BERNSTEIN'S THEOREM AND OTHER BASICS OF MINIMAL SURFACE THEORY

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Abstract. The theory of minimal surfaces is an area of mathematical research that boasts both a deep history and immense influence on the rest of mathematics. In this paper, we establish some basic facts about minimal surfaces, and prove the famous Bernstein's theorem concerning them.

Contents

1. Introduction

The history of minimal surface theory begins with a memoir published by Lagrange in 1762, which considered the problem of finding surfaces of least area with a given boundary. Lagrange's studies soon turned into a body of mathematics that drew in many of the greatest mathematicians of the 19th and 20th centuries, including Euler, Legendre, Catalan, and Weierstrass. Measure theory, a prominent area of analysis, was developed by Lebesgue to study Plateau's problem, one of the central problems of minimal surface theory. The great impact that minimal surface theory had on mathematics is undeniable.

Intuitively speaking, a minimal surface is one that connects a given boundary most economically: any distortion to a minimal surface only increases its area. In this paper, we will focus on minimal surfaces that are graphs of smooth functions $u : \mathbb{R}^2 \to \mathbb{R}$. While such functions defined over some bounded open set Ω can be very 'curvy', as we increase the size of Ω these surfaces must necessarily become more and more flat. Indeed, there is a famous result in minimal surface theory, named Bernstein's theorem, formalizing this observation. It asserts that any u defined over all of $\mathbb{R}^2$ that is minimal must be a plane. While seemingly obvious, the generalizations of this theorem to functions $u : \mathbb{R}^n \to \mathbb{R}$ fail for $n \geq 8$. In this

way, the fact that this result holds true for surfaces in three dimensions is quite remarkable. The goal of this paper is to work towards its proof.

It is expected that the reader of this paper has a firm grasp of undergraduate real analysis and linear algebra, and is also comfortable with the concepts of differentiable manifolds, tangent spaces, differential forms, and their integration. Minimal surface theory borrows a lot of machinery from differential geometry of surfaces, so it is introduced in the first section of this paper. The primary source for this section's material, and a great introduction to the subject in general, is [1]. The discussion of minimal surfaces in the following sections is based heavily on the material from the first chapter of [2].

2. Preliminaries

2.1. **Two Facts about Differential Forms.** First, we will remind ourselves of Poincaré's lemma and Stokes' theorem in the context of differentiable manifolds embedded in the Euclidean space $\mathbb{R}^3$, both of which will be used heavily in the discussion to follow.

Definition 2.1. *Let $\Sigma \subset \mathbb{R}^3$ be a differentiable manifold, and let ω be a differential form on Σ. Then, ω is* ***exact*** *if there exists a form α so that $d\alpha = \omega$ (where d is the exterior differention operator). Also, ω is* ***closed*** *if $d\omega = 0$.*

Because $d(d\alpha) = 0$ for any differential form α, all exact forms are closed. The Poincaré lemma tells us that in contractible manifolds, the converse is also true.

Lemma 2.2 (Poincaré Lemma)**.** *Let ω be a closed differential form on a contractible manifold. Then ω is exact.*

All manifolds that we will be talking about, specifically submanifolds of the Euclidean space $\mathbb{R}^3$, are contractible.

Definition 2.3. *Let $\Sigma \subset \mathbb{R}^3$ be a differentiable manifold, and let p be a point on Σ. Then, p is an* ***interior point*** *of Σ if there exists a neighborhood U of p in $\mathbb{R}^3$ so that $U \cap \Sigma$ is homeomorphic to an open unit ball in $\mathbb{R}^2$. All points in Σ that are not interior points of it comprise the* ***boundary*** *of Σ, denoted $\partial\Sigma$.*

In the case of manifolds which are images of subsets of $\mathbb{R}^2$, the boundary will be the image of the boundary (in the topological sense) of the domain set. Thus, we can introduce Stokes' theorem, a fundamental tool for integration of differential forms.

Theorem 2.4 (Stokes' Theorem)**.** *Let ω be a differential form on some orientable differentiable manifold Σ. Then*

$$\int_\Sigma d\omega = \int_{\partial\Sigma} \omega$$

2.2. **Differential Geometry of Surfaces.** In order to talk about minimal surfaces, we also need to introduce some tools used in the differential geometry of surfaces. Their common goal is to extract information about the curvature or shape of some differentiable surface. Most of the surfaces we will be talking about will be graphs, as defined below.

Definition 2.5. *Let Ω be an open subset of $\mathbb{R}^2$ and $u : \mathbb{R}^2 \to \mathbb{R}$ be a twice-differentiable function on Ω. Then, the* ***graph of*** *u is a subset of $\mathbb{R}^3$ defined as follows:*

$$\mathrm{Graph}_u = \{(x, y, u(x, y)) | (x, y) \in \Omega\}$$

One way to think about graphs is as smooth distortions of the $\mathbb{R}^2$ plane. Then, a way to get information about their shape is to consider how much the transformation stretches small patches on the surface of graph around some point. This idea is formalized in the notion of the first fundamental form.

Let $\gamma : [0, 1] \to \Omega$ be a smooth curve, parameterized with respect to the usual basis as $\gamma(t) = (\gamma_x(t), \gamma_y(t))$. Suppose we want to find the length of the image $u(\gamma) \subset \mathrm{Graph}_u$ of this curve. To do so, integrate the magnitude of the curves' derivative, $|\gamma'| = \sqrt{\gamma' \cdot \gamma'}$ from $t = 0$ to 1. Differentiating the parameterized curve, and simplifying, we get that the length of the curve is

$$\int_0^1 \sqrt{E\left(\frac{d\gamma_x}{dt}\right)^2 + 2F\left(\frac{d\gamma_x}{dt}\right)\left(\frac{d\gamma_y}{dt}\right) + G\left(\frac{d\gamma_y}{dt}\right)^2}\, dt \tag{2.6}$$

where

$$E(x, y) = \left(\frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial x}\right)(x, y) \tag{2.7}$$

$$F(x, y) = \left(\frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y}\right)(x, y) \tag{2.8}$$

$$G(x, y) = \left(\frac{\partial u}{\partial y} \cdot \frac{\partial u}{\partial y}\right)(x, y) \tag{2.9}$$

The first fundamental form is defined as the object in the inside of the square root in the integrand, evaluated at a particular point on the surface, without the dependence on the time variable of some curve.

Definition 2.10. *Let $p = (x, y, u(x, y))$ be a point on Graph_u. Then, the* ***first fundamental form*** *is denoted:*

$$\mathrm{I} = E dx^2 + 2F dx\, dy + G dy^2$$

for E, F, G defined as in (2.7), (2.8), (2.9) at (x, y).

Take a tangent space to the graph, $T_p \mathrm{Graph}_u$, and its basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$. Then we can rewrite the first fundamental form as a bilinear form of two vectors $a, b \in T_p \mathrm{Graph}_u$ with respect to that basis:

$$\mathrm{I}(a, b) = (a_1, a_2) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Now we see that the first fundamental form I also defines a positive-definite inner product on the tangent space $T_p \mathrm{Graph}_u$.

We can define another useful inner product on the surface, called the second fundamental form.

Definition 2.11. *Let $U(t)$ be a map from an interval $(-\epsilon, \epsilon)$ to a function $u_t : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ defined as:*

$$u_t(x, y) = u(x, y) - tn(x, y)$$

where $n(x,y)$ is the normal vector to the surface of Graph_u *at $(x,y,u(x,y))$. The* ***second fundamental form*** *is the following time derivative of the first fundamental form:*

$$\frac{1}{2}\frac{\partial}{\partial t}(Edx^2 + 2Fdx\,dy + Gdy^2)\Big|_{t=0} = Ldx^2 + 2Mdx\,dy + Ndy^2$$

where

$$\begin{aligned} L &= -\frac{\partial u}{\partial x}\cdot\frac{\partial n}{\partial x} \\ 2M &= -\left(\frac{\partial u}{\partial x}\cdot\frac{\partial n}{\partial y} + \frac{\partial u}{\partial y}\cdot\frac{\partial n}{\partial x}\right) \\ L &= -\frac{\partial u}{\partial y}\cdot\frac{\partial n}{\partial y} \end{aligned}$$

We can also rewrite the second fundamental form in matrix form:

$$\mathrm{II}(a,b) = (a_1, a_2)\begin{pmatrix} L & M \\ M & N \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Both fundamental forms are examples of a more general inner product on differentiable manifolds, namely the Riemannian metric.

Definition 2.12. *Let Σ be a differentiable manifold. Then, g is a* ***Riemannian metric*** *if it is a smoothly chosen function from points p on Σ to bilinear forms $g_p : T_p\Sigma \times T_p\Sigma \to \mathbb{R}$ that are positive-definite inner products. Namely, for all $a, b \in T_p\Sigma$:*

(1) $g_p(a,b) = g_p(b,a)$
(2) $g_p(a,a) \geq 0$
(3) $g_p(a,a) = 0$ if and only if $a = 0$.

When there is no confusion about what p is, we can also denote $g_p(a,b)$ as $\langle a, b\rangle$.

While we derived the first fundamental form using a parameterization of our curve, we did not have to do so. In fact, we could have first specified a Riemannian metric and constructed its differentiable manifold without the need to first immerse it in some other manifold, like $\mathbb{R}^3$. Thus, the first fundamental form is an *intrinsic* object to the manifold. There are also *extrinsic* objects that give us information about the shape of a surface. The most familiar is the Gauss map.

Definition 2.13. *Let $\Sigma \subset \mathbb{R}^3$ be a two-dimensional differentiable manifold. Then, the continuos map $N : \Sigma \to \mathbb{S}^2$ giving the unit normal vector to a point on Σ, is called the* ***Gauss map****.*

Another example of an extrinsic object is the second fundamental form we discussed. Of course, the Gauss map by itself does not give us much information about the curvature of the surface at a point, and the second fundamental form is too indirect to use by itself. Thus, we use the shape operator (otherwise known as the Weingarten map) that tells us how much the normal vector to the surface will change if we go in a certain direction along the surface. To formalize that idea, we need to define a covariant derivative.

Definition 2.14. *Let p be a point on* Graph_u*, and let $v_p \in T_p\,\mathrm{Graph}_u$, and let $f : \mathrm{Graph}_u \to \mathbb{R}$ be a differentiable function. Then, let $\alpha : [0,1] \to \mathrm{Graph}_u \subset \mathbb{R}^3$*

be a differentiable curve such that $\alpha(0) = p$ and $\alpha'(0) = v_p$. Then, the ***covariant derivative*** *may be characterized as follows:*

$$\nabla_{v_p} f = \frac{d}{dt}(f \circ \alpha)(t)\big|_{t=0} \tag{2.15}$$

Note that this vector does not depend on the choice of α.

Definition 2.16. *The* ***shape operator*** $S : T_p \operatorname{Graph}_u \to T_p \operatorname{Graph}_u$ *is the covariant derivative of the Gauss map with respect to the input vector:*

$$S(v_p) = \nabla_{v_p} N \tag{2.17}$$

This map can be identified with the differential of the Gauss map, dN, and with the matrix of the second fundamental form II. If $\{e_1, e_2\}$ is a basis of the tangent space of $T_p \operatorname{Graph}_u$, the element S_{ij} of the shape operator is:

$$\langle dN(e_i), e_j \rangle = \langle \nabla_{e_i} N, e_j \rangle = \langle N, \nabla_{e_i} e_j \rangle \tag{2.18}$$

The product and sum of the eigenvalues of this matrix turn out to be extremely important in investigating minimal surfaces.

Definition 2.19. *Let κ_1, κ_2 be the eigenvalues of the shape operator S at $p \in T_p \operatorname{Graph}_u$. Then, $\kappa_1 + \kappa_2$, or otherwise the trace of the shape operator, is the* ***mean curvature*** *of* Graph_u*. The value $\kappa_1 \kappa_2$ is the* ***Gaussian curvature****. Finally, the value*

$$|A|^2 = \kappa_1^2 + \kappa_2^2 \tag{2.20}$$

is called the ***total curvature*** *of* Graph_u *at p.*

Notably, Gaussian curvature at point p also equals the determinant of $S(p)$.

3. Minimal Surfaces

3.1. **Minimal Surface Equation.** We are now ready to begin talking about minimal surfaces. Roughly speaking, minimal surfaces are those with locally minimal area: when distorted slightly, their area only increases. To determine the area of a particular surface that is a graph of some function u, we can integrate the magnitude of the cross product $(1, 0, \frac{du}{dx}) \times (0, 1, \frac{du}{dy})$, representing the area of the 'infinitesimal' rectangle formed by $\frac{du}{dx}$ and $\frac{du}{dy}$, over the domain Ω. Simplifying, we can get the following expression for the area of the surface:

Proposition 3.1. *The* ***area of*** Graph_u *is*

$$\operatorname{Area}(\operatorname{Graph}_u) = \int_\Omega \sqrt{1 + |\nabla u|^2} \tag{3.2}$$

Now consider the family of 'distortions' of u, namely functions $u + t\eta$, with $t \in (0, 1)$, and $\eta : \Omega \to \mathbb{R}$ a twice differentiable function that is 0 on the boundary $\partial\Omega$. The area of a surface in this family is

$$\operatorname{Area}(\operatorname{Graph}_{u+t\eta}) = \int_\Omega \sqrt{1 + |\nabla u + t\nabla\eta|^2}$$

Now, we can say that an area is minimal with respect to distortions if

$$\frac{d}{dt}(\operatorname{Area}(\operatorname{Graph}_{u+t\eta}))|_{t=0} = \int_\Omega \frac{\langle \nabla u, \nabla \eta \rangle}{\sqrt{1 + |\nabla u|}^2} = 0 \tag{3.3}$$

for all η. Because the last integral equals

$$\int_\Omega \eta \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$$

we know that the above is true if and only if the divergence of $\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$ is 0. We take this as the defining feature of minimal surfaces.

Definition 3.4. *The* ***minimal surface equation*** *is*

(3.5)
$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \left(1+\frac{\partial u}{\partial y}^2\right)\frac{\partial^2 u}{\partial x \partial x} + \left(1+\frac{\partial u}{\partial x}^2\right)\frac{\partial^2 u}{\partial y \partial y} - 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} = 0$$

Definition 3.6. *A surface* Graph_u *is* ***minimal*** *if it satisfies the minimal surface equation (3.5).*

Remarkably, graphs that satisfy the minimal surface equation have the smallest area out of all surfaces with same boundary (this is not true for more general surfaces). We prove a formal version of this fact in the lemma below:

Lemma 3.7. *If* $u : \Omega \to \mathbb{R}$ *is minimal, and* Σ *is any surface contained in the cylinder* $\Omega \times \mathbb{R}$ *so that the boundaries* $\partial\Sigma$ *and* $\partial \operatorname{Graph}_u$ *are the same, then*

$$\operatorname{Area}(\operatorname{Graph}_u) \leq \operatorname{Area}(\Sigma)$$

Proof. Define a *calibration* ω on $\Omega \times \mathbb{R}$ for two vectors X, Y in $\mathbb{R}^3$ both originating in some $p = (x, y, z)$ as follows:

$$\omega(X, Y) = \det(X, Y, N)$$

where N is the normal vector to the surface Graph_u:

$$N = \frac{(-\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}, 1)}{\sqrt{1+|\nabla u|^2}}$$

Notice that for any two orthogonal unit vectors X, Y we have

$$\omega(X, Y) \leq 1$$

with equality if and only if X and Y lie on $T_p \operatorname{Graph}_u$. Essentially, the calibration acts as a detector of when X, Y, and N are actually orthogonal. Also notice that because we can write ω as

$$\omega = \frac{dx \wedge dy - \frac{\partial u}{\partial x} dy \wedge dz - \frac{\partial u}{\partial y} dz \wedge dx}{\sqrt{1+|\nabla u|^2}}$$

and then differentiate it

$$d\omega = \frac{\partial}{\partial x}\left(\frac{-\frac{\partial u}{\partial x}}{\sqrt{1+|\nabla u|^2}}\right) + \frac{\partial}{\partial x}\left(\frac{-\frac{\partial u}{\partial y}}{\sqrt{1+|\nabla u|^2}}\right) = 0$$

we know ω must be a closed form. But since both Graph_u and Σ are contractible, by the Poincaré lemma we have that ω is also exact. Let α be a one-form with the property that $d\alpha = \omega$. Then by Stokes' theorem:

$$\int_{\operatorname{Graph}_u} \omega = \int_{\partial \operatorname{Graph}_u} \alpha = \int_\Sigma \omega$$

Then, because

$$\text{Area}(\text{Graph}_u) = \int_{\text{Graph}_u} 1 \qquad \text{Area}(\Sigma) = \int_{\Sigma} 1$$

and $\omega = 1$ on Graph_u but $\omega \leq 1$ on Σ, we have our result. □

3.2. **Mean Curvature of Minimal Surfaces.** A very important property of minimal surfaces is that their mean curvature is zero everywhere. In fact, minimal surfaces are often defined as any surface with such a property. We will devote this section to proving the correspondence between minimal surfaces as we defined them, namely as critical points of the area functional, and surfaces of zero mean curvature.

Theorem 3.8. *If* Graph_u *is a minimal surface, then at any point* p *on it, the mean curvature is* 0.

Proof. Let $F : \text{Graph}_u \times (-\epsilon, \epsilon) \to M$ be a function on a minimal surface Graph_u, that is the identity on Graph_u for $t = 0$ and on $\partial\,\text{Graph}_u$ for all t. In other words, F represents a slight pertubation of Graph_u into a surface with the same boundary. We now want to compute the differential of the area of $F(\text{Graph}_u, t)$ with respect to t at 0.

For convenience, define a matrix G and a function ν:

$$G_{ij} = \left\langle \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \right\rangle \tag{3.9}$$

$$\nu(t) = \sqrt{\det(G(t))}\sqrt{\det(G^{-1}(0))} \tag{3.10}$$

where $x_1 = x, x_2 = y$. Notice that unlike the usual Riemannian volume form of $F(\text{Graph}_u, t)$, namely $\sigma = \sqrt{\det(G(t))}$, $\nu(t)$ is a quantity that is independent of the coordinate system we pick for the relevant tangent spaces. This makes it useful to us for determining the area of $F(\text{Graph}_u, t)$, which can then be expressed as follows:

$$\text{Area}(F(\text{Graph}_u, t)) = \int_{\text{Graph}_u} \nu(t)\sqrt{\det G(0)}$$

Note that we can pick orthonormal bases at each $x \in \text{Graph}_u$ such that G (as well as G^{-1}) is the identity at $t = 0$. The basis then becomes $\{\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\}$. In this case,

$$\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x} \right\rangle = \left\langle \frac{\partial F}{\partial y}, \frac{\partial F}{\partial y} \right\rangle = 1$$
$$\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right\rangle = 0$$

These expressions, along with chain and product rules, will help us evaluate the derivative of the area functional

$$\frac{d}{dt}\Big|_{t=0}\text{Area}(F(\text{Graph}_u, t)) = \int_{\text{Graph}_u} \frac{d}{dt}\Big|_{t=0}\nu(t)\sqrt{\det G(0)} \tag{3.11}$$

since now, $\sqrt{\det G(0)} = 1$, and

$$\frac{d}{dt}\Big|_{t=0}\nu(t) = \left\langle \frac{\partial F}{\partial x}, \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial x}\right)\right\rangle + \left\langle \frac{\partial F}{\partial y}, \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial x}\right)\right\rangle \tag{3.12}$$

$$= \left\langle \frac{\partial F}{\partial x}, \nabla_{\frac{\partial F}{\partial t}} \frac{\partial F}{\partial x}\right\rangle + \left\langle \frac{\partial F}{\partial y}, \nabla_{\frac{\partial F}{\partial t}} \frac{\partial F}{\partial y}\right\rangle \tag{3.13}$$

The derivatives with respect to t and x_i commute, so we know (3.13) is

$$\left\langle \frac{\partial F}{\partial x}, \nabla_{\frac{\partial F}{\partial x}} \frac{\partial F}{\partial t}\right\rangle + \left\langle \frac{\partial F}{\partial y}, \nabla_{\frac{\partial F}{\partial y}} \frac{\partial F}{\partial t}\right\rangle \tag{3.14}$$

But since $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ are the basis vectors of the tangent plane (equivalent to e_1, e_2), by definition, the above equals to $\mathrm{div}_{\mathrm{Graph}_u} \frac{\partial F}{\partial t}$. Then, this can be expressed as a sum of components that are parallel and perpendicular to the normal vector. Since the normal component of a vector v is $\langle v, N\rangle N$, we can express the normal component of (3.14) as

$$\left\langle \frac{\partial F}{\partial t}, N\right\rangle \left(\left\langle \frac{\partial F}{\partial x}, \nabla_{\frac{\partial F}{\partial x}} N\right\rangle + \left\langle \frac{\partial F}{\partial y}, \nabla_{\frac{\partial F}{\partial y}} N\right\rangle\right) \tag{3.15}$$

$$= \left\langle \frac{\partial F}{\partial t}, N\right\rangle \left(\left\langle \frac{\partial}{\partial x}, \nabla_{\frac{\partial}{\partial x}} N\right\rangle + \left\langle \frac{\partial}{\partial y}, \nabla_{\frac{\partial}{\partial y}} N\right\rangle\right) \tag{3.16}$$

$$= \left\langle \frac{\partial F}{\partial t}, H\right\rangle \tag{3.17}$$

where H is the mean curvature vector, i.e., the inner product of the normal vector and the diagonal of the shape operator. Then, we can express (3.14) as follows:

$$\left\langle \frac{\partial F}{\partial t}, H\right\rangle + \mathrm{div}_{\mathrm{Graph}_u} \frac{\partial F}{\partial t} \tag{3.18}$$

We now have an alternate expression for the derivative of the area functional, (3.11):

$$\frac{d}{dt}\Big|_{t=0}\mathrm{Area}(F(\mathrm{Graph}_u, t)) = \int_{\mathrm{Graph}_u} \left\langle \frac{\partial F}{\partial t}, H\right\rangle + \int_{\mathrm{Graph}_u} \mathrm{div}_{\mathrm{Graph}_u} \frac{\partial F}{\partial t} \tag{3.19}$$

Notice that the form $\left(\mathrm{div}_{\mathrm{Graph}_u} \frac{\partial F}{\partial t}\right) dx \wedge dy$ is the exterior derivative of the form

$$\frac{\partial F_1}{\partial t} dy - \frac{\partial F_2}{\partial t} dx$$

Then, by Stokes' theorem,

$$\int_{\mathrm{Graph}_u} \mathrm{div}_{\mathrm{Graph}_u} \frac{\partial F}{\partial t} = \int_{\partial\,\mathrm{Graph}_u} \frac{\partial F_1}{\partial t} dy - \frac{\partial F_2}{\partial t} dx$$

However, since F is constant on $\partial\,\mathrm{Graph}_u$, $\frac{\partial F_1}{\partial t} = \frac{\partial F_2}{\partial t} = 0$, so

$$\int_{\mathrm{Graph}_u} \mathrm{div}_{\mathrm{Graph}_u} \frac{\partial F}{\partial t} = 0 \tag{3.20}$$

So,

$$\frac{d}{dt}\Big|_{t=0}\mathrm{Area}(F(\mathrm{Graph}_u, t)) = \int_{\mathrm{Graph}_u} \left\langle \frac{\partial F}{\partial t}, H\right\rangle \tag{3.21}$$

But by (3.3), this integral is also 0. This is true only if H also vanishes (in the basis we have been working in). Thus, we have our result. □

4. Bernstein's Theorem

We are now ready to take the final steps to proving Bernstein's theorem. Recall from the introduction that the intuitive explanation for Bernstein's result is that as the set Ω on which u is defined gets larger, the surface defined by the image of u gets locally less 'curvy'. We prove the theorem by making this intuition rigorous: we find a bound on the total curvature of a graph that goes to 0 as Ω grows to encompass all of $\mathbb{R}^2$. For this, we introduce a logarithmic cutoff technique commonly used in results about minimal surfaces.

Firstly, however, we need to introduce a embedded surface version of a gradient.

Definition 4.1. *Recall that ∇f is the gradient of some real-valued function $f : \mathbb{R}^n \to \mathbb{R}$. Then, if Σ is a smooth, proper subset of $\mathbb{R}^n$, $\nabla_\Sigma f$ denotes the gradient of f restricted to Σ:*

$$\nabla_\Sigma f = \sup_{v \in T_p\Sigma,\, |v|=1} \nabla_v f \tag{4.2}$$

It can be seen that

$$\nabla_\Sigma f \leq \nabla f \tag{4.3}$$

Lemma 4.4. *If $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a solution to the minimal surface equation, then for all non-negative Lipschitz functions η with support contained in $\Omega \times \mathbb{R}$,*

$$\int_{\mathrm{Graph}_u} |A|^2 \eta^2 \leq C \int_{\mathrm{Graph}_u} |\nabla_{\mathrm{Graph}_u} \eta|^2 \tag{4.5}$$

where

$$|A|^2 = \kappa_1^2 + \kappa_2^2 \tag{4.6}$$

is the magnitude of the ***total curvature*** *at p, with respect to an orthogonal basis $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$ of $T_p \,\mathrm{Graph}_u$.*

Proof. Let ω be the area two-form on the unit sphere S^2. Since the upper hemisphere of S^2 is contractible, by Poincare's lemma, we know there is a one-form α so that $d\alpha = \omega$. Now, consider the quantity $|A|^2 d\,\mathrm{Area}$. Using the change of basis formula for Riemannian volume forms (which both $d\,\mathrm{Area}$ and $N^*\omega$ are), we have

$$2N^*\omega = \sqrt{\det(S)}\, d\,\mathrm{Area} = |A|^2 d\,\mathrm{Area} \tag{4.7}$$

Now, because the derivative operation commutes with the pullback operation, we have

$$2N^*\omega = 2dN^*\alpha$$

As such,

$$\int_{\mathrm{Graph}_u} \eta^2 |A|^2 d\,\mathrm{Area} = 2\int_{\mathrm{Graph}_u} \eta^2 dN^*\alpha \tag{4.8}$$

Then, using the identity

$$d(\eta^2 \wedge N^*\alpha) = 2\eta d\eta \wedge N^*\alpha + \eta^2 \wedge dN^*\alpha$$

We express Stokes' Theorem for the form $\eta^2 \wedge N^*\alpha$:

$$\int_{\mathrm{Graph}_u} \eta^2 dN^*\alpha + 2\eta d\eta \wedge N^*\alpha = \int_{\partial\,\mathrm{Graph}_u} \eta^2 \wedge N^*\alpha$$

But because η has compact support on $\Omega \times \mathbb{R}$, $\eta = 0$ on $\partial \operatorname{Graph}_u$. Thus,

$$\int_{\operatorname{Graph}_u} \eta^2 dN^*\alpha = -2 \int_{\operatorname{Graph}_u} \eta d\eta \wedge N^*\alpha \tag{4.9}$$

Now, note that

$$(N^*\alpha)_p = \alpha_{N(p)}(dN_p)$$

where subscripts indicate the tangent plane the form is evaluated over. Also, because $\alpha_{N(p)}$ is a smooth function with compact support, we have that $\alpha_{N(p)}(T_p \operatorname{Graph}_u)$ is bounded above, so there is some C_α such that

$$|N^*\alpha| \leq C_\alpha |dN| \tag{4.10}$$

And since N gives unit vectors,

$$C_\alpha |dN| = C_\alpha |A| \tag{4.11}$$

We can extend (4.8) and (4.9) (using the fact that $\nabla\eta = d\eta$):

$$\int_{\operatorname{Graph}_u} \eta^2 |A|^2 d\operatorname{Area} = 2 \int_{\operatorname{Graph}_u} \eta^2 dN^*\alpha \tag{4.12}$$

$$= -4 \int_{\operatorname{Graph}_u} \eta d\eta \wedge N^*\alpha \tag{4.13}$$

$$\leq 4C_\alpha \int_{\operatorname{Graph}_u} \eta |\nabla_{\operatorname{Graph}_u} \eta| |A| d\operatorname{Area} \tag{4.14}$$

Now, using the Cauchy-Schwartz inequality:

$$\int_{\operatorname{Graph}_u} \eta^2 |A|^2 d\operatorname{Area} \leq 4C_\alpha \left(\int_{\operatorname{Graph}_u} \eta^2 |A|^2 d\operatorname{Area} \right)^{\frac{1}{2}} \left(\int_{\operatorname{Graph}_u} |\nabla\eta|^2 d\operatorname{Area} \right)^{\frac{1}{2}} \tag{4.15}$$

Squaring both sides and dividing by $\int_{\operatorname{Graph}_u} \eta^2 |A|^2$ gives us out result. □

Before we apply this lemma and derive the curvature bound, we need to show a consequence of a result from the previous section.

Lemma 4.16. *If $u : \Omega \to \mathbb{R}$ satisfies the minimal surface equation and the disk with radius r, D_r, is contained in Ω, then*

$$\operatorname{Area}(B_r \cap \operatorname{Graph}_u) \leq 2\pi r^2 \tag{4.17}$$

Proof. Notice that $\partial B_r \cap \operatorname{Graph}_u$ divides B_r into two surfaces with the same boundary as $B_r \cap \operatorname{Graph}_u$. One of these surfaces has to have an area that is smaller than the area of a hemisphere of B_r, namely $2\pi r^2$. Thus, by 3.7, we have our result. □

Lemma 4.18. *If $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is a solution to the minimal surface equation, $\kappa > 1$, and Ω contains a ball of radius κR centered at the origin, then*

$$\int_{B_{\sqrt{\kappa}R} \cap \operatorname{Graph}_u} |A|^2 \leq \frac{C}{\log \kappa} \tag{4.19}$$

Proof. Define the cutoff function $\eta : \mathbb{R}^3 \to \mathbb{R}$ as follows:

$$\eta(x) = \begin{cases} 1 & r^2 \leq \kappa R^2 \\ 2 - \frac{2\log(rR^{-1})}{\log \kappa} & \kappa R^2 < r^2 \leq \kappa^2 R^2 \\ 0 & r^2 > \kappa^2 R^2 \end{cases} \tag{4.20}$$

with $r = |x|$. By (4.3), we have

$$|\nabla_\Sigma \eta| \leq \frac{2}{r \log \kappa} \tag{4.21}$$

Now, we can extend the result of 4.4 using the preceding lemmas to give

$$\int_{B_{\sqrt{\kappa}R} \cap \mathrm{Graph}_u} |A|^2 \leq \int_{\mathrm{Graph}_u} \eta |A|^2 \tag{4.22}$$

$$\leq C \int_{\mathrm{Graph}_u} |\nabla_{\mathrm{Graph}_u} \eta|^2 \tag{4.23}$$

$$\leq \frac{4C}{(\log \kappa)^2} \int_{B_{\kappa R} \cap \mathrm{Graph}_u} \frac{1}{r^2} \tag{4.24}$$

We introduce the so-called 'logarithmic cutoff trick'. Instead of integrating over all of $B_{\kappa R \cap \mathrm{Graph}_u}$, we can integrate over the annuli $(B_{e^l R} \setminus B_{e^{l-1} R}) \cap \mathrm{Graph}_u$ for some integers l. Set $L_{\min} = \lfloor \frac{\log \kappa}{2} \rfloor$, and $L_{\max} = \lceil \log \kappa \rceil$. We get

$$\frac{4C}{(\log \kappa)^2} \int_{B_{\kappa R} \cap \mathrm{Graph}_u} \frac{1}{r^2} \leq \frac{4C}{(\log \kappa)^2} \sum_{l=L_{\min}}^{L_{\max}} \int_{(B_{e^l R} \setminus B_{e^{l-1} R}) \cap \mathrm{Graph}_u} \frac{1}{r^2} \tag{4.25}$$

Using the fact $\frac{1}{r^2} \leq \frac{1}{e^{2(l-1)}}$, we have

$$\int_{(B_{e^l R} \setminus B_{e^{l-1} R}) \cap \mathrm{Graph}_u} \frac{1}{r^2} \leq \frac{1}{e^{2l-2}} \int_{B_{e^l R} \cap \mathrm{Graph}_u} \leq 2\pi e^2 \tag{4.26}$$

Plugging this into (4.25), we get

$$\frac{4C}{(\log \kappa)^2} \int_{B_{\kappa R} \cap \mathrm{Graph}_u} \frac{1}{r^2} \leq \frac{4C}{(\log \kappa)^2} \sum_{l=L_{\min}}^{L_{\max}} 2\pi e^2 \tag{4.27}$$

$$\leq \frac{4C}{(\log \kappa)^2} (2\pi e^2) \left(\log \kappa - \frac{1}{2} \log \kappa \right) \tag{4.28}$$

$$= \frac{4\pi C e^2}{\log \kappa} \tag{4.29}$$

This proves the result. □

Using the preceding lemmas, we can prove the central theorem of this paper.

Theorem 4.30 (Bernstein's Theorem)**.** *If $u : \mathbb{R}^2 \to \mathbb{R}$ is an entire solution to the minimal surface equation, then $u(\mathbb{R}^2)$ is a plane.*

Proof. By 4.18, for any $R > 1$ we have

$$\int_{B_{\sqrt{R}} \cap \mathrm{Graph}_u} |A|^2 \leq \frac{C}{\log R} \tag{4.31}$$

As we let $R \to \infty$, we see that $\frac{C}{\log R} \to 0$, and since $|A|^2 \geq 0$, we must have that $|A|^2 = \kappa_1^2 + \kappa_2^2 = 0$. But then, the shape operator of Graph_u at any p is equivalent to 0, so the surface will have the same normal vector everywhere. This implies Graph_u is a plane. □

Acknowledgments

I would like to thank my advisor, Judson Kuhrman, for his useful advice and valuable discussions. Additional thanks to Dr. Peter May for organizing the University of Chicago 2022 REU, the Jeff Metcalf Internship Program for sponsoring me on this opportunity, and Patrick Borse for his invaluable feedback and assistance.

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