ALGORITHMIC INSOLUBILITY OF MANIFOLD AND CW-COMPLEX SIMPLE CONNECTEDNESS

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Abstract. In this paper, we explore two insoluble problems in topology: simple connectedness of manifolds and CW-complexes. First, we briefly introduce Turing machines, decidability, undecidable problems, and Turing-reduction. Then, we connect these concepts to algebra by introducing the word problem for finitely presented groups. Finally, we use the undecidability of word problem for finitely presented groups to prove the undecidability of simple connectedness of CW-complexes and manifolds.

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1. Introduction

An undecidable problem is a decision problem for which it is proved that it is impossible to construct an algorithm that always leads to a correct yes or no answer. The idea of undecidable problems first appeared in a list of problems presented by David Hilbert in 1900. He presented a list of problems [1] that he thought to be examples of the most important problems to be solved in mathematics. However, most of these problems were proven to be unsolvable. In 1936, Alonzo Church presented the first explicit example of an undecidable problem [2]. In 1937, Alan Turing formalized the notion of an algorithm by introducing a mathematical model of computing, which we call now Turing machines. He proved that the halting problem, to decide if a Turing machine will stop on an input, is unsolvable in finite steps by a Turing machine. [3].

In 1910, before Gödel's incompleteness theorem, Max Dehn wanted to extend Hilbert's ideas of algorithmic decidability to geometric problems. He proposed three problems in group theory, each with a geometric interpretation [4]. These problems were the word problem, the conjugacy problem, and the isomorphism problem. The geometric motivations are to decide if a loop in a topological space is contractible, to decide whether two loops are freely homotopic, and to decide if

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one can algorithmically distinguish topological spaces based on their fundamental groups.

Until 1947, the only undecidability results were in Logic. At that time, the first unsolvable problem of regular mathematics was shown. Emil Post [5] and Andrei Markov [6] showed the insolubility of the word problem for semi-groups. In 1954-1958, Novikov [7], Boone [8], and Britton [9] all independently succeeded in extending this result to groups.

In this paper, we aim to show the proof of the insolubility of two topological problems: the simple connectedness of CW-complexes and manifolds. In the first section we present fundamental examples of undecidable problems in logic and prove the undecidability of the halting problem. In the second section we prove the undecidability of the word problem for finitely presented semi-groups to show how to apply the idea of Turing machines in algebra. In the third section we present the topological background that we need to prove the main result in the fourth section. This paper assumes some basic knowledge of algebra and topology and intermediate knowledge of manifolds.

2. Turing machines and undecidability

To be able to talk about the algorithmic insolubility of a problem, we need to formalize the notions of an algorithm and solvability. An algorithm is known to be a set of instructions that can run on a computer. So in order for this to make sense in a mathematical way, we need a mathematical model of computation. In 1936, Alan Turing proposed a strong model of computation which we call nowadays Turing machine. Informally, a Turing machine is a box with a tape running through it. The tape consists of a series of an infinite numbers of squares. The box can print a finite number of symbols, say $s_1, s_2, \cdots, s_M$, and can be in one of finite number of states, say $q_0, q_1, q_2, \cdots, q_N$. At each step of the computation, the machine is in some state q_i and reading some symbol s_j, then it decides to perform one of the following operations:

(i) Replace the symbol s_j by the symbol s_k and stay at the same square.
(ii) Move one square to the right and read that square.
(iii) Move one square to the left and read that square.

Now we can formalize this notion of a Turing machine.

Definition 2.1. A ***quadruple*** is a 4-tuple of one of the following three types:

$$q_i s_j s_k q_l,$$
$$q_i s_j R q_l,$$
$$q_i s_j L q_l;$$

Definition 2.2. A ***Turing Machine*** T is a finite set of quadruples, not two of which have the same first two letters. The ***alphabet*** of T is the set $\{s_0, s_1, \cdots, s_M\}$ of all s-letters occurring in its quadruples.

These three types of quadruples represent the three possible operations of a Turing machine that we mentioned above. For example, the quadruple $q_i s_j s_k q_l$ means when the machine is in the state q_i and reading s_j, then it writes s_k and moves to the state q_l. The quadruple $q_i s_j R q_l$ means when the machine is in the state q_i and reading the symbol s_j, then it moves to the right square and moves to the state q_l.

Definition 2.3. A ***configuration*** α is a positive word of the form $\alpha = \sigma q_i \tau$, where σ, τ are s-words and τ is not empty.

A configuration describes the state of the machine at a certain step of its operation. For example, the configuration $\alpha = s_0 s_1 s_2 q_1 s_3 s_4$ means that the machine is in state q_1 and reading the symbol s_3.

Definition 2.4. A configuration α is ***terminal*** if there is no configuration α' such that the machine moves from α to α'.

Definition 2.5. Let ω be a positive word on the alphabet of a Turing machine T. We say T ***computes*** ω if there is a sequence of configurations $\alpha_1 = q_1\omega, \alpha_2, \cdots, \alpha_n$ such that α_i moves to α_{i+1} for all $i \leq n-1$ and α_n is a terminal configuration.

Intuitively, this means that if a word ω is on the tape and the machine T is in starting state q_1 while reading the first square, T might run forever with an infinite sequence of configurations $\alpha_1, \alpha_2, \cdots$. This sequence stops if and only if T computes ω; otherwise, the machine does not terminate. Moreover, the machine can "accept" or "reject" an input. This is done by just having two states $q_{\text{accept}}, q_{\text{reject}}$. If a machine accepts an input ω, then it computes it and terminates with state q_{accept}. The situation is similar to that for rejecting an input.

Definition 2.6. If T is a Turing machine whose alphabet contains S, define

$$e(T) = \{\omega \in S^* : T \text{ computes } \omega\},$$

and we say that T ***enumerates*** $e(T)$. A subset E of S^* is called ***recognizable*** if there exists a Turing machine T that enumerates E.

This means that if we have a set E that is recognizable, then we can find a Turing machine T that halts on each element of E. However, if the machine T was given an element outside E, it does not necessarily halt. This idea of some inputs leading the Turing machine to not terminate motivates the following definition.

Definition 2.7. A set E is ***co-recognizable*** if $S^* \setminus E$ is recognizable.

Definition 2.8. A set E is ***decidable*** if it's both recognizable and co-recognizable.

This means that if a set E is decidable, then there is a Turing machine that *decides* it i.e. the machine halts on all possible inputs in S^*. And the intuition behind the word *decide* here is that the machine takes a word $\omega \in S^*$ and *decides* whether $\omega \in E$ or not. On the other hand, merely *recognizing* E means that the machine takes $\omega \in S^*$, and halts if $\omega \in E$. Thus, if $\omega \neq E$, we might never know, because T might never halt.

A *decision problem* is a problem where we are given a set $E \subset S^*$ and a word ω, and we are asked to *decide* whether $\omega \in E$ or not. Answering a decision problem is equivalent to showing that the set E is decidable. Now we see the intuitive relation between the *solvability* of a problem and the notion of *decidability.*

Now comes up an interesting question. Do there exist unsolvable decision problems? Formally, does there exist sets that are not decidable? The following theorem answers this question affirmatively. Let us call any subset of S^* a *language.*

Theorem 2.9. *Some languages are not Turing-recognizable and hence not Turing-decidable.*

Proof. Fix an alphabet S. Notice that since every Turing machine is a finite set of quadruples, every Turing machine T can be encoded as a finite binary sequence $\langle T\rangle$. Eliminating all the finite binary sequences that do not encode an actual Turing machine, we see that the set of all Turing machines over an alphabet S is countable. Now we show that the set of all languages $\mathcal{L}$ is uncountable. This is done by showing a bijection between the set of all infinite binary sequences $\mathcal{B}$ and $\mathcal{L}$. Since S^* is countable, then we can list its elements in a sequence $S = \{s_1, s_2, \cdots\}$. Now, for any $A \in \mathcal{L}$, consider the characteristic sequence χ_A which is the binary sequence whose i^{th} element is 1 if $s_i \in A$ and 0 if $s_i \notin A$. The function $f : \mathcal{L} \to \mathcal{B}$, defined by $f(A) = \chi_A$ is surjective by definition. It is injective because if $f(A_1) = f(A_2)$, then A_1, A_2 contains the same words, hence $A_1 = A_2$. Since the set of all infinite binary sequences $\mathcal{B}$ is uncountable, then $\mathcal{L}$ is uncountable as well. Since the set of Turing machines is countable and $e(T)$ is countable, then there must exist some language that is not recognized by any Turing machine.

□

Now that we have proved the existence of such languages, we give two fundamental examples of undecidable languages. The first example is the problem of deciding whether a Turing machine computes an input or not. We call it A_{TM}. The problem is the following: given a Turing machine M and a word ω; can we decide if M accepts ω? Formally this problem is described as the language:

$$A_{TM} = \{\langle M, \omega\rangle : M \text{ is a Turing machine and } M \text{ accepts } \omega\}$$

and solving this problem means finding a Turing machine that decides this language. The following theorem proves that such a Turing machine doesn't exist.

Theorem 2.10. *A_{TM} is undecidable*

Proof. Assume that A_{TM} is decidable. Suppose that H is a Turing machine that decides A_{TM}. Given an input $\langle M, \omega\rangle$, H halts accepts if M halts and accepts ω. Moreover, H halts and rejects if M rejects ω. Now we construct a new Turing machine D, that works as follows:

$$\begin{aligned} D = &\text{ On input } \langle M\rangle, \text{ where } M \text{ is a Turing machine :}\\ &\quad 1. \text{ Run } H \text{ on } \langle M, \langle M\rangle\rangle \ .\\ &\quad 2. \text{ Accept if } H \text{ rejects, and Reject if } H \text{ accepts.} \end{aligned}$$

Now notice what happens when we run D on its own description. If we run D on input $\langle D\rangle$, then D will accept if D rejects and D will reject when D accepts, which is a complete nonsense; hence, we have our contradiction. □

Another fundamental example of undecidable decision problems is the halting problem. The problem is as follows: given a Turing machine M and an input ω, does M halt on ω ? Formally,

$$HALT_{TM} = \{\langle M, \omega\rangle : M \text{ is a Turing machine and halts on } \omega\}$$

Theorem 2.11. *$HALT_{TM}$ is undecidable*

Proof. Assume $HALT_{TM}$ is decidable. Let H be the Turing machine that decides $HALT_{TM}$. Consider the following Turing machine:

$D =$ On input $\langle M, \omega \rangle$, where M is a Turing machine :

1. Run H on $\langle M, \omega \rangle$.
2. If H rejects, then Reject.
3. If H accepts, then simulate M on ω .
4. If M accepts, then Accept. Otherwise, Reject.

Now notice that this machine decides A_{TM}, which we proved to be undecidable. This is a contradiction. □

Notice that in the proof of the last theorem, we proved that the halting problem is undecidable by using the fact that A_{TM} is undecidable. We ***reduced*** the A_{TM} problem to the halting problem.

Definition 2.12. A language A is ***Turing-redeucible*** to a language B, written $A \leq_T B$, if given a decider for B, we can decide A.

Corollary 2.13. $A_{TM} \leq_T HALT_{TM}$.

Turing reduction is a fundamental concept when dealing with undecidable decision problems and will be used multiple times throughout this paper.

3. The word problem for finitely presented semi-groups

Now, we apply these ideas to algebra. Let Γ be a semi-group generated by $X = \{x_1, x_2, \cdots, x_n\}$. Let Ω be the set of all positive words on X. We say Γ has a ***solvable word problem*** if there is a process to decide whether a given ω is identical to some word $\omega' \in \Omega$ in Γ.

If ω and ω' are words on an alphabet X, then we write $\omega \equiv \omega'$ if they have the same spelling. Let Γ be the semi-group with presentation:

$$\Gamma = \langle X | \alpha_j = \beta_j, \forall j \in J \rangle$$

If ω, ω' are two positive words on X, then $\omega = \omega'$ in Γ if and only if there exists a finite sequence $\omega_1, \omega_2, \cdots, \omega_n$ of positive words on X such that

$$\omega \equiv \omega_1 \to \omega_2 \to \cdots \to \omega_n \equiv \omega'$$

where $\omega_i \to \omega_{i+1}$ means $\omega_i \equiv \sigma \alpha_j \tau$ (resp. $\sigma \beta_j \tau$) and $\omega_{i+1} \equiv \sigma \beta_j \tau$ (resp. $\sigma \alpha_j \tau$). Intuitively, this means that two words are equal in Γ if they can be reduced to each other through the using the set of rules of the group presentation.

The main result that we will prove in this section is that the word problem is unsolvable for finitely presented semi-groups. The plan here is to reduce the word problem for semi-groups to the halting problem which we proved to be undecidable in the previous section. The general idea is that given a Turing machine we construct some kind of corresponding semi-group presentation. Then our argument should go as follows. If we can decide the word problem for this semi-group presentation, then we can decide if the machine halts. This idea is motivated by the fact that if two words are equal in a group, then this is equivalent to finding some "decision process" to show it using the rules of the semi-group itself. So this motivates us to encode the quadruples of a given Turing machine into the rules the semi-group presentation.

Let's now encode a Turing machine T with letters $s_1, \cdots, s_M$ and states $q_0, \cdots, q_N$ as a semi-group. Let h, q be new letters.

Definition 3.1. If T is a Turing machine with a stopping state q_0, then its ***associated semi-group*** $\Gamma(T)$ is the semi-group with the presentation:

$$\Gamma(T) = \langle q, h, s_0, \cdots, s_M, q_0, q_1, \cdots, q_N | R(T) \rangle$$

where the relations $R(T)$ are:

$$q_i s_j = q_l s_k \quad \text{if} \quad q_i s_j s_l q_l \in T$$

for all $k = 0, 1, \cdots, M$:

$$\begin{aligned} q_i s_j s_k &= s_j q_l s_k \quad \text{if} \quad q_i s_j R q_l \in T; \\ q_i s_j h &= s_j q_l s_0 h \quad \text{if} \quad q_i s_j R q_l \in T; \\ s_k q_i s_j &= q_l s_k s_j \quad \text{if} \quad q_i s_j L q_l \in T; \\ h q_i s_j &= h q_l s_0 s_j \quad \text{if} \quad q_i s_j L q_l \in T; \\ q_0 s_k &= q_0; \\ s_k q_0 h &= q_0 h; \\ h q_0 h &= q. \end{aligned}$$

The new letter h in the definition is more or less used to mark the ends of the tape. We call a word ***h-special*** if it has the form $h\alpha h$, where α is a configuration. Let T be a Turing machine with terminating state q_0 and let α be a terminating configuration. Then notice that $h\alpha h$ can be written as $h\sigma q_0 \tau h$ where $\sigma, \tau \in S^*$ and τ is non-empty word. Then notice that the last three rules in the previous definition allow us to write $h\alpha h = q$ in $\Gamma(T)$ when α is terminating.

Lemma 3.2. *Let T be a Turing machine with stopping state q_0 and associated semi-group*

$$\Gamma(T) = \langle q, h, s_0, s_1, s_2, \cdot, s_M, q_0, q_1, \cdots, q_N | R(T) \rangle$$

(i) *Let ω, ω' be words on $\{s_0, s_1, \cdots, s_M, q_0, q_1, \cdots, q_N\}$ with $\omega \not\equiv q$. If $\omega \to \omega'$, then ω is h-special if and only if ω' is h-special.*
(ii) *If $\omega = h\alpha h$ is h-special, $\omega' \not\equiv q$, and $\omega \to \omega'$, then $\omega' = h\beta h$, where the either T moves from α to β or from β to α.*

Proof. (i) This follows from the fact that the only relation that creates or removes the letter h is the relation $hq_0 h = q$.
(ii) By the first part, ω' is h-special. Then there exists a configuration β such that $\omega' = h\beta h$. Since $\omega \to \omega'$, then $\alpha = \beta$ in $R(T)$. Since $\omega' \not\equiv q$, then $\alpha = \beta$ is one of the first 5 types of relations in $R(T)$. Hence, $\alpha = \beta$ represents a quadruple in T that represents a basic move of T. Therefore, T moves from $\alpha \to \beta$ or the other way around.

□

Now we get to an important lemma. The following lemma relates the halting problem of Turing machines and word problem for semi-groups which is the main step in the proof of the undecidability of the word problem for semi-groups.

Lemma 3.3. *Let T be a Turing machine over alphabet S with stopping state q_0. T halts on $\omega \in S^*$ if and only if $hq_1\omega h = q$ in $\Gamma(T)$*

Proof. Suppose T halts on ω. This implies that there is a finite sequence of configurations $\alpha_1 = q_1\omega, \alpha_2, \cdots, \alpha_n$ such that T moves from α_i to α_{i+1} and a_n has q_0 in it. Since each machine move represents a quadruple that corresponds to one of the first types of relations in $G(T)$, then we can use these operations to show that $hq_1\omega h = h\alpha_2 h = \cdots = h\alpha_n h$ in $\Gamma(T)$. Since α_n involves q_0, using the last three relations we see that $hq_1\omega h = h\alpha_n h = q$ in $\Gamma(T)$.

Conversely, suppose $hq_1\omega h = q$ in $\Gamma(T)$. Then there is a finite sequence of words $\omega_1, \cdots, \omega_n$ on $\{h, s_0, s_1, \cdots, s_M, q_0, q_1, \cdots, q_N\}$ with

$$hq_1\omega h \equiv \omega_1 \to \omega_2 \to \cdots \to \omega_n \equiv hq_0h \to q$$

By Lemma 3.2(i), each ω_i is h-special. Let $\omega_i = h\alpha_i h$ for some configuration α_i. By Lemma 3.2(ii), for each i, either T moves from α_i to α_{i+1} or from α_{i+1} to α_i. We prove by contradiction that for all i, T moves from α_i to α_{i+1}. Notice that it's always true that T moves from α_{n-1} to α_n since α_n is terminal. Now assume the contrary. Let k be the largest index such that T moves from α_{k+1} to α_k. This means that at configuration α_{k+1}, T should move to α_{k+2} *and* α_k which cannot happen in a Turing machine; contradiction. It follows that T moves from $q_1\omega$ to α_1 to $\cdots$ α_n, which is terminal. Hence, T halts on ω. □

Now, the main theorem of the section follows.

Theorem 3.4. *The word problem for finitely presented semi-groups is undecidable.*

Proof. We proceed in the same way as in the proof of Theorem 2.11. Suppose that the word problem is decidable for finitely presented semi-groups. Let W be a Turing machine that decides it. This machine takes $\langle G, \omega, \omega' \rangle$, where G is a finite presentation of a semi-group and ω, ω' are two words over the letters of G. W decides whether $\omega = \omega'$ in G or not. Now consider the following Turing machine :

$H =$ On input $\langle M, \omega \rangle$, where M is a Turing machine :

1. Run W on $\langle \Gamma(M), hq_1\omega h, q \rangle$.
2. If W accepts, then Accept; otherwise, Reject

By lemma 3.3, the machine H decides $HALT_{TM}$; contradiction. □

We have just proved the unsolvablity of the word problem for finitely presented *semi*-gorups; however, the main result that we will need later is the unsolvablity of the word problem for finitely presented groups. Novikov, et. al. proved this result (see [7] [8] [9]) by algebraically reducing the problem in groups to the problem in semi-groups which is outside the scope of this paper. An important corollary of the unsolvability of the word problem for finitely presented groups is the undecidability of deciding whether a finite presentation of a group is the trivial group (see [15]).

4. CW-complixes and Van-Kampen's Theorem

Now we go through the topological background here. We start with the concept of CW-complecies. A CW-complex is a space built out of smaller spaces, iteratively by a process called ***attaching cells***. A k***-cell*** is $k-$dimensional disc

$$D^k = \{x \in \mathbb{R}^k : |x| \leq 1\}$$

Intuitively, attaching a $k-$cell to another space X means the union of X and D^k with *gluing* the boundary of D^k to X. The following definition formalizes these notions.

Definition 4.1. Let X be a space and let D^k be a $k-$dimensional disc. Let $\phi : \partial D^k \to X$ be a continuous map. Consider the space $X \cup_\phi D^k = (X \amalg D_k)/\sim$, where the equivalence relation is $x \sim \phi(x) \in X$ for $x \in \partial D^k$. We call $X \cup_\phi D^k$ the result of attaching a k-cell to X along the map ϕ

Now, we formally define CW-complecies.

Definition 4.2. A ***CW-complex*** is any topological space X built in the following way:

(i) Start with a discrete space X^0, called the *0-skeleton*.

(ii) Attach 1-cells e^1 by attaching maps $\partial e^1 \to X^0$. The result is the *1-skeleton* X^1.

(iii) Attach 2-cells e^2 by attaching maps $\partial e^2 \to X^1$. The result is the *2-skeleton* X^2.

(iv) Continue in this way, we get a sequence of skeletons

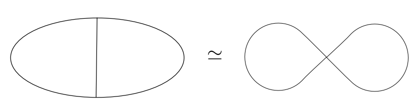
$$X^0 \subset X^1 \subset \cdots \subset X^n \subset \cdots$$

(v) Now let $X = \bigcup_{n \geq 0} X^n$ and equip it with the *weak topology*.

Notice that it's possible to add no k-cells for a certain k. For example, a sphere is a 2-CW-complex formed by attaching a sphere, a 2-cell, with a point.

CW-complexes have a very important topological property that we will use, the homotopy extension property. Using this property, we can show that different spaces are homotopy equivalent. For example, the figure eight and theta shape are homotopically equivalent by "squeezing" the middle line in the theta shape (see Fig. 1.). First we define this property.

FIGURE 1. Homotopy equivalence of the figure eight and theta shape



Definition 4.3. Given a space X and a subspace A, we say that the pair (X, A) has the homotopy extension property HEP if for every continuous map $F : X \to Y$ and every homotopy $h : A \times [0, 1] \to Y$ with $h(x, 0) = F(x)$ there exists a homotopy $H : X \times [0, 1] \to Y$ such that $H(x, 0) = F(x)$ and $H|_{A\times[0,1]} = h$.

The previous definition just says that if we have a homotopy on a subspace, then we can extend this homotopy to the entire space. The following formalizes the idea of proof by "squeezing" that we used in the previous example.

Lemma 4.4. *If (X, A) has the HEP and A is contractible, then $X \simeq X/A$.*

Proof. To show that $X \simeq X/A$, we need to find two functions $p : X \to X/A$ and $q : X/A \to X$ such that $p \circ q = \text{id}_X$ and $q \circ p = \text{id}_Y$. Let p be the canonical projection into the quotient space. Now we find the function q. Since the space

A is contractible, then there exists a point $a \in A$ and a homotopy $h_t : A \to A$ such that $h_0(x) = \mathrm{id}_A$ and $h_1(x) = a$ for all $x \in A$. Using the HEP of (X, A), there exists a homotopy $H_t : X \to X$ such that $H_0 = \mathrm{id}_X$ and $H_t|_A = h_t$. Since $h_1(A) = \{a\}$, then $H_1(A) = \{a\}$. This means that H_1 can be written as $H_1 = q \circ p$ (i.e. H_1 descends to the quotient) for some continuous map $q : X/A \to X$. Now notice that $q \circ p = H_1 \simeq H_0 = \mathrm{id}_X$. Moreover, notice that for all $t \in [0, 1]$

$$(p \circ H_t)(A) = p(h_t(A)) \subset p(A)$$

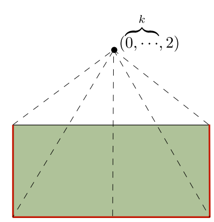
where $p(A)$ is a single point. Therefore, $p \circ H_t$ descends to the quotient for all $t \in [0, 1]$. This means that for each t, there exists a continuous map H'_t such that $p \circ H_t = H'_t \circ p$. Then for $t = 1$, $p \circ q \circ p = H'_1 \circ p$ which implies $p \circ q = H'_1$ since p is surjective. Hence, $p \circ q = H'_1 \simeq H'_0 = \mathrm{id}_{X/A}$ which completes the proof. □

Now to completely justify the proof by "squeezing" that we used in the previous example, we show that CW-complexes have the HEP.

Lemma 4.5. *If e is a $k-$dimesnioal disc then there is a continuous map $e \times [0, 1] \to (\partial e \times [0, 1]) \cup (e \times \{0\})$.*

Proof. We embed $e \times [0, 1]$ into $\mathbb{R}^{k+1}$ as the subset $\{(x, y) \in \mathbb{R}^k \times \mathbb{R} : |x| \leq 1, y \in [0, 1]\}$. Look at figure 2 for an example. Our desired mapping is the linear projection from a point $(\underbrace{0, \cdots,}_{k} 2)$ to $(\partial e \times [0, 1]) \cup (e \times \{0\})$ □

Figure 2. Example of the linear projection for $k = 1$



Lemma 4.6. *If X is a space obtained from A by attaching a $k-$cell e, then the pair (X, A) has the HEP.*

Proof. Notice that $X = A \cup_\phi e$. Given a homotopy $h_t : A \to A$ and continuous mapping $F : X \to Y$ such that $h_0 = F|_A$. We want to extend h_t to a homotopy $H_t : X \to X$ such that $H_t|_A = h_t$ and $H_0 = F$. It's only left to define H on $e \times [0, 1]$. Let $G : e \times [0, 1] \to (\partial e \times [0, 1]) \cup (e \times \{0\})$ be the map from the previous lemma. Then define H on $e \times [0, 1]$ as $h \circ G$. Notice that G takes points of $e \times [0, 1]$ into the region $(\partial e \times [0, 1]) \cup (e \times \{0\})$ which is attached to $(A \times [0, 1]) \cup (e \times \{0\})$ on which H is already defined. This completes the proof. □

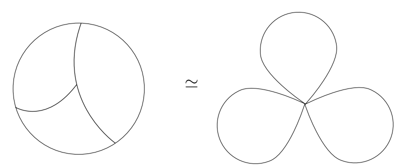
Theorem 4.7. *If X is a CW-complex and A is a subcomplex then the pair (X, A) has the HEP.*

Proof. The theorem follows from the previous two lemmas inductively by the construction of X from A by adding cells. □

Corollary 4.8. *Any connected 1-dimensional CW-complex X is homotopy equivalent to a wedge of circles i.e. a single 0-cell with a bunch of 1-cells attached.*

Proof. The strategy is to find a closed subcomlpex A that contains all the 0-cells. First we order contractible subcomlpexes partially by inclusion. By Zorn's lemma, there is a maximal contractible subcomlpex with respect to this partial order. Notice that a maximal contractible subcomplex A should pass through all 0-cells; otherwise, we could add an edge connecting A to a 0-cell it doesn't connect to obtain a bigger contractible subcomlpex. Now, by previous theorem and HEP, $X \simeq X/A$ which is a wedge of circles. □

FIGURE 3. A 2-CW-complex homotopically equivalent to a wedge of 3 circles.



Now we introduce Van Kampen's Theorem. It's one of the important results in algebraic topology. It expresses the structure of the fundamental group of a topological X space in terms of the fundamental groups of two open subspaces that cover X. This theorem will be used multiple times in the next section to prove the main results of the paper.

Theorem 4.9. *Let X be a topological space and let $U, V \subset X$ be open subsets such that $U \cap V$ is non-empty and path-connected. Let $x \in U \cap V$ be a base point. Then*

$$\pi_1(X, x) = \pi_1(U, x) \star_{\pi_1(U \cup V, x)} \pi_1(V, x).$$

Here $A \star_C B$ is the amalgmated product. Suppose we have A, B, C groups and homomorphism $f : C \to A, g : C \to B$, then

$$A \star_C B = \langle \text{generators of } A, \text{ generators of } B | R(A), R(B), \text{amalgmated relations} \rangle$$

The amalgamated relations are the relations that come from elements $c \in C$: each $c \in C$ induces a relation $f(c) = g(c)$. In our case, $A = \pi_1(U, x), B = \pi_1(V, x)$, and $C = \pi_1(U \cap V, x)$. The homomorphisms $i : \pi_1(U \cap V, x) \to \pi_1(U, x), j : \pi_1(U \cap V, x) \to \pi_1(V, x)$ are the homomorphisms induced by inclusions.

5. UNDECIDABILITY OF SIMPLE CONNECTEDNESS OF MANIFOLDS AND CW-COMPLEXES

First we prove our first interesting result: the problem of deciding which 2-CW-complexes are simply connected is undecidable. We start with the following lemma about the fundamental group of CW-complecies.

Lemma 5.1. *If X is a CW-complex with one 0-cell then*

$$\pi_1(X) = \langle \textit{generators of 1-cells} \mid \textit{relations induced by 2-cells} \rangle$$

Proof. Let X^1 be the 1-skeleton of X. It is obtained by attaching 1-cells to a point, so X^1 is just a wedge of 1-cells $X^1 = \bigvee_{\text{1-cells}} S^1$. Since $\pi_1(S^1) = \mathbb{Z}$, then by induction $\pi_1, (X^1) = \bigstar_{\text{1-cells}}\mathbb{Z}$ (This is just the group with one generator for each 1-cell and no relations). Now we notice what happens when we attach 2-cells.

Claim. *If Y is any space and Z is obtained from Y by attaching a 2-cell e with some map ϕ, then $\pi_1(Z) = \pi_1(Y)/N(\partial e)$, where $N(\partial e)$ is the normal subgroup generated by the boundary of e (This is equivalent to adding the relation $\partial e = 1$ to the relations of Y).*

Proof of claim. Decompose $Z = U \cup V$, where $U = int(e)$ and $V = Y \cup (e/\{0\})$ (V is everything except a point in the interior of e). Notice that: (i) $U \cap V = e\ \{0\}$, a punctured 2-cell, which is homotopically equivalent to S^1, (ii) U is contractible, and (iii) V is homotopy equivalent to Y by retracting the punctured 2-cell down into its boundary which is attached to Y. Now by applying Van-Kampen's theorem we get that $\pi_1(Z) = \pi_1(Y) \star_{\mathbb{Z}} \{1\} = \pi_1(Z)/N(\partial e)$, in other words $\pi_1(Z)$ has the same group presentation as Y but with the relation $\partial e = 1$ added.

Moreover, adding higher dimensional cells doesn't change the fundamental group. This follows from the same argument in the claim, when e is n-cell with $n > 2$ we get that $\pi_1(Z) = \pi_1(Y) \star_{\{1\}} \{1\} = \pi_1(Y)$ because $U \cap V$, in this case, is homotopy equivalent to S^{n-1} which is simply connected for $n > 2$. □

Corollary 5.2. *Given a finitely presented group G, we can construct a 2-dimensional CW-complex whose fundamental group is G.*

Proof. The construction is basically reverse engineering of the result of the previous lemma. Let $G = \langle s_1, s_2, \cdots, s_n | r_1, r_2, \cdots, r_m\rangle$. We construct the CW-complex by taking a wedge of circles, one for each $s_i \in S$. For each relation $r_l = s_1^{k_1} s_2^{k_2} \cdots s_n^{k_n}$, we attach a 2-cell with an attachment map that goes as follows: it should attach the 2-cell to the boundary of s_1 loop such that the boundary of the 2-cell goes around the loop s_i k_i times in one direction if $k_i > 0$ and in the opposite direction if $k_i < 0$. □

Now we prove the undecidability of the simple connectedness of 2-dimensional CW-complecies.

Theorem 5.3. *The set of simply connected 2-dimensional CW-complecies is undecidable.*

Proof. Suppose it's decidable. Let C be a Turing machine that decides it. Now consider the following turing machine.

$W =$ On input $\langle G\rangle$, where G is a finite presentation of a group :

1. Run C on the 2-CW-complex X whose fundamental group is G.
2. If C accepts, then Accept; otherwise, Reject

If C accepts, this means that X is simply connected which implies that G is the trivial group. This means that W decides the triviality problem; contradiction. □

Now we move on to prove our next interesting result: the undecidability of manifold simple connectedness. We will do the same thing as we did with CW-complecies. We will show that given a finite presentation of a group G, we can construct a manifold whose fundamental group is G. Then using the same argument in the previous theorem, the undecidablility follows. First we need to define connected sums of manifolds.

Definition 5.4. Let A^n, B^n be $n-$manifolds. Let D^n be a closed $n-$disc. Let $\alpha : D^n \to A^n, \beta : D^n \to B^n$ be continuous injections. Define $S = (A^n \setminus int(\alpha(D^n))) \sqcup (B^n \setminus int(\alpha(D^n)))$. Define the following equivalence relation $\sim$ on S as :

$$x \sim y \iff ((x = y) \text{ or } (\alpha^{-1}(x) = \beta^{-1}(y)))$$

Since the interiors of the coordinate disks were removed from the manifold, then it follows that $\alpha^{-1}(x), \beta^{-1}(y) \in \partial D^n$. The connected sum $A^n \# B^n = S/\sim$.

Now we prove the following lemma about the fundamental group of the connected sum of two manifolds.

Lemma 5.5. *Let M, N be to $n-$manifolds for $n > 2$. The fundamental group of their connected sum is $\pi_1(M\#N) = \pi_1(M) \star \pi_1(N)$*

Proof. We can write $M = U_1 \cup U_2$, where U_1 is M without a point inside the open ball we removed to perform the connected sum and U_2 is the open ball removed for the connected sum. Similarly, We can write $N = V_1 \cup V_2$. Moreover, we can write $M\#N = W_1 \cup W_2$, where W_1 is homotopically equivalent to U_1 and W_2 is homotopically equivalent to V_1. Additionally, W_1 (or W_2) covers U_1 (or V_1) plus a small collar neighborhood of the glued sphere which is the boundary of V_1 (or U_1). Notice that, $\pi_1(U_1 \cap U_2) \simeq \pi_1(V_1 \cap V_2) \simeq \pi_1(W_1 \cap W_2) \simeq \pi_1(S^{n-1}) \simeq 0$ because they are all homotopically equivalent. Applying Van-Kampen's we get that, $\pi_1(U_1) \simeq \pi_1(M)$ and $\pi_1(V_1) \simeq \pi_1(N)$. Applying Van-Kampen's again we get

$$\begin{aligned}\pi_1(M\#N) &\simeq \pi_1(W_1) \star \pi_1(W_2) \\ &\simeq \pi_1(U_1) \star \pi_1(V_1) \\ &\simeq \pi_1(M) \star \pi_1(N)\end{aligned}$$

□

Now we present the main construction. Given a group presentation $G = \langle S|R\rangle$, we want to construct a manifold whose fundamental group is G. The main idea here is that we construct a manifold whose fundamental group is the free group on the generators S. Then, we start "killing" the loops that represent the relations R using a technique called manifold surgery.

Surgery is a procedure for changing manifold into another (of the same dimension n) by excising a copy of $S^r \times D^{n-r}$ for some r, and replacing it by $D^{r+1} \times S^{n-r-1}$, which has the same boundary, $S^r \times S^{n-r-1}$. Now the intuition of "killing" a loop is the following. Let $w_j \in R$ that's represented by some loop b_j. If we perform surgery on some tubular neighborhood around b_j, then b_j will live in a simple connected region in the new manifold, hence it will be a nullhomotopic loop. Now we carry out this intuition rigorously.

Theorem 5.6. *Given a group presentation $G = \langle S|R\rangle$, we can construct an n-manifold with $n \geq 4$ whose fundamental group is G.*

Proof. We start with the n-manifold $X = \overbrace{(S^{n-1} \times S^1)\# \cdots \#(S^{n-1} \times S^1)}^{|S|}$. Using the previous lemma, we have that $\pi_1(X) = \langle a_1, a_2, \cdots, a_{|S|}|\rangle$. Each relation can be represented by a loop in X. Let $r_j = a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n} \in R$. This relation is represented by a loop that goes as follows: for each i, the loop goes around the S^1-factor of the i^{th} $S^{n-1} \times S^1$ summand k_i times (we should pick an orientation for each a_i so that if k_i is negative then the loop goes in the opposite direction). Let b_j be the loop that represents the relation r_j. Each loop b_j has a tubular neighborhood N_j, which is a copy of $S^1 \times D^{n-1}$ embedded in X. The boundary of N_j is homeomorphic to $S^1 \times S^{n-2}$. Note that $D^2 \times S^{n-2}$ has boundary $S^1 \times S^{n-2}$ as well. By cutting out the open neighborhood N_j and attaching $D^2 \times S^{n-2}$ along the boundary $S^1 \times S^{n-2}$, we get a new manifold $X_j = (X \setminus int(N_j)) \cup_{S^1 \times S^{n-2}} (D^2 \times S^{n-2})$. Then applying Van-Kampen's we get,

$$\pi_1(X_j) \simeq \pi_1(X \setminus int(N_j)) \star_{\pi_1(S^1 \times S^{n-2})} \pi_1(D^2 \times S^{n-2})$$

Claim. $\pi_1(X) \simeq \pi_1(X \setminus int(N_j)) \simeq \langle a_1, \cdots, a_{|S|}|\rangle$.

Proof of claim. Consider a regular collar neighborhood K of the boundary $\partial(S^1 \times D^{n-1})$ from inside such that it deformation retracts onto the boundary. Let $U = X \setminus int(S^1 \times D^{n-1}) \cup K$, and $V = int(S^1 \times D^{n-1})$. Now applying Van-Kampen's theorem,

$$\begin{aligned}\pi_1(X) &\simeq \pi_1(X \setminus int(S^1 \times D^{n-1})) \star_{\pi_1(S^1 \times S^{n-2})} \pi_1(int(S^1 \times D^{n-1})) \\ &\simeq \pi_1(X \setminus int(S^1 \times D^{n-1})) \star_{\mathbb{Z}} \mathbb{Z}\end{aligned}$$

But since $n \geq 4$, then the generator of $\pi_1(S^1 \times S^{n-2})$ is the same as the generator of $\pi_1(int(S^1 \times D^{n-1}))$, which implies

$$\pi_1(X) \simeq \pi_1(X \setminus int(S^1 \times D^n - 1)) \simeq \pi_1(X \setminus int(N_j))$$

. Hence, the claim.

Then we have,

$$\begin{aligned}\pi_1(X_j) &\simeq \pi_1(X \setminus int(N_j)) \star_{\pi_1(S^1 \times S^{n-2})} \pi_1(D^2 \times S^{n-2}) \\ &\simeq \langle a_1, \cdots, a_{|S|}|\rangle \star_{\langle b_j|\rangle} \{1\} \\ &\simeq \langle a_1, \cdots, a_{|S|}|b_j\rangle\end{aligned}$$

Doing the same process for all relations $r_j \in R$, we obtain a new manifold such that

$$\pi_1(X') \simeq \langle a_1, \cdots a_{|S|}|b_1, \cdots, b_{|R|}\rangle \simeq G$$

□

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