and 2.6 thus suggest a general method for determining whether an immediate basin is simply connected or multiply connected—in other words, whether the complement of the immediate basin is connected or disconnected. This method works for any rational map f with an attracting fixed point p. First, choose a simply connected attracting neighborhood U_0 of p for f. Then, for each $k > 0$, examine the component U_k of $f^{-k}(U)$ that contains p. If each U_k is simply connected, then the immediate basin is simply connected. If instead U_k is multiply connected for some $k > 0$, then the immediate basin is multiply connected. In fact, the immediate basin must be uncountably connected in this case. Thus the immediate basin is either simply connected or uncountably connected, and the connectedness is detected by taking preimages of an attracting neighborhood. See [4, §8] for more details.

The remainder of this paper is devoted to proving this result solely for polynomials. We actually prove a stronger and more specific result. For one, we show that if p is any attracting fixed point for a polynomial f *other than* the superattracting fixed point ∞, then the basin of p simply connected. The main concern of this paper, however, is the superattracting fixed point ∞. In addition to proving that the basin of ∞ is either simply connected or uncountably connected, we show that this dichotomy is governed completely by the dynamics of the critical points of f. Specifically, we prove

Theorem 2.7. *The filled Julia set of a nonlinear polynomial f is connected if and only if it contains all finite critical points of f. Moreover, if the filled Julia set is not connected, then it has uncountably many connected components.*

By "finite critical point" we simply mean a critical point not equal to ∞. Note that since the basin of ∞ is an open neighborhood of ∞, the filled Julia set is a bounded subset of the complex plane. Thus, since the filled Julia set is totally invariant, it is precisely the set of points whose forward orbit is bounded. We can then rephrase Theorem 2.7 as follows: *The filled Julia set of f is connected if and only if the orbit of each finite critical point of f is bounded.*

The proof of Theorem 2.7 is long and at times technical, but the main ideas are contained entirely in Examples 2.5 and 2.6. First, note the role of the unique finite critical point 0 of f_{-3} in Example 2.5. Both the critical point 0 and the critical value $f_{-3}(0) = -3$ belong to the basin of ∞. Moreover, the critical value -3 belongs to the boundary of the attracting neighborhood $\hat{\mathbb{C}} \setminus \overline{D}_3$. Now, since

-3 is a critical value of the quadratic f_{-3}, it has exactly one preimage, the critical point 0. On the other hand, every other point of $\mathbb{C}$, in particular every other point of $\overline{D}_3$, has *two* preimages under f_{-3}. Thus the two components of $f_{-3}^{-1}(D_3)$, which are topological disks, have 0 as a unique shared boundary point. In other words, $f_{-3}^{-1}(\overline{D}_3)$ is a figure "8". As we saw above, the fact that $f^{-1}(\overline{D}_3)$ has disconnected interior implies that the filled Julia $K(-3)$ is disconnected. Thus the presence of the critical point 0 in the basin—more precisely, the existence of an attracting neighborhood containing the critical value -3 on its boundary—is the source of the disconnectedness of $K(-3)$. In the proof of Theorem 2.7, a figure "8" will arise as it did here whenever a critical point belongs to the basin of ∞ of a nonlinear polynomial.

Secondly, note the role of the conjugacy g in Example 2.6. Note in particular that the existence of the conjugacy g between f_{-2} and $w \mapsto w^2$ defined on the *entire* basin of ∞ is precisely what allowed us to find an attracting neighborhood U of ∞ each of whose preimages $f_{-2}^{-k}(U)$ is simply connected. We expect more generally, then, that if f is a polynomial conjugate to a power map $w \mapsto w^n$ on some neighborhood V of ∞, then there is an attracting neighborhood U of ∞ whose preimages under f *in* V are all topological disks. We will see in §3 that such a conjugacy actually exists for *any* polynomial in a neighborhood of ∞. Thus, for any polynomial, there is an attracting neighborhood of ∞ whose preimages are all simply connected near ∞. The union of these preimages moreover is simply connected, and the complement of this union is therefore connected.

Thus Example 2.6 suggests that, whenever a polynomial f is conjugate to a power map on the *entire* basin of ∞, then its complement, the filled Julia set, is connected. Example 2.5 suggests, on the other hand, that whenever the basin of ∞ for f contains a finite critical point, the filled Julia set is disconnected. The remarkable fact connecting these two observations is that *the presence of a finite critical point in the basin of* ∞ *is the only way the conjugacy can fail to be defined on the entire basin.* Thus the presence of a critical point in the basin is the only way there can fail to exist an attracting neighborhood whose preimages under f are all simply connected, hence the only reason the entire basin can fail to be simply connected. In other words, the presence of a critical point in the basin is the only way the filled Julia set can be disconnected.

In the next section, §3 , we construct the map that conjugates f to a power map $w \mapsto w^n$ in a neighborhood of ∞. This conjugacy in fact exists for any rational map in a neighborhood of a superattracting fixed point, not just near the point ∞ for polynomials. The conjugacy is called the *Böttcher map.* We then show that the Böttcher map extends to the entire superattracting basin exactly when the basin contains no critical point besides the fixed point itself. Then, in §4, we use the result about extending the Böttcher map to conclude that the superattracting basin of ∞ for a polynomial contains a finite critical point if and only if it is uncountably connected.

3. The Böttcher Map

In this section we introduce the main technical tool for understanding the dynamics of rational maps near superattracting fixed points: the Böttcher map. If $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational map with a superattracting fixed point p, then the *Böttcher map* for p is a holomorphic change of coordinates defined in a neighborhood of p

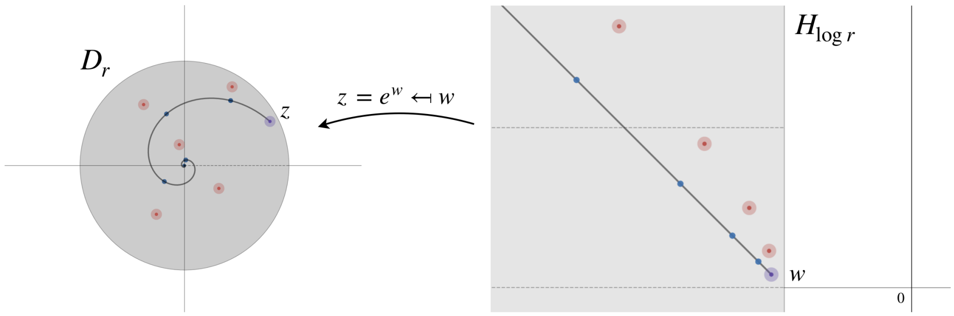

FIGURE 3. An illustration of the lift used in the construction of the Böttcher map in Theorem 3.1. On the left, the orbit of $z = e^w$ under f is shown in red, and its orbit under $z \mapsto z^n$ is shown in blue. The lifted orbits appear on the right, with the orbit of w under the lift F shown in red, and its orbit under $w \mapsto nw$ shown in blue.

that conjugates f to a map of the form $w \mapsto w^n$, for some $n \geq 2$.* The conjugacy maps this neighborhood of p biholomorphically onto a neighborhood of 0. Thus the dynamical behavior of f near p resembles the behavior of the power map $w \mapsto w^n$ near 0. Our first aim is show that the Böttcher map always exists in a neighborhood of a superattracting fixed point.

Then we turn our attention to the problem of analytically continuing the Böttcher map beyond the neighborhood on which it is originally defined. By examining the maximal neighborhood of the superattracting fixed point to which the Böttcher map can be analytically continued, we then show that the Böttcher map either extends to the entire immediate basin of the superattracting fixed point, or else is obstructed from extending to the whole immediate basin by the presence of a critical point of f (besides the fixed point itself). Thus the Böttcher map, besides giving us a precise description of the dynamics of f where it is defined, also detects the presence of critical points where it *cannot* be defined.

Later, in Section 4, we will use the Böttcher map to understand the dynamics of nonlinear polynomials near the superattracting fixed point at ∞. We will see that the Böttcher map extends to the entire basin of ∞ precisely when the filled Julia set is connected.

Now we proceed with the construction of the Böttcher map near superattracting fixed points of holomorphic functions. In general, if a map $f : N \to N$ is analytic in a neighborhood $N \subset \hat{\mathbb{C}}$ of a superattracting fixed point p, then as we saw in the proof of Proposition 2.4, there is a local holomorphic change of coordinates α mapping N onto a neighborhood of 0 such that $\alpha(p) = 0$. Thus, α conjugates f to a map that has 0 as a superattracting fixed point. For simplicity, we state Böttcher's Theorem for the case of a superattracting fixed point at 0.

Theorem 3.1 (Böttcher's Theorem)**.** *Suppose $f : N \to N$ is analytic in a neighborhood of 0 and has 0 as a superattracting fixed point. In particular, suppose the*

*The language "the Böttcher map" is justified because, as it turns out, the conjugacy is unique up to scaling by an $(n-1)$th root of unity [4].

Taylor expansion of f at 0 takes the form

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + \dots, \ \textit{for } a_n \neq 0, n \geq 2.$$

Then there are neighborhoods U and V of 0, with $U \subset N$, and a biholomorphic change of coordinates $\phi : U \to V$ such that

$$\phi \circ f \circ \phi^{-1}(w) = w^n, \tag{3.2}$$

for all $w \in V$.

Proof. We may assume without loss of generality that $a_n = 1$, since if $a_n \neq 1$, then we can consider the conjugate map $g(z) = cf(z/c)$, where the constant $c \neq 0$ satisfies $c^{n-1} = a_n$. The Taylor series of g then has the form $g(z) = z^n + \dots$ at 0.

Naively, we would like to define the conjugacy $\phi : U \to V$ as the limit

$$\phi(z) = \lim_{k\to\infty} \sqrt[n^k]{f^k(z)},$$

so that

$$\phi(f(z)) = \lim_{k\to 0} \sqrt[n^k]{f^{k+1}(z)} = \left(\lim_{k\to 0} \sqrt[n^{k+1}]{f^{k+1}(z)}\right)^n = (\phi(z))^n,$$

yielding (3.2) as long as ϕ is invertible. A major issue with this definition of ϕ, however, is that the n^kth root is multivalued, with n^k holomorphic branches to choose from. For each k, we evidently must choose one of these n^k branches so that the resulting limit ϕ is holomorphic. We will not attempt this. Instead, we make a certain "change of coordinates" that eliminates the multivalued-ness of the nth root altogether. In particular, in these new "coordinates," the map $z \mapsto z^n$, is represented by a *biholomorphic* function G. Thus, if F is the representation of f in these new "coordinates," then we can define a semiconjugacy Φ between F and G by $\Phi(z) = \lim_{k\to\infty} G^{-k} \circ F^k(z)$. Indeed, we have $\Phi(F(z)) = G(\Phi(z))$, assuming the limit exists. Then, by "undoing" this "change of coordinates," we obtain a semiconjugacy between f and $z \mapsto z^n$.

Formally, this "change of coordinates" is a lift. In particular, we lift f and $z \mapsto z^n$ through the exponential map $z \mapsto e^z$, thereby making a "logarithmic change of coordinates." Recall that the exponential map $z \mapsto e^z$ is a covering map of $\mathbb{C}$ onto the punctured plane $\mathbb{C} \setminus \{0\}$. As we will see, the *lift* of the map $z \mapsto z^n$ is biholomorphic, unlike the map $z \mapsto z^n$ itself. Thus, we can define a conjugacy between the *lift* of f and the *lift* of $z \mapsto z^n$ using the limit trick described above. This conjugacy between lifts then descends to the desired conjugacy between f and $z \mapsto z^n$. Intuitively, by lifting through $z \mapsto e^z$, we are "unwrapping" the orbits of f and of $z \mapsto z^n$, which wind in tight spirals about the origin. See Figure 3. We now proceed with the construction of these conjugacies.

First, we must choose a suitable punctured neighborhood of 0 to lift. Pick a small open disk D_r centered at 0 such that D_r is contained in the basin of 0, and so that f maps D_r into itself. Since the zeros of f are isolated (as f is holomorphic), we can choose the radius r small enough so that 0 is the only root of f in D_r. Thus f sends the punctured disk $D_r \setminus \{0\}$ into itself. Note that the power map $z \mapsto z^n$ also sends $D_r \setminus \{0\}$ into itself. Thus, we can lift both f and $z \mapsto z^n$ to the universal cover of $D_r \setminus \{0\}$, which is the left half-plane

$$H_{\log r} = \{z \in \mathbb{C} : \mathrm{Re}(z) < \log r\}.$$

The covering map is the exponential function $z \mapsto e^z$. See Figure 3.

Now we express f and $z \mapsto z^n$ in the new logarithmic coordinates. Write $f(z) = z^n(1+\eta(z))$, where $\eta(z) \to 0$ as $z \to 0$. Take r sufficiently small so that $1+\eta(z)$ belongs to the cut plane $\mathbb{C}\setminus(-\infty, 0]$ for every z in D_r. Consider the lift $F : H_{\log r} \to H_{\log r}$ of f given by

$$F(w) = nw + \log(1+\eta(e^w)).$$

Here we have chosen the principal branch of log, which is analytic on the region $\mathbb{C}\setminus(-\infty, 0]$. Thus the map F is a well-defined holomorphic function on $H_{\log r}$. Note that F is in fact a lift of f because

$$e^{F(w)} = e^{nw+\log(1+\eta(e^w))} = e^{nw}(1+\eta(e^w)) = f(e^w),$$

for all $w \in H_{\log r}$. Similarly, the map $w \mapsto nw$ is a lift of $z \mapsto z^n$ since

$$e^{nw} = (e^w)^n.$$

We now show that the lift F of f is semiconjugate to the lift $w \mapsto nw$ of $z \mapsto z^n$ on the half-plane $H_{\log r}$. Since $w \mapsto nw$ is biholomorphic, we can construct a well-defined semiconjugacy $\Phi : H_{\log r} \to H_{\log r}$ between $w \mapsto nw$ and F as the limit

$$\Phi(w) = \lim_{k\to\infty} F^k(w)/n^k.$$

If this limit Φ exists then it satisfies the functional equation

$$\Phi(F(w)) = \lim_{k\to\infty} F^{k+1}(w)/n^k = n\left(\lim_{k\to\infty} F^{k+1}(w)/n^{k+1}\right) = n\Phi(w).$$

We must show that the limit $\Phi(w)$ exists for each $w \in H_{\log r}$ and that the sequence of holomorphic functions $\Phi_k(w) = F^k(w)/n^k$ converges uniformly to Φ (since we would like the limit function to be holomorphic). To do this, we prove that the sequence of holomorphic functions Φ_k is uniformly Cauchy on $H_{\log r}$.

First, since

$$F(w) - nw = \log(1+\eta(e^w)) \to 0 \text{ as } \operatorname{Re}(w) \to -\infty,$$

we can take r sufficiently small so that $|F(w) - nw| < 1$ for all $w \in H_{\log r}$. Thus, since F maps $H_{\log r}$ into itself, for all $k > 0$ and all $w \in H_{\log r}$, we have

$$|F^{k+1}(w) - nF^k(w)| < 1.$$

Dividing this inequality by n^k, we find

$$|\Phi_{k+1}(w) - \Phi_k(w)| = \frac{1}{n^{k+1}}|F^{k+1}(w) - nF^k(w)| < \frac{1}{n^{k+1}}.$$

Thus, writing the difference $\Phi_{k+\ell}(w) - \Phi_k(w)$ as a telescoping sum, we obtain the bound

$$|\Phi_{k+\ell}(w) - \Phi_k(w)| \le \sum_{i=0}^{\ell-1} |\Phi_{k+i+1}(w) - \Phi_{k+i}(w)| < \frac{1}{n^{k+1}} \sum_{i=0}^{\ell-1} \frac{1}{n^i} < \frac{2}{n^{k+1}}.$$

Since $1/n^{k+1} \to 0$ as $k \to \infty$ independently of ℓ and w, this shows that the sequence of holomorphic functions Φ_k is uniformly Cauchy on $H_{\log r}$. Thus the maps Φ_k converge uniformly to Φ, and Φ is holomorphic. By construction, Φ satisfies the functional equation

$$\Phi(F(w)) = n\Phi(w)$$

and hence is a semiconjugacy between F and the map $w \mapsto nw$.

Next we show that the semiconjugacy $\Phi : H_{\log r} \to H_{\log r}$ descends to a map $\phi : D_r \setminus \{0\} \to D_r \setminus \{0\}$, and that ϕ is a semiconjugacy between f and $z \mapsto z^n$

on $D_r \setminus \{0\}$. We then fill in the value of ϕ at 0 by showing that 0 is a removable singularity for ϕ.

To verify that Φ descends to a map $\phi : D_r \setminus \{0\} \to D_r \setminus \{0\}$, it suffices to show that $\Phi(w + 2\pi i) - \Phi(w)$ is an integer multiple of $2\pi i$ for every $w \in H_{\log r}$. (Note that $w + 2\pi i$ belongs to $H_{\log r}$ as long as w does.) In that case, the formula

$$\phi(e^w) = e^{\Phi(w)}$$

determines a well-defined function $\phi : D_r \setminus \{0\} \to D_r \setminus \{0\}$.

First note that, for all $w \in H_{\log r}$,

$$e^{F(w+2\pi i)} = f(e^{w+2\pi i}) = f(e^w) = e^{F(w)}.$$

Thus $F(w)$ and $F(w + 2\pi i)$ differ by an integer multiple of $2\pi i$. In fact, we have

$$F(w + 2\pi i) = F(w) + 2\pi n i.$$

To see this, recall that $|F(w) - nw| < 1$ for all $w \in H_{\log r}$. Thus we have

$$|F(w + 2\pi i) - F(w) - 2\pi n i| \leq |F(w + 2\pi i) - n(w + 2\pi i)| + |F(w) - nw| < 2.$$

Since the only integer multiple of $2\pi i$ with absolute value smaller than 2 is equal to 0, we conclude that $F(w + 2\pi i) = F(w) + 2\pi n i$. Applying this identity iteratively, we see that

$$F(w + 2\pi m i) = F(w) + 2\pi n m i, \text{ for } m \in \mathbb{Z}.$$

Thus, for the second iterate of F, we obtain

$$F^2(w + 2\pi i) = F(F(w) + 2\pi n i) = F^2(w) + 2\pi n^2 i,$$

and so by induction we deduce

$$F^k(w + 2\pi i) = F^k(w) + 2\pi n^k i, \text{ for all } k \geq 0.$$

Dividing this equation by n^k then gives

$$\Phi_k(w + 2\pi i) = \Phi_k(w) + 2\pi i.$$

Finally, taking the limit of this equation as $k \to \infty$, we find

$$\Phi(w + 2\pi i) = \Phi(w) + 2\pi i.$$

Thus Φ descends to a map $\phi : D_r \setminus \{0\} \to D_r \setminus \{0\}$ which satisfies the equation $\phi(e^w) = e^{\Phi(w)}$, for all $w \in H_{\log r}$.

We now show that ϕ is a semiconjugacy between f and $z \mapsto z^n$ on the punctured disk $D_r \setminus \{0\}$; that is, we show $\phi(f(z)) = (\phi(z))^n$ for all $z \in D_r \setminus \{0\}$. Indeed, since $\exp : H_{\log r} \to D_r \setminus \{0\}$ is surjective, for each $z \in D_r \setminus \{0\}$ there is $w \in H_{\log r}$ such that $z = e^w$. Thus the equation $\Phi(F(w)) = n\Phi(w)$, together with the fact that F is a lift of f and Φ is a lift of ϕ, implies that

$$\phi(f(z)) = \phi(f(e^w)) = \phi(e^{F(w)}) = e^{\Phi(F(w))} = e^{n\Phi(w)} = (e^{\Phi(w)})^n = (\phi(e^w))^n = (\phi(z))^n.$$

This shows that ϕ is a semiconjugacy between f and $z \mapsto z^n$ on $D_r \setminus \{0\}$.

To complete the proof of Theorem 3.1, it remains to define ϕ at 0 and to show that this extension has a local inverse at 0 satisfying (3.2). First, note that ϕ is bounded on the punctured disk $D_r \setminus \{0\}$ and hence that its "singularity" at 0 is removable. Thus ϕ can be extended to a holomorphic map defined on the entire disk D_r by setting

$$\phi(0) = \lim_{z \to 0} \phi(z) = \lim_{\mathrm{Re}(w) \to -\infty} e^{\Phi(w)}.$$

To evaluate this limit, we study the asymptotic behavior of Φ. Now, to study the asymptotic behavior of Φ, we first study the asymptotic behavior of the iterates of F. We prove by induction that $F^k(w) - n^k w \to 0$ as $\mathrm{Re}(w) \to -\infty$. The base case was proved above; that is, we showed there is a holomorphic function ξ_1 defined on $H_{\log r}$ such that

$$F(w) = nw + \xi_1(w),$$

and such that $\xi_1(w) \to 0$ as $\mathrm{Re}(w) \to -\infty$. In particular,

$$\mathrm{Re}(F(w)) \to -\infty \text{ as } \mathrm{Re}(w) \to -\infty.$$

Now suppose that for some $\ell \geq 1$, there is a holomorphic function ξ_ℓ defined on $H_{\log r}$ such that

$$F^\ell(w) = n^\ell w + \xi_\ell(w)$$

and such that $\xi_\ell(w) \to 0$ as $\mathrm{Re}(w) \to -\infty$. Then, for the next iterate $F^{\ell+1}$, we have

$$F^{\ell+1}(w) = F^\ell(F(w)) = n^\ell F(w) + \xi_\ell(F(w)) = n^{\ell+1} w + \big(n^\ell \xi_1(w) + \xi_\ell(F(w))\big).$$

Since

$$n^\ell \xi_1(w) + \xi_\ell(F(w)) \to 0 \text{ as } \mathrm{Re}(w) \to -\infty,$$

this completes the inductive step. We conclude that each iterate F^k of F may be written as

$$F^k(w) = n^k w + \xi_k(w),$$

where $\xi_k(w) \to 0$ as $\mathrm{Re}(w) \to -\infty$. In particular, we have

$$\Phi_k(w) = w + \frac{1}{n^k}\xi_k(w).$$

It follows that $\Phi_k(w) - w \to 0$ as $\mathrm{Re}(w) \to -\infty$ for all $k > 0$. Since the maps Φ_k converge uniformly to Φ on $H_{\log r}$, we conclude that

$$\Phi(w) - w \to 0 \text{ as } \mathrm{Re}(w) \to -\infty.$$

Thus we define

$$\phi(0) = \lim_{\mathrm{Re}(w) \to -\infty} e^{\Phi(w)} = \lim_{\mathrm{Re}(w) \to -\infty} e^w = 0,$$

which implies that ϕ satisfies the equation $\phi(f(z)) = (\phi(z))^n$ for all $z \in D_r$.

Using $\phi(0) = 0$, we also see that

$$\phi'(0) = \lim_{z \to 0} \frac{\phi(z)}{z} = \lim_{\mathrm{Re}(w) \to -\infty} \frac{e^{\Phi(w)}}{e^w} = \lim_{\mathrm{Re}(w) \to -\infty} e^{\Phi(w) - w} = 1.$$

Thus ϕ is invertible in a neighborhood U of 0 and has a holomorphic inverse defined on $V = \phi(U)$. Since ϕ satisfies $\phi(f(z)) = (\phi(z))^n$ for z in D_r, we obtain (3.2) for all $w = \phi(z)$ in V. This completes the proof. □

Böttcher's Theorem tells us that if p is a superattracting fixed point of a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, then f is holomorphically conjugate to a power map in a neighborhood of p, and the Böttcher map ϕ maps this neighborhood of p biholomorphically onto a neighborhood of 0.

Now that we have constructed the Böttcher map ϕ, we turn to the problem of analytically continuing ϕ to larger and larger neighborhoods of p. Our goal is to find a critical point of f (other than p itself) that belongs to the immediate basin Ω_0 of p. The idea is that, if ϕ is defined on some neighborhood U of p in Ω_0, then ϕ can be extended to a neighborhood of U if and only if the boundary of U does not contain a critical point of f. Indeed, the conjugacy ϕ can never be defined at a critical point of f other than p, since the image of this critical point under ϕ would be a critical point of $w \mapsto w^n$ which is attracted to 0 under iteration but which is not equal to 0. No such critical point of $w \mapsto w^n$ exists.

To see, conversely, that ϕ can be extended beyond U whenever ∂U does not contain a critical point, suppose that V is an open set containing U such that $f(V) \subset U$. We now attempt, informally, to define a map $\overline{\phi}$ extending ϕ to V. If a point $z \in V$ already belongs to U, then define $\overline{\phi}(z) = \phi(z)$. If z belongs to $V \setminus U$, then we would like to define $\overline{\phi}(z)$ to be an nth root of $\phi(f(z))$, so that the extension of ϕ satisfies $\phi(f(z)) = (\phi(z))^n$. Just as when we constructed the Böttcher map in the proof of Theorem 3.1, the multivalued-ness of the nth root prohibits us from defining $\overline{\phi}$ in this way. But we can again circumvent this multivalued-ness by defining the extension $\overline{\phi}$ as a lift. The absence of critical points on ∂U is precisely what allows us to perform this lift construction. The construction is depicted in Figure 4 below, where the map being extended is denoted ψ, and the lift/extension is denoted Ψ.

This lift construction takes advantage of the following connection between critical points and covering maps: If a nonconstant holomorphic map $g : A \to \mathbb{C}$ defined on a domain $A \subset \mathbb{C}$ has no critical points, then g is automatically a covering map of A onto $g(A)$. Indeed, for each $w \in g(A)$, the preimages of w under g are isolated points in A, and by the Inverse Function Theorem, g maps a neighborhood of each preimage diffeomorphically (in fact biholomorphically) onto a neighborhood of w.

Before tackling the continuation of ϕ, which may or may not be possible throughout the entire immediate basin of p, we first show that the modulus of ϕ, that is, the map $z \mapsto |\phi(z)|$, always extends to the entire immediate basin. This fact will be helpful for proving properties of the extension of ϕ to larger neighborhoods of p.

Lemma 3.3. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map with a superattracting fixed point p, and let Ω_0 be the immediate basin of p. Let $\phi : N \to \mathbb{C}$ be the Böttcher map defined on a neighborhood $N \subset \Omega_0$ of p, which conjugates f to $w \mapsto w^n$, $n \geq 2$. Then there is a continuous map $M : \Omega_0 \to [0, 1)$ such that*

$$M(z) = |\phi(z)| \text{ for all } z \in N.$$

Proof. Fix $z \in \Omega_0$. There is a positive integer k (depending on z) such that $f^k(z)$ belongs to N, and hence so that $\phi(f^k(z))$ is defined. We define the value of M at z by

$$M(z) = |\phi(f^k(z))|^{1/n^k}.$$

Note that the multivalued-ness of the nth root poses no issue here, since we are fixing the branch that takes $[0, \infty)$ to itself. Nonetheless, we still must show that M is well-defined. Suppose ℓ is another positive integer such that $f^\ell(z)$ belongs to N. Assume $\ell > k$ without loss of generality. Then since $f^k(z)$ belongs to N, we have $\phi(f^{\ell-k}(f^k(z))) = (\phi(f^k(z)))^{n^{\ell-k}}$. Hence,

$$|\phi(f^\ell(z))|^{n^{-\ell}} = |\phi(f^{\ell-k}(f^k(z))|^{n^{-\ell}} = |\phi(f^k(z))|^{n^{-\ell} \cdot n^{\ell-k}} = |\phi(f^k(z))|^{n^{-k}}.$$

This shows that M is well-defined.

Now we show that M is continuous on Ω_0. It suffices to show that M is continuous throughout a neighborhood of every point of Ω_0. For $z_0 \in \Omega_0$, let U be a neighborhood of z_0 in Ω_0 such that $\overline{U}$ is compact. Then there is $j > 0$ such that

$f^j(z)$ belongs to N for *all* $z \in U$. Thus M is equal to the continuous function $z \mapsto |\phi(f^j(z))|^{1/n^j}$ throughout U. Thus M is continuous on Ω_0.

Finally, we show that $M(z)$ is contained in $[0, 1)$ for all z in Ω_0. First note that, since $\phi(N)$ is a neighborhood of 0, the intersection $\phi(N) \cap D$ of $\phi(N)$ with the open unit disk D is still a neighborhood of 0. Let $N' \subset N$ be the preimage of $\phi(N) \cap D$ under ϕ. The open set N' is a neighborhood of p. For each $z \in \Omega_0$, since the orbit of z converges to p, there is $m > 0$ such that $f^m(z)$ belongs to N'. By definition of N', the image $\phi(f^m(z))$ belongs to D. Thus $|\phi(f^m(z))| < 1$. Taking the n^mth root of this number then gives $M(z) < 1$. Thus $M(z)$ belongs to the interval $[0, 1)$ for all $z \in \Omega_0$. □

One consequence of Lemma 3.3 is that any holomorphic extension $\overline{\phi}$ of the Böttcher map ϕ to a larger neighborhood $N_0 \supset N$ of p always takes values in the open unit disk D. For, if $z \in N_0$, then for some $k > 0$ the iterate $f^k(z)$ belongs to N. Thus, since $(\overline{\phi}(z))^{n^k} = \overline{\phi}(f^k(z)) = \phi(f^k(z))$, by taking the modulus of both sides and then taking the n^kth root, we obtain

$$|\overline{\phi}(z)| = M(z) \in [0, 1).$$

This also shows that $z \mapsto |\overline{\phi}(z)|$ coincides with M wherever $\overline{\phi}$ is defined. In particular, the extension $\overline{\phi}$ can only take values in the open unit disk. Thus the *inverse* of $\phi : N \subset \Omega_0 \to \phi(N) \subset D$ *can only be extended as far as the open unit disk* D. Thus there must be some *maximal* open disk $D_r \subseteq D$ on which the inverse ψ of the Böttcher map can be defined. In the next theorem, we show that the presence of a critical point of f near p is determined solely by whether $r < 1$ or $r = 1$. Our proof broadly follows [4, §9].

Theorem 3.4. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map with a superattracting fixed point p, and let Ω_0 be the immediate basin of p. Let ϕ be a conjugacy between f and $w \mapsto w^n, n \geq 2$, defined on a neighborhood of p in Ω_0. Let $0 < r \leq 1$ be the radius of the maximal open disk D_r on which there exists a holomorphic map $\psi : D_r \to \Omega_0$ extending ϕ^{-1} and satisfying*

$$f(\psi(w)) = \psi(w^n) \tag{3.5}$$

for all $w \in D_r$. Then the map ψ is actually a biholomorphism onto its image, and the only critical point of f in $\psi(D_r)$ is p.

In addition, if $r = 1$, then ψ is a conformal isomorphism $D \to \Omega_0$, and so $\Omega_0 = \psi(D)$ contains no critical points of f besides p. On the other hand, if $r < 1$, then Ω_0 contains a critical point of f lying on the boundary of $\psi(D_r)$.

Proof. First we show that ψ is a biholomorphism onto its image. By the Inverse Function Theorem, it suffices to show that ψ is injective and has no critical points in D_r. We first show that ψ has no critical points. Suppose on the contrary that ψ has a critical point w_0 in D_r. Then w_0 is not 0, since ψ is invertible in a neighborhood of 0. But since w_0 is a critical point of ψ, the functional equation $f(\psi(w)) = \psi(w^n)$ implies that $w_0^{n^k}$ is a critical point of ψ for each $k \geq 0$, as

$$\psi'(w_0^{n^k}) \cdot n^k w_0^{n^k - 1} = (f^k)'(\psi(w_0)) \cdot \psi'(w_0) = 0.$$

Note that each point $w_0^{n^k}$ belongs to D_r since $|w_0| < r \leq 1$. It follows that ψ has an infinite sequence of critical points $w_0, w_0^n, w_0^{n^2}, \ldots$, converging to 0, contradicting the fact that ψ is conformal near 0. We conclude that ψ has no critical points.

Now, since ψ has no critical points in D_r, ψ is a covering map of D_r onto $\psi(D_r)$. Moreover, $\psi^{-1}(p) = \{0\}$, since if there were $w_1 \neq 0$ such that $\psi(w_1) = p$, then, by (3.5), $\psi^{-1}(p)$ would contain the infinite sequence of points $w_1, w_1^n, w_1^{n^2}, \ldots$, which converge to zero, contradicting the fact that ψ is injective near 0. Thus $\psi^{-1}(p) = \{0\}$. The cardinality of the preimage $\psi^{-1}(z)$ is thus equal to 1 for all z in $\psi(D_r)$, and it follows that ψ is injective. Hence ψ is injective and holomorphic, which implies that ψ is a biholomorphism onto its image.

We now show that f has no critical points in $\psi(D_r)$ besides 0. Indeed, if a point $\psi(w) \neq 0$ were a critical point of f for some $w \in D_r \setminus \{0\}$, then w^n would be a critical point of ψ, since

$$\psi'(w^n) \cdot nw^{n-1} = f'(\psi(w))\psi'(w) = 0.$$

Since w^n cannot be a critical point of ψ, we conclude that f has no critical points in $\psi(D_r)$ besides 0.

Now suppose $r = 1$. This means ψ is a map from the open unit disk $D = D_1$ into the immediate basin Ω_0. We now show that ψ is a conformal isomorphism between D and Ω_0. Since ψ is a biholomorphism onto its image, we must show that $\psi(D) = \Omega_0$. To prove $\psi(D) = \Omega_0$, it suffices to show that Ω_0 and $\partial\psi(D)$ are disjoint, since in this case Ω_0 is contained in the disjoint union of $\psi(D)$ with its exterior $\mathrm{ext}(\psi(D))$. Since Ω_0 is connected and intersects the connected open set $\psi(D)$, it then follows that Ω_0 is contained entirely in $\psi(D)$, and so $\psi(D) = \Omega$.

It remains to show that Ω_0 is disjoint from $\partial\psi(D)$. Suppose, on the contrary, that there is a point z belonging to $\Omega_0 \cap \partial\psi(D)$. Then there is a sequence $\{w_i\}$ in D such that $\psi(w_i) \to z$. Since the w_i are bounded in D, we may choose a subsequence $\{w_{i_k}\}$ that converges to a point $w \in \overline{D}$. The limit w cannot belong to D, since in that case continuity of ψ on D would imply $z = \psi(w)$, contradicting the fact z does not lie in $\psi(D)$. So w belongs to ∂D. In particular, the sequence of norms $|w_{i_k}|$ converges to $|w| = 1$. On the other hand, by Lemma 3.3 and the discussion following the proof of the lemma, continuity of M at z implies that the norms $|w_{i_k}| = M(\psi(w_{i_k}))$ converge to $M(z) < 1$. This contradiction implies $\partial\psi(D)$ and Ω_0 are disjoint, so we conclude that $\Omega_0 = \psi(D)$. Thus $\psi : D_r \to \Omega_0$ is a conformal isomorphism.

Now suppose $r < 1$. This means ψ is a map from the open disk $D_r \subset D$ into the immediate basin Ω_0. Set $U = \psi(D_r)$. We now show that ∂U is contained in Ω_0 and that ∂U contains a critical point of f. To see that ∂U is contained in Ω_0, it suffices to show that $f(\partial U)$ is contained in Ω_0, since this then implies that ∂U is contained in some component of the attracting basin Ω of p. We can then conclude that ∂U belongs to $\overline{\Omega_0} \cap \Omega = \Omega_0$. It remains to show that $f(\partial U)$ is contained in Ω_0. Using (3.5) and the fact that f and ψ are open maps, we have

$$f(\partial U) = \partial f(U) = \partial f(\psi(D_r)) = \partial\psi(D_{r^n}) = \psi(\partial D_{r^n}).$$

Since ∂D_{r^n} is contained in D_r, it follows that $f(\partial U)$ is contained in $\psi(D_r) \subset \Omega_0$. Thus ∂U is contained in Ω_0.

We now show that ∂U contains a critical point of f. We prove this by contradiction. The idea is that if ∂U does not contain a critical point of f, then we can extend ψ to a holomorphic map Ψ defined outside D_r and satisfying (3.5), contradicting the fact that D_r is the maximal open disk to which the inverse of the Böttcher map can be analytically continued. Intuitively, we extend ψ to a neighborhood of D_r as follows: If w lies outside D_r, then first map w into D_r by $w \mapsto w^n$. Next, map w^n

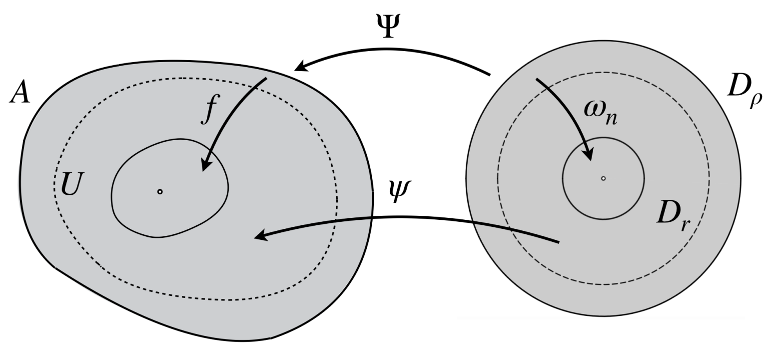

FIGURE 4. Extension of the Böttcher inverse $\psi : D_r \to U$ to a holomorphic map $\Psi : \overline{D}_\rho \to A$. The map Ψ is a lift of $\psi \circ \omega_n$ through f.

by ψ into U, and define $\Psi(w)$ to be the image of $\psi(w^n)$ through some branch of f^{-1}. This branch exists because f has no critical points near ∂U. By construction, Ψ satisfies $\Psi(w^n) = f(\Psi(w))$ outside D_r. See Figure 4. A priori, the function Ψ just described is not well-defined, since f may have many holomorphic branches to choose from. To account for this, we construct Ψ as a lift of ψ, just as we lifted $z \mapsto z^n$ in the proof of Theorem 3.1 to account for the multiple branches of the nth root. We now proceed with the construction of Ψ.

Suppose ∂U does not contain a critical point of f. To carry out the argument described above, we first pick a suitable neighborhood of U to which we can lift the map ψ, and pick a suitable disk containing D_r to which ψ can be extended. Since $f(\overline{U}) \subset U$, there is a connected open set $V \subset \Omega_0$ containing $\overline{U}$ such that V contains no critical points of f other than p, and so that $f(V) \subset U = \psi(D_r)$. In particular, $f(V)$ is an open neighborhood of $\overline{f(U)}$. By taking V to be a sufficiently small neighborhood of $\overline{U}$, we may also ensure that the closure of $f(V \setminus U)$ does not contain p. This follows from the fact that the compact set $f(\partial U) = \psi(\partial D_{r^n})$ does not contain p.

Now, since $f(V)$ contains $f(\overline{U}) = \psi(\overline{D}_{r^n})$, for each $w_0 \in \partial D_r$ there is an open ball B_{w_0} centered at w_0 such that $\overline{B_{w_0}}$ is mapped into $f(V)$ by $w \mapsto \psi(w^n)$. Since ∂D_r is compact, there are finitely many of these balls $B_1, \ldots, B_N$ that cover ∂D_r. Using the fact that ∂D_r is contained in the interior of the balls B_i, it then follows that there is $r < \rho < 1$ such that the closure of the larger disk D_ρ is contained in the union $D_r \cup B_1 \cup \ldots \cup B_N$. In particular, $\overline{D}_\rho$ is mapped by $w \mapsto \psi(w^n)$ into $f(V)$.

Let A be the connected component of the preimage $f^{-1}(\psi(\overline{D}_{\rho^n}))$ that contains p (see Figure 4). Since $\psi(\overline{D}_{\rho^n})$ is contained in $f(V)$, and $V \ni p$ is connected, A is contained in V. Thus, since $V \setminus \{p\}$ contains no critical points of f, neither does $A \setminus \{p\}$. It follows that f is a covering map of $A \setminus \{p\}$ onto $f(A) \setminus \{p\} = \psi(\overline{D}_{\rho^n} \setminus \{0\})$.

In fact, $f : A \setminus \{p\} \to f(A) \setminus \{p\}$ is an n-fold cover. To see this, note that every point of $f(U) \setminus \{p\} \subset f(A) \setminus \{p\}$ has exactly n-preimages under f in U. Indeed, if $z_0 \in f(U) \setminus \{p\}$, then the preimages of z_0 under f in U are exactly the images under ψ of the nth roots of $\psi^{-1}(z_0) \in D_{r^n}$. Moreover, if z_0 is sufficiently close to p, then z_0 has no additional preimages contained in $A \setminus U$, since $f(A \setminus U)$ is compact and does not contain p. It follows that z_0 has exactly n preimages under f in $A \setminus \{p\}$. Since $A \setminus \{p\}$ is a connected covering space of $f(A) \setminus \{p\}$, the cardinality of the preimage $f^{-1}(z) \cap A \setminus \{p\}$ is constant as z varies in $f(A) \setminus \{p\}$. Thus f is precisely an n-fold cover of $A \setminus \{p\}$ onto $f(A) \setminus \{p\}$.

We now show that the map $w \mapsto \psi(w^n)$, which sends $\overline{D}_\rho \setminus \{0\}$ to $f(A) \setminus \{p\}$, lifts to a map $\Psi : \overline{D}_\rho \setminus \{0\} \to A \setminus \{p\}$. By definition, the lift Ψ will satisfy the equation

$$f \circ \Psi(w) = \psi(w^n),$$

and hence will be a holomorphic extension of ψ to the disk D_ρ.

Let ω_n denote the nth power map $w \mapsto w^n$. To show that $\psi \circ \omega_n : \overline{D}_\rho \setminus \{0\} \to f(A) \setminus \{p\}$ lifts to $A \setminus \{p\}$, we must show

$$(\psi \circ \omega_n)_*(\pi_1(\overline{D}_\rho \setminus \{0\})) \subset f_*(\pi_1(A \setminus \{p\})),$$

where $(\psi \circ \omega_n)_*$ is the map $\pi_1(\overline{D}_\rho) \to \pi_1(f(A) \setminus \{p\})$ induced by $\psi \circ \omega_n$, and f_* is the map $\pi_1(A \setminus \{p\}) \to \pi_1(f(A) \setminus \{p\})$ induced by f.

Let $\alpha : [0,1] \to \overline{D}_\rho \setminus \{0\}$ be the path given by $\alpha(t) = \rho_0 e^{2\pi i t}$, for $0 < \rho_0 < \rho^n$, and let $\beta : [0,1] \to f(A) \setminus \{p\}$ be the path $\beta(t) = \psi(\alpha(t))$. The path α generates the fundamental groups of the two punctured disks $\overline{D}_\rho \setminus \{0\}$ and $\overline{D}_{\rho^n} \setminus \{0\}$.

We now show that β is a generator of $\pi_1(A \setminus \{p\})$. First, since α generates $\pi_1(\overline{D}_{\rho^n} \setminus \{0\})$ and ψ is a biholomorphism from $\overline{D}_{\rho^n} \setminus \{0\}$ onto $f(A) \setminus \{p\}$, it follows that $\beta = \psi \circ \alpha$ generates $\pi_1(f(A) \setminus \{p\})$. Thus to show that β generates the fundamental group of $A \setminus \{p\}$, which contains $f(A) \setminus \{p\}$, it suffices to show that $A \setminus \{p\}$ deformation retracts onto $f(A) \setminus \{p\}$. For the inclusion of $f(A) \setminus \{p\}$ into $A \setminus \{p\}$ then induces an isomorphism on the respective fundamental groups.

To prove that $A \setminus \{p\}$ deformation retracts onto $f(A) \setminus \{p\}$, we first show that $f(A) \setminus \{p\}$ deformation retracts onto $f^2(A) \setminus \{p\}$. Since $f(A) \setminus \{p\}$ is equal to $\psi(\overline{D}_{\rho^n} \setminus \{0\})$, and $f^2(A) \setminus \{p\}$ is equal to $\psi(\overline{D}_{\rho^{n^2}} \setminus \{0\})$, we obtain a deformation retract of $f(A) \setminus \{p\}$ onto $f^2(A) \setminus \{p\}$ by precomposing ψ with a deformation retract of $\overline{D}_{\rho^n} \setminus \{0\}$ onto $\overline{D}_{\rho^{n^2}} \setminus \{0\}$. Then, to obtain a deformation retract of $A \setminus \{p\}$ onto $f(A) \setminus \{p\}$ we lift the deformation retract of $f(A) \setminus \{p\}$ onto $f^2(A) \setminus \{p\}$ through the covering map f.

To be explicit, let $r_t : f(A) \setminus \{p\} \to f(A) \setminus \{p\}$ be a deformation retract of $f(A) \setminus \{p\}$ onto the subspace $f^2(A) \setminus \{p\}$. Then r_t is a homotopy relative to $f^2(A) \setminus \{p\}$ from the identity r_0 on $f(A) \setminus \{p\}$ to a retraction r_1 of $f(A) \setminus \{p\}$ onto $f^2(A) \setminus \{p\}$. Since $f \circ r_0 = f$ lifts to the identity on $A \setminus \{p\}$, the homotopy lifting property implies there is a unique homotopy $\widetilde{r}_t : A \setminus \{p\} \to A \setminus \{p\}$ lifting $f \circ r_t$ such that $\widetilde{r}_0$ is the identity on $A \setminus \{p\}$.

To show that $\widetilde{r}_t$ is a deformation retract of $A \setminus \{p\}$ onto $f(A) \setminus \{p\}$, it remains to show that the homotopy $\widetilde{r}_t$ fixes $f(A) \setminus \{p\}$ and that the image $\widetilde{r}_1(A \setminus \{p\})$ is contained in $f(A) \setminus \{p\}$. Indeed, since for each z in $f(A) \setminus \{p\}$, the path $t \mapsto \widetilde{r}_t(z)$ is a lift of the constant path $t \mapsto r_t(f(z)) = f(z)$, it follows that $t \mapsto \widetilde{r}_t(z)$ is also a constant path, with image equal to $\widetilde{r}_0(z) = z$. Thus $\widetilde{r}_t$ fixes $f(A) \setminus \{p\}$.

Now we show that $\widetilde{r}_1$ maps $A \setminus \{p\}$ into $f(A) \setminus \{p\}$. First, since $r_1(f(A) \setminus \{p\})$ is contained in $f^2(A) \setminus \{p\}$, and since $f \circ \widetilde{r}_1 = r_1 \circ f$, it follows that f maps $\widetilde{r}_1(A \setminus \{p\})$ into $f^2(A) \setminus \{p\}$. Thus each point of $\widetilde{r}_1(A \setminus \{p\})$ is a lift of a point in $f^2(A) \setminus \{p\}$. Now, each point z_0 of $f^2(A) \setminus \{p\}$ has n lifts in $A \setminus \{p\}$, corresponding under ψ^{-1} to the nth roots of $\psi^{-1}(z_0)$, which are contained in D_{ρ^n}. Thus the n lifts of any point z_0 in $f^2(A) \setminus \{p\}$ are contained in $\psi(D_{\rho^n} \setminus \{0\}) = f(A) \setminus \{p\}$. So any point in $f^2(A) \setminus \{p\}$ lifts to $f(A) \setminus \{p\}$. Since each point of $\widetilde{r}_1(A \setminus \{p\})$ is a lift of a point in $f^2(A) \setminus \{p\}$, it follows that $\widetilde{r}_1(A \setminus \{p\})$ is contained in $f(A) \setminus \{p\}$. We conclude that $\widetilde{r}_t$ is a deformation retract of $A \setminus \{p\}$ onto $f(A) \setminus \{p\}$.

Now, since $A \setminus \{p\}$ deformation retracts onto $f(A) \setminus \{p\}$, the inclusion map from $f(A) \setminus \{p\}$ into $A \setminus \{p\}$ induces an isomorphism on their fundamental groups. In particular, since the path β is a generator of $\pi_1(f(A) \setminus \{p\})$, it follows that β is also a generator of $\pi_1(A \setminus \{p\})$.

Now recall that β and α are related by the equation

$$f \circ \beta = f \circ \psi \circ \alpha = (\psi \circ \omega_n) \circ \alpha.$$

Since β generates the fundamental group of $A \setminus \{p\}$ and α generates the fundamental group of $\overline{D}_\rho \setminus \{0\}$, we conclude that

$$(\psi \circ \omega_n)_*(\pi_1(\overline{D}_\rho \setminus \{0\})) = f_*(\pi_1(A \setminus \{p\})).$$

Thus the map $\psi \circ \omega_n$ lifts to a map $\Psi : \overline{D}_\rho \setminus \{0\} \to A \setminus \{p\}$ satisfying

$$f \circ \Psi = \psi \circ \omega_n.$$

Moreover, since f is a local biholomorphism throughout $A \setminus \{p\}$, this implies Ψ is holomorphic throughout $D_\rho \setminus \{0\}$.

We now show that Ψ is actually an extension of ψ. Notice that on the punctured disk $D_r \setminus \{0\}$, where both ψ and Ψ are defined, we have

$$f \circ \Psi = \psi \circ \omega_n = f \circ \psi.$$

Thus Ψ and ψ are both lifts of the map $\psi \circ \omega_n : D_r \setminus \{0\} \to f(U) \setminus \{p\}$ to the covering space $U \setminus \{p\}$. Let w_0 be a point of $D_r \setminus \{0\}$. Since $f(\Psi(w_0)) = f(\psi(w_0))$, the points $\Psi(w_0)$ and $\psi(w_0)$ are lifts of the same point in $f(U) \setminus \{p\}$. Thus, by replacing Ψ with $\gamma \circ \Psi$, for some deck transformation $\gamma : A \setminus \{p\} \to A \setminus \{p\}$, we may assume $\Psi(w_0) = \psi(w_0)$. (Note that γ is holomorphic throughout the interior of $A \setminus \{p\}$, since $f \circ \gamma = f$ on $A \setminus \{p\}$ and f is a local biholomorphism.) But since $\Psi|_{D_r \setminus \{0\}}$ and ψ are both lifts of $\psi \circ \omega_n$ to $U \setminus \{p\}$, and two lifts of the same map (from a connected space) are determined by where they send a single point, it follows that Ψ and ψ coincide on all of $D_r \setminus \{0\}$. Moreover, since ψ is continuous at 0 with $\psi(0) = 0$, the map Ψ extends to a holomorphic function defined on all of D_ρ with $\Psi(0) = 0$. Thus Ψ is a holomorphic extension of ψ to the disk D_ρ containing D_r, and since w^n belongs to D_r for all $w \in D_\rho$, we have

$$f \circ \Psi(w) = \psi(w^n) = \Psi(w^n), \text{ for all } w \in D_\rho.$$

This contradicts maximality of the disk D_r. We conclude that f has a critical point on $\partial U \subset \Omega_0$, which completes the proof. $\square$

Theorem 3.4 tells us that the immediate basin of a superattracting fixed point for a rational map contains no additional critical points of f precisely when f is conjugate to a power map on the whole immediate basin. In particular, when the immediate basin contains no additional critical points, it is conformally isomorphic to a disk. In the next section, we show that this implies the complement of the immediate basin is connected. On the other hand, when the immediate basin does contain an additional critical point, there is a neighborhood of the fixed point that contains a critical point on its boundary. In the proof of Theorem 3.4, this neighborhood was $U = \psi(D_r)$. We will see that, when f is a nonlinear polynomial and U is the neighborhood of ∞ containing a critical point on its boundary, the exterior of U is always disconnected. The Böttcher map gives us the geometric means to understand how this disconnectedness arises.

4. Connectedness of the Filled Julia Set

The goal of this section is to prove the following theorem, which relates the connectedness of the filled Julia set of a nonlinear polynomial to the dynamics of that polynomial's critical points. The theorem is a mild extension of Theorem 2.7 stated in Section 2.

Theorem 4.1. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a polynomial of degree $d \geq 2$. Let Ω be the superattracting basin of ∞. Then $K := \mathbb{C} \setminus \Omega$ is connected if and only if every finite critical point of f has bounded forward orbit. Moreover, if K is connected, then so is its boundary $J := \partial K$, and the basin Ω is conformally isomorphic to the open unit disk. If K is not connected, then neither is J, and both K and J have uncountably many components.*

The proof of Theorem 4.1 will follow the approach suggested by Examples 2.5 and 2.6. In particular, if Ω contains a finite critical point of f, then, just as in Example 2.5, we consider an attracting neighborhood of ∞ that contains the critical value on its boundary. Such an attracting neighborhood exists by Theorem 3.4. Then we show that the boundary of the preimage of this neighborhood separates K into two nonempty disjoint open subsets, just as the figure "8" did in Example 2.5. If, on the other hand, Ω does not contain a finite critical point, then, just as in Example 2.6, the polynomial f is conjugate to a power map on all of Ω, and this conjugacy is a conformal isomorphism between Ω and the open unit disk. The fact that the conjugacy extends to the entire basin of ∞ in this case follows from Theorem 3.4.

Note that Theorem 4.1 concerns the whole superattracting basin of ∞, while Theorem 3.4, which we hope to apply in the proof of Theorem 4.1, concerns only the *immediate* basin of superattracting fixed points. Indeed, the basin of ∞ for a polynomial f is actually the same as its immediate basin. To see this, note that, since f is a polynomial, $f^{-1}(\infty) = \{\infty\}$. Thus, since every component of $f^{-1}(\Omega_0)$ contains a different preimage of ∞, it follows that $f^{-1}(\Omega_0) = \Omega_0$. Hence $\Omega = \Omega_0$, since Ω is the union of the iterated preimages of Ω_0. Thus we can apply Theorem 3.4 to the basin of ∞ for a nonlinear polynomial. The proof of Theorem 4.1 given below follows [4, §9].

Proof of Theorem 4.1. First suppose Ω contains no finite critical points of f. Then, by Theorem 3.4, the Böttcher map ϕ extends to all of Ω and is a conformal isomorphism between Ω and the open unit disk D. The fact that $\partial\Omega$ and $\mathbb{C} \setminus \Omega$ are connected now follows as a purely topological consequence of this isomorphism.

Indeed, to see that $\partial\Omega$ is connected, we approximate it by a sequence of closed topological annuli, which are images of annuli under the inverse ψ of ϕ. For all $n \geq 1$, let $A_n \subset D$ be the annulus

$$A_n = \{z \in D : 1 - 1/n < |z| < 1\}.$$

Then consider the images $\psi(A_n)$ of these annuli in Ω. For each n, the boundary $\partial\Omega$ of the basin Ω is contained in $\overline{\psi(A_n)}$. Indeed, if $z \in \partial\Omega$, then there is a sequence of points $\{z_i\} \subset \Omega$ that converges to z. Since the sequence of images $\phi(z_i)$ is contained in the open unit disk D, there is a subsequence $\{z_{i_k}\}$ such that $\phi(z_{i_k})$ converges to a point $w \in \overline{D}$. The limit w cannot belong to D, since continuity of ψ would

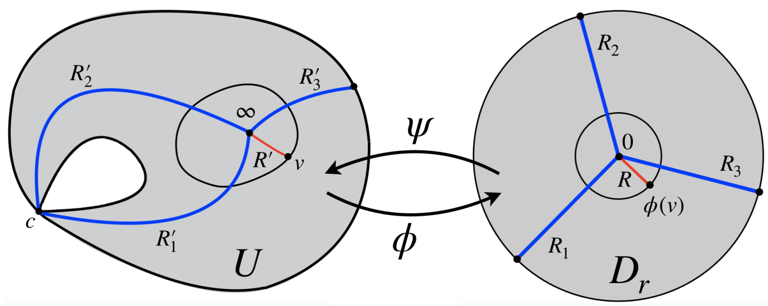

FIGURE 5. The proof of Theorem 4.1, pictured for $n = 3$.

then imply that the unique limit z of the points $z_{i_k} = \psi(\phi(z_{i_k}))$ belongs to Ω, contrary to our assumption. Thus $w \in \partial D$. In particular, for i_k sufficiently large, $\phi(z_{i_k})$ belongs to the annulus A_n, and so $z_{i_k} = \psi(\phi(z_{i_k}))$ belongs to $\psi(A_n)$. Thus z belongs to $\overline{\psi(A_n)}$. Since this holds for every $n \geq 0$ and every $z \in \partial\Omega$, we conclude that $\partial\Omega$ is contained in the intersection $\bigcap_{n\geq 0} \overline{\psi(A_n)} \subset \overline{\Omega}$.

In fact, $\partial\Omega$ is precisely equal to this intersection, since any $z \in \Omega$ is contained in the image of the disk $D_{|\phi(z)|}$ of radius $|\phi(z)|$ under ψ, and $D_{|\phi(z)|}$ is disjoint from $\overline{A_n}$ when n is sufficiently large. Thus we have

$$\partial\Omega = \bigcap_{n\geq 0} \overline{\psi(A_n)}.$$

In particular, $\partial\Omega$ is the intersection of a descending sequence of compact connected subsets of $\mathbb{C}$, which implies $\partial\Omega$ is connected. It then follows that $K = \mathbb{C} \setminus \Omega$ is connected, since it is a closed subset with connected boundary $\partial K = \partial\Omega$. This completes the case when Ω contains no finite critical points.

Now suppose Ω contains a finite critical point of f. We show that K and its boundary $J := \partial K$ are disconnected. Then we use the total invariance of J and K (which follows from the total invariance of Ω) to show that they both in fact have *uncountably many* components. The disconnectedness of K will arise in precisely the same way it did in Example 2.5. Namely, we will take the preimage of a neighborhood of ∞ containing the critical value on its boundary and show that the complement of this preimage has disconnected interior. After our work in the previous section studying the Böttcher map, we now have the tools to follow this procedure for a general polynomial f.

Since Ω contains a finite critical point of f, Theorem 3.4 implies there is a neighborhood U of ∞ whose boundary $\partial U \subset \Omega$ contains a critical point c of f, and the Böttcher map extends to a conformal isomorphism from U to an open disk D_r centered at the origin, $0 < r < 1$. We now show that the interior of $\hat{\mathbb{C}} \setminus U$ is disconnected. To do this, we study how $f(U)$ transforms into U under inverse iteration of f. Since $f(U)$ contains the critical value $v := f(c)$ on its boundary, and since v has fewer distinct preimages than all points of $f(U) \setminus \{\infty\}$, we will see that $\overline{U}$ resembles a closed disk with finitely many points on its boundary identified. See Figure 5.

To see geometrically how this identification takes place, consider the ray $R = \{t\phi(v) : 0 \leq t \leq 1\}$ in $\overline{D}_{r^n}$ connecting $\phi(v)$ to 0, and consider the corresponding ray $R' = \psi(R)$ in $\overline{f(U)}$ connecting v to ∞ (see Figure 5). In D_r, the preimage of the ray R under $w \mapsto w^n$ consists of n rays $R_1, \dots, R_n$, each connecting 0 to a different nth root of $\phi(v)$. The rays R_i intersect at ∞ but are otherwise disjoint. Thus, since the conjugacy ψ is an isomorphism of D_r onto U, the corresponding preimages $R'_1, \dots, R'_n$ of R' under f meet at ∞, but are otherwise disjoint *in* U. On the *boundary* of U, however, at least two of the rays R'_i will intersect (see Figure 5). Indeed, each of the n rays R'_i connects ∞ to a different preimage of v. And since v is a critical value, v *has fewer than* n *preimages*. Indeed, c is a multiple root of the polynomial $z \mapsto f(z) - v$. Thus at least two of the rays R'_i intersect on ∂U. In fact two of the rays must intersect at the critical point c, since f sends a punctured neighborhood of c to a punctured neighborhood of v by an $(m+1)$-fold cover, where m is the multiplicity of c as a root of f'. So in fact there are m rays R'_i that meet at c.

Let R'_1 and R'_2 be two of the rays that meet at c. The rays have the same endpoints, ∞ and c, but are otherwise disjoint. Thus the rays together form a simple closed curve that separates the sphere $\hat{\mathbb{C}}$ into two disjoint open disks V_0 and V_1. To show that K is disconnected, we show that K is contained in the union $V_0 \cup V_1$ and that K has nonmempty intersection with both V_0 and V_1. First note that the boundary of each V_j is the union $R'_1 \cup R'_2$. Thus the boundary of $f(V_j)$ is equal to the single ray R'. But this means that the complement $\hat{\mathbb{C}} \setminus R'$ is contained in the union of $f(V_j)$ with its exterior $\mathrm{ext}(f(V_j))$. Since $\hat{\mathbb{C}} \setminus R'$ is connected and has nonempty intersection with $f(V_j)$, it follows $\hat{\mathbb{C}} \setminus R'$ is precisely equal to $f(V_j)$. In particular, $f(V_j)$ contains K. The total invariance of K then implies that V_j and K have nonempty intersection. Thus K is disconnected since K is equal to the union of the nonempty disjoint open subsets $K \cap V_0$ and $K \cap V_1$. Since K is closed, this implies that $J = \partial K$ is also disconnected, also separated by V_0 and V_1.

We now show that J and K both have uncountably many components. Let $L \subset \hat{\mathbb{C}}$ be either K or J. The important property of L, here, is that $f(L) = L = f^{-1}(L)$. We have seen that L be can written as the union of two nonempty disjoint open subsets $L_0 := L \cap V_0$ and $L_1 := L \cap V_1$. Now, if $z \in L$, then each of its iterates $f^k(z)$ belongs to L; in particular, $f^k(z)$ belongs either to L_0 or to L_1. For example, there might be some $z \in L$ such that z belongs to L_0, while its first iterate $f(z)$ belongs to L_1, and its second iterate $f^2(z)$ belongs to L_0, and so on. Thus, to each $z \in L$ we can associate a sequence $s(z) = (s_0 s_1 s_2 s_3 \cdots)$ of 0s and 1s such that the kth iterate $f^k(z)$ of z belongs to the component L_{s_k} for each $k \geq 0$. In a sense the sequence $(s_k)_{k \geq 0}$ tracks the orbit of z in L.

Now let z and z' be elements of L whose orbits correspond to different sequences $s(z) = (s_0 s_1 s_2 s_3 \cdots)$ and $s(z') = (s'_0 s'_1 s'_2 s'_3 \cdots)$. Two such elements exist since, for example, we can take z in L_0 and z' in L_1, so that $s_0 \neq s'_0$. We show that z and z' belong to different components of L. Indeed, since the sequences $s(z)$ and $s(z')$ are not the same, there must be some index $k \geq 0$ such that $s_k \neq s'_k$. By definition, this means $f^k(z)$ belongs to L_{s_k} while $f^k(z')$ belongs to $L_{s'_k}$. Equivalently, z belongs to $L \cap f^{-k}(V_{s_k})$ while z' belongs to $L \cap f^{-k}(V_{s'_k})$. Since $L \cap f^{-k}(V_{s_k})$ and $L \cap f^{-k}(V_{s'_k})$ are disjoint open subsets, this shows that z and z' do not belong to the same connected component of L.

Let $\mathcal{S}$ be the set of sequences $(s_0 s_1 s_2 s_3 \cdots)$ of 0s and 1s such that $(s_0 s_1 s_2 s_3 \cdots) = s(z)$ for some z in L. We have shown that every element of $\mathcal{S}$ corresponds to a different connected component of L. To show that L has uncountably many components, it suffices to show that $\mathcal{S}$ is uncountable. We show in fact that $\mathcal{S}$ contains *every* possible sequence of 0s and 1s; that is, we show that for every sequence $(s_0 s_1 s_2 s_3 \cdots)$ there is a point z in L with $s(z) = (s_0 s_1 s_2 s_3 \cdots)$.

First we show that for every *finite* sequence $(s_0 s_1 \dots s_k)$, there is a point z in L such that $f^\ell(z) = s_\ell$ for all $\ell = 0, \dots k$. Thus we must show that the intersection

$$L_{s_0 s_1 \dots s_k} := L_{s_0} \cap f^{-1}(L_{s_1}) \cap \dots \cap f^{-k}(L_{s_k})$$

is nonempty for every finite sequence $(s_0 s_1 \dots s_k)$. We induct on k. We have already seen that L_{s_0} is nonempty and satisfies $f(L_{s_0}) = L$ for all $s_0 \in \{0, 1\}$. Now assume that for every sequence $(s_0 s_1 \dots s_m)$ of length $m+1$ the set $L_{s_0 s_1 \dots s_m}$ is nonempty. Let $(t_0 t_1 \dots t_{m+1})$ be an arbitrary sequence of 0s and 1s of length $m+2$. By hypothesis $L_{t_1 t_2 \dots t_{m+1}}$ is nonempty. Thus since $f(L_{t_0}) = L$ contains the set $L_{t_1 t_2 \dots t_{m+1}}$, the formula

$$L_{t_0 t_1 \dots t_{m+1}} = L_{t_0} \cap f^{-1}(L_{t_1 t_2 \dots t_{m+1}})$$

implies that $L_{t_0 t_1 \dots t_{m+1}}$ is nonempty. This completes the induction and proves that the set $L_{s_0 s_1 \dots s_k}$ is nonempty for every finite sequence $(s_0 s_1 \dots s_k)$, for all $k \geq 0$.

Now, an infinite sequence $(s_0 s_1 s_2 \cdots)$ of 0s and 1s belongs to $\mathcal{S}$ if there is a point z in L such that $f^k(z)$ belongs to L_{s_k} for all $k \geq 0$. Equivalently, the sequence $(s_0 s_1 s_2 \dots)$ belongs to $\mathcal{S}$ if the intersection

$$\bigcap_{k=0}^{\infty} f^{-k}(L_{s_k}) = \bigcap_{k=0}^{\infty} L_{s_0 s_1 \dots s_k}$$

is nonempty. But, for every sequence $(s_0 s_1 s_2 \dots)$, the sets $L_{s_0 s_1 \dots s_k}$, $k \geq 0$, are nonempty and compact and form a descending sequence

$$L_{s_0} \supset L_{s_0 s_1} \supset L_{s_0 s_1 s_2} \supset \cdots .$$

Thus the intersection $\bigcap_{k=0}^{\infty} f^{-k}(L_{s_k})$ is nonempty for every sequence $(s_0 s_1 s_2 \dots)$, which implies $\mathcal{S}$ contains every infinite sequence of 0s and 1s. In particular, $\mathcal{S}$ is uncountable. We conclude that L has uncountably many components. $\square$

Theorem 4.1 shows that the attracting basin of ∞ for a polynomial f is simply connected if and only if it does not contain a finite critical point of f. An important special case of the theorem is when f is a quadratic polynomial. Since quadratics have a unique finite critical point, the basin of ∞ for a quadratic is simply connected if and only if the orbit of the finite critical point is bounded. In particular, the topology of the basin is determined by the orbit of a single point!

The strong connection given in Theorem 4.1 between the topology of the basin of ∞ and the critical orbits of a polynomial does not extend more generally to the immediate basins of *finite* attracting fixed points. In fact, the immediate basin of a finite attracting fixed point for a polynomial is always simply connected. We prove this below. We also give an example of a (necessarily simply connected) finite superattracting basin that contains multiple critical points, in direct contrast with Theorem 4.1.

First we prove the simple connectivity of finite attracting basins. The key is to use the Maximum Modulus Principle.

Proposition 4.2. *Suppose $p \in \mathbb{C}$ is an attracting fixed point for a polynomial f of degree $d \geq 2$. Then the immediate basin Ω_0 of p is simply connected.*

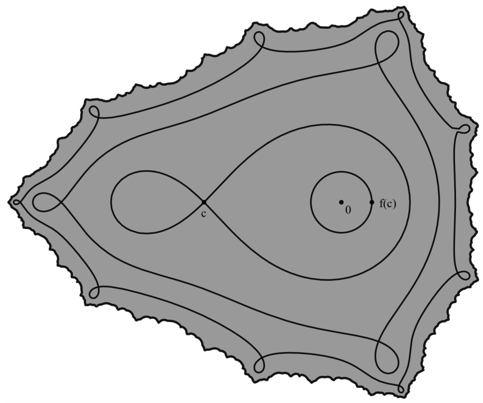

FIGURE 6. Immediate basin of 0 for $f : z \mapsto z^3 + z^2$. The curves are the iterated preimages of $\partial B = \partial D_{4/27}$. See Example 4.3.

Proof. To prove that Ω_0 is simply connected, we show that $K = \hat{\mathbb{C}} \setminus \Omega_0$ is connected. To prove that K is connected, we show that there is no simple closed curve contained in Ω_0 that separates K. Suppose γ is the image of a simple closed curve contained in Ω_0. Since the orbit of every $z \in \gamma$ converges to p, and γ is compact, there is a sequence of positive numbers $r_k \to 0$ such that $|f^k(z) - p| \leq r_k$ for every $k \geq 0$ and $z \in \gamma$. Let U be the bounded component of $\hat{\mathbb{C}} \setminus \gamma$. Since the boundary of U is precisely γ, and f is holomorphic throughout a neighborhood of $\overline{U}$, the Maximum Modulus Principle implies that $|f^k(z) - p| \leq r_k$ for all z in U. Since $r_k \to 0$, this implies that U is contained in Ω_0. Thus K is contained entirely in the unbounded component of $\hat{\mathbb{C}} \setminus \gamma$, which means γ does not separate K. Since this is true for every simple closed curve contained in Ω_0, it follows that K is connected. Thus Ω_0 is simply connected. $\square$

Proposition 4.2 shows that, in contrast to the superattracting basin of ∞, finite basins of attraction for a polynomial are always simply connected. The contrast is even sharper for a finite *geometrically* attracting fixed point, since it turns out its immediate basin *always* contains a critical point (see [4, §8]*). Moreover, the topology of a finite superattracting basin is not determined as in Theorem 4.1 by whether or not it contains several critical points. We now give an example of a simply connected superattracting basin that contains multiple critical points.

Example 4.3. Consider the cubic $f(z) = z^3 + z^2$. The map f has a superattracting fixed point at 0. The orbit of the critical point $c = -2/3$ of f converges to 0, so c belongs to the basin Ω of 0. In fact, since the whole interval $[-2/3, 0]$ is contained in the basin (see Figure 6), it follows that the critical point c is contained in the immediate basin. Nonetheless, Proposition 4.2 implies that the immediate basin of 0 is simply connected. The immediate basin of 0 is pictured in Figure 6. The curves drawn in the basin are the successive preimages of the $\partial D_{4/27}$, which contains the critical value $f(c) = 4/27$ on its boundary. The disk B bounded by this circle is

*The proof of this fact in [4, §8] is similar to the proof of Theorem 3.4 given above. The main idea again is to extend a conjugacy until its image hits a critical point.

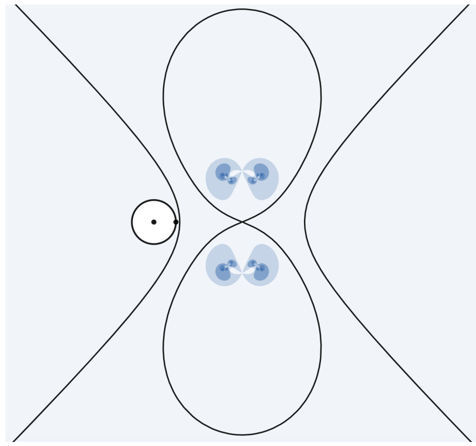

FIGURE 7. An attracting neighborhood (the white disk) of 0 for the map f of Example 4.4, and the iterated preimages of this neighborhood. The attracting neighborhood for 0 contains the critical value 1/8 on its boundary. The complement of the basin of 0 is contained in the darker region in the middle of the picture.

an attracting neighborhood of 0. Note that, for each iterated preimage of $\overline{B}$, the component containing 0 is simply connected.

Note also that every finite critical point of f is contained in the basin of 0, so the basin of ∞ for $f(z) = z^3 + z^2$ is simply connected by Theorem 4.1.

Though for polynomials, the immediate basin of a geometrically attracting fixed point is always simply connected, this is false for general rational functions. We now give an example of a rational function that has a geometrically attracting fixed point whose immediate basin is not simply connected.

Example 4.4. Consider the rational function

$$f(z) = \frac{z(z-1)}{2(2z-1)^2}.$$

The map f has 0 as a geometrically attracting fixed point, with $f'(0) = -1/2$. The only finite critical point of f is $z = 1/2$, but ∞ is also a critical point, since the derivative of $1/f(1/z)$ vanishes at 0. The whole extended real line $\mathbb{R} \cup \{\infty\}$ is contained in the basin of 0, so both critical points $z = \infty$ and $z = 1/2$ are contained in the immediate basin. See Figure 7.

To show that the immediate basin Ω of 0 is multiply connected, we first pick an attracting neighborhood of 0 containing the critical value $f(\infty) = 1/8$. The open disk $D_{1/8}$ of radius 1/8 about 0 works. The preimage of this disk has two simply connected components, which are the two regions in Figure 7 bounded by the two branches of a hyperbola. Each of these two regions is actually the lobe of a figure "8", where the two lobes meet at ∞. Let L be the open lobe containing $D_{1/8}$. Note that the critical value $f(1/2) = \infty$ lies on the boundary of L (see Figure 7). The complement of $f^{-1}(L)$ is a vertical figure "8" and has disconnected interior. Since the interior of each lobe of $\hat{\mathbb{C}} \setminus f^{-1}(L)$ meets $\hat{\mathbb{C}} \setminus \Omega$, we conclude that $\hat{\mathbb{C}} \setminus \Omega$ is

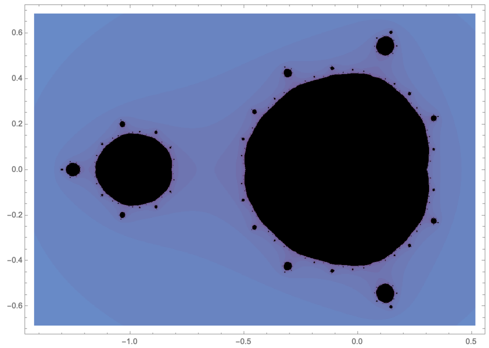

FIGURE 8. The filled Julia set for $z \mapsto \frac{5}{2}(z^3 + z^2)$. See Example 4.5. This picture was generated using Mathematica.

disconnected. Thus the basin of 0 is multiply connected, and it appears to be the complement of a Cantor set. See Figure 7.

A less pictorial way to see that the immediate basin is multiply connected is to note that f has additional fixed points $z = (9 \pm i\sqrt{15})/16$, neither of which belongs to the immediate basin of 0. Since the immediate basin contains $\mathbb{R} \cup \{\infty\}$, the complement of the basin of 0 has at least two connected components contained in the upper and lower half-planes, respectively.

All of the multiply connected immediate basins pictured above have appeared to be complements of Cantor sets. We conclude now with an example of a polynomial whose superattracting basin is multiply connected but is not the complement of a Cantor set.

Example 4.5. Consider the cubic $f(z) = \frac{5}{2}(z^3 + z^2)$, which has a superattracting fixed point at 0 and repelling fixed points at $z = (-5 \pm \sqrt{65})/10$. The map f has another critical point $c = -2/3$, which belongs to the basin of ∞. Note for example that $f(c) = 10/27 = 0.\overline{370}$ is greater than the repelling fixed point $(-5+\sqrt{65})/10 \approx 0.30622\ldots$, so it is clear from the graph of f on $\mathbb{R}$ that the orbit of c diverges. Since the basin of ∞ contains a critical point, its complement K has uncountably many components. Yet K has nonempty interior, since it contains the attracting basin of 0. In fact, K has infinitely many components with nonempty interior. See Figure 8.

Acknowledgments

I would like to thank first and foremost my mentor, Meg Doucette, for her detailed feedback on all drafts of this paper. It is my pleasure also to thank Professor Amie Wilkinson for encouraging me to scrap the original topic I had chosen—the original paper I'd had in mind would not have been as fun to write. Finally, I am grateful to Professor Peter Pay for organizing the University of Chicago REU and for scheduling engaging speakers throughout the program.

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