ERDÖS DISTANCE PROBLEMS

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Abstract. Consider n points in the plane. In 1946, Paul Erdös proposed two questions relating to distances between pairs of points. What is the maximum amount of pairs that are exactly one unit distance apart? What is the minimum amount of distinct distances between pairs of points. We will discuss the current best known bounds on these questions.

Contents

1. Preliminaries

Consider n points in a plane. What is the maximum amount of pairs of points that are one unit distance apart? What is the minimum amount of distinct distances between pairs of points? The former question is referred to as the unit distance problem and the latter is referred to as the distinct distances problem. These questions are considered twin problems proposed by Paul Erdös in 1946 [1]. Although simple questions, the search for answers has generated years research into fields like discrete geometry, combinatorics, and number theory.

Notation 1.1. This paper will use $\lesssim$, $\gtrsim$, and $\sim$ to denote inequalities up to multiplicative absolute constants. For example, $X \lesssim Y$ implies that there exists an absolute constant C such that $X \leq CY$. Occasionally, the standard asymptotic notation from computer science of $O(\cdot), \Theta(\cdot)$, and $\Omega(\cdot)$ will also be used. That is to say that $X = O(Y)$ is equivalent to $X \lesssim Y$, $X = \Omega(Y)$ is equivalent to $X \gtrsim Y$, and $X = \Theta(Y)$ is equivalent to $X \sim Y$. This allows for statements such as $X = 2^{O(Y)}$ to imply that there exists an absolute constant C such that $X \leq 2^{CY}$.

Within this paper, we will discuss known proofs on the current bounds of $O(n^{4/3})$ proven by Spencer, Szemerédi, and Trotter [2] on the unit distance question and

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$\Theta(\frac{n}{\log n})$ proven by Guth and Katz [3] on the distinct distance problem. We will begin with an introduction to incidence theory and the methods used to show the bounds mentioned above, which include Szekely's crossing inequality method [4] and Guth and Katz's polynomial method.

Before we get into the problems, it is important to note this history of the problems and how its previous bounds were improved. When Erdös first proposed the questions, he proved a lower bound of $n^{1+c/\log\log n}$ and an upper bound of $O(n^{3/2})$ on the unit distance problem and conjectured a lower bound of $\Theta(\frac{n}{\sqrt{\log n}})$. His approach includes first understanding the combinatorial nature of the question and then associating a graph to the geometric situation. Then, he considers a *forbidden subgraph*, which is a subgraph that does not follow the characterization of a graph. A well known example of a forbidden subgraph argument is Kuratowski's Theorem, which characterizes planar graphs through the exclusion of the forbidden subgraphs of K_5 or $K_{3,3}$. Then, Erdös applies known results to bound the edges of the associated graph. Erdös's approach created a tremendous impact on combinatorial geometry and has motivated many new problems and results. Today, the bound on the unit distance problem has still not been improved and Erdös distinct distance conjecture has still not been proven. We will not contribute anything to the conversation, but we will provide a nice little walk on the journey to the best bounds we know so far.

2. Introduction to Incidences

Definition 2.1. Let P denote a finite set of points and let V denote a finite set of geometric objects (e.g. lines, circles, polygons, etc). An incidence is a pair $(p, v) \in P \times V$ such that the point p is contained in the geometric object v. The set of incidences for a finite set of points P and finite geometric objects V will be denoted by $I(P, V)$.

Proofs for both distance problems rely on bounding incidences. We use incidences for the distance problems because of the ability to impose combinatorial and algebraic properties on them. Therefore, understanding incidences and their nature becomes crucial to understanding the distance problems. Classical incidence problems are concerned with incidences derived from a set of n points and m lines.

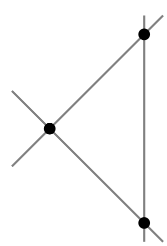

Figure 1. Six incidences between three points and three lines.

To motivate an incidence approach to the unit distance problem, consider a set P of n points and denote the number of unit distances that occur as $u(n)$. Now draw a unit circle (a circle of radius one) around each point of P. For points p and q in P that determine a unit distance, we count two incidences: the circle around p incident to q and vice versa. Therefore, we can rephrase the unit distance question in the following way: Given a set of n points and n unit circles, what is the maximum amount of incidences that can occur?

2.1. **The Szemerédi-Trotter Theorem.**

The Szemerédi-Trotter Theorem gives us a bound on incidences between points and lines. It is the most fundamental bound to incidence geometry and was partly inspired by the distance problems.

Theorem 2.2. *(Szemerédi-Trotter) For a finite set of points P and a finite set of lines L both in $\mathbb{R}^2$, $|I(P,L)| \lesssim (|P| \cdot |L|)^{2/3} + |P| + |L|$.*

For the purposes of this paper, we will only consider incidences in $\mathbb{R}^2$, but we note that Szemerédi-Trotter does not hold in finite fields. We come across a counterexample by considering incidences in $\mathbb{F}^2$. The original proof presented by Szemerédi and Trotter [2] is complicated and lengthy. The general idea behind the proof is to create cells (or convex open sets) in the plane $\mathbb{R}^2$ and bound incidences within each cell to amplify the overall bound. We will attempt a similar proof in spirit of the original by partitioning the plane through polynomials. However, we simplify the original proof through a method attributed to Guth-Katz by using a more trivial bound and invoking a Polynomial Ham Sandwich. Then, we will prove this theorem in a very elegant graph theory approach that we can attribute to Székely. Both methods will mirror how we will approach the distinct and unit distance problems respectively.

To first gain a general intuition about incidences and counting them, we will provide an example to show that this bound is tight. Let L be the set of $N \sim M^3$ lines defined by the form $\{(x,y) \in \mathbb{R}^2 : y = ax + b\}$, where a is in the set of integers $[M]$ and b is in the set of integers $[M^2]$. Now, let P be the set of $N \sim M^3$ points such that $P = \{(x,y) \in \mathbb{R}^2 : x \in [M], y \in [2M^2]\}$. Notice that for each $x \in [M]$, we have that $ax + b \leq 2M^2$ for $a \in [M]$ and $b \in [M^2]$. Therefore, if we let $y = ax + b$ (i.e. (x,y) is on a line $l \in L$) we have $(x,y) \in P$. Therefore, each line $l \in L$ intersects P in at least M points. This gives a total of $M^4 \sim N^{4/3}$ incidences. And thus, our bound is tight.

Lemma 2.3. $|I(P,L)| \lesssim |P| \cdot |L|^{1/2} + |L|$ *and* $|I(P,L)| \lesssim |L| \cdot |P|^{1/2} + |P|$

Proof. We will only prove $I(P,L) \lesssim |P| \cdot |L|^{1/2} + |L|$ as the proof for the other statement can be acheived symmetrically. First, we can note that

$$I(P,L) \leq |P|^2 + |L|.$$

We arrive at this by first counting incidences on lines with at most one point on them and then counting all incidences on other lines with at least 2 points on them. Since there are $|L|$ lines, there are at most $|L|$ incidences contributed by lines with at most one point on them. Then, note that the maximum amount of times a point $p \in P$ can be incident to a line with at least two points on it is $|P|$. If not, then each of the lines p is incident to contains at least one other point (and these points are distinct because lines in Euclidean geometry intersect in at most one point), implying that there are more than $|P|$ points. Therefore the maximum incidences contributed by lines with at least two points on them is $|P|^2$.

Now, let $1_{p\in l}$ be an indicator function having value 1 if p is incident to line l and 0 otherwise. Then, we have that

$$|I(P,L)|^2 = \left(\sum_{l\in L}\sum_{p\in P} 1_{p\in l}\right)^2$$

Now, using the Cauchy-Schwartz inequality which states that

$$\left(\sum_{i=1}^{k} a_i \cdot b_i\right)^2 \leq \left(\sum_{i=1}^{k} a_i^2\right)\left(\sum_{i=1}^{k} b_i^2\right)$$

if we let $a_i = 1$ and $b_i = 1_{p\in l}$, then we have that

$$\begin{aligned}
|I(P,L)|^2 &\leq |L| \cdot \sum_{l\in L}\left(\sum_{p\in P} 1_{p\in l}\right)^2 \\
&= |L| \cdot \sum_{p_1,p_2\in P}\sum_{l\in L} 1_{p_1\in l}\cdot 1_{p_2\in l} \\
&= |L| \cdot \left(\sum_{p_1=p_2\in P}\sum_{l\in L} 1_{p_1\in l}\cdot 1_{p_2\in l} + \sum_{p_1\neq p_2\in P}\sum_{l\in L} 1_{p_1\in l}\cdot 1_{p_2\in l}\right)
\end{aligned}$$

Note here that $|I(P,L)| = \sum_{l\in L}\sum_{p\in P} 1_{p\in l} = \sum_{p_1=p_2\in P}\sum_{l\in L} 1_{p_1\in l}\cdot 1_{p_2\in l}$. And the other summation where $p_1 \neq p_2$ is equivalent to finding all pairs of distinct points in P such that they are incident to the same line. Notice that this is equivalent to counting incidences on lines with at least two points incident to them, and by the proof of Lemma 2.4, we have that this is bounded above by $|P|^2$. Therefore, we have that

$$\begin{aligned}
|I(P,L)|^2 &\leq |L| \cdot (I(P,L) + |P|^2) \\
&\leq 2|P|^2|L| + |L|^2
\end{aligned}$$

Therefore, this proves our bound $|I(P,L)| \lesssim |P||L|^{1/2} + |L|$. □

We will use this bound again in our first proof of the Szemerédi-Trotter Theorem. However, we mention this proof prematurely to familiarize the reader with bounding incidences. Within this proof, we use a common combinatorial strategy of *double counting* to bound incidences. By counting incidences in two ways and comparing these two bounds, we are able to find a stronger bound. This strategy will continue rather frequently throughout this paper.

2.2. **The Polynomial Method and Cell Partitioning.** A common method to count incidences in a field or plane called the **polynomial method** imposes an algebraic structure on geometric incidences. By considering n points in a plane, we introduce polynomials to our problem by finding polynomials that vanish at those points. We gravitate towards this method because the point-line configurations that determine incidences might lead us to expect some sort of lattice structure. The basis of the polynomial method involves bounding the attributes of the polynomial (like the degree and roots), and thus bounding incidences represented by the polynomial. We are interested in *nonzero* polynomials, meaning polynomials with at least one nonzero coefficient.

Formally, we can sum up the core polynomial method into the following claim:

Claim 2.4. Let $P \subset \mathbb{R}^n$ be a finite set. If $|P| < \binom{n+d}{d}$, then there exists a nonzero polynomial $g \in \mathbb{R}[x_1, ..., x_n]$ of degree at most d such that $g(p) = 0$ for all $p \in P$.

Proof. Note that for a degree d polynomial in n variables, each monomial term must have a degree term at most d. Therefore, if we let d_i denote the degree of x_i term in a monomial, then $d_1 + d_2 + ... + d_n \leq d$. Therefore, by a stars and bars combinatorial argument we have that there are exactly $\binom{n+d}{d}$ coefficients in our polynomial. Let f be an arbitrary polynomial in n variables with degree d. Consider the map that goes from the coefficients of f to $(f(p_1), f(p_2), ..., f(p_{|P|}))$ where $p_1, p_2, ..., p_{|P|}$ are the elements of P. Thus, since the map is linear, the dimension of the domain is $\binom{n+d}{d}$ and the dimension of the range space is $|P| < \binom{n+d}{d}$. Therefore, by the Rank-Nullity theorem, the dimension of the null space is greater than 1, implying that there is a non-trivial null space. Thus, there exists a a polynomial g of degree d in n variables that vanishes for all $p \in P$ $\square$

In relation to the distinct distances problem, the polynomial method becomes useful because we create what we call *cell partitions* of the plane. These are open, convex sets where we can apply more trivial, well-known incidence bounds to amplify the bound over the entire plane. We will do this through Guth and Katz's method of creating a Polynomial Ham Sandwich, originally proved by Stone-Turkey. We will use *hypersurfaces*, which are geometric objects defined by polynomials, in order to create the 'walls' of our cells.

Definition 2.5. A **hypersurface** is a set $H = \{x \in \mathbb{R}^n : h(x) = 0\}$ where $h(x)$ is a polynomial in n variables $x_1, x_2, ..., x_n$ of arbitrary degree d.

Example 2.6. The set $H = \{x \in \mathbb{R}^n : h(x) = 0\}$ where $h(x)$ is a degree 1 polynomial is called a **hyperplane**.

Example 2.7. The set $H = \{x \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + ... + x_{n+1}^2 = 1\}$ defines a hypersurface called the **unit n-sphere**.

Property 2.8. A hyperplane divides $\mathbb{R}^n$ into two 'halves', which we will denote with H^+ and H^- defined as:

$$H^+ = \{x \in \mathbb{R}^n : h(x) > 0\}$$
$$H^- = \{x \in \mathbb{R}^n : h(x) < 0\}$$

We apply this property of hypersurfaces dividing the plane to divide a finite amount of sets. These next series of partitioning theorems are called 'ham sandwich theorems,' cleverly named by their discoverers, Stone and Turkey.

Theorem 2.9. (Ham Sandwich Theorem) *Let $U_1, U_2, ..., U_n \in \mathbb{R}^n$ be open bounded sets. Then, there exists a hyperplane such that for all $i \in [n]$, the sets $U_i \cap H^+$ and $U_i \cap H^-$ have the same volume.*

To prove the Ham Sandwich Theorem, we invoke the Borsuk-Ulam Theorem.

Theorem 2.10. (Borsuk-Ulam) *Let $S^n \subset \mathbb{R}^n$ be the unit n-sphere. Let $f : S^n \to \mathbb{R}^n$ be a continuous map such that for all $x \in S^n$, we have that $f(-x) = -f(x)$. Then, there exists some y such that $f(y) = 0$.*

A proof for the Borsuk-Ulam theorem may be found in [5], and uses some basic tools from algebraic topology.

Proof of Ham Sandwich. Let $v_h = (h_1, h_2, ..., h_n)$ be a vector such that $h_1^2 + h_2^2 + ... + h_n^2 = 1$. Therefore, $v_h \in S^n$. The components of v_h define the coefficients of a degree one polynomial $h(x) = h_0 + h_1x_1 + ... + h_nx_n$ that defines a hyperplane

$$H = \{x \in \mathbb{R}^n : h(x) = 0\}.$$

Let us define $f : S^n \to \mathbb{R}^n$ by

$$f(v_h) = \sum_{i=1}^{n}(|U_i \cap H^+| - |U_i \cap H^-|)e_i.$$

Note here that the vector $-v_h$ defines a hyperplane that flips the two partitions of the hyperplane defined by v_h. Therefore, we have that

$$\begin{aligned} f(-v_h) &= \sum_{i=1}^{n}(|U_i \cap H^-| - |U_i \cap H^+|)e_i \\ &= -\sum_{i=1}^{n}(|U_i \cap H^+| - |U_i \cap H^-|)e_i \\ &= -f(v_h). \end{aligned}$$

Additionally, f is continuous because volume varies continuously. Thus, we can apply the Borsuk-Ulam and note that there exists some $v \in S^n$ such that $f(v) = 0$. Therefore the hyperplane H_0 defined by v satisfies

$$|U_i \cap H_0^-| = |U_i \cap H_0^+|$$

for all $i \in \{1, 2, ..., n\}$. Thus, we have proven the Ham Sandwich Theorem. □

Now, that we have proven the Ham-Sandwich theorem, we can derive the Polynomial Ham Sandwich Theorem to create cell partitions to count incidences in the plane.

Theorem 2.11. (Polynomial Ham Sandwich) *Let $U_1, U_2, ..., U_t \in \mathbb{R}^n$ where $t < \binom{n+d}{d}$ be open bounded sets. Then, there exists a degree at most d hypersurface H such that for each $i \in [t]$ the sets $U_i \cap H^+$ and $U_i \cap H^-$ have the same volume.*

Proof. The proof for this is similar to the proof for the Ham Sandwich Theorem. Note that by a stars and bars combinatorial argument used in Claim 2.4, for a polynomial in n variables of degree at most d we have that there are at most $\binom{n+d}{d}$ coefficients.

Thus, we can utilise a similar argument to the previous proof. We will let vector $v_h \in S^a$ where $a = \binom{n+d}{d}$ define a hypersurface from a polynomial with its coefficients corresponding to v_h. Then, we define a function f that finds the difference between each set's two sides with respect the the hypersurface. Because $t < \binom{n+d}{d}$ we can append extra zeros at the end of our output vector such that $f : S^a \to \mathbb{R}^b$ where $b = a$. Then, we can apply the Borsuk-Ulam Theorem again in a similar fashion to find a hypersurface H such that $|U_i \cap H^+| = |U_i \cap H^-|$ for all $i \in [t]$. □

Now, using the Polynomial Ham Sandwich, we are able to prove the discrete case, which we can apply to finite sets of objects (i.e. incidences in the plane).

Definition 2.12. A hypersurface $H = \{x \in \mathbb{R}^n : h(x) = 0\}$ **bisects** a finite set A if $A \cap H^+$ and $A \cap H^-$ both have cardinality that does not exceed $\frac{|A|}{2}$.

Theorem 2.13. (Discrete Polynomial Ham Sandwich) *Let $A_1, A_2, ..., A_t \subset \mathbb{R}^n$ be finite sets of points where $t < \binom{n+d}{d}$. Then there exists a degree d hypersurface H that bisects each A_i for $i \in [t]$.*

Proof. Let us denote $U_{i,\epsilon}$ as the union of open epsilon balls around each point of A_i. Then, since all $U_{i,\epsilon}$ are open, we can apply the Polynomial Ham Sandwich to find a hyperplane H_ϵ such that $|U_{i,\epsilon} \cap H_\epsilon^+| = |U_{i,\epsilon} \cap H_\epsilon^-|$ for all $i \in [t]$. Now, consider a sequence $\epsilon_1, \epsilon_2, ..., \epsilon_i, ...$ such that

$$\epsilon_i = \frac{1}{3^i}.$$

For all i, we use the Polynomial Ham Sandwich to find a hypersurface H_{ϵ_i} such that $|U_{i,\epsilon} \cap H_{\epsilon_i}^+| = |U_{i,\epsilon} \cap H_{\epsilon_i}^-|$ for all $i \in [t]$. From each of these hypersurfaces, we obtain a polynomial h_{ϵ_i}. Scaling does not affect the roots of h_{ϵ_i}, thus scaling h_{ϵ_i} will still define the same hypersurface. Now, we rescale the coefficients of h_{ϵ_i} such that the vector v_{ϵ_i} constructed from the coefficients of each h_{ϵ_i} has a magnitude of 1. Therefore, all v_{ϵ_i} lie on the surface of the unit hypersphere. Thus, since this space is compact, there exists a subsequence $v_{\epsilon_{k_i}}$ that converges to a vector v on the unit hypersphere. Let h be the polynomial with its coefficients as the components of v, then let us define a hypersurface H such that

$$H = \{x \in \mathbb{R}^n : h(x) = 0\}.$$

Now, we claim that H bisects each A_i. For the sake of contradiction, suppose not. Then, there exists some m such that

$$|A_m \cap H^+| > \frac{|A_m|}{2}.$$

Since h is a polynomial, it is continuous. Therefore, there exists a sufficiently small $\delta > 0$ such that $h > \delta$ on the δ-ball around each point in $A_m \cap H^+$. Then, by the convergence of $v_{\epsilon_{k_i}}$, there exists a sufficiently large enough k_j such that $v_{\epsilon_{k_j}}$ defines polynomial $h_{\epsilon_{k_j}} > 0$ on the δ-ball around each point in $A_m \cap H^+$. Note that for all $k_i > k_j$ this property holds as well. Thus, if $\epsilon_{k_j} > \delta$, by how we defined ϵ_i, we can choose a k_i large enough such that $\epsilon_{k_i} < \delta$. Therefore, we have that more than half of the ϵ_{k_i}-balls in U_{i,k_i} have $h_{\epsilon_{k_i}} > 0$ which is a contradiction. □

Finally, with our Discrete Polynomial Ham Sandwich, we can create our cell partitions of our finite set of points P. The idea for creating our cells is that we will be able to repeatedly bisect subsets of our set and define a hypersurface in this way.

Lemma 2.14. (Polynomial Cell Partition) *Let $A \subset \mathbb{R}^2$ be a finite set and let $t > 1$. Then, there exists a hypersurface H that decomposes $\mathbb{R}^2$ into $O(t)$ cells and H has degree $d = O(\sqrt{t})$ and each cell contains at most $\frac{|A|}{t}$ points from $|A|$.*

We note here that it might happen that $\frac{|A|}{t} < 1$ or all the cells are empty, meaning most points lie on H. For example, if A contains only points on the x-axis, then H must contain the x-axis, or else H intersects with the x-axis at most $O(\sqrt{t})$ times and therefore each cell has more than $\frac{|A|}{t}$ points. We account for this case in bounding incidences by bounding incidences on H as well as within cells.

Proof of Polynomial Cell Partition. We will inductively construct collections of sets $\mathcal{A}_0, \mathcal{A}_1, ...$ that are a collection of disjoint subsets of A. We begin with $\mathcal{A}_0 = \{A\}$. Then, assume we have $\mathcal{A}_j$ that consists of at most 2^j sets each of size at most $\frac{|A|}{2^j}$. Then, by the Discrete Polynomial Ham Sandwich, there exists a hypersurface H_j that bisects the sets of $\mathcal{A}_j$. For an arbitrary $P \in \mathcal{A}_j$, let $P^+ = P \cap H_j^+$ and $P^- = P \cap H_j^-$. Now, let $\mathcal{A}_{j+1} = \bigcup_{P \in \mathcal{A}_j} \{P^+, P^-\}$ (empty sets ignored). Thus, $\mathcal{A}_{j+1}$ has at most 2^{j+1} sets with size at most $\frac{|A|}{2^{j+1}}$.

Thus, if we let $k = \lceil \log_2(t) \rceil$, and we apply the process described above k times, we obtain a collection of disjoint subsets of A, $\mathcal{A}_k$, that each have size of at most $\frac{|A|}{t}$ and form $A_1, A_2, ..., A_s$ where $s \leq 2^k \leq 2t$. Therefore, if we define a polynomial $h := h_1 h_2 ... h_k$ such that h_i are the polynomials that define hypersurfaces H_i, then we have a hypersurface H defined by h such that it partitions $\mathbb{R}^2$ as we desire.

Now, we must show that h is a polynomial of degree $O(\sqrt{t})$. By the discrete polynomial ham sandwich theorem, for bisecting at most 2^j sets, the polynomial h_j suffices to have degree $O(\sqrt{2^j})$. Therefore, $deg(h) = O(\sum_{j=1}^{k} O(\sqrt{2^j})) = O(\sqrt{t})$ and, thus, the degree of H is $O(\sqrt{t})$. □

As a consequence of the lemma, we have created our cells that partition the $\mathbb{R}^2$ plane. We can then use the bound in Lemma 2.3 to find a bound on incidences in each cell and on the hypersurface to achieve the desired bound from Szemerédi-Trotter.

2.3. **The Crossing Lemma.**

The geometric and combinatorial nature of incidences also introduces a very elegant graph theory approach to the unit distance question which we attribute to Székely. Instead of bounding incidences through polynomial partitions, we instead bound them through crossings on a graph. Within this section we will give some preliminary definitions and theorems used to prove The Crossing Lemma, which we will then later use to construct a graph to prove the Szemerédi-Trotter theorem.

For the purpose of this paper, we will only be considering *undirected graphs*, meaning that if an edge is determined by two vertices, v_1 and v_2, then the order of the vertices does not matter. Before we begin proving the crossing lemma, we will introduce some preliminary graph theory definitions and theorems. The *degree* of a vertex to be the number of edges that come out of it. A *path* within a graph is a series of connected edges to travel from one vertex to another. And a *cycle* is any path that starts and ends on the same vertex.

Definition 2.15. The **crossing number** of a graph, denoted by $cr(G)$ is the smallest integer k such that a planar drawing of G can have k edges crossings.

Example 2.16. The complete, bipartite graph of 6 nodes, $K_{3,3}$, has a crossing number of 1. Here is a planar drawing of the graph below in figure 2.

Definition 2.17. A graph is said to be **planar** if $cr(G) = 0$.

Because planar graphs have no crossings, there are areas within planar graphs that are bounded by its edges. These areas, and the plane itself, are called *faces.* To prove the crossing inequality for graphs, we rely on Euler's Formula, a fundamental equation in geometry that relates the vertices, faces, and edges of planar graphs.

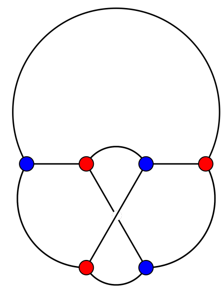

FIGURE 2. $K_{3,3}$ with one crossing.

Theorem 2.18. (Euler's Formula) *Let $G = (V, E)$ be a connected planar graph. Then $|V| - |E| + |F| = 2$.*

Lemma 2.19. *Let $G = (V, E)$ be a connected planar graph that is connected and contains no cycles. Then $|V| - |E| + |F| = 2$.*

Proof. Since our graph contains no cycles, it does not divide the plane into multiple regions. Therefore, the number of faces is always 1. Thus, we must only prove that $|V| - |E| = 1$

We will induct on the edges of the graph to show this lemma. First, our base case is when $|E| = 0$. Then our graph is one vertex and one face (the region around the vertex). Therefore, $|V| - |E| = 1 - 0 = 1$ holds. Now, assume that our claim holds for n edges.

Consider a graph with $n + 1$ edges. Now, find a vertex of degree one. We accomplish this by starting at any vertex in our graph and travelling along any path until we reach a dead end. Because our graph has no cycles and finite amount of vertices, no vertex will be travelled to more than once, and therefore this dead end is guaranteed. Call this vertex p. Remove p and the edge attached to it. Now, we have a graph with n vertices and by induction hypothesis, this graph has $n + 1$ vertices. However, this implies our original graph has $n+2$ vertices and $n+1$ edges, and our claim holds. □

Proof of Euler's Formula. We will induct on edges again like above. Our base case is when $|E| = 0$, and by the same argument from the above Lemma, our formula holds. Assume that for a graph with n edges, $|V| - |E| + |F| = 2$. Consider a graph G with $n + 1$ edges.

Case 1: G has no cycles. By Lemma 2.17, then Euler's Formula holds.

Case 1: G has at least one cycle. Remove one edge from a cycle and call this new graph G'. By induction hypothesis, $|V'| - |E'| + |F'| = 2$. However, note that this cycle partitioned the plane into two regions, so by removing an edge we merge these two regions, and remove a face. Therefore, $|V'| = |V|$, $|E'| = |E| - 1$, and $|F'| = |F| - 1$. Therefore, we have that

$$\begin{aligned} 2 &= |V'| - |E'| + |F'| \\ &= |V| - (|E| - 1) + (|F| - 1) \\ &= |V| - |E| + |F|. \end{aligned}$$

□

We will now proceed into a series of preliminary inequalities derived from Euler's formula that will be used to show the result of the Crossing Lemma. The idea will be to achieve a more trivial bound on crossings, and then use this bound on a series of subgraphs.

Lemma 2.20. *Let $G = (V, E)$ be a planar graph. Then, we have that $|E| \leq 3|V| - 6$.*

Proof. Note that every edge either lies on the boundary between two faces or lies on the boundary of the same face. Additionally, every face has at least three edges. Thus, $2|E| \geq 3|F|$. Plugging this into Euler's, we have that $|V| - |E| + \frac{2|E|}{3} \geq 2$, and therefore, $|E| \leq 3|V| - 6$. □

Lemma 2.21. *Let $G = (V, E)$ be a graph. Then $cr(G) > |E| - 3|V|$.*

Proof. Suppose G has minimal crossings. Let E' be the maximum subset of edges that such that no two edges of E' intersect each other. Therefore, by Lemma 2.22, we have that $|E'| \leq 3|V| - 6$. Then, since every edge in $E \setminus E'$ crosses an edge of E' at least once we have that $cr(G) \geq |E \setminus E'| \geq |E| - 3|V| + 6$, and therefore $cr(G) > |E| - 3|V|$ □

Definition 2.22. For $0 < p \leq 1$, a **random subgraph** of $G = (V, E)$ is generated by selecting each vertex of G with probability p. An edge is selected if its two endpoints are selected with probability p, otherwise it is deleted. We will denote the subgraph induced by the selected vertices as G_p.

The Székely argument relies on the crossing lemma (or sometimes called the crossing number inequality) of graphs. We introduce the idea of a *random subgraph* because, if we have $f(G_p)$ be a function on the set of random subgraphs, then $f(G_p)$ is a random variable, and therefore we can take take its expectation. Thus, if we use functions on G_p that output the number of crossings, edges, and vertices, we are able to apply the inequality from Lemma 2.22 on random subgraphs to find a bound on crossings on the parent graph and arrive at the following.

Theorem 2.23. (The Crossing Lemma) *Let $G = (V, E)$ be a graph. If $|E| \geq 4|V|$, then*

$$cr(G) \geq \frac{|E|^3}{64|V|^2}.$$

Proof. Let $p = \frac{4|V|}{|E|}$. Since $|E| \geq 4|V|$, then $0 < p \leq 1$. Thus, we can generate a random subgraph G_p of G. We denote the edges of G_p to be $E(G_p)$ and the vertices to be $V(G_p)$. By Lemma 2.23, we have that $cr(G_p) > |E(G_p)| - 3|V(G_p)|$. Then, we can take the expectation of this inequality to get that

$$\mathbf{E}[cr(G_p)] > \mathbf{E}[|E(G_p)|] - 3\mathbf{E}[|V(G_p)|].$$

It is easy to see that $\mathbf{E}[|V(G_p)|] = p|V|$. Then, note that an edge of G belongs to G_p if and only if both endpoints have been selected. So $\mathbf{E}[|E(G_p)|] = p^2|E|$. Finally, we have that a crossing of G belongs to G_p if and only if both edges that create the crossing have been selected. Therefore, $\mathbf{E}[cr(G_p)] = p^4 cr(G)$. Thus, we have that

$$p^4 cr(G) > p^2|E| - 3p|V|.$$

Then, substituting $p = \frac{4|V|}{|E|}$, we have that

$$\begin{aligned} cr(G) &> \frac{|E|}{p^2} - \frac{3|V|}{p^3} \\ &= \frac{|E|^3}{16|V|^2} - \frac{3|E|^3}{|V|^2} \\ &= \frac{|E|^3}{64|V|^2}. \end{aligned}$$

□

Now that we have shown an established bound on crossings within graphs, a clever construction of a graph that captures incidences on points and lines will give the Szemerédi-Trotter bound in Theorem 2.2 on incidences.

3. The Unit Distance Problem

Suppose we have n points in the plane $\mathbb{R}^2$. We define $u(n)$ to be the maximum number of pairs of points that are a unit distance from each other. In this section, we will prove the Szemerédi-Trotter Theorem with Szekely's method using the crossing inequality, Ultimately, we will show that Szemerédi-Trotter implies the following:

Proposition 3.1. *For a finite set of points P and a finite set of unit circles Γ, we have $I(P, \Gamma) \lesssim (|P| \cdot |\Gamma|)^{2/3} + |P| + |\Gamma|$.*

Note here, that for n points and n unit circles, the unit distance bound immediately follows. This proposition is simply the Szemerédi-Trotter Theorem for points and circles of the same radii. Thus, to prove it, we make slight changes to our proof of Szemerédi-Trotter, but generally the same ideas hold.

Proof of Szemeredi-Trotter. Consider a graph $G = (V, E)$ where the vertices correspond the points P and an edge is defined by two consecutive points along a line $l \in L$. Let us denote by $|P_l|$ the points along line l. Then, we have that $|E| = \sum_{l \in L}(|P_l| - 1)$. Additionally, we can note here that $I(P, L) = \sum_{l \in L} |P_l|$. Therefore, we have that $|E| = I(P, L) - |L|$.

If $|E| < 4|V|$, then since $|V| = |P|$, we have that $I(P, L) - |L| < 4|P|$, and $I(P, L) \lesssim |P| + |L|$ and the bound from Szemeredi-Trotter follows.

If $|E| \geq 4|V|$, then we can invoke the Crossing Lemma such that

$$\begin{aligned} cr(G) &\geq \frac{|E|^3}{|V|^2} \\ &= \frac{(I(P, L) - |L|)^3}{|P|^2}. \end{aligned}$$

Now, consider the drawing of graph G such that each vertex corresponds to a point p in P and each edge corresponds to a line segment l in L. Then, since every crossing in G corresponds to an intersection of two lines in L and two lines can intersect at most once, we have that $cr(G) \leq \binom{|L|}{2} \lesssim |L|^2$. Combining this with the above equation, we get that

$$\begin{aligned}|L|^2 &\gtrsim \frac{(I(P,L)-|L|)^3}{|P|^2}\\ (|P|\cdot|L|)^{2/3}+|L| &\gtrsim I(P,L)\\ (|P|\cdot|L|)^{2/3}+|L|+|P| &\gtrsim I(P,L).\end{aligned}$$

And thus, we have our desired bound for the Szemeredi-Trotter Theorem. □

Now, to prove Proposition 3.1, we use the same logic, however we create slight changes to accomodate for the use of circles instead of lines in our incidences. Instead, we create a graph where edges are arcs of our set of circles instead of line segments. Then, we similarly will create bounds using the crossing inequality and relating edges to incidences.

Proof of Proposition 3.1. We will mirror the proof above, but due to finding incidences between points and circles instead of points and lines, we will define a new graph $G=(V,E)$ in a slightly different manner. Like before, we let the vertices of G correspond to the points of P. We define edges to be the arc between 2 consecutive points p_1 and p_2 in P along the same circle γ in Γ with no other points between. Then, we delete circles with at most two points incident to them, and note that these arcs contribute at most $2|\Gamma|$ incidences. Therefore, among the circles we keep, we have that each circle contributes $|P_\gamma|$ edges if P_γ denotes the points of P on a circle γ.

Now, we have a graph G such that $|V|=|P|$ and $|E|\geq\left(\sum_{\gamma\in\Gamma}|P_\gamma|\right)-2|\Gamma|$. And since $|I(P,\Gamma)|=\sum_{\gamma\in\Gamma}|P_\gamma|$, then we have that $|E|\geq|I(P,\Gamma)|-2|\Gamma|$. However, we may have *multi-edges* within graph G, meaning we have multiple edges coming from the same two vertices, if two circles in Γ intersect along two points in P. Since circles can only intersect in at most two points, there are at most two edges for every pair of points. We will delete one of these multi-edges and this reduces the number of edges by at most a factor of two. Thus, our revised version of G has that $|E|\geq\frac{|I(P,\Gamma)|}{2}-|\Gamma|$.

If $|E|<4|V|$, we have that $\frac{|I(P,\Gamma)|}{2}-|\Gamma|<4|P|$ and our bound directly follows. If we have that $|E|\geq 4|V|$, then we can invoke The Crossing Lemma again, to get that

$$\begin{aligned}cr(G) &\gtrsim \frac{|E|^3}{|V|^2}\\ &\geq \frac{(\frac{|I(P,\Gamma)|}{2}-|\Gamma|)^3}{|P|^2}.\end{aligned}$$

However, note that a crossing occurs within our graph G if we have an intersection between two circles. And thus, since circles intersect at most twice, we have that $cr(G)\leq 2\binom{|\Gamma|}{2}\lesssim|\Gamma|^2$. Therefore,

$$|\Gamma|^2\gtrsim\frac{(\frac{|I(P,\Gamma)|}{2}-|\Gamma|)^3}{|P|^2},$$

and the bound from Proposition 3.1 immediately follows. □

As a consequence of this property, when we consider $|P| = |\Gamma| = n$, then we immediately have that $|I(P, \Gamma)| \lesssim n^{4/3}$, and thus we have shown the best known bound on the unit distance problem.

4. The Distinct Distances Problem

Often paired with the unit distance problem is its more complicated and arguably more famous twin question, the distinct distance problem. The problem is stated as follows: for n points in the plane, what is the minimum amount of distinct distances between pairs of points? That is, for a finite set P of n points in the plane, we denote $d(P) = \{|p - q| : p, q \in P\}$ as the set of all distances between points in P. We wish to find a lower bound on $|d(P)|$.

First, we prove Szemerédi-Trotter through polynomial cell partitions of the plane. This method is attributed to Guth and Katz. We include another proof of Szemerédi-Trotter because this method will clarify how Guth and Katz were able to bound the distinct distances in the plane. Before we begin proving Szemerédi-Trotter, we will prove some preliminary results about hypersurfaces and lines.

Lemma 4.1. *Given any line $l \subset \mathbb{R}^2$ and hypersurface $H \subset \mathbb{R}^2$ with degree d. Either $l \subset H$ or $|l \cap H| \leq d$.*

Proof. Let h be the polynomial describing H and let g be the restriction of h to line g. Either g is uniformly 0, or not. If it is, then this implies that $l \subset H$. If not, then we have that $deg(g) \leq d$ since the restriction to a line does not make a higher degree polynomial. Then, since l contains at most $deg(g) \leq d$ roots of g and the roots of g are the intersection of l and H, we have that l intersects h in at most d points. Therefore, $|l \cap H| \leq d$. □

Lemma 4.2. *For a hypersurface $H \subset \mathbb{R}^2$ of degree d, there are at most d lines contained in H.*

Proof. Suppose that there are $d+1$ distinct lines in H. Since $d+1$ is finite, then we can choose a line l in $\mathbb{R}^2$ that is not parallel to $d+1$ lines. Then, we have that l intersects $d+1$ lines and l intersects H at least $d+1$ times, which is a contradiction to Lemma 3.2. □

Notation 4.3. We use notation $A << B$ to imply that $A \leq cB$ for an arbitrarily small c. Intuitively, we can think of $<<$ meaning 'much less than.'

The idea for the cell-partitioning proof of Szemeredi-Trotter is that we will be able to use the bound in Lemma 2.3 within each cell to amplify the overall bound. Then, we will count incidences on the hypersurface and combine these two results to achieve the bound in Szemerédi-Trotter.

Proof of Szemeredi-Trotter. We have two cases, either $|P|^{1/2} << |L| << |P|^2$ holds, or it does not. Suppose that $|P|^{1/2} << |L| << |P|^2$ is not true. By Lemma 2.3 we have that

$$\begin{aligned} |I(P, L)| &\lesssim |P| \cdot |L|^{1/2} + |L| + |L| \cdot |P|^{1/2} + |P| \\ &\lesssim (|P| \cdot |L|)^{2/3}(|P|^{1/3}|L|^{-1/6} + |L|^{1/3}|P|^{-1/6}) + |L| + |P| \end{aligned}$$

By our assumption, we can bound the expression $(|P|^{1/3}|L|^{-1/6} + |L|^{1/3}|P|^{-1/6})$ by a constant, and the bound from Szemeredi-Trotter is immediately implied.

Thus, we will assume $|P|^{1/2} << |L| << |P|^2$. Then we apply the Polynomial Cell Partition in Lemma 2.14 to find a hypersurface H of degree $d \lesssim t^{1/2}$ that decomposes $\mathbb{R}^2$ into cells $C_1, C_2, ..., C_{O(t)}$ such that each cell contains at most $\frac{|P|}{t}$ points from P. For the sake of simplicity, we will consider cells $C_1, C_2, ..., C_t$ and note that if we created Kt cells, the proof is identical due to the constant K being absorbed.

Then, we bound incidences in each cell and then incidences on the hyperplane. To do this, we introduce notation $P_0 = P \cap H$ to denote points on H, $L_0 = L \cap H$ to denote lines that intersect H, $P_i = P \cap C_i$ to denote points within the cell C_i, and $L_i = L \cap C_i$ to denote lines that intersect cell C_i. Here, lines are double counted, as lines in L_0 can occur in multiple L_i, but since we are finding an upper bound, this is okay.

Therefore, we have that

$$|I(P, L)| \leq |I(P_0, L_0)| + \sum_{i=1}^{t} |I(P_i, L_i)|.$$

Then, by Lemma 2.3, we have that

$$\begin{aligned} |I(P_i, L_i)| &\lesssim |P_i||L_i|^{1/2} + |L_i| \\ &\lesssim \frac{|P|}{t}|L_i|^{1/2} + |L_i|. \end{aligned}$$

We observe that for a line l in L_i, it is not contained in H. If $l \subset H$ then we have that it cannot intersect a cell. Thus, by Lemma 3.3, we have that a line in L_i can intersect H at most $d \lesssim t^{1/2}$ times. Therefore, each line in L_i can intersect at most d cells because intersecting H implies crossing from one cell to another. Therefore,

$$\sum_{i=1}^{t} |L_i| \lesssim t^{1/2}|L|.$$

Then, by applying the Cauchy-Schwartz inequality, we have that

$$\begin{aligned} \left(\sum_{i=1}^{t} |L_i|^{1/2} \right)^2 &\leq \sum_{i=1}^{t} |L_i| \sum_{i=1}^{t} 1 \\ &\lesssim (t^{1/2}|L|) \cdot t \\ &= t^{3/2}|L|. \end{aligned}$$

Therefore, we have that $\sum_{i=1}^{t} |L_i|^{1/2} \lesssim t^{3/4}|L|^{1/2}$. Now, we can bound incidences in the cells by putting these two together, such that

$$\begin{aligned} \sum_{i=1}^{t} |I(P_i, L_i)| &\leq \sum_{i=1}^{t} \left(\frac{|P|}{t}|L_i|^{1/2} + |L_i| \right) \\ &\lesssim \frac{|P|}{t} t^{3/4}|L|^{1/2} + t^{1/2}|L| \\ &= t^{-3/4}|P||L|^{1/2} + t^{1/2}|L|. \end{aligned}$$

Now, we will bound incidences found on our hypersurface. We will do this by splitting L_0 into two sets: L_0' to denote lines in L_0 that are on H and L_0'' to denote lines that only intersect H at some points. Note that for a line l that is in L_0'', by Lemma 3.2, we have that l intersects H in at most $d \lesssim t^{1/2}$ points. Therefore,

$$I(P_0, L_0'') \lesssim t^{1/2}|L_0''| \lesssim t^{1/2}|L|.$$

Then, by Lemma 3.3, we have that $|L_0'| \lesssim t^{1/2}$. Applying the bound in Lemma 2.3, we have that

$$\begin{aligned} |I(P_0, L_0')| &\lesssim |L_0||P_0|^{1/2} + |P_0| \\ &\lesssim t^{1/2}|P|^{1/2} + |P|. \end{aligned}$$

Therefore, putting these two bounds together, we have that

$$\begin{aligned} |I(P_0, L_0)| &= |I(P_0, L_0')| + |I(P_0, L_0'')| \\ &\lesssim t^{1/2}|P|^{1/2} + |P| + t^{1/2}|L| \\ &\lesssim t^{1/2}|L|^{1/2} + |P| + t^{1/2}|L| \text{ By assuming } |P|^{1/2} << |L| \\ &\lesssim t^{1/2}|L|^{1/2} + |P|. \end{aligned}$$

Thus, we have successfully bounded incidences on H and within cells, therefore, we put this all together to get that

$$\begin{aligned} |I(P, L)| &\leq |I(P_0, L_0)| + \sum_{i=1}^{t} |I(P_i, L_i)| \\ &\lesssim t^{1/2}|L|^{1/2} + |P| + t^{-3/4}|P||L|^{1/2} + t^{1/2}|L| \\ &\lesssim t^{1/2}|L|^{1/2} + |P| + t^{-3/4}|P||L|^{1/2}. \end{aligned}$$

Now, we can let $t = \frac{|P|^{4/3}}{|L|^{2/3}}$ to achieve our desired bound and since we assumed $|L| << |P|^2$ then $t \geq 1$ and we are done. □

Using the bound on unit distances, we are able to find a maximal bound on the pairs of points that are a distance r apart, where r is any real number. Therefore, we immediately are able to obtain a lower bound on the number of distinct distances. Since we arrive at a bound of $\lesssim |P|^{4/3}$ on the number of *identical* distances, then we have that there must be a minimal bound of $\gtrsim |P|^{2/3}$ of distinct distances because there are $\sim |P|^2$ total pairs of points.

Erdös posed the distinct distance problem in the same 1946 paper in which he introduced the unit distance question. He made the following conjecture that

Conjecture 4.4. *For a finite set of n points P, we have that $|d(P)| \gtrsim \frac{|P|}{(\log |P|)^{1/2}}$.*

If we consider n points on a square $\sqrt{n} \times \sqrt{n}$ integer grid, then this bound is tight. This conjecture on a generalization of a lower bound remains an open problem. However, Guth and Katz obtained a very close bound of $\gtrsim \frac{|P|}{\log |P|}$. We will not extensively go into detail about the Guth-Katz proof, but the original paper is very well explained and largely self-contained so we strongly recommend referencing the original argument. We will briefly review the key ideas.

First, we motivate an incidence approach to the question. Guth and Katz reduced the question into a linear problem by considering the Elekes and Sharir framework [6] which relies mostly on some euclidean geometry. To see how this works, first assume by contradiction that $|d(P)| \lesssim \frac{|P|}{\log |P|}$ for some large set of points P. We note here that there are $\binom{|P|}{2} \sim |P|^2$ different line segments. Then, through an application of the Cauchy-Schwartz inequality, they find that there must be many distinct but congruent pairs of line segments $\overline{p_1p_2}$, $\overline{p_3p_4}$ (i.e. line segments of equal length $|p_1 - p_2| = |p_3 - p_4|$). In fact they find that the number must be

$$\frac{\binom{|P|}{2}^2}{|d(P)|} - \binom{|P|}{2} \gtrsim |P|^3 \cdot \log |P|.$$

Therefore, there are $\gtrsim |P|^3 \cdot \log |P|$ pairs of congruent line segments.

Then, Guth and Katz re-framed the question in terms of *rigid motions*. Rigid motion refers to any movement between points such that the relative distance and position between the points are conserved. In other words, two line segments are congruent if there exists a rigid motion relative to the endpoints of the segments. Therefore, our bound of $\gtrsim |P|^3 \cdot \log |P|$ equivalently describes a bound sets of four points and a rigid motion R. Then, they observed that the set of rigid motions between two points x and y in $\mathbb{R}^2$ form a line in $\mathbb{R}^3$ through the euclidean geometry fact that a rigid motion can be described through a rotation centered at a point along the perpendicular bisector of x and y. Thus, a pair of points within our collection creates a distinct line in $\mathbb{R}^3$ and we now have a collection $\mathcal{L}$ of $\sim |P|^2$ lines. Using the bound above, since there are $\gtrsim |P|^3 \cdot \log |P|$ quintuples (p_1, p_2, p_3, p_4, R), this gives rise to $\gtrsim |P|^3 \cdot \log |P|$ intersections between two lines l, l' in $\mathcal{L}$. Therefore, we can rephrase the distinct distance problem into an incidence question: is it possible for $\sim n^2$ lines in $\mathbb{R}^3$ to generate $\gtrsim n^3 \cdot \log n$ pairs of intersecting lines? If the answer is 'no' then the bound of $\Theta(\frac{n}{\log n})$ is proven. However, the answer is 'yes' but Guth and Katz were able to find the following restrictions in order to prove the theorem.

Theorem 4.5 (Guth-Katz). *Let L be a set of N^2 lines in $\mathbb{R}^3$ such that no more than N lines intersect at a single point and no plane or double ruled surface contains more than N lines. Then the number of incidences of lines in L, $|I(L)|$, is at most $\lesssim N^3 \cdot \log N$.*

The proof by Guth and Katz uses the polynomial method and cell partitions in the same spirit of the first proof of Szemerédi-Trotter that we presented. An outline of the proof is as follows. They bound the set of all points of concurrency for the set of lines, S and do this by contradiction. First, they separate the problem into two cases: if S contains points with least 2 lines going through it or if S contains points with more than 2 lines going through it. Then, through a generalization of the polynomial cell partition lemma, they obtain a hypersurface of degree $O(t^{1/3})$ that creates $O(t)$ cells with at most $\frac{|S|}{t}$ points of S in each cell.

Beginning with the latter case, they obtain two more subcases: the cellular case and the algebraic case. The cellular case is when there are more points within the cells than there are on the surface. In this case, the proof will use a generalization of Székmeredi-Trotter in three dimensions and the proof for the bound is similar to the proof we presented on Szémeredi-Trotter. The algebraic case is when there are more points on the algebraic surface than within the cells. Through a degree argument, the algebraic surface defining the cells must contain 'many' lines in L. Then, by the assumption that there are at least 3 lines going through each point of S, they argue that these lines must be contained in a plane, which gives a contradiction.

Then, assuming that S contains points with at least 2 lines going through each point, Guth and Katz invoke the polynomial method to find a hypersurface that vanishes at all points of S. Then, many lines of L will intersect this hypersurface many times, and they will be essentially contained on the hypersurface. However, this forces components of the hypersurface to be single ruled sufaces, (a surface in which every point has one line passing through it), double rules surfaces (a surface in which every point has two lines passing through it), or planes. By assumption, the latter two cannot contain many lines, so the majority of lines must come from the single rules surfaces. But, the number of times lines from a single ruled surfaces can intersect other lines from single rules surfaces can be controlled, and this gives a contradiction.

Therefore, as a result of this theorem, we find a contradiction and the bound $|d(P)| \gtrsim \frac{|P|}{\log |P|}$ holds.

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