FLOER HOMOLOGIES AND THE ARNOLD CONJECTURES

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ABSTRACT. Floer theory has quickly become the focus of modern symplectic geometry. Its applications are powerful and far-reaching, spanning from low-dimensional topology to string theory. In this paper we introduce the origins of this theory in Andreas Floer's work on the Hamiltonian and Lagrangian Arnold conjectures for symplectic manifolds.

CONTENTS

1. INTRODUCTION

The utility of homology in stating and proving fixed-point theorems for topological spaces is well-documented by classical examples such as the Lefschetz-Hopf fixed-point theorem:

$$\sum_{f(x)=x} i(f,x) = \Lambda_f = \sum_{k\geq 0}(-1)^k \operatorname{Tr}\left(f_*\big|_{H_k(X;\mathbb{Q})}\right) \text{ for } f: X \to X \text{ continuous}$$

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and Brouwer's fixed-point theorem:

Any continuous function $f : \mathbb{D}^n \to \mathbb{D}^n$ has a fixed point.

It was in the context of Vladimir Arnold's symplectic fixed point conjecture that Andreas Floer (1956-1991) developed his homology theories for Hamiltonian diffeomorphisms and intersections of Lagrangian submanifolds ([3], [4], [5], [6], [7]).

Conventions 1.1. Manifolds are taken to be smooth (Hausdorff and second-countable), real, finite-dimensional, connected, and boundaryless unless otherwise stated. All vector fields are smooth. We adopt the convention of identifying S^1 with $\mathbb{R}/\mathbb{Z}$ (that is, $\mathbb{R}$ modulo integer addition), so that closed paths are parameterized by reals and 1-periodic. Homology is computed over the integers when the coefficients are unspecified.

Conjecture 1.2 (Arnold)**.** [1] *Let ϕ be a Hamiltonian diffeomorphism of a closed symplectic $2n$-manifold M such that the fixed points of ϕ are non-degenerate. Then*

$$|\{p \in M \mid \phi(p) = p\}| \geq \sum_{k=0}^{2n} \operatorname{rank}(H_k(M)).$$

In Section 5.1, we will see that this conjecture is closely related to a statement about the Lagrangian submanifolds of M. This perspective leads to another homology theory known as Lagrangian Floer homology, or Lagrangian intersection homology.

Conjecture 1.3 (Arnold)**.** *Let L be a compact Lagrangian submanifold of a closed symplectic $2n$-manifold (M, ω) with $\int_{S^2} f^*\omega = 0$ for all $[f] \in \pi_2(M, L)$ and ϕ a Hamiltonian diffeomorphism of M such that $L \pitchfork \phi(L)$. Then*

$$|L \cap \phi(L)| \geq \sum_{k=0}^{2n} \operatorname{rank}(H_k(L)).$$

We will refer to Conjecture 1.2 and Conjecture 1.3 as the Hamiltonian and Lagrangian Arnold conjectures, respectively.

For this paper we assume that the reader is familiar with elementary algebraic topology (mainly singular (co)homology) and differential geometry of smooth manifolds. However, we introduce the necessary notions from symplectic geometry and Morse theory. Many of Floer's proofs and subsequent generalizations involve difficult analytical subtleties; it is not our goal to go into this detail. Instead, we aim to provide a geometrically focused overview of the constructions of Hamiltonian Floer homology and Lagrangian intersection homology with intuition from Morse homology.

2. Symplectic and Almost Complex Structures

A *symplectic manifold* is a pair (M, ω), where M is a manifold and ω is a closed, non-degenerate 2-form on M. By "non-degenerate" we mean as a bilinear map on the tangent spaces: for all $p \in M$ and $v \in T_pM$, there exists $w \in T_pM$ such that $\omega_p(v, w) \neq 0$. We then say that ω is a *symplectic form* or *symplectic structure* on M. We will often abuse notation by saying that M is a symplectic manifold

[1] Technically, these are not Arnold's original conjectures and have been influenced by the field's history. See e.g. Abbondandalo and Schlenk's survey [1].

if it admits a symplectic form. The following is obtained as a special case of the canonical form of an antisymmetric bilinear map; see [2] for details.

Proposition 2.1. *Suppose (M, ω) is a symplectic manifold. Then M is even-dimensional. Moreover, if $2n = \dim M$ then on each tangent space T_pM there exists a basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ such that*

$$\begin{aligned}\omega_p(e_i, e_j) &= 0,\\ \omega_p(f_i, f_j) &= 0,\\ \omega_p(e_i, f_i) &= 1,\end{aligned}$$

for all i, j, and

$$\omega_p(e_i, f_j) = 0$$

for distinct i, j. □

The next proposition summarizes the basic properties of M guaranteed by the non-degeneracy of ω.

Proposition 2.2. *Let (M, ω) be a symplectic $2n$-manifold. Then*

(i) M is orientable with the n-fold wedge product $\omega^{\wedge n}$ as a volume form.
(ii) If M is closed, $H^{2k}(M; \mathbb{R})$ is non-trivial for all $k = 1, \dots, n$.
(iii) ω determines a natural bundle isomorphism $TM \cong T^*M$ given by

$$v \mapsto \iota_v \omega_p = \omega_p(v, \cdot)$$

on the fibers.

Proof. Let $p \in M$ and choose local coordinates $x_1, \dots, x_n, y_1, \dots, y_n$ around p such that

$$e_1 = \left.\frac{\partial}{\partial x_1}\right|_p, \dots, e_n = \left.\frac{\partial}{\partial x_n}\right|_p, f_1 = \left.\frac{\partial}{\partial y_1}\right|_p, \dots, f_n = \left.\frac{\partial}{\partial y_n}\right|_p,$$

where $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ is the basis of T_pM given by Proposition 2.1.

(i) In local coordinates we have

$$\omega_p = \sum_{i=1}^{n} dx_i \wedge dy_i,$$

and so

$$\begin{aligned}(\omega_p)^{\wedge n} &= \sum_{i_1, i_2, \dots, i_n = 1}^{n} (dx_{i_1} \wedge dy_{i_1}) \wedge \cdots \wedge (dx_{i_n} \wedge dy_{i_n})\\ &= n!(dx_1 \wedge dy_1) \wedge \cdots \wedge (dx_n \wedge dy_n)\\ &\neq 0.\end{aligned}$$

(ii) Using the orientation given by $\omega^{\wedge n}$, we have $\int_M \omega^{\wedge n} > 0$, so by Stoke's theorem, $\omega^{\wedge n}$ is not exact. Thus $[\![\omega]\!]^{\smile n} \neq 0$, so $[\![\omega]\!]^{\smile k} \neq 0$ for $k = 1, \dots, n$.

(iii) Linearity and naturality are immediate; an inverse of the stated map is given locally by

$$\sum_{i=1}^{n} (a_i e_i^* + b_i f_i^*) \mapsto \sum_{i=1}^{n} (b_i e_i - a_i f_i).$$ □

Examples 2.3.

- $\mathbb{R}^{2n}$ has a canonical symplectic form given by

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i,$$

  where $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ is the standard coordinate system. Under the identification $z_i = x_i + iy_i$, we obtain a symplectic form on $\mathbb{C}^n$:

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i = \frac{i}{2} \sum_{i=1}^{n} dz \wedge d\bar{z}.$$

- Any orientable surface (S^2 or Σ_g) with area form ω is a symplectic manifold. An even-dimensional product of S^1's and $\mathbb{R}$'s has a symplectic form analogous to that of $\mathbb{R}^{2n}$; in particular, the even-dimensional toruses $\mathbb{T}^{2n} = (S^1)^{2n}$ are symplectic. Note, however, that by (ii) of Proposition 2.2 the 2-sphere is the only sphere which admits a symplectic form.
- Let M be an n-manifold. Then the cotangent bundle $\pi : T^*M \to M$ has a natural 1-form given by

$$\lambda(p, v) = v \circ d\pi_p$$

  called the *Liouville 1-form*. In standard local coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$, its derivative is given by

$$\omega = d\lambda = \sum_{i=1}^{n} dx_i \wedge d\xi_i.$$

  Thus ω is a symplectic form on T^*M, known as the *canonical symplectic form*.
- For any finite collection $\{(M_1, \omega_1), (M_2, \omega_2), \ldots, (M_N, \omega_N)\}$ of symplectic manifolds, consider the 2-form

$$\omega = \sum_{i=1}^{N} c_i \pi_i^* \omega_i$$

  where $\pi_i : M = \Pi_{i=1}^N M_i \to M_i$ are the projections and the c_i are nonzero reals. Under the canonical isomorphism

$$T_pM \cong T_{\pi_1(p)}M_1 \oplus T_{\pi_2(p)}M_2 \oplus \cdots \oplus T_{\pi_N(p)}M_N$$

  we see that ω is non-degenerate and

$$d\omega = \sum_{i=1}^{N} c_i d(\pi_i^* \omega_i) = \sum_{i=1}^{N} c_i \pi_i^*(d\omega_i) = 0.$$

  Then (M, ω) is a symplectic manifold.

Definition 2.4. For a manifold M, an *almost complex structure* is a collection $J = \{J_p\}_{p \in M}$ of linear transformations $J_p : T_pM \to T_pM$ that vary smoothly with p such that $J_p^2 = -\operatorname{id}_{T_pM}$. If ω is a symplectic form on M, J is compatible with ω if $g_p : T_pM \times T_pM \to \mathbb{R}$ given by

$$g_p(v, w) = \omega_p(v, J_p w)$$

defines a Riemannian metric. The collection of compatible almost complex structures is denoted by $\mathcal{J}(M, \omega)$ and is given the subspace topology.

Remark 2.5. Note that J is compatible with ω if and only if $\omega_p(J_p v, J_p w) = \omega_p(v, w)$ for all $v, w \in T_pM$ and $\omega_p(v, J_p v) > 0$ for nonzero $v \in T_pM$. The term "almost complex" corresponds to the existence of a canonical almost complex structure on any complex manifold given by multiplication by i, and so almost complex structures are also commonly called "almost Kähler."

The following result is crucial to the construction of Floer chain complexes.

Lemma 2.6. *Let (M, ω) be a symplectic manifold. Then $\mathcal{J}(M, \omega)$ is non-empty and path-connected.*

Proof. We follow [2] and [11]. Let g be a Riemannian metric on M. By non-degeneracy of g and ω, there exist unique isomorphisms $A_p : T_pM \xrightarrow{\cong} T_pM$ such that

$$\omega_p(v, w) = g_p(A_p v, w)$$

for all $v, w \in T_pM$. Then

$$g_p(A_p v, w) = \omega_p(v, w) = -\omega_p(w, v) = g_p(v, -A_p w),$$

so $-A_p$ is the adjoint of A_p with respect to g_p. Thus $A_p^* A_p = -A_p^2$ has a unique self-adjoint and positive-definite (with respect to g_p) square root P_p. This construction is canonical, so P_p varies smoothly with p, and P_p is an isomorphism by positive-definiteness. Then P_p and A_p commute, so

$$J_p = P_p^{-1} A_p$$

is orthogonal:

$$J_p^* J_p = (P_p^{-1} A_p)^* (P_p^{-1} A_p) = A_p^* P_p^{-1} P_p^{-1} A_p = \mathrm{id}_{T_pM}$$

and skew-adjoint:

$$J_p^* = (P_p^{-1} A_p)^* = A_p^* P_p^{-1} = -P_p^{-1} A_p = -J_p.$$

Thus $J_p^2 = -\mathrm{id}_{T_pM}$. Finally, we have

$$\omega_p(J_p v, J_p w) = g_p(A_p J_p v, J_p w) = g_p(J_p A_p v, J_p w) = g_p(A_p v, w) = \omega_p(v, w)$$

for $v, w \in T_pM$ and

$$\omega_p(v, J_p v) = g_p(A_p v, J_p v) = g_p(-J_p A_p v, v) = g_p(P_p v, v) > 0$$

for nonzero $v \in T_pM$, so that J is compatible with ω.

If g was induced by an almost complex structure J, then

$$\omega(v, w) = g_p(v, J_p^{-1} w) = g_p(J_p v, w)$$

so $A_p = J_p$ in the construction above. Then $-A_p^2 = \mathrm{id}_{T_pM}$ has unique square root $P_p = \mathrm{id}_{T_pM}$, and so we retrieve the original almost complex structure. Hence we have constructed a retraction of the space $\mathfrak{Met}(M)$ of Riemannian metrics on M onto $\mathcal{J}(M, \omega)$ (where metrics and almost complex structures are identified if they are compatible). But $\mathfrak{Met}(M)$ is convex, so this implies $\mathcal{J}(M, \omega)$ is path-connected. $\square$

Multiplication by an almost complex structure J_p gives T_pM the structure of a complex vector space of n dimensions, allowing us to define the Chern classes $c_k(TM, J) \in H^{2k}(M)$. By homotopy invariance of Chern classes and Lemma 2.6, these are independent of the almost complex structure, and so we denote them by $c_k(TM)$. Additionally, if M is compact then

$$\left\{ \int_{S^2} f^* c_1(TM) \;\middle|\; [f] \in \pi_2(M) \right\} = N\mathbb{Z}$$

for some non-negative integer N. If $N > 0$ it is called the *minimal Chern number* of M, otherwise the minimal Chern number is defined to be ∞ ([10]).

2.1. **Diffeomorphisms.** Applying (iii) of Proposition 2.2 to the differential of a smooth function $H : M \to \mathbb{R}$, we have

Corollary 2.7. *For any $H \in C^\infty(M)$, there exists a unique vector field X_H on M such that*

$$dH = \iota_{X_H} \omega.$$

Then X_H is called a Hamiltonian vector field with Hamiltonian function H, and the maximal flow $\psi_H : U \to M$ of X_H is called the Hamiltonian flow of H (with U a neighborhood of $\{0\} \times M$ in $\mathbb{R} \times M$). □

Definition 2.8. Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds and $\phi : M_1 \to M_2$ a diffeomorphism. Then ϕ is a *symplectomorphism* if $\phi^* \omega_2 = \omega_1$. If $(M, \omega) = (M_1, \omega_1) = (M_2, \omega_2)$, ϕ is a *Hamiltonian diffeomorphism* (or an *exact diffeomorphism*) if it is generated by the flow of a smooth family $H = \{H_t\}_{t \in \mathbb{R}/\mathbb{Z}}$ of Hamiltonian functions (a "1-periodic time-dependent" Hamiltonian). That is, $\phi = \phi_1$ where ϕ is the unique $\mathbb{R}$-action on M defined by $\phi_0 = \mathrm{id}_M$ and

$$\frac{d\phi_t}{dt} = X_{H_t} \circ \phi_t$$

for all t.

We denote the groups of Hamiltonian diffeomorphisms and symplectomorphisms $(M, \omega) \to (M, \omega)$ by $\mathrm{Ham}(M, \omega)$ and $\mathrm{Symp}(M, \omega)$, respectively. An important property of symplectomorphisms $\mathrm{Symp}(M, \omega)$ is that they are area-preserving (using the volume form $\omega^{\wedge n}$):

$$\phi^*(\omega^{\wedge n}) = (\phi^* \omega)^{\wedge n} = \omega^{\wedge n}.$$

Proposition 2.9. *Any Hamiltonian diffeomorphism on (M, ω) is a symplectomorphism between (M, ω) and (M, ω), that is, $\mathrm{Ham}(M, \omega) \subset \mathrm{Symp}(M, \omega)$.*

Proof. Using the notation of Definition 2.8, we have that $\phi_t^* \omega$ is constant:

$$\frac{d(\phi_t^* \omega)}{dt} = \phi_t^* \mathcal{L}_{X_{H_t}} \omega = \phi_t^* (d\iota_{X_{H_t}} \omega + \iota_{X_{H_t}} d\omega) = \phi_t^* (d(dH_t) + \iota_{X_{H_t}}(0)) = 0.$$

Thus $\phi_1^* \omega = \omega$. □

Definition 2.10. A fixed point $p \in M$ of a Hamiltonian diffeomorphism $\phi : M \to M$ is *non-degenerate* if 1 is not an eigenvalue of $d\phi_p$:

$$\det(d\phi_p - \mathrm{id}_{T_p M}) \neq 0.$$

Note that any periodic time-dependent Hamiltonian function on a closed symplectic manifold generates a Hamiltonian diffeomorphism, and conversely every Hamiltonian diffeomorphism has a unique time-dependent Hamiltonian. We record now some observations about the non-degenerate Hamiltonian Arnold conjecture Conjecture 1.2.

Examples 2.11.

- We will prove in Proposition 5.3 that non-degenerate fixed points are isolated. In particular, there are finitely many on a closed manifold.
- We have
$$\sum_{k=0}^{2n} \operatorname{rank}(H_k(M)) = \sum_{k=0}^{2n} \dim(H^k(M;\mathbb{R})) \geq n+1$$
by (ii) of Proposition 2.2 and the Universal Coefficient Theorem.
- Complex projective space $\mathbb{C}P^n$ has homology groups $H_k(\mathbb{C}P^n) \cong \mathbb{Z}$ for even k, and zero otherwise. Thus the Arnold conjecture states there are at least $n+1$ fixed points of a Hamiltonian diffeomorphism of $\mathbb{C}P^n$.[2]
- Recall that the torus $\mathbb{T}^{2n}$ has homology groups $H_k(\mathbb{T}^{2n}) \cong \mathbb{Z}^{\binom{2n}{k}}$ by the Künneth formula. Then the number of fixed points of a non-degenerate Hamiltonian diffeomorphism of $\mathbb{T}^{2n}$ is at least 2^{2n}.

3. Integral Morse Homology

Floer's technique is based on an elegant proof of the weak Morse inequalities for closed manifolds using Morse homology. Indeed, we will see that the constructions of Floer homologies closely follow those of Morse homology. We now explain this theory, primarily following [18] and [17].

3.1. **Morse Functions.** Let M be an n-manifold.

Definition 3.1. A critical point p of $f \in C^\infty(M)$ is said to be *non-degenerate* if the Hessian d^2f_p is non-singular, and *degenerate* otherwise. A function is *Morse* if all its critical points are non-degenerate. The *Morse index* $\operatorname{ind}_f(p)$ of a non-degenerate critical point $p \in M$ is defined as the dimension of the negative eigenspace of d^2f_p (and so the dimension of the positive eigenspace is $n - \operatorname{ind}_f(p)$). We denote the set of index k critical points of a Morse function f by $\operatorname{Crit}_k(f)$ and the set of all critical points by $\operatorname{Crit}(f)$.

The following result, known as the Morse lemma, is at the foundation of Morse theory and states that the canonical form of d^2f_p describes f near a non-degenerate critical point p.

Lemma 3.2 ([14], Lemma 2.2)**.** *If $f \in C^\infty(M)$ has a non-degenerate critical point p, there exists a chart in a neighborhood $U(p)$ of p such that p has coordinates $(0,\dots,0)$ and the local representation of f is*
$$\tilde{f}(x_1,\dots,x_n) = f(p) - \sum_{i=1}^{\operatorname{ind}_f(p)} x_i^2 + \sum_{i=\operatorname{ind}_f(p)+1}^{n} x_i^2.$$
Such a chart is called a Morse chart for p. □

By Lemma 3.2, a Morse function $f : M \to \mathbb{R}$ has isolated critical points. Let M be closed. Then f has finitely many critical points (and at least one of index 0 and one of index n), and we define the *gradient flow* of f as the global flow $\psi : \mathbb{R} \times M \to M$ of $-\nabla f$ with respect to some Riemannian metric g.[3] A *gradient flow line* is an integral curve of $-\nabla f$.

[2] A symplectic structure of $\mathbb{C}P^n$ can be seen by defining a symplectic form compatible with the Fubini-Study metric and the canonical almost complex structure.

[3] It is conventional to use the flow of the negative gradient rather than that of the gradient.

Proposition 3.3 ([18], Proposition 2.1.6)**.** *Any gradient flow line $\gamma : \mathbb{R} \to M$ of a Morse function f on a closed manifold M converges to critical points as $t \to \pm\infty$.*

Proof. Suppose for sake of contradiction that γ does not converge to a critical point as $t \to \infty$. Let

$$U = \bigcup_{p \in \mathrm{Crit}(f)} U(p);$$

then $\gamma(t) \notin U$ for large t. We have that $M - U$ is compact and $g_p((\nabla f)_p, (\nabla f)_p) > 0$ for all $p \in M - U$, so there exists $\epsilon > 0$ such that

$$\frac{d(f \circ \gamma)}{dt}(t) = (df)_{\gamma(t)}(-\nabla f)_{\gamma(t)} = g_{\gamma(t)}((\nabla f)_{\gamma(t)}, -(\nabla f)_{\gamma(t)}) < -\epsilon$$

for sufficiently large t. But then $\lim_{t \to \infty}(f \circ \gamma)(t) \to -\infty$, contradicting the compactness of M. ☐

3.2. **Stable and Unstable Manifolds.** For $p \in \mathrm{Crit}(f)$, we define the *stable and unstable manifolds* by

$$W_f^s(p) := \{x \in M \mid \lim_{t \to \infty} \psi_t(x) = p\}$$
$$\text{and } W_f^u(p) := \{x \in M \mid \lim_{t \to -\infty} \psi_t(x) = p\},$$

respectively. The following result follows from analysis of sub-level sets $M^a = f^{-1}((-\infty, a])$ in the spirit of classical Morse theory.

Proposition 3.4 ([18], Proposition 2.1.5)**.** *The stable and unstable manifolds of a Morse function $f : M \to \mathbb{R}$ at a critical point p are embedded submanifolds of M such that (where $\mathbb{B}$ represents the open ball)*

$$W_f^s(p) \approx \mathbb{B}^{n - \mathrm{ind}_f(p)}$$
$$\text{and } W_f^u(p) \approx \mathbb{B}^{\mathrm{ind}_f(p)}.$$ ☐

Suppose that for all $p, q \in \mathrm{Crit}(f)$, $W_f^u(p)$ and $W_f^s(q)$ meet transversally. Then (f, g) is called a *Morse-Smale pair*. It can be shown by the Sard-Smale theorem that Morse-Smale pairs are residual in the Banach manifold[4] of Morse functions and Riemannian metrics on M; in particular, a Morse-Smale pair always exists ([17]). Moreover, given a Morse function and any Riemannian metric, the pair can be perturbed to become Morse-Smale while not increasing the number of critical points. Let

$$\widehat{\mathcal{M}}(p, q) = W_f^u(p) \cap W_f^s(q)$$

for all $p, q \in \mathrm{Crit}(f)$. By considering the flow lines $t \mapsto \psi_t(x)$ for $x \in \widehat{\mathcal{M}}(p, q)$, we see that

$$\widehat{\mathcal{M}}(p, q) = \bigcup \{\mathrm{im}(\gamma) \mid \gamma : \mathbb{R} \to M \text{ is a flow line and } \lim_{t \to -\infty} \gamma(t) = p, \lim_{t \to \infty} \gamma(t) = q\}.$$

The Morse-Smale condition implies that $\widehat{\mathcal{M}}(p, q)$ is a submanifold of M with

$$\dim(\widehat{\mathcal{M}}(p, q)) = \dim(W_f^u(p)) + \dim(W_f^s(q)) - n = \mathrm{ind}_f(p) - \mathrm{ind}_f(q).$$

Observation 3.5. This submanifold is empty if $\mathrm{ind}_f(p) < \mathrm{ind}_f(q)$ or (by the following result) $\mathrm{ind}_f(p) = \mathrm{ind}_f(q)$ and $p \neq q$. Intuitively, "the Morse index decreases along non-trivial flow lines."

[4]A *Banach manifold* is a topological space with smoothly compatible local charts to Banach spaces. We will see them again in Section 4.

Proposition 3.6. *For $p \neq q$, the gradient flow is a smooth, free, and proper $\mathbb{R}$-action on $\widehat{\mathcal{M}}(p,q)$.*

Proof. The gradient flow is a smooth $\mathbb{R}$-action on M, and we have

$$\lim_{t\to\pm\infty} \psi_t(\psi_s(x)) = \lim_{t\to\pm\infty} \psi_{t+s}(x) = \lim_{t\to\pm\infty} \psi_t(x) \text{ for all } x \in M$$

so that $\psi_s\left(\widehat{\mathcal{M}}(p,q)\right) \subset \widehat{\mathcal{M}}(p,q)$ for all $s \in \mathbb{R}$. Then the gradient flow is a smooth $\mathbb{R}$-action on $\widehat{\mathcal{M}}(p,q)$. For $x \in \widehat{\mathcal{M}}(p,q)$, we have

$$\frac{d(f \circ \psi_t(x))}{dt} = g(\nabla f_{\psi_t(x)}, -\nabla f_{\psi_t(x)}) \leq 0.$$

Then if $\psi_s(x) = x$ for some $s \in \mathbb{R}$ we have $-\nabla f_{\psi_t(x)} = 0$ for all t between 0 and s. Thus the gradient flow is constant between 0 and s, and is hence constant for all time. Finally, suppose $K \subset \widehat{\mathcal{M}}(p,q)$ is compact. Then f obtains a minimum a and a maximum b on K, and (as in Proposition 3.3) there exists $\epsilon > 0$ such that

$$\frac{d(f \circ \psi_t(x))}{t} < -\epsilon$$

for all $x \in K$ and $t \in \mathbb{R}$ such that $\psi_t(x) \in K$ and so

$$T := \{t \in \mathbb{R} \mid \psi_t(K) \cap K \neq \emptyset\} \subset \left[-\frac{b-a}{\epsilon}, \frac{b-a}{\epsilon}\right]$$

is bounded. Additionally, if $\{t_i\}_{i\in\mathbb{N}} \subset T$ converges to $t \in \mathbb{R}$ then there exists a sequence $\{x_i\}_{i\in\mathbb{N}} \subset K$ such that $\psi_{t_i}(x_i) \in K$ for all i. By compactness of K, an infinite set $I \subset \mathbb{N}$ exists so that $\{x_i\}_{i\in I}$ and $\{\psi_{t_i}(x_i)\}_{i\in I}$ converge to some $x, y \in K$, respectively. Then

$$y = \psi_t\left(\lim_{i\to\infty, i\in I} \psi_t^{-1}(\psi_{t_i}(x_i))\right) = \psi_t\left(\lim_{i\to\infty, i\in I} \psi_{t_i - t}(x_i)\right) = \psi_t(x),$$

so $t \in T$. Thus T is also closed in $\mathbb{R}$, so ψ is a proper action. □

Now the orbit space $\widehat{\mathcal{M}}(p,q)/\mathbb{R}$ has a natural structure as a smooth manifold of dimension $\mathrm{ind}_f(p) - \mathrm{ind}_f(q) - 1$ (see e.g. [13], Theorem 21.10). This manifold is denoted by $\mathcal{M}(p,q)$ and is called the *moduli space of flow lines*. Since $x_1, x_2 \in \widehat{\mathcal{M}}(p,q)$ lie in the same orbit space if and only if they lie on the same flow line, we interpret $\mathcal{M}(p,q)$ as the space of unparameterized flow lines.

We wish now to orient these orbit spaces to define boundary operators $\partial_k^M : \bigoplus_{\mathrm{Crit}_k(f)} \mathbb{Z} \to \bigoplus_{\mathrm{Crit}_{k-1}(f)} \mathbb{Z}$, assuming that M is orientable.[5] Then it follows from Proposition 3.4 that the stable and unstable manifolds are orientable, moreoever, there exist orientations which are compatible in the sense that $T_pW_f^u(p) \oplus T_pW_f^s(p)$ and T_pM have the same orientation for all $p \in \mathrm{Crit}(f)$. The Morse-Smale condition yields a split exact sequence

$$0 \longrightarrow T_x\widehat{\mathcal{M}}(p,q) \xrightarrow{i\oplus j} T_xW_f^u(p) \oplus T_xW_f^s(q) \longrightarrow T_xM \longrightarrow 0$$

for all $x \in \widehat{\mathcal{M}}(p,q)$ with $i \oplus j$ the direct sum of the inclusions. This sequence induces an orientation of $\widehat{\mathcal{M}}(p,q)$ for each choice of compatible orientations on the stable and unstable manifolds ([20]). Taking the canonical orientation of each flow line

[5] If M is not orientable, the complex can still be constructed with coefficients in $\mathbb{Z}_2$. This method will be used in Section 5.

$\gamma \in \mathcal{M}(p,q)$ to be given by the nonzero vector field $-\nabla f$, we then have a natural orientation of $\mathcal{M}(p,q)$ when $p \neq q$.

3.3. Chain Complex.

To define the Morse complex, we require one more result.

Lemma 3.7 ([18], Theorem 3.2.2)**.** *There exists a compactification of $\mathcal{M}(p,q)$ to a smooth manifold with corners*[6] $\overline{\mathcal{M}}(p,q)$ *so that*

$$\overline{\mathcal{M}}(p,q)_j = \coprod_{\substack{r_1,\ldots,r_j \in \mathrm{Crit}(f) \\ p,r_1,\ldots,r_j,q \text{ distinct}}} \mathcal{M}(p,r_1) \times \mathcal{M}(r_1,r_2) \times \cdots \times \mathcal{M}(r_j,q). \qquad \square$$

Interpreting elements of $\mathcal{M}(p,r_1) \times \mathcal{M}(r_1,r_2) \times \cdots \times \mathcal{M}(r_i,q)$ as collections of $i+1$ flow lines joined at their common limits, we call Lemma 3.7 "compactification by broken flow lines." In the case that $\mathrm{ind}_f(p) - \mathrm{ind}_f(q) = 1$, $\dim \mathcal{M}(p,q) = 0$ and so $\mathcal{M}(p,q)$ is a compact 0-dimensional manifold. Then there are finitely many flow lines from p to q. The orientation on $\mathcal{M}(p,q)$ discussed above assigns a value $\epsilon(\gamma) = \pm 1$ to each. Define the signed count $\#(p,q)$ of flow lines between p and q by summing over these orientations:

$$\#(p,q) := \sum_{\gamma \in \mathcal{M}(p,q)} \epsilon(\gamma).$$

Finally, we have

Definition 3.8. For a closed n-manifold M with Morse-Smale pair (f,g), the integral *Morse-Smale-Witten complex*[7] is the graded free abelian group $CM_*(M,f,g)$ with

$$CM_k(M,f,g) := \bigoplus_{p \in \mathrm{Crit}_k(f)} \mathbb{Z} \cdot p$$

and boundary homomorphisms $\partial_k^M : CM_k(M,f,g) \to CM_{k-1}(M,f,g)$ generated by

$$p \mapsto \sum_{q \in \mathrm{Crit}_{k-1}} \#(p,q)q.$$

Proposition 3.9. *The Morse complex is a chain complex.*

Proof. We wish to show $\partial_{k-1}^M \circ \partial_k^M = 0$. For $p \in \mathrm{Crit}_k(f)$, we have

$$\partial_{k-1}^M \circ \partial_k^M(p) = \sum_{\substack{r \in \mathrm{Crit}_{k-1}(f) \\ q \in \mathrm{Crit}_{k-2}(f)}} \#(p,r)\#(r,q) \cdot q. \tag{1}$$

[6] That is, there are compatible charts to $\mathbb{R}^{n-j} \times [0,\infty)^j$ around each point.

[7] The complex is named after Marston Morse, Stephen Smale, and Edward Witten. The modern formulation in terms of gradient flow lines is largely due to Witten's highly influential 1982 paper [12] as a consequence of his study of Hodge theory using a conjugated differential operator, though similar ideas were introduced by René Thom and Stephen Smale. The reader is referred to Raoul Bott's survey [20] for a discussion of the development of Morse theory in the 20th century.

From Lemma 3.7 and Observation 3.5, we have for $p \in \mathrm{Crit}_k(f)$ and $q \in \mathrm{Crit}_{k-2}(f)$ that

$$\begin{aligned}
\overline{\mathcal{M}}(p,q)_0 &= \mathcal{M}(p,q) \\
\overline{\mathcal{M}}(p,q)_1 &= \coprod_{r \in \mathrm{Crit}_{k-1}(f)} \mathcal{M}(p,r) \times \mathcal{M}(r,q) \\
\overline{\mathcal{M}}(p,q)_j &= \emptyset \text{ for } j \geq 2.
\end{aligned}$$

By the classification of compact 1-manifolds with boundary (namely, every such manifold is a disjoint union of closed intervals and copies of S^1), the sum of the induced orientations of the boundary of $\mathcal{M}(p,q)$ is 0. But then

$$\sum_{q \in \mathrm{Crit}_{k-2}(f)} \left(\sum_{r \in \mathrm{Crit}_{k-1}(f)} \#(p,r)\#(r,q) \right) \cdot q = \sum_{q \in \mathrm{Crit}_{k-2}(f)} 0 \cdot q = 0$$

and so the right-hand side of (1) is zero. □

We define the *Morse homology* of M as the homology of its Morse complex,

$$HM_k(M,f,g) := \frac{\ker(\partial_k^M)}{\mathrm{im}(\partial_{k-1}^M)}.$$

Theorem 3.10. *For a closed manifold M with Morse-Smale pair (f,g), we have*

$$HM_*(M,f,g) \cong H_*(M).$$

Sketch of proof. We briefly describe the proof contained in Section 4.9 of [18]. Recalling from Proposition 3.4 that the unstable manifolds $W_f^u(p)$ are diffeomorphic to open disks and cover M by Proposition 3.3, we aim to produce a cellular decomposition of M given by compactifications of $W_f^u(p)$ for each $p \in \mathrm{Crit}(f)$. Indeed, this can be achieved by defining a suitable topology on

$$\overline{W}_f^u(p) := W_f^u(p) \cup \left(\bigcup_{\substack{q \in \mathrm{Crit}(f) \\ \mathcal{M}(p,q) \neq \emptyset}} \left(\overline{\mathcal{M}}(p,q) \times W_f^u(q)\right) \right).$$

Then the dimension of $\overline{W}_f^u(p)$ is still the index $\mathrm{ind}_f(p)$, and moreover, the incidence number of a cell corresponding to $p \in \mathrm{Crit}_k(f)$ and one to $q \in \mathrm{Crit}_{k-1}(f)$ is equal to the signed count of flow lines $\#(p,q)$. Thus the Morse complex and cellular complex are naturally isomorphic chain complexes so they have the same homology. □

Theorem 3.10 allows one to study singular homology by Morse theory.[8] The promised "weak Morse inequalities" are immediate from Theorem 3.10.

Corollary 3.11. *For a closed n-manifold M and Morse function $f : M \to \mathbb{R}$,*

$$|\,\mathrm{Crit}_k(f)| \geq \mathrm{rank}(H_k(M))$$

for all $0 \leq k \leq n$. Summing over k,

$$|\,\mathrm{Crit}(f)| \geq \sum_{k=0}^{n} \mathrm{rank}(H_k(M)).$$ □

[8] Interested readers are encouraged to read Chapter 4 of [18] for interpretations of classical results in singular homology over $\mathbb{Z}_2$.

4. Hamiltonian Floer Homology

We now describe Floer's constructions for the proof of Conjecture 1.2, developed in [3], [4], [5], [6], [7], and [8]. Our introduction is strongly influenced by Dietmar Salamon's lectures ([10]). For the remainder of this paper, (M, ω) is a closed symplectic $2n$-manifold with minimal Chern number N.

Assumptions 4.1. Floer worked under the assumption of *monotonicity*, that is,

$$\int_{S^2} f^*c_1(TM) = \tau \int_{S^2} f^*\omega$$

for all $[f] \in \pi_2(M)$ and some $\tau > 0$. This includes the cases $\pi_2(M) = 0$ and

$$\int_{S^2} f^*\omega = 0$$

for all $[f] \in \pi_2(M)$, termed *symplectic asphericity*. However, we note that generalizations of these constructions exist which remove the monotonicity restriction. Most notably, Kenji Fukaya and Kaoru Ono constructed a Hamiltonian Floer homology with rational coefficients to prove the Hamiltonian Arnold conjecture for all closed symplectic manifolds in [15].

For simplicity, we also assume that $\int_{S^2} f^*\omega \in \mathbb{Z}$ for all $[f] \in \pi_2(M)$. Note that this can always be achieved by normalization of ω.

4.1. Symplectic Action and Pseudoholomorphic Curves.

If p is a fixed point of $\phi \in \mathrm{Ham}(M, \omega)$ generated by $H = \{H_t\}_{t\in\mathbb{R}/\mathbb{Z}}$ then $x(t) = \phi_t(p)$ defines a smooth closed path $x : S^1 \to M$ satisfying the differential equation

$$\frac{dx}{dt}(t) = X_{H_t} \circ \phi_t(x(0)) = X_{H_t}(x(t)).$$

Conversely, any closed $(x(0) = x(1))$ solution to this equation corresponds to a fixed point of f by $x \mapsto x(0)$. Therefore, there is a natural bijection between the fixed points of ϕ and the set of 1-periodic integral curves of X_H.

Hence Conjecture 1.2 can be equivalently stated as a property of the (smooth) free loop space $\mathcal{L}M := C^\infty(S^1, M)$. This space carries a Banach manifold structure such that tangent vectors to smooth loops $x : S^1 \to M$ are vector fields along the loop, that is, $T_x\mathcal{L}M = x^*TM$. Denote the submanifold of $\mathcal{L}M$ consisting of contractible loops by $\mathcal{L}_{\mathrm{contr}}M$, and the space of contractible 1-periodic integral curves of X_H by $\mathcal{P}(H)$. We wish to define a function on $\mathcal{L}_{\mathrm{contr}}M$ which takes the role of the Morse function in Section 3, followed by corresponding notions of index and moduli spaces.

Definition 4.2. The *symplectic action functional* on $\mathcal{L}_{\mathrm{contr}}M$ is defined by

$$a_H(x) = \int_{\mathbb{D}^2} \tilde{x}^*\omega + \int_0^1 H_t(x(t))dt$$

where $\tilde{x}$ is a smooth map $\mathbb{D}^2 \to M$ such that $\tilde{x}\big|_{\partial\mathbb{D}^2=S^1} = x$. The real value of this action may depend on the choice of u, but by Assumptions 4.1 the dependence lies in $\mathbb{Z}$. Thus $a_H : \mathcal{L}_{\mathrm{contr}}M \to \mathbb{R}/\mathbb{Z}$ is well-defined.

Now we compute the gradient of the action functional with respect to a canonical Riemannian metric on the free loop space. Namely, let J be a time-dependent 1-periodic family of compatible almost complex structures on (M, ω) that induces

Riemannian metrics $g = \{g_t\}_{t\in\mathbb{R}/\mathbb{Z}}$. Then the metric on the loop space is given by integration: for $\xi, \eta \in T_x\mathcal{L}M$ define

$$\langle \xi, \eta\rangle := \int_0^1 (g_t)_{x(t)}(\xi(t), \eta(t))dt.$$

Then symmetry, linearity, and non-degeneracy of $\langle\cdot,\cdot\rangle$ are immediate from those of g_t.

Proposition 4.3. *The gradient of the action functional at $x \in \mathcal{L}_{\mathrm{contr}}M$ is the tangent vector at x given by*

$$(\nabla a_H)_x(t) = (J_t)_{x(t)}\frac{dx}{dt}(t) + (\nabla H_t)_{x(t)}(t).$$

Proof. For $\xi \in T_x\mathcal{L}_{\mathrm{contr}}M$, we have

$$(da_H)_x\xi = \int_0^1 \left(\omega_{x(t)}\left(\frac{dx}{dt}(t), \xi(t)\right) + (dH_t)_{x(t)}\xi(t)\right) dt.$$

Then

$$\begin{aligned}(da_H)_x\xi &= \int_0^1 \omega_{x(t)}\left(\frac{dx}{dt}(t), \xi(t)\right)\, dt + \int_0^1 (dH_t)_{x(t)}\xi(t)\, dt\\ &= \int_0^1 (g_t)_{x(t)}\left(J_t(x(t))\frac{dx}{dt}(t), \xi(t)\right)\, dt + \int_0^1 (g_t)_{x(t)}((\nabla H_t)_{x(t)}, \xi(t))\, dt\\ &= \int_0^1 (g_t)_{x(t)}\left((J_t)_{x(t)}\frac{dx}{dt}(t) + (\nabla H_t)_{x(t)}(t), \xi(t)\right)\, dt.\end{aligned}$$

□

We see that $(\nabla a_H)_x(t) = 0$ if and only if

$$(J_t)_{x(t)}\frac{dx}{dt}(t) = -(\nabla H_t)_{x(t)}(t).$$

This is equivalent to the statement that, for all $v \in T_{x(t)}M$,

$$\omega_{x(t)}\left(v, \frac{dx}{dt}(t)\right) = (g_t)_{x(t)}(v, (\nabla H_t)_{x(t)}(t)) = (dH_t)_{x(t)}v,$$

that is,

$$\frac{dx}{dt}(t) = (X_{H_t})_{x(t)}.$$

Then the critical points of the action functional are exactly the contractible fixed point loops $\mathcal{P}(H)$. We again consider the integral curves of the negative gradient of a_H; by Proposition 4.3 these are functions $u : \mathbb{R}\times(\mathbb{R}/\mathbb{Z}) \to M$ satisfying

$$\frac{\partial u}{\partial s}(s,t) = -(\nabla a_H)_{u(s,\cdot)}(t) = -J_t(u(s,t))\frac{\partial u}{\partial t}(s,t) - (\nabla H_t)_{u(s,t)}$$

for $s \in \mathbb{R}$ and $t \in \mathbb{R}/\mathbb{Z}$. Such a function u is called a *pseudoholomorphic curve* or *trajectory* for H and J. It is shown in [5] (Lemma 2.1) that all pseudoholomorphic curves are indeed smooth.

4.2. **Moduli Spaces of Trajectories.** The trajectories of a Hamiltonian behave poorly compared to the gradient flow lines of a Morse function. Even on closed symplectic manifolds, trajectories do not necessarily converge (as $s \to \pm\infty$) to fixed point loops of the Hamiltonian flow (cf. Proposition 3.3). The following result, however, gives a simple condition for convergence.

Proposition 4.4 ([5])**.** *Define the energy of a pseudoholomorphic curve* $u : \mathbb{R} \times (\mathbb{R}/\mathbb{Z}) \to M$ *by*

$$E(u) := \int_0^1 \int_{-\infty}^{\infty} \left(\left| \frac{\partial u}{\partial s}(s,t) \right|^2 + \left| \frac{\partial u}{\partial t}(s,t) - X_{H_t}(u(s,t)) \right|^2 \right) ds\, dt.$$

Then there exist $x^+, x^- \in \mathcal{P}(H)$ *such that*

$$\lim_{s \to \pm\infty} u(s,t) = x^{\pm}(t)$$

for all $t \in \mathbb{R}/\mathbb{Z}$ *if and only if* $E(u)$ *is finite.* □

Definition 4.5. The *space of trajectories* between $x^{\pm} \in \mathcal{P}(H)$ with respect to H and J is defined as the union of the images in $\mathcal{L}M$ of all pseudoholomorphic curves $u : \mathbb{R} \times (\mathbb{R}/\mathbb{Z}) \to M$ such that

$$\lim_{s \to \pm\infty} u(s,t) = x^{\pm}(t).$$

The space is denoted by $\widehat{\mathcal{M}}((x^-, x^+), H, J)$.

The Morse-Smale condition in Morse homology transfers to Hamiltonian Floer homology as "regularity" of Hamiltonians.

Theorem 4.6 ([7], Proposition 1b)**.** *For any periodic compatible almost complex structure* J *there exists a dense subset* $\mathcal{H}_{reg}(J) \subset C^{\infty}(\mathbb{R}/\mathbb{Z} \times M)$ *of regular Hamiltonians such that for all* $H \in \mathcal{H}_{reg}(J)$ *there exists a function*

$$\mu_H : \mathcal{P}(H) \to \mathbb{Z}_{2N}$$

so that for all $x^{\pm} \in \mathcal{P}(H)$*,* $\widehat{\mathcal{M}}((x^-, x^+), H, J)$ *is a submanifold of* $\mathcal{L}M$ *with*

$$\dim(\widehat{\mathcal{M}}((x^-, x^+), H, J)) \equiv \mu_H(x^-) - \mu_H(x^+) \ \mathit{mod}\ 2N.$$ □

As in Proposition 3.6, it can be shown that reparameterization of pseudoholomorphic curves, that is,

$$\psi_r(u)(s,t) = u(r+s,t)$$

for $r \in \mathbb{R}$, is a smooth, free, and proper $\mathbb{R}$-action on $\widehat{\mathcal{M}}((x^-, x^+), H, J)$ for distinct $x^-, x^+ \in \mathcal{P}(H)$. Then for $H \in \mathcal{H}_{reg}(J)$ we have that the quotient space

$$\mathcal{M}((x^-, x^+), H, J) := \widehat{\mathcal{M}}((x^-, x^+), H, J)/\mathbb{R}$$

is a manifold with

$$\dim(\mathcal{M}((x^-, x^+), H, J) \equiv \mu_H(x^-) - \mu_H(x^+) - 1 \bmod 2N.$$

This manifold is the *moduli space of trajectories* from x^- to x^+, relative to H and J.

Theorem 4.7 ([7])**.** *If $x^{\pm} \in \mathcal{P}(H)$ and $H \in \mathcal{H}_{reg}(J)$ such that*

$$\mu_H(x^-) - \mu_H(x^+) \equiv 1 \ mod\ 2N,$$

then $\mathcal{M}((x^-, x^+), H, J)$ is compact. If

$$\mu_H(x^-) - \mu_H(x^+) \equiv 2 \ mod\ 2N,$$

then there exists a compactification of $\mathcal{M}((x^-, x^+), H, J)$ to a 1-manifold with boundary $\overline{\mathcal{M}}((x^-, x^+), H, J)$ such that

$$\partial\overline{\mathcal{M}}((x^-, x^+), H, J) = \coprod_{\substack{y \in \mathcal{P}(H) \\ \mu_H(x^-) - \mu_H(y) \equiv 1 \ mod\ 2N}} \mathcal{M}((x^-, y), H, J) \times \mathcal{M}((y, x^+), H, J).$$

□

Moreover, Floer and Helmut Hofer ([7], [8]) showed that there exist "coherent orientations" on $\widehat{\mathcal{M}}((x^-, x^+), H, J)$ that induce orientations on the moduli spaces such that the equalities above preserve orientation. In the case that

$$\mu_H(x^-) - \mu_H(x^+) \equiv 1 \bmod 2N,$$

there are finitely many points in $\mathcal{M}((x^-, x^+), H, J)$ and so its orientation yields values $\epsilon(u) = \pm 1$ for each trajectory u from x^- to x^+.

4.3. **Chain Complex.** We are now ready to construct the Floer chain complex. For $x^{\pm} \in \mathcal{P}(H)$ such that $\mu_H(x^-) - \mu_H(x^+) \equiv 1 \bmod 2N$, we count the trajectories from x^- to x^+ by summing their orientation values:

$$\#(x^-, x^+) = \sum_{u \in \mathcal{M}((x^-, x^+), H, J)} \epsilon(u).$$

Definition 4.8. Suppose $J = \{J_t\}_{t \in \mathbb{R}/\mathbb{Z}}$ is compatible with (M, ω) and $H \in \mathcal{H}_{reg}(J)$. The (integral) *Hamiltonian Floer complex* on M is the graded free abelian group $CF_*((M, \omega), H, J)$ defined by

$$CF_k((M, \omega), H, J) = \bigoplus_{\substack{x \in \mathcal{P}(H), \\ \mu_H(x) \equiv k \bmod 2N}} \mathbb{Z} \cdot x$$

with boundary homomorphisms $\partial_k^F : CF_k((M, \omega), H, J) \to CF_{k-1}((M, \omega), H, J)$ generated by

$$\partial_k^F(x^-) = \sum_{\substack{x^+ \in \mathcal{P}(H), \\ \mu_H(x) \equiv k-1 \bmod 2N}} \#(x^-, x^+) \cdot x^+$$

Proposition 4.9. *The Hamiltonian Floer complex is a chain complex.*

Proof. By Theorem 4.7, whenever $x^{\pm} \in \mathcal{P}(H)$ with $\mu_H(x^-) - \mu_H(x^+) \equiv 2 \bmod 2N$,

$$\sum_{\substack{y \in \mathcal{P}(H) \\ \mu_H(x^-) - \mu_H(y) \equiv 1 \bmod 2N}} \#(x^-, y)\#(y, x^+) = 0.$$

Then for $x^- \in \mathcal{P}(H)$ with $\mu_H(x^-) \equiv k \pmod{2N}$, we see that

$$\partial_{k-1}^F \circ \partial_k^F(x^-) = \sum_{\substack{x^+, y \in \mathcal{P}(H) \\ \mu_H(y) \equiv k-1 \pmod{2N} \\ \mu_H(x^+) \equiv k-2 \pmod{2N}}} \#(x^-, y)\#(y, x^+) \cdot x^+ = 0.$$

□

The *Hamiltonian Floer homology* is defined as the homology of $CF_*((M,\omega),H,J)$ with $\mathbb{Z}_{2N}$-grading:

$$HF_k((M,\omega),H,J) := \frac{\ker(\partial_k^F)}{\operatorname{im}(\partial_{k-1}^F)}.$$

Theorem 4.10 ([7])**.** *If J is a 1-periodic almost complex structure on (M,ω) and $H \in \mathcal{H}_{reg}(J)$, then*

$$HF_k((M,\omega),H,J) \cong \bigoplus_{i \equiv k \pmod{2N}} H_i(M).$$

Sketch of proof. Floer proved this result by constructing "continuation maps"

$$HF_k((M,\omega),H,J) \to HF_k((M,\omega),H',J')$$

to show that the homology is independent of the regular Hamiltonian and almost complex structure chosen. Then Theorem 4.10 follows from his result ([6], Theorem 3) that there exists compatible $J = \{J_t\}_{t\in\mathbb{R}/\mathbb{Z}}$ such that for sufficiently small $H = \{H_t\}_{t\in\mathbb{R}/\mathbb{Z}}$ in the C^2-topology, the fixed point loops of H are constant and the pseudoholomorphic curves are precisely the gradient flow lines of Morse homology. Since $\mathcal{H}_{reg}(J)$ is dense in $C^\infty(\mathbb{R}/\mathbb{Z}\times M)$, there is a C^2-small regular Hamiltonian H. Then the isomorphism holds for H, J, so it holds for all almost complex structures and regular Hamiltonians. □

As an immediate corollary, we have

$$\begin{aligned}|\mathcal{P}(H)| &\geq \sum_{k=0}^{2N} \operatorname{rank}(HF_k((M,\omega),H,J)) \\ &= \sum_{k=0}^{2N} \sum_{i\equiv k \pmod{2N}} \operatorname{rank}(H_i(M)) \\ &= \sum_{i=0}^{2n} \operatorname{rank}(H_i(M))\end{aligned}$$

for $H \in \mathcal{H}_{reg}(J)$. But the time-dependent Hamiltonian generating a Hamiltonian diffeomorphism with non-degenerate fixed points can always be perturbed to some regular Hamiltonian with at most an equal number of fixed points. Thus we have established Conjecture 1.2 in the case that (M,ω) is monotone.

5. Lagrangian Intersection Homology

In this section we describe the relationship between Conjecture 1.2 and Conjecture 1.3, and the homology theory used to prove Conjecture 1.3 for $\mathbb{Z}_2$ coefficients ([3], [4], [5], [6]).

5.1. **Lagrangian Submanifolds.** A submanifold L of M is *Lagrangian* if $\omega|_L = 0$ and L has the maximum possible dimension. By Proposition 2.1, $\dim L = \frac{1}{2}\dim M$.

Examples 5.1.

- The zero section is trivially a Lagrangian submanifold of T^*M for any manifold M.

- The real projective space $\mathbb{R}P^n$ can be embedded as a Lagrangian submanifold of $\mathbb{C}P^n$.
- Lagrangian submanifolds are preserved under symplectomorphism, in particular, they are preserved under Hamiltonian diffeomorphisms.

Proposition 5.2. *Suppose (M_1, ω_1) and (M_2, ω_2) are symplectic n-manifolds and $\phi : M_1 \to M_2$ a diffeomorphism. Then ϕ is a symplectomorphism if and only if the graph of ϕ is a Lagrangian submanifold of $(M_1 \times M_2, \pi_1^*\omega_1 - \pi_2^*\omega_2)$ where $\pi_1, \pi_2 : M_1 \times M_2 \to M_1, M_2$ are the projections.*

Proof. Let $M = M_1 \times M_2$ and $\omega = \pi_1^*\omega_1 - \pi_2^*\omega_2$. As in Examples 2.3, ω is a symplectic form on M and so (M, ω) is a symplectic $2n$-manifold. Since $\operatorname{graph}(\phi)$ is an n-dimensional submanifold of M, it suffices to show that $\phi^*\omega_2 = \omega_1$ if and only if ω is zero on $\operatorname{graph}(\phi)$. Indeed,

$$\phi^*\omega_2 = i^*\pi_2^*\omega_2 = i^*(\pi_1^*\omega_1 - \omega) = \omega_1 - i^*\omega$$

where i is the embedding $\mathrm{id}_{M_1} \times \phi$. □

In particular, the diagonal $\Delta \subset M \times M$ is a Lagrangian submanifold of $(M \times M, \pi_1^*\omega - \pi_2^*\omega)$ for any symplectic manifold M, and the graph of any Hamiltonian diffeomorphism $\phi \in \operatorname{Ham}(M, \omega)$ is a Lagrangian submanifold of $(M \times M, \pi_1^*\omega - \pi_2^*\omega)$. This leads to a convenient way to interpret non-degeneracy of fixed points.

Proposition 5.3. *If $\phi \in \operatorname{Ham}(M, \omega)$, then a fixed point p of ϕ is non-degenerate if and only if the intersection of $\operatorname{graph}(\phi)$ and Δ at (p, p) is transversal.*

Proof. If $\phi(p) = p$ then $(p, \phi(p)) = (p, p)$ lies on both $\operatorname{graph}(\phi)$ and Δ. The intersection is transversal if and only if $T_{(p,p)} \operatorname{graph}(\phi)$ and $T_{(p,p)}\Delta$ span $T_{(p,p)}(M \times M)$. Since $\dim(\operatorname{graph}(\phi)) + \dim \Delta = \dim M$, this is equivalent to

$$T_{(p,p)} \operatorname{graph}(\phi) \cap T_{(p,p)}\Delta = \{0\}.$$

We have that

$$T_{(p,p)} \operatorname{graph}(\phi) = \{(v, d\phi_p(v)) \mid v \in T_{(p,p)}M\}$$

and

$$T_{(p,p)}\Delta = \{(v, v) \mid v \in T_{(p,p)}M\}.$$

Hence the intersection is nonempty if and only if $d\phi_p(v) \neq v$ for all v, that is, 1 is not an eigenvalue of $d\phi_p$. □

In particular, Proposition 5.3 implies that the fixed points of ϕ are isolated and therefore there are only finitely many (for closed M).

If $\phi \in \operatorname{Ham}(M, \omega)$ is generated by a periodic Hamiltonian $H = \{H_t\}_{t \in \mathbb{R}/\mathbb{Z}}$, then $\{H_t' = H_t \circ \pi_2\}_{t \in \mathbb{R}/\mathbb{Z}}$ is a periodic Hamiltonian on $M \times M$. Moreover, $X_{H_t'} = (0, X_{H_t})$ with respect to $\pi_1^*\omega - \pi_2^*\omega$. Then $\mathrm{id}_M \times \phi \in \operatorname{Ham}(M \times M, \pi_1^*\omega - \pi_2^*\omega)$. Hence we have shown that Conjecture 1.3 generalizes Conjecture 1.2 in the case that M is symplectically aspherical:

$$|\{p \in M \mid \phi(p) = p\}| = |\Delta \cap \operatorname{graph}(\phi)| = |\Delta \cap (\mathrm{id}_M \times \phi)(\Delta)| \geq \sum_{k=0}^{2n} \operatorname{rank}(H_k(M))$$

for all $\phi \in \operatorname{Ham}(M, \omega)$ with non-degenerate fixed points, where we have used that $\Delta \approx M$. Indeed, this is the form of the Hamiltonian Arnold conjecture proved by Floer in [3] with coefficients in $\mathbb{Z}_2$.

5.2. **Chain Complex.** Let (M,ω), ϕ, and L satisfy the hypotheses of Conjecture 1.3.

Definition 5.4. Suppose $p, q \in L \cap \phi(L)$ and $J \in \mathcal{J}(M,\omega)$. A *$J$-holomorphic curve* from p to q relative to L and $\phi(L)$ is a smooth function $u : \mathbb{R} \times [0,1] \to M$ such that

$$\begin{aligned}\lim_{s\to-\infty} u(s,t) &= p,\\ \lim_{s\to\infty} u(s,t) &= q,\\ u(s,0) &\in L,\\ u(s,1) &\in \phi(L),\\ \frac{\partial u}{\partial s}(s,t) &= -(J_t)_{u(s,t)}\frac{\partial u}{\partial t}(s,t)\end{aligned}$$

for all $s \in \mathbb{R}$ and $t \in [0,1]$.

As in Proposition 4.3, we have that J-holomorphic curves are flow lines of the negative gradient of an action functional. Define Ω to be the subset of $C^\infty([0,1], M)$ of paths such that $x(0) \in L$, $x(1) \in \phi(L)$, and there exists $\tilde{x} \in C^\infty([0,1]\times[0,1], M)$ such that

$$\begin{aligned}\tilde{x}(s,0) &\in L,\\ \tilde{x}(s,1) &\in \phi(L),\\ \tilde{x}(1,t) &= x(t),\end{aligned}$$

for all $s,t \in [0,1]$. The action functional $a : \Omega \to \mathbb{R}$ is given by

$$a(x) = \int_0^1 \int_0^1 \tilde{x}^*\omega \; ds\, dt,$$

so that the gradient is

$$(\nabla a)_x(t) = (J_t)_{x(t)}\frac{dx}{dt}(t).$$

Theorem 5.5 ([3], Proposition 2.4)**.** *There exists a dense subset $\mathcal{J}_{reg}(L,\phi(L))$ of $\mathcal{J}(M,\omega)$ and a function*

$$\mu : L \cap \phi(L) \to \mathbb{Z}$$

such that the spaces $\widehat{\mathcal{M}}((p,q),J)$ for $p,q \in L\cap\phi(L)$ are submanifolds of Ω with

$$\dim(\widehat{\mathcal{M}}((p,q),J)) = \mu(p) - \mu(q)$$

for all $J \in \mathcal{J}_{reg}(L,\phi(L))$. □

The *moduli space of J-holomorphic curves* between p and q is given by

$$\mathcal{M}((p,q),J) = \widehat{\mathcal{M}}((p,q),J))/\mathbb{R}$$

where $\mathbb{R}$ acts on $\widehat{\mathcal{M}}((p,q),J))$ by translation in the first component. Thus

$$\dim(\mathcal{M}((p,q),J)) = \mu(p) - \mu(q) - 1.$$

If $\mu(p) = \mu(q)+1$, then $\mathcal{M}((p,q),J)$ is finite. Define the $\mathbb{Z}_2$ count of J-holomorphic curves from p to q by

$$\#(p,q) := |\mathcal{M}((p,q),J)| \bmod 2.$$

Definition 5.6. The *Lagrangian Floer complex* $CF_*((L,\phi(L)),J)$ is the graded abelian group given by

$$CF_k((L,\phi(L)),J) = \bigoplus_{\substack{p\in L\cap\phi(L)\\ \mu(p)=k}} \mathbb{Z}_2 \cdot p$$

with boundary homomorphisms $\partial_k^F : CF_k((L,\phi(L)),J) \to CF_{k-1}((L,\phi(L)),J)$ given by

$$\partial_k^F(p) = \sum_{\substack{q\in L\cap\phi(L)\\ \mu(q)=k-1}} \#(p,q)\cdot q.$$

Then the Lagrangian Floer complex is a chain complex, and the proof is identical to that of Proposition 3.9 or Proposition 4.9 given a suitable compactification of the 1-dimensional moduli spaces ([3], Section 4). The *Lagrangian Floer homology* $HF_*((L,\phi(L)),J)$ of L and $\phi(L)$ is the homology of $CF_*((L,\phi(L)),J)$. Finally, Conjecture 1.3 for $\mathbb{Z}_2$ coefficients follows from

Theorem 5.7 ([6], Theorem 1)**.** *If (M,ω), ϕ, and L are as above then*

$$HF_*((L,\phi(L)),J) \cong H_*(L,\mathbb{Z}_2)$$

for all $J \in \mathcal{J}_{reg}(L,\phi(L))$. □

6. A_∞-Categories

In this final section, we put aside the Arnold conjectures to discuss a topic at the center of 21$^{\text{st}}$ century symplectic geometry. In 1993, Fukaya ([16]) introduced an A_∞-category $\mathrm{Fuk}(M,\omega)$ as a method for storing algebraic data about transversal intersections of Lagrangian submanifolds in a manner analogous to generalized cup products on Morse-Smale-Witten complexes. Lagrangian submanifolds and pseudoholomorphic curves have found interpretations in string theory as branes and world sheets, respectively, and the Fukaya category associated with Lagrangian cohomology is crucial in framing the homological mirror symmetry conjecture of Maxim Kontsevich.[9]

6.1. A_∞-Relations.

The following is a precise (cohomological) version of Fukaya's original definition of an A_∞-category ([16], Definition 1.15).

Definition 6.1 ([9], Proposition 2.7)**.** An *A_∞-category* $\mathcal{A}$ is a collection of objects $\mathrm{Ob}(\mathcal{A})$, a cochain complex of R-modules (over some commutative ring R) "morphisms" $\mathrm{Hom}(a,b) = \mathrm{Hom}^*(a,b)$ with coboundary operator η_1 for each $a,b \in \mathrm{Ob}(\mathcal{A})$, and higher composition maps

$$\eta_k : \mathrm{Hom}^*(a_0,a_1)\otimes_R \mathrm{Hom}^*(a_1,a_2)\otimes_R \cdots \otimes_R \mathrm{Hom}^*(a_{k-1},a_k) \to \mathrm{Hom}^*(a_0,a_k)$$

of degree $2-k$ for all $k \geq 1$ that satisfy the *A_∞-relations*

$$\sum_{0\leq i<j\leq k} (-1)^{\bullet}\eta_{k-j+i+1}(x_1\otimes\cdots\otimes x_i\otimes \eta_{j-i}(x_{i+1}\otimes\cdots\otimes x_j)\otimes x_{j+1}\otimes\cdots\otimes x_k) = 0$$

for each $k \geq 1$, where $\bullet = i + \deg(x_1) + \cdots + \deg(x_i)$.

It is instructive to consider these compatibility relations for small values of k.

[9] For an introduction to the string theoretic perspective on Floer cohomology and Fukaya categories, see Wang Yao's 2021 REU paper [22].

Examples 6.2.

- For $k = 1$, the series contains only one nonzero term for $(i, j) = (0, 1)$, so that
$$(-1)^0 \eta_1(\eta_1(x_1)) = 0$$
for all morphisms x_1. Hence $\eta_1 \circ \eta_1 = 0$, and so the A_∞-relations state that η_1 is a valid coboundary.
- For $k = 2$, we have terms for $(i, j) = (0, 1), (0, 2)$, and $(1, 2)$:
$$(-1)^0 \eta_2(\eta_1(x_1) \otimes x_2) + (-1)^0 \eta_1(\eta_2(x_1 \otimes x_2)) + (-1)^{1+\deg(x_1)} \eta_2(x_1 \otimes \eta_1(x_2)) = 0,$$
so
$$\eta_1(\eta_2(x_1 \otimes x_2)) = \eta_2(\eta_1(x_1) \otimes x_2) + (-1)^{\deg(x_1)} \eta_2(x_1 \otimes \eta_1(x_2)).$$
Then η_2 defines a chain map of degree 0,
$$\eta_2 : (\mathrm{Hom}(a_0, a_1) \otimes_R \mathrm{Hom}_*(a_1, a_2))_* \to \mathrm{Hom}_*(a_0, a_2)$$
for all $a_0, a_1, a_2 \in \mathrm{Ob}(\mathcal{A})$.
- When $k = 3$, there are terms for $(i, j) = (0, 1), (0, 2), (0, 3), (1, 2), (1, 3)$, and $(2, 3)$:
$$\begin{aligned} 0 &= (-1)^0 \eta_3(\eta_1(x_1) \otimes x_2 \otimes x_3) + (-1)^0 \eta_2(\eta_2(x_1 \otimes x_2) \otimes x_3) \\ &\quad + (-1)^0 \eta_1(\eta_3(x_1 \otimes x_2 \otimes x_3)) + (-1)^{1+\deg(x_1)} \eta_3(x_1 \otimes \eta_1(x_2) \otimes x_3) \\ &\quad + (-1)^{1+\deg(x_1)} \eta_2(x_1 \otimes \eta_2(x_2 \otimes x_3)) + (-1)^{2+\deg(x_1)+\deg(x_2)} \eta_3(x_1 \otimes x_2 \otimes \eta_1(x_3)), \end{aligned}$$
so
$$\begin{aligned} &\eta_2(\eta_2(x_1 \otimes x_2) \otimes x_3) - (-1)^{\deg(x_1)} \eta_2(x_1 \otimes \eta_2(x_2 \otimes x_3)) \\ &= -\eta_3(\eta_1(x_1) \otimes x_2 \otimes x_3) - \eta_1(\eta_3(x_1 \otimes x_2 \otimes x_3)) \\ &+ (-1)^{\deg(x_1)} \eta_3(x_1 \otimes \eta_1(x_2) \otimes x_3) + (-1)^{1+\deg(x_1)+\deg(x_2)} \eta_3(x_1 \otimes x_2 \otimes \eta_1(x_3)). \end{aligned}$$
Thus the compatibility relations state that η_2 is not in general associative.

6.2. **Fukaya Categories.** For a closed symplectic manifold (M, ω), Fukaya's A_∞-category $\mathrm{Fuk}(M, \omega)$ has as objects the compact Lagrangian submanifolds L of M such that $[\![\omega]\!]$ is zero on $\pi_2(M, L)$, and as hom-sets the "Lagrangian Floer cochain complexes"
$$\mathrm{Hom}^*(L_0, L_1) = CF^*((L_0, L_1), J)$$
where $J = \{J_t\}_{t \in \mathbb{R}/\mathbb{Z}}$ is compatible with (M, ω). We briefly describe this cohomology for coefficients in $\mathbb{Z}_2$, a generalization of the constructions in Section 5, and the higher composition maps of $\mathrm{Fuk}(M, \omega)$ following [9].

The cochain complex $CF^*((L_0, L_1), J)$ for $L_0, L_1 \in \mathrm{Ob}(\mathrm{Fuk}(M, \omega))$ is comprised of groups generated by the intersections of L_0 and L_1 after they have been perturbed by a generic Hamiltonian diffeomorphism so that $L_0 \pitchfork L_1$. Then $L_0 \cap L_1$ is a 0-dimensional submanifold of M, a finite set of isolated points. The coboundary operators δ_*^F will be defined using J-holomorphic curves between intersection points as in Section 5, but we reverse the grading from Section 5 by considering a J-holomorphic u from p to q to be one for which $\lim_{s \to -\infty} u(s, t) = q$ and $\lim_{s \to \infty} u(s, t) = p$.

Under assumptions,[10] the dimension of $\widehat{\mathcal{M}}((p, q), J)$ is given by $\mu(q) - \mu(p)$ for some $\mu : L_0 \cap L_1 \to \mathbb{Z}$, so that the unparameterized moduli space $\mathcal{M}((p, q), J)$ has dimension $\mu(q) - \mu(p) - 1$. If u is a J-holomorphic curve in $\mathcal{M}((p, q), J)$, then there

[10] Specifically, $2c_1(TM) \in H^2(M)$ and the Maslov class of L in $H^1(L)$ both vanish.

is a corresponding homotopy class $[u] \in \pi_2(M, L_0 \cup L_1)$. An index of $[u]$ can be defined so that $\mathrm{ind}([u]) = \mu(q) - \mu(p)$.

In general we have that the moduli spaces $\mathcal{M}((p,q),J)$ for $\mu(q) - \mu(p) = 1$ are infinite, that is, there may be infinitely many curves u between $p, q \in L_0 \cap L_1$ with $\mathrm{ind}([u]) = 1$. This problem is solved by allowing infinite sums in $CF^*((L_0, L_1), J)$ using coefficients in a Novikov field.

Definition 6.3. For a field k, the *Novikov field over k* is the field of generalized polynomials

$$\Lambda := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid a_i \in k, \lambda_i \in \mathbb{R}, \lim_{i\to\infty} \lambda_i = \infty \right\}$$

with the expected operations

$$\sum_{i=0}^{\infty} a_i T^{\lambda_i} + \sum_{i=0}^{\infty} b_i T^{\mu_i} = \sum_{i=0}^{\infty} (a_i T^{\lambda_i} + b_i T^{\mu_i})$$

and

$$\left(\sum_{i=0}^{\infty} a_i T^{\lambda_i}\right) \cdot \left(\sum_{i=0}^{\infty} b_i T^{\mu_i}\right) = \sum_{k=0}^{\infty} \sum_{i+j=k} a_i b_j T^{\lambda_i + \mu_j}.$$

For simplicity we take coefficients in the Novikov field Λ over $\mathbb{Z}_2$ to avoid the question of orientability of moduli spaces.

Definition 6.4. The *Lagrangian intersection cochain complex* of L_0 and L_1 is the graded free Λ-module $CF^*((L_0, L_1), J)$ with

$$CF^k((L_0, L_1), J) = \bigoplus_{\substack{p \in L_0 \cap L_1 \\ \mu(p) = k}} \Lambda \cdot p$$

and coboundary homomorphisms $\delta_F^k : CF^k((L_0, L_1), J) \to CF^{k+1}((L_0, L_1), J)$ given by

$$\delta_F^k(p) := \sum_{\substack{q \in L_0 \cap L_1 \\ [u] \in \pi_2(M, L_0 \cup L_1) \\ \mathrm{ind}([u]) = 1}} \#\mathcal{M}((p,q),[u],J) T^{E([u])} \cdot q$$

where $\#\mathcal{M}((p,q),[u],J)$ is the count of J-holomorphic curves from p to q corresponding to $[u]$, modulo 2, and $E([u])$ is the "energy"

$$E([u]) := \int_{S^2} u^* \omega.$$

Standard arguments using Gromov's compactness theorem ([19], 1.5.B) yield compactifications of the moduli spaces of J-holomorphic curves to show that $\delta_F^{k+1} \circ \delta_F^k = 0$ for all k. The *Lagrangian Floer cohomology* is defined as the cohomology of this cochain complex.

Definition 6.5. The higher composition maps of $\mathrm{Fuk}(M, \omega)$ are the Λ-linear maps

$$\mu_k : CF^*(L_0, L_1) \otimes_\Lambda CF^*(L_1, L_2) \otimes_\Lambda \cdots \otimes_\Lambda CF^*(L_{k-1}, L_k) \to CF^*(L_0, L_k)$$

for $L_0, L_1, \dots, L_k \in \mathrm{Ob}(\mathrm{Fuk}(M, \omega))$ given by

$$\mu_k(p_1, \dots, p_k) := \sum_{\substack{q \in L_0 \cap L_k \\ [u] \in \pi_2(M, L_0 \cup L_1 \cup \cdots \cup L_k) \\ \mathrm{ind}([u]) = 2-k}} \#\mathcal{M}((p_1, \dots, p_k, q),[u],J) T^{E([u])} \cdot q,$$

where $\mathcal{M}((p_1, \ldots, p_k, q), [u], J)$ is the moduli space of J-holomorphic curves corresponding to a homotopy class $[u] \in \pi_2(M, L_0 \cup L_1 \cup \cdots \cup L_k)$ given by a map $f : \mathbb{D}^2 \to M$ for which $f(z) \in L_i$ for $z = e^{i\theta}$ and $\frac{2\pi i}{k+1} < \theta < \frac{2\pi(i+1)}{k+1}$ and

$$f(0) = p_1, f\left(e^{\frac{2\pi i}{k+1}}\right) = p_2, \ldots, f\left(e^{\frac{2\pi i k}{k+1}}\right) = q.$$

For $k = 1$ the composition μ_1 is simply the coboundary operator δ_F^*. For all $k \geq 1$, we have

$$\operatorname{ind}([u]) = \mu(q) - \mu(p_1) - \cdots - \mu(p_k),$$

and so when $\#\mathcal{M}((p_1, \ldots, p_k, q), [u], J) \neq 0$ with $\operatorname{ind}([u]) = 2 - k$ we must have

$$\mu(q) = (\mu(p_1) + \cdots + \mu(p_k)) + (2 - k).$$

Then μ_k has degree $2 - k$. That μ_k satisfy the A_∞-relations is sketched in [9], Proposition 2.7.

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