RICCI FLOW AND THE GEOMETRIZATION CONJECTURE

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Abstract. In this expository paper, we present the basic ideas behind Thurston's Geometrization conjecture, along with an introduction to Ricci flow in order to outline Perelman's proof of the Poincaré conjecture and how it relates to the Geometrization conjecture. Additionally, we first present a brief review of curvature to emphasize its importance in the study of manifolds.

Contents

1. Curvature

First, let us present a brief introduction to a curvature, and to how Riemannian metrics surprisingly endow us with all the information needed to determine how a manifold is different from the Euclidean space; *i.e.* how curved it is. In order to do so, let us recall that connections (denoted $\nabla_X Y$) on Riemannian manifolds are the machinery we use to speak about a notion of derivatives of vector fields on manifolds, and also give rise to fundamental ideas like curvature and geodesics, and as such are one of the cornerstones of Riemannian geometry (the reader is highly encouraged to see doCarmo's [1] to review the formal definition of connections and a construction of the Levi-Civita connection).

Now, in order to introduce and briefly motivate the idea of curvature, let us first present the following construction. Consider a Riemannian manifold M with the Levi-Civita connection ∇. Given any point $p \in M$, let $z \in T_pM$, and given a choice of local coordinates $(x_1, \dots, x_n)$, let us parallel-translate z along the curve x_1. Moreover, at each point along x_1, let us also parallel-translate z along x_2. Now, if $M = \mathbb{R}^n$, it would follow that the vectors, would continue to be parallel to

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each other when translated along x_2. Nevertheless, if we consider a space which is intuitively curved, say $\mathbb{S}^{n-1}$ (Figure 1), we can see that this may not be the case. In terms of the metrics, we can express this condition by checking whether

$$\nabla_{\partial_1} Z = 0, \tag{1.1}$$

where $Z \in \mathfrak{X}(M)$[1] is the smooth vector field associated with z and its parallel transports. Now, note that (1.1) is true is whenever $x_2 = 0$ by construction.

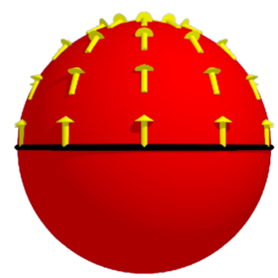

FIGURE 1. Result of the construction above on $\mathbb{S}^2$.

By uniqueness of parallel transport, this condition is equivalent to having

$$\nabla_{\partial_2}\nabla_{\partial_1} Z = 0,$$

so if it were true that

$$\nabla_{\partial_1}\nabla_{\partial_2} Z = \nabla_{\partial_2}\nabla_{\partial_1} Z,$$

then (1.1) would be immediately satisfied since $\nabla_{\partial_2} Z = 0$ by construction. This last equation is clearly satisfied in $\mathbb{R}^n$, however, it is seen by a few computations that this is not true of every manifold. Now, for arbitrary vector fields $X, Y, Z \in \mathfrak{X}(\mathbb{R}^n)$, one can check that

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_{[X,Y]} Z.$$

In this way, we arrive at the following definition.

Definition 1.2. The (Riemannian) **Curvature tensor** is the $(3,1)$-tensor (via an isomorphism) $R(X,Y)Z : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$\boxed{R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.} \tag{1.3}$$

Conveniently, we can turn R into a $(0,4)$-tensor using the Riemannian metric g in the following way:

$$R(X,Y,Z,W) = g(R(X,Y)Z, W).$$

Proposition 1.4. *The Curvature tensor satisfies:*

(1) Skew-symmetry in the first two and last two entries:

$$R(X,Y,Z,W) = -R(Y,X,Z,W) = R(Y,X,W,Z).$$

[1]We adopt the notation that $\mathfrak{X}(M)$ denotes the space of smooth vector fields on M.

(2) *Symmetry around the first two and the last two entries:*

$$R(X, Y, Z, W) = R(Z, W, X, Y).$$

(3) *Satisfies a cyclic permutation (**Bianchi's first identity**):*

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0.$$

(4) *(**Bianchi's second identity**):*

$$(\nabla_Z R)(X, Y)W + (\nabla_X R)(Y, Z)W + (\nabla_Y R)(Z, X)W = 0.$$

Proof. These can be checked by direct computation. □

1.1. Ricci and scalar curvatures.

Definition 1.5. If the vectors $e_1, \dots, e_n \in T_pM$ form an orthonormal basis, then we define

$$\begin{aligned}\mathrm{Ric}(v, w) &= \mathrm{tr}\,(x \mapsto R(x, v)w) \\ &= \sum_{i=1}^{n} g\,(R(e_i, v)w, e_i) \\ &= \sum_{i=1}^{n} g\,(R(v, e_i)e_i, w) \\ &= \sum_{i=1}^{n} g\,(R(e_i, w)v, e_i)\,.\end{aligned}$$

Ric is called the **Ricci curvature** and is a symmetric bilinear form (since it is the trace or contraction of R). It can also be equivalently defined as the $(1,1)$-tensor

$$\boxed{\mathrm{Ric}(v) = \sum_{i=1}^{n} R(v, e_i)e_i.} \tag{1.6}$$

Definition 1.7. The **scalar curvature** is defined by $\mathrm{tr}(\mathrm{Ric}) = g^{ij}\mathrm{Ric}_{ij}$.[2]

Intuitively, the scalar curvature measures the first order deviation of volumes in a manifold (as specified by the metric) from the volumes in Euclidean space. For instance, given a point p in a 2-manifold, we can specify a disk $B(p, r)$ centered a p and of radius r, and the scalar curvature at p would be precisely given by

$$R(p) = \lim_{r \to 0} \frac{\pi r^2 - |\mathrm{vol}(B(p, r))|}{\pi r^4/24}.$$

That is, when the volume of a disk of radius r grows slower (resp. faster) at a point in the manifold, the scalar curvature is positive (resp. negative). Consider, for instance, the sphere, where the volume grows slower than in Euclidean space, or conversely, a saddle, where the volume grows faster (Figure 2).

Similarly, the Ricci curvature measures this deviation but in a particular direction. Again, if we consider a 2-manifold as an example, for a point $p \in M$, we can consider a unit vector $v \in T_pM$ and a angle θ away from it–and this allows us to

[2]Recall that given a choice of coordinates we can locally write $g = g_{ij}dx^i dx^j$, and g^{ij} is defined as the inverse of the matrix g_{ij}.

measure the area of a sector of a disk. In this way, the Ricci curvature measures the deviation of areas of sectors, and can be precisely written as

$$\mathrm{Ric}(x)(v,v) = \lim_{r\to 0}\lim_{\theta\to 0}\frac{\frac{1}{2}\theta r^2 - |A(p,r,\theta,v)|}{\theta r^4/24}.$$

The reader is encouraged to see [12] for a more detailed explanation of the discussion above.

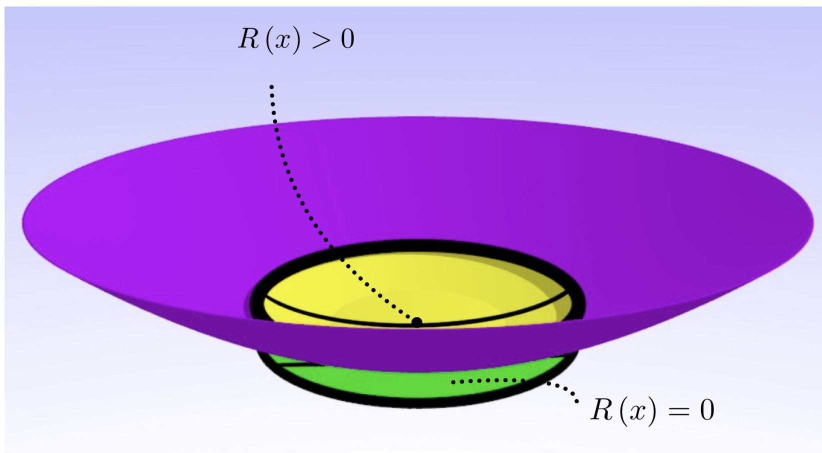

FIGURE 2. Point in surface with positive scalar curvature.

This idea will be fundamental for our understanding of Ricci flow, which will be explored in section 3. For now, let us explore a seemingly unrelated question, but which turns out to have a solution (Perelman's proof of Poincaré conjecture!) in terms of Ricci flow.

2. Geometrization Conjecture

Let us recall the following definition.

Definition 2.1. Given two disjoint connected n-manifolds M_1 and M_2, their **connected sum**, denoted by $M_1 \# M_2$, can be constructed by deleting the interiors of closed n-balls $B_1 \subset M_1$ and $B_2 \subset M_2$ and identifying the resulting ∂B_1 and ∂B_2 via a homeomorphism.

It is natural ask how this notion may allow us to reduce the study of manifolds (in particular, 3-manifolds, (see section 3)) to simpler classes of manifolds. Indeed, this motivates the following definition.

Definition 2.2. We say that an orientable closed 3-manifold M^3 is **prime** if M^3 is not diffeomorphic to the 3-sphere $\mathbb{S}^3$ and if a connected sum decomposition $M^3 = M_1^3 \# M_2^3$ is possible if and only if M_1^3 or M_2^3 is itself diffeomorphic to $\mathbb{S}^3$.

As the definition suggests, the following theorem provides such a way to reduce the study of 3-manifolds.

Theorem 2.3 (Prime Decomposition of 3-manifolds)**.** *Let M be a 3-manifold. Then M admits a (unique) decomposition*

$$M \cong M_1 \# M_2 \# \dots \# M_n$$

where each M_i is prime.

Sketch of proof. Recall that for a compact 3-manifold M, its fundamental group $\pi_1 M$ is finitely generated. We denote by $n(M)$ the minimal number of generators of $\pi_1 M$. The proof follows by induction on $n(M)$: if $n(M) = 0$ or M is prime, then there is nothing to prove. Now, if M is not prime, then it admits a decomposition $M = M_1 \# M_2$ where neither M_1 nor M_2 is diffeomorphic to $\mathbb{S}^3$. Thus, if M_1 or M_2 admit prime factorizations, we can repeat the process and the result follows by induction after proving the fact that $n(M_1 \# M_2) = n(M_1) + n(M_2)$. This last result follows from van Kampen's theorem and the following more general theorem:

Theorem 2.4 (Grushko)**.** *Let F be a finitely generated free group, and let $\phi : F \to G \star H$ be a surjection of groups. Then F can be decomposed as a free product $F_0 \star F_1$ so that ϕ is a free product of maps $\phi_0 : F_0 \to G$, $\phi_1 : F_1 \to H$.*

The reader is encouraged to see [13] for the full details of the proof (of Theorem 2.3, also containing the proof of Theorem 2.4), including the uniqueness portion of it. □

While there are many interesting questions that can subsequently be studied around 3-manifolds arising from this result, we will present a slight refinement of it for the purposes of presenting Thurston's Geometrization Conjecture, so we require to introduce a few more definitions.

Definition 2.5. Let M be a 3-manifold which is not a 3-sphere. We say that M is **irreducible** if every embedded 2-sphere $\mathbb{S}^2 \hookrightarrow M$ bounds a 3-ball. (More generally, an irreducible n-manifold is one in which for every $n-1$-sphere $S \hookrightarrow M^n$ there exists an n-ball B_n such that $\partial B_n = S$.)

Example 2.6. $\mathbb{R}^3 \backslash \{0\}$ is not irreducible.

Remark 2.7. It is well-known that the only orientable 3-manifold that is prime but not irreducible is $\mathbb{S}^2 \times \mathbb{S}^1$. Indeed, any 2-sphere $\mathbb{S}^2 \times \{p\}$ (for $p \in \mathbb{S}^1$) has a connected complement which is not a ball (rather, it is the product of the 2-sphere and a line).

A consequence of the Prime Decomposition Theorem is that an orientable closed 3-manifold M^3 can be decomposed into a finite connected sum of prime factors

$$M^3 \cong \left(\#_j \mathcal{X}_j^3\right) \# \left(\#_k \mathcal{Y}_k^3\right) \# \left(\#_l \mathbb{S}^2 \times \mathbb{S}^1\right),$$

where each $\mathcal{X}_j^3$ is irreducible with a finite fundamental group and a homotopy 3-sphere as universal cover; each $\mathcal{Y}_k^3$ is irreducible with infinite fundamental group and a contractible universal cover. The prime decomposition is unique up to re-ordering and orientation-preserving diffeomorphisms of the factors.

Definition 2.8. Let Σ^2 be a two-sided compact properly embedded surface in a manifold $\mathcal{N}^3$ with boundary. Assume that Σ^2 has no components diffeomorphic to the 2-disk $\mathcal{D}^2$, and that Σ^2 either lies in $\partial\mathcal{N}^3$ or intersects $\partial\mathcal{N}^3$ only in $\partial\Sigma^2$. We say that Σ^2 is **incompressible** if for each $\mathcal{D}^2 \subset \mathcal{N}^3$ with $\mathcal{D}^2 \cap \Sigma^2 = \partial\mathcal{D}^2$ there exists a disk $\mathcal{D}_* \subset \Sigma^2$ with $\partial\mathcal{D}_* = \partial\mathcal{D}^2$.

Intuitively, an incompressible surface is a non-trivial surface (*i.e.* not diffeomorphic to a disk) which cannot be simplified within the manifold. For example, the surface of a suitcase is compressible since the handle can be shrinked into the rest of the surface, and all of this within $\mathbb{R}^3$.

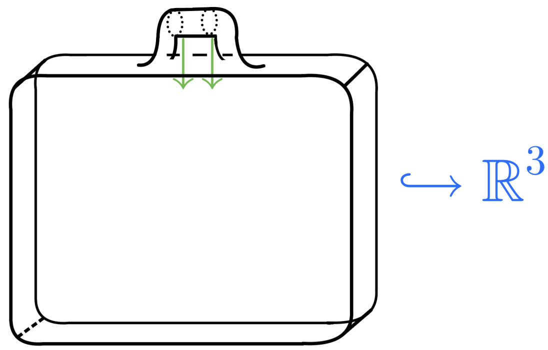

FIGURE 3. Compressibility (or *non-incompressibility*) of a suitcase.

Now, let us recall the following result.

Theorem 2.9 (Classification theorem)**.** *Let Σ be a connected oriented surface. Then Σ can be obtained as a connected sum*

$$\mathbb{T}\#\mathbb{T}\#\ldots\#\mathbb{T}$$

of g copies of the torus $\mathbb{T}$, for some $g \geq 0$ (the genus of the surface).

Remark 2.10. If $g = 0$, this means $\Sigma \cong \mathbb{S}^2$.

Definition 2.11. An irreducible manifold $\mathcal{N}^3$ with boundary is said to be **geometrically atoroidal** if every incompressible torus $\mathbb{T}^2 \subset \mathcal{N}^3$ is isotopic[3] to a component of $\partial\mathcal{N}$.

Definition 2.12. One says that a compact manifold $\mathcal{N}^3$ is a **Seifert fiber space** if it admits a foliation by $\mathbb{S}^1$ fibers.

Let us state a result from which Thurston's Geometrization Conjecture can arise more naturally.

Theorem 2.13 (Torus Decomposition Theorem)**.** *Let M^3 be irreducible. There exists a finite collection of disjoint incompressible 2-tori $\mathbb{T}_i^2$ such that each component $\mathcal{N}^3$ of $M^3 \backslash \cup_i \mathbb{T}_i^2$ is either geometrically atoroidal or a Seifert fiber space. and a minimal such collection $\{\mathbb{T}_i^2\}$ is unique up to homotopy.*

Finally, we need to state one more condition which is usually not necessary but which turns out to be true very frequently.

Definition 2.14. One says M^3 is **Haken** if it is prime and contains an incompressible surface other than $\mathbb{S}^2$.

In [7], Thurston proved the following theorem.

[3]Recall that an *isotopy* is a homotopy H such that for each fixed t, $H(x,t)$ gives an embedding.

Theorem 2.15. *If M^3 is Haken (in particular, if M^3 has a nontrivial torus decomposition) then M^3 admits a canonical decomposition into finitely many pieces $\mathcal{N}_i^3$ such that each one possesses a unique geometric structure (*i.e. *each $\mathcal{N}_i^3$ possesses a unique complete locally homogeneous Riemannian metric g_i).*

Definition 2.16. One says that g is **locally homogeneous** if for every pair of points $x, y \in M^n$ there exist neighborhoods $U_x \subset M^n$ of x and $U_y \subset M^n$ of y and a g-isometry $\gamma_{xy} : U_x \to U_y$ such that $\gamma_{xy}(x) = y$.

For the purposes of this paper we will not go into full detail, but there are eight such geometric structures for 3-manifolds given by the locally homogeneous metrics (sometimes called *Thurston's geometries*). Here we will list only three of these model geometries (see [4] for the complete list):

(1) Spherical geometry $\mathbb{S}^3$,
(2) Euclidean geometry $\mathbb{E}^3$,
(3) Hyperbolic geometry $\mathbb{H}^3$.

While we will not prove this result directly (instead, we will explore the fundamentals of the techniques Perelman used to prove it), Thurston conjectured in [7] that this is not true only for Haken manifolds but more generally. This is his celebrated Geometrization Conjecture (which, incidentally, proves the Poincaré conjecture).

Theorem 2.17 (Geometrization Conjecture)**.** *The interior of every compact* 3*-manifold has a canonical decomposition into pieces which have unique geometric structures.*

In 2003, Perelman announced a full proof of the Geometrization Conjecture using powerful and novel developments he made within the theory of Ricci Flow, which was first developed by Hamilton.

3. Ricci Flow

Definition 3.1. A 1-parameter family of metrics $g(t)$ on M^n, for t in an interval I, is said to obey **Ricci flow** if it satisfies the equation

$$\partial_t g(t) = -2\mathrm{Ric}(t). \tag{3.2}$$

First, let us note that this equation makes sense since both Ric and g are symmetric rank-2 tensors. Now, given the discussion in Section 1, we can infer that what Ricci flow typically does is shrink the length and volume of the manifold at points with positive curvature, and expand them in places with negative curvature. In a way, this smooths out the geometry of the manifold and makes it more symmetric.

Definition 3.3. Given a Riemannian manifold (M^n, g) the curvature equation given by

$$\mathrm{Ric}_g = \lambda g$$

is called the **Einstein equation**, which is in a way a simpler version of Ricci flow.

Example 3.4. Say $\mathrm{Ric}_g = \lambda g$. Then a solution to Ricci flow is given by

$$g(t) = \begin{cases} -2\lambda t g \text{ for } t > 0 \text{ if } \lambda < 0, \\ g \text{ for } t \in \mathbb{R} \text{ if } \lambda = 0, \\ -2\lambda t g \text{ for } t < 0 \text{ if } \lambda > 0, \end{cases}$$

where $g = g(0)$.

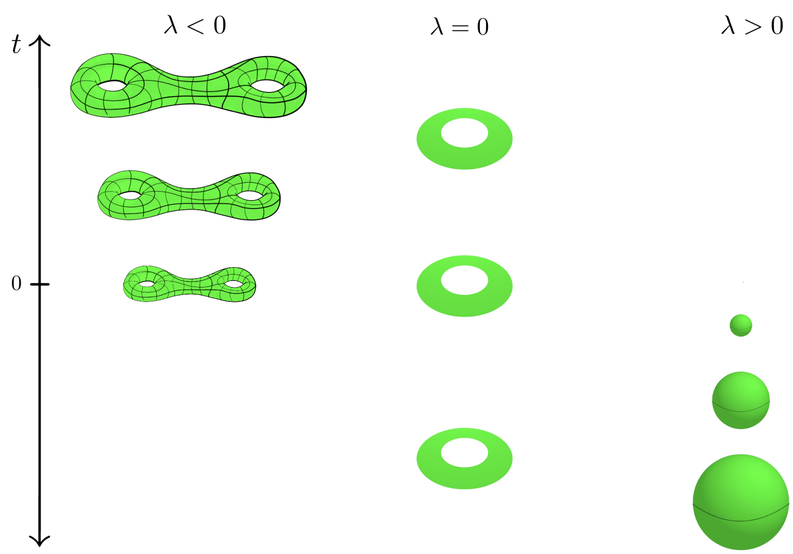

Figure 4. Evolution under the Ricci flow above of a Riemann surface with negative Ricci curvature (left); of flat torus with zero Ricci curvature (middle); of a sphere (positive Ricci curvature) (right).

Example 3.5. If $\left(M_i, (g_i(t))_{t\in I}\right),\ \ i = 1, 2$ are Ricci flows, then

$$\left(M_1 \times M_2, (g_1(t) + g_2(t))_{t\in I}\right)$$

is also a Ricci flow.

Even though the Ricci flow equation initially makes sense, it is not clear whether solutions exist or to what extent. Thus, we first need to establish some properties of Ricci flow in terms of the metric and connections.

3.1. **Evolution of geometric quantities.**

Lemma 3.6 (Variation of Christoffel symbols)**.** *If $g(s)$ is a 1-parameter family of metrics with $\frac{\partial}{\partial s} g_{ij} := v_{ij}$, then*

$$\frac{\partial}{\partial s}\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\nabla_i v_{jl} + \nabla_j v_{il} - \nabla_l v_{ij}\right).$$

Proof. For some $p \in M$, let us consider consider normal coordinates [4] centered at p, so $\Gamma_{ij}^k(p) = 0$. Note that also $\frac{\partial}{\partial x^i} g_{jk}(p) = 0$. Therefore, in these coordinates

$$\nabla_k a_{i_1,\ldots,i_r}^{j_1,\ldots,j_q}(p) = \frac{\partial}{\partial x^k} a_{i_1,\ldots,i_r}^{j_1,\ldots,j_q}(p)$$

for any tensor (r,q)-tensor a. Thus, at p we have

$$\frac{\partial}{\partial s}\Gamma_{ij}^k = \frac{1}{2} g^{kl}\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial s} g_{jl} + \frac{\partial}{\partial x^j}\frac{\partial}{\partial s} g_{il} - \frac{\partial}{\partial x^l}\frac{\partial}{\partial s} g_{ij}\right),$$

and the result follows by noting that $\nabla_i v_{jl}(p) = \frac{\partial}{\partial x^i} v_{jl}(p)$. Now, since this holds as a tensor equation, this is true for any coordinate system and not just normal coordinates. □

Remark 3.7. In coordinate-free notation, this expression becomes

$$\left\langle \left(\frac{\partial}{\partial s}\nabla\right)(X,Y), Z\right\rangle = \frac{1}{2}\left(\left(\nabla_X v\right)(Y,X) + \left(\nabla_Y v\right)(X,Z) - \left(\nabla_Z v\right)(X,Y)\right).$$

Corollary 3.8 (Evolution of Christoffel symbols under Ricci Flow)**.** *Under the Ricci flow $\partial_t g_{ij} = -2R_{ij}$, then*

$$\frac{\partial}{\partial t}\Gamma_{ij}^k = g^{kl}\left(\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij}\right). \tag{3.9}$$

Lemma 3.10 (Evolution of Laplacian under Ricci flow)**.** *If $(M^n, g(t))$ is a solution to Ricci flow, then*

$$\frac{\partial}{\partial t}\left(\Delta_{g(t)}\right) = 2R_{ij}\cdot\nabla_i\nabla_j,$$

where $\Delta_{g(t)}$ is the Laplacian acting on functions.[5] *In particular, when $n = 2$,*

$$\frac{\partial}{\partial t}(\Delta) = R\Delta.$$

Proof. We can compute it directly:

$$\frac{\partial}{\partial t}\left(g^{ij}\nabla_i\nabla_j\right) = -\frac{\partial}{\partial t}\left(g_{ij}\cdot\nabla_i\nabla_j\right) - g^{ij}\left(\frac{\partial}{\partial t}\Gamma_{ij}^k\right)\nabla_k$$

and

$$g^{ij}\left(\frac{\partial}{\partial t}\Gamma_{ij}^k\right) = -g^{kl}\left(2g^{ij}\nabla g^{ij}\nabla_i R_{jl} - \nabla_l R\right) = 0$$

using the Second Bianchi's identity. □

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[4]Recall that there exist coordinates $(x^1,\ldots,x^n)$ satisfying $\partial_i = e_i$ and $\nabla\partial_i = 0$ at p. These are called **normal coordinates**, and their definition implies that we have $\partial_k g_{ij} = 0$ at p and $g_{ij} = \delta_{ij}$.

[5]Recall that the Laplacian (Laplace-Beltrami operator) acting on functions is defined in local coordinates by

$$\Delta := \operatorname{div}\nabla = g^{ij}\nabla_i\nabla_j = g^{ij}\left(\frac{\partial^2}{\partial x^i \partial x^j} - \Gamma_{ij}^k\frac{\partial}{\partial x^k}\right).$$

Now, we can write the components of the Riemannian Curvature tensor (1.3) as

$$R\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = R^l_{ijk}\frac{\partial}{\partial x^l}$$

and

$$R^l_{ijk} = \partial_j\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} + \Gamma^p_{jk}\Gamma^l_{ip} - \Gamma^p_{ik}\Gamma^l_{jp}, \tag{3.11}$$

so the Ricci curvature (or **Ricci tensor**) is given by $R_{ij} = R^p_{pij}$. Using this, we calculate the variation of the **Ricci tensor in terms of the variation of the connection**:

$$\frac{\partial}{\partial s}R_{ij} = \nabla_p\left(\frac{\partial}{\partial s}\Gamma^p_{ij}\right) - \nabla_i\left(\frac{\partial}{\partial s}\Gamma^p_{pj}\right) \tag{3.12}$$

which can be obtained using the same method as in Lemma 3.6. Now, note that if $\frac{\partial}{\partial s}g_{ij} = v_{ij}$, then

$$\frac{\partial}{\partial s}R_{ij} = \frac{1}{2}\nabla_l\left(\nabla_i v_{jl} + \nabla_j v_{il} - \nabla_l v_{ij}\right) - \frac{1}{2}\nabla_i\nabla_j V, \tag{3.13}$$

where we define $V := g^{ij}v_{ij} = \text{trace}(v)$. Taking the trace (or, equivalently, contracting),

$$\begin{aligned}\frac{\partial}{\partial s}R &= g^{ij}\left(\frac{\partial}{\partial s}R_{ij}\right) - \frac{\partial}{\partial s}g_{ij}\cdot R_{ij}\\ &= \nabla_l\nabla_i v_{i,l} - \Delta V - v_{ij}\cdot R_{ij}.\end{aligned}$$

Recall that $R_{kijl} = g_{lp}R^p_{kij}$, so

$$\begin{aligned}\nabla_l\left(\nabla_i v_{jl} + \nabla_j v_{il}\right) &= \nabla_i\nabla_l v_{jl} - R_{ljim}v_{ml} - R_{lilm}v_{jm}\\ &\quad + \nabla_j\nabla_l v_{il} - R_{ljim}v_{ml} - R_{ljlm}v_{im}\\ &= \nabla_i\left(\text{div } v\right)_j + \nabla_j(\text{div } v)_i - 2R_{lijm}v_{lm}\\ &\quad + R_{im}v_{jm} + R_{jm}v_{im},\end{aligned}$$

(recall that $(\text{div } v)_k := g^{ij}\nabla_i v_{jk}$) which yields the **variation of Ricci formula**:

$$\frac{\partial}{\partial s}R_{ij} = -\frac{1}{2}\Bigg(\Delta_L v_{ij} + \nabla_i\nabla_j V - \nabla_i(\text{div } v)_j - \nabla_j(\text{div } v)_i\Bigg), \tag{3.14}$$

where Δ_L denotes the **Lichnerowicz Laplacian**, which we define by

$$\Delta_L v_{ij} := \Delta v_{ij} + 2R_{kijl}v_{kl} - R_{ik}v_{jk} - R_{jk}v_{ik}.$$

3.2. **Short time existence.** First let us introduce a few very important definitions for PDEs.

Definition 3.15. A partial differential equation (take, for instance, the Ricci flow equation) is said to be **strictly parabolic** if the principal symbol of the second-order operator on the right-hand side is positive definite.

These PDEs are well know to have local, unique solutions, which motivates the proof that will be presented below. The reader is encouraged to see [10] to explore more on this topic.

Theorem 3.16 (Hamilton, DeTurck - Short time existence and uniqueness)**.** *If M^n is a closed Riemannian manifold and if g_0 is a smooth Riemannian metric, then there exists a unique smooth solution $\bar{g}(t)$ to the Ricci flow defined on some interval $[0, \varepsilon)$, $\varepsilon > 0$, with $\bar{g}(0) = g_0$.*

This theorem was originally proved by Hamilton, and while his proof relied on the Nash-Mosher theorem, here we present a later proof by DeTurck, in which he reduces the problem of existence and uniqueness to that of a local existence problem for parabolic PDEs.

Proof. DeTurck's trick consists in the following: let us consider a background metric $\tilde{g}$ and with it a fixed background connection $\tilde{\nabla}$, with Christoffel symbols $\tilde{\Gamma}$. For convenience, we assume this to be the Levi-Civita connection. We now define the **Ricci-DeTurck flow** by

$$\begin{aligned} \frac{\partial}{\partial t} g_{ij} &= -2R_{ij} + \nabla_i W_j + \nabla_j W_i, \\ g(0) &= g_0, \end{aligned}$$

where the time-dependent 1-form $W = W(g)$ is defined by

$$W_j := g_{jk} g^{pq} \left(\Gamma^k_{pq} - \tilde{\Gamma}^k_{pq} \right).$$

Note that if $g(s)$ is a 1-parameter family of metrics with $g(0) = g$ and

$$\left. \frac{\partial}{\partial s} \right|_{s=0} g_{ij} = v_{ij},$$

then

$$\left. \frac{\partial}{\partial s} \right|_{s=0} W(g(s))_j = -X_j + \text{zeroth-order terms in X},$$

where $X := \frac{1}{2} \nabla V - \operatorname{div} v$ as above. Moreover, we obtain

$$\left. \frac{\partial}{\partial s} \right|_{s=0} (-2R_{ij} + \nabla_j + \nabla_j W_i) = \Delta_L v_{ij} + \text{ first-order terms in } V,$$

from the variation of Ricci formula (3.14). From this, it follows that the Ricci-DeTurck flow is strictly parabolic, and therefore that given any smooth metric g_0 on a closed manifold, there exists a unique solution $g(t)$ to the Ricci-DeTurck flow, with $g(0) = g_0$.

Now, given such a solution to the Ricci-DeTurck flow, we can further solve the following ODE at any point $p \in M$:

$$\begin{aligned} \frac{\partial}{\partial t} \varphi_t &= -W^*, \\ \varphi_0 &= \mathrm{id}, \end{aligned}$$

where W^* is the vector field dual to $W(t)$ with respect to $g(t)$. We can thus pull back $g(t)$ by the diffeomorphisms φ_t, and we obtain a solution

$$\bar{g}(t) := \varphi_t^* g(t)$$

to Ricci Flow, with $g(\bar{0}) = g_0$. Indeed, computing directly:

$$\begin{aligned}\frac{\partial}{\partial t}\bar{g}(t) &= \varphi_t^*\left(\frac{\partial}{\partial t}g(t)\right) + \frac{\partial}{\partial s}\bigg|_{s=0}\left(\varphi_{t+s}^* g(t)\right)\\ &= -2\mathrm{Ric}\left(\varphi_t^* g(t)\right) + \varphi_t^*\left(\mathcal{L}_w(t)g(t)\right) - \mathcal{L}_{\left(\varphi_t^{-1}\right)_* W(t)}\left(\varphi_t^* g(t)\right)\\ &= -2\mathrm{Ric}\left(\bar{g}(t)\right),\end{aligned}$$

where $\mathcal{L}$ denotes the Lie derivative. The uniqueness part of the proof follows from studying the harmonic map heat flow, and the reader is encouraged to see [5] to look at the details of the rest of the proof. □

3.3. Some special solutions.

Definition 3.17. Suppose $(M^n, g(t))$ is a solution to Ricci Flow on a time interval (α, β), $\alpha < 0$, $\beta > 0$, and set $g(0) = g_0$. We say that $g(t)$ is a **self-similar solution** of the Ricci flow if there exist scalars $\sigma(t)$ and diffeomorphisms φ_t of M^n such that

$$g(t) = \sigma(t)\varphi_t^*(g_0) \tag{3.18}$$

for all $t \in (\alpha, \beta)$.

Definition 3.19. Suppose $(M^n, g(0))$ is a fixed Riemannian manifold. We say that g_0 is **Ricci soliton** if

$$-2\mathrm{Ric}_{g_0} = \mathcal{L}_X(g_0) + 2\lambda g_0 \tag{3.20}$$

holds for some $\lambda \in \mathbb{R}$ and some complete vector field $X \in \mathfrak{X}(M)$.

Remark 3.21. Note that if $X \equiv 0$, this is just an Einstein metric.

Let us state the following result without proof, which broadly tells us that the above two definitions are equivalent.

Lemma 3.22. *If $(M^n, g(t))$ is a self-similar solution to Ricci flow, then there exists a vector field $X \in \mathfrak{X}(M)$ that solves (3.20). Conversely, given a soliton $(M^n, g(t), X)$, there exists a 1-parameter family of scalars $\sigma(t)$ and diffeomorphisms ψ_t of M^n such that $(M^n, g(t))$ is a solution to Ricci flow if $g(t)$ is defined by (3.18).*

Definition 3.23. A solution to Ricci flow is said to be **eternal** if it is defined for all time.

3.3.1. *The Cigar soliton.*

Definition 3.24. We define Hamilton's **cigar soliton**[6] as the complete Riemann surface $(\mathbb{R}^2, g_\Sigma)$, where

$$g_\Sigma = \frac{dx \otimes dx + dy \otimes dy}{1 + x^2 + y^2}.$$

Computing the Christoffel symbols with respect to the coordinates $(x^1 = x, x^2 = y)$, we obtain

$$\begin{aligned}\Gamma_{11}^1 &= -\frac{x}{1+r^2}, \quad \Gamma_{12}^1 = -\frac{y}{1+r^2}, \quad \Gamma_{22}^1 = \frac{x}{1+r^2},\\ \Gamma_{11}^2 &= \frac{y}{1+r^2}, \quad \Gamma_{12}^2 = \frac{-x}{1+r^2}, \quad \Gamma_{22}^2 = -\frac{y}{1+r^2},\end{aligned}$$

[6]In physics, this is also known as **Witten's black hole**.

where $r := \sqrt{x^2+y^2}$. Let us note that g_Σ is rotationally symmetric, and it is therefore natural to write it in polar coordinates:

$$g_\Sigma = \frac{dr^2 + r^2 d\theta^2}{1+r^2}.$$

The scalar curvature of g_Σ is

$$R_\Sigma = \frac{4}{1+r^2}.$$

Because $\frac{r^2}{1+r^2} \to 1$ as $r \to \infty$, we can see that the metric is asymptotic at infinity to a cylinder of radius 1 (and moreover, one can check that the scalar curvature of a cylinder $\mathbb{R} \times \mathbb{S}^1$ is equal to 0). Now let us note that the geodesics in the cigar metric are the curves $\gamma(t) = (r(t), \theta(t))$ which are solutions to the system

$$\theta'' + \frac{2}{r(1+r^2)}\theta' r' = 0$$
$$r'' - \left[(r')^2 + (\theta')^2\right]\frac{r}{1+r^2} = 0.$$

Thus, we can write

$$\theta(t) = a + bt + b\int_0^t \frac{d\tau}{r^2(\tau)},$$

where $a, b \in \mathbb{R}$. In particular, when $r(0) = r_0$ is large, a Euclidean circle of radius r^2 is close to being a geodesic for a short period of time. With a little bit of more work, one can check that the cigar is actually a solution to Ricci flow, and the reader is encouraged to see [4] to look at the computation of this fact.

3.3.2. *The round sphere.*

Definition 3.25. An **ancient solution** of the Ricci flow is one which exists for $-\infty < t < \alpha$ for $\alpha < \infty$.

Let g_{st} be the standard round metric on $\mathbb{S}^n$, and let

$$g(t) := r(t)^2 g_{st},$$

where $r(t)$ is to be determined. We can see that $g(t)$ is a solution to Ricci flow if and only if

$$2r\frac{dr}{dt} \cdot g_{st} = \frac{\partial}{\partial s} g = -2\mathrm{Ric}_g = -2\mathrm{Ric}_{g_{st}} = -2(n-1)g_{st}.$$

Moreover, the above is true if and only if $r(t)$ is a solution to

$$\frac{dr}{dt} = -\frac{n-1}{r}.$$

Therefore, setting

$$r(t) = \sqrt{r_0^2 - 2(n-1)t} = \sqrt{2(n-1)} - \sqrt{T-t},$$

where $\infty < t < T$, $T < \infty$, yields an ancient solution $(\mathbb{S}^n, g(t))$ to Ricci flow, and T is the singularity time defined by

$$T = \frac{r_0^2}{2(n-1)}.$$

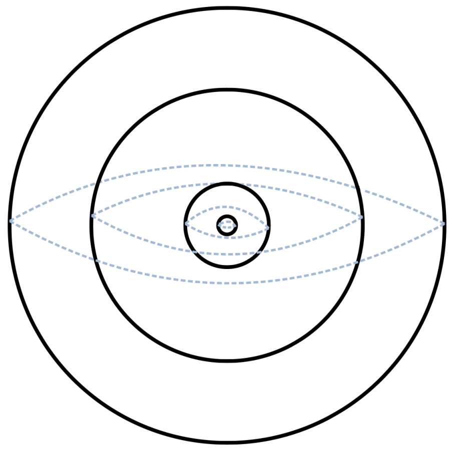

FIGURE 5. Shrinking sphere.

Motivated by this example, it is natural to study singularities in Ricci flow.

Definition 3.26. One says $(M^n, g(t))$ of the Ricci flow encounters a **local singularity** at $T < \infty$ if there exists a compact $K \subset M^n$ such that

$$\sup_{K\times[0,T)} |R| = \infty,$$

but

$$\sup_{(M^n\setminus K)\times[0,T)} |R| < \infty$$

where $|R|$ is the norm of the Riemannian curvature tensor. This is also called **pinching behavior**.

This is the fundamental obstacle when considering a manifold under Ricci flow. For instance, in [14], Hamilton established the following theorem.

Theorem 3.27 (Rounding theorem)**.** *If a compact Riemannian manifold has everywhere positive Ricci curvature, then Ricci flow with these initial conditions encounters a local singularity in a finite time.*

Nevertheless, before Perelman there was a general lack of deep understanding of singularities, and around the time he presented his results he also completely classified all possible types of singularities for 3 manifolds.

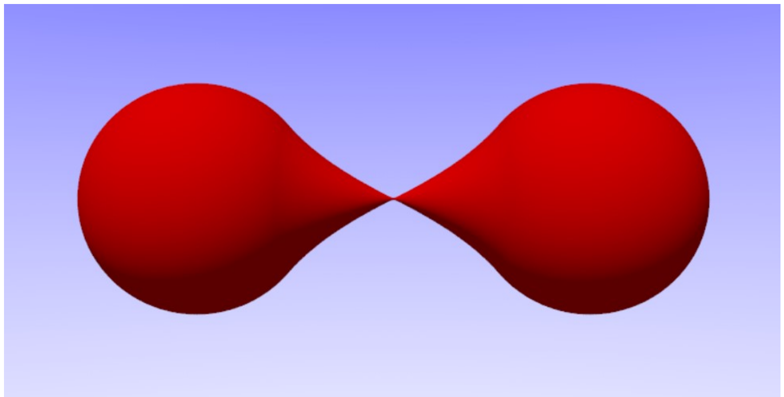

FIGURE 6. Pinching behavior forming a local singularity.

3.4. **Perelman's approach to Poincaré conjecture.** First, let us recall the statement of the theorem.

Theorem 3.28 (Poincaré Conjecture)**.** *Let M be a closed, simply-connected* 3*-manifold. Then M is homeomorphic to the* 3*-sphere* $\mathbb{S}^3$.

When using Ricci flow, the idea is as follows: if under Ricci flow, the metric on a manifold shrinks uniformly to 0, then one expects that the manifold was initially (or at some finite point in time) homeomorphic to the 3-sphere, and ideally, the converse should be true, thus proving Poincaré conjecture. Nevertheless, manifolds that develop singularities in finite times, are not necessarily homeomorphic to $\mathbb{S}^3$. To address this problem, Hamilton developed a technique called **surgery**, which, as its name suggests, is a very clever means for removing these problematic portions of manifolds while allowing the rest of the manifold to continue transforming under Ricci flow. This led to a systematic theory of Ricci flow with surgery, and which Perelman used in his groundbreaking work. While here we will not delve into the details of Ricci flow with surgery due to the highly technical background required, let us roughly state some of Perelman's most notable advancements.

Theorem 3.29 (Global existence of Ricci flow with surgery)**.** *Let (M, g) be a simply connected, compact* 3*-manifold. Then there exists a* ***Ricci flow with surgery*** $t \mapsto (M(t), g(t))$ *which assigns a compact manifold $(M(t), g(t))$ to each time $t \in [0, +\infty)$, and a closed set $T \subset (0, +\infty)$ of* ***surgery times*** *such that*

(1) $M(0) = M$ and $g(0) = g$,
(2) If I is a connected component of $(0, \infty) \backslash T$, then $t \mapsto (M(t), g(t))$ is a solution to Ricci flow on I,
(3) If $t \in T$, there exists a sufficiently small $\varepsilon > 0$ such that the connected components of $M(t - \varepsilon)$ are each homeomorphic to a connected sum of finitely many connected components of $M(t)$, together with a finite number of spherical space-forms.

There is another implication (item) in this theorem, but the main idea here is contained in item 3, which implies a topological compatibility of the manifold at surgery times and at non-surgery times.

Theorem 3.30 (Finite time extinction)**.** *Let (M, g) be a simply connected, compact 3-manifold. If $t \mapsto (M(t), g(t))$ is an associated Ricci flow with surgery, then $M(t)$ is empty for all sufficiently large times t.*

These last few theorems, along with some other Riemannian geometry machinery (such as triangulations of manifolds), imply the Poincaré conjecture (the proofs can be found in [11], but they are quite involved for the purposes of this paper). As mentioned before, Perelman actually presented a full proof of the Geometrization conjecture, and hopefully this provides a sense for how the Poincaré conjecture follows from it.

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References

[1] doCarmo, Manfredo P. Riemannian Geometry. Springer. 1992.
[2] John M. Lee. Introduction to Smooth Manifolds. Springer Graduate Texts in Mathematics. 2013.
[3] John M. Lee. Introduction to Riemannian Manifolds. Springer Graduate Texts in Mathematics. 2018.
[4] Chow & Knopf. The Ricci Flow: An Introduction. American Mathematical Society. 2004.
[5] B. Chow, P. Lu & L. Ni. Hamilton's Ricci Flow. American Mathematical Society. 2006.
[6] Thurston, William. Three-dimensional Geometry and Topology. Princeton University Press. 1997.
[7] Thurston, William. Three Dimensional Manifolds, Kleinian groups and hyperbolic geometry. Bulletin of the American Mathematical Society. 1982.
[8] Hatcher. Algebraic Topology. Cambridge University Press. 2002.
[9] Peter Petersen. Riemannian Geometry. Springer. 2006.
[10] Lawrence C. Evans. Partial Differential Equations. American Mathematical Society. 2010.
[11] Terence Tao. Lecture Notes for 285G - Perelman's proof of the Poincaré Conjecture. https://terrytao.wordpress.com/category/teaching/285g-poincare-conjecture/
[12] Terence Tao. Ricci flow. https://terrytao.files.wordpress.com/2008/03/ricci1.pdf
[13] Jacob Lurie. Lecture notes for topics in Geometric Topology. https://www.math.ias.edu/~lurie/937.html
[14] Richard S. Hamilton. Three-manifolds with positive Ricci curvature. J. Differential Geometry. 1982.
[15] Richard Bamler. Lecture notes and recordings for Math 277: Ricci flow. https://math.berkeley.edu/~rbamler/rfclass.html