ON THE COCOMPLETENESS OF *Cat*

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Abstract. This paper provides a self-contained account of the cocompleteness of *Cat*, the category of small categories, by embedding it in the presheaf category of simplicial sets.

Contents

1. Introduction

This paper will serve as a brief exploration of basic categorical concepts as we prove the completeness and cocompleteness of *Cat*. We will prove in Section 5 that it suffices to show that a category contains all its (co)products and (co)equalizers to prove (co)completeness.

The completeness of *Cat* will be immediate, as we can explicitly construct products and equalizers. However, the explicit construction of coequalizers is not as straightforward and we choose a different route to prove cocompleteness.

In Section 6, we explore the functor category of simplicial sets and show that it is cocomplete by virtue of being a presheaf category. We prove that right adjoints preserve limits and the dual and use these results to show that if a category is cocomplete, its reflective subcategories would also be cocomplete.

It only remains to prove that *Cat* is a reflective subcategory of **sSet**. We explicitly construct the nerve and Π functors in Section 7 to do so, after which we obtain a complete proof of cocompleteness of *Cat*.

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2. Basic Category Theory

Category theory provides a language which allows us to abstract out some of the contextually negligible details to establish similarities between concepts in different areas of mathematics. In a sense, it is a zooming-out, where the resolution is just good enough to view critical details, allowing us to look at a bigger picture. It deals with objects on a larger scale than what one usually does, and builds an environment to observe how objects interact with other objects in that environment.

It also provides important proof-writing tools like diagram chasing and ideas of duality, 'abstract nonsense' as Steenrod puts it, that provide us with a concrete language to prove abstract ideas of category theory.

Saunders MacLane and Samuel Eilenberg discovered the universal coefficient theorem of algebraic topology that relates homology and cohomology groups of a space X via a group extension:

$$0 \to \mathrm{Ext}(H_{n-1}(X), G) \to H^n(X, G) \to \mathrm{Hom}(H_n(X), G) \to 0$$

To generalize this theorem, it needed to be shown that the above group homomorphisms are natural between continuous maps of topological spaces. Mathematicians have used the term 'natural' colloquially to indicate that something was 'defined without making any arbitrary choices.' In trying to rigorously prove this generalization, Eilenberg and MacLane set to formalize this notion of naturality. To do that, they introduced the concept of natural transformations. They then introduced the concept of functors to describe the source and target of these transformations and categories to describe the target and source of functors.

2.1. **Categories, Functors and Natural Transformations.**

Sets are the most basic collection of items in mathematics, but depending on what we need the elements for, we often need more sophisticated versions than simple collections of items. Enforcing an operation on a set gives a group and imposing a set of open sets, a topology, on a set gives a topological space. The different structures make these resultant collections of objects useful in different ways.

Often, it is useful to look at how objects relate to each other. It then makes sense to look at collections of objects and the morphisms between them.

Definition 2.1. A category $\mathcal{C}$ consists of:

- a collection of objects $ob(\mathcal{C})$
- for all $A, B \in ob(\mathcal{C})$, a collection of morphisms $Hom(A, B)$
- a **composition** function that defines for any two morphisms $f \in Hom(A, B)$ and $g \in Hom(B, C)$, a morphism $g \circ f \in Hom(A, C)$
- there exists an **identity** morphism id_A for all $A \in ob(\mathcal{C})$

satisfying the following axioms:

- composition is **associative**, meaning for any three morphisms $f \in Hom(A, B)$, $g \in Hom(B, C)$, $h \in Hom(C, D)$, $(h \circ g) \circ f = h \circ (g \circ f)$
- for any morphism $f : X \to Y$, $f \circ id_X = f = id_Y \circ f$

We can construct categories out of sets as objects with functions as the morphisms, topological spaces with continuous functions, groups with group homomorphisms and rings with ring homomorphisms etc. In this way, categories offer a great deal of abstraction and allow us to study large collections of objects at the same time.

Definition 2.2. A functor $\mathcal{F}$ defined from category $\mathcal{A}$ to $\mathcal{B}$ consists of:

- $Fa \in ob(\mathcal{B})$ for all $a \in ob(\mathcal{A})$
- $Ff \in mor(\mathcal{B})$ for all $f \in mor(\mathcal{A})$

satisfying the following axioms:

- composition is preserved, meaning for morphisms $f : A \to B$ and $g : B \to C$, $Fg \circ Ff = F(g \circ f)$
- identities are preserved, meaning for any object $a, F_{id_a} = id_{Fa}$

It is natural to think about categories as objects in their own right, so we could try to construct categories that have categories as objects and functors as the morphisms between them.

Definition 2.3. A category C is called small if $ob(C)$ and $mor(C)$ are sets. A category C is locally small if for any two objects A, B in C, $Hom(A, B)$ is a set.

Definition 2.4. We define Cat as the category of small categories and functors between them.

Such a category, whose objects are categories is called a 2-category. We can keep building up in a similar manner and eventually reach an $\infty-$category.

Definition 2.5. A natural transformation η between functors $\mathcal{F}, \mathcal{G} : \mathcal{A} \to \mathcal{B}$ is defined as a collection of morphisms η_x for each $x \in \mathcal{A}$, such that for any $f : X \to Y$ in $\mathcal{A}$, the following diagram commutes:

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ GX & \xrightarrow[Gf]{} & GY \end{array}$$

Natural transformations generalize the idea of homotopies to the categorical level.

2.2. **Duality.**
The notion of an opposite category equips us with a concrete way to talk about the principle of duality in category theory.

Definition 2.6. Let $\mathcal{C}$ be any category. Then, the opposite category $\mathcal{C}^{op}$ is defined as:

- $ob(\mathcal{C}) = ob(\mathcal{C}^{op})$
- $mor(\mathcal{C}^{op})$ is obtained by reversing all arrows in $mor(\mathcal{C})$. For every $f \in mor(\mathcal{C})$, $mor(\mathcal{C}^{op})$ contains f^{op} such that the domain of f is the codomain of f^{op} and the codomain of f is the domain of f^{op}.

$$f^{op} : X \to Y \in \mathcal{C}^{op} \rightsquigarrow f : Y \to X \in \mathcal{C}$$

Whenever we have a theorem that is true for all categories, it is necessarily true for the opposite categories. So, the statement for each theorem gives us the statement for a dual theorem obtained by reversing all the morphisms and the proof for any theorem gives us a proof for the dual theorem, similarly obtained by reversing all the arrows.

Definition 2.7. A contravariant functor $\mathcal{F}$ from $\mathcal{C}$ to $\mathcal{D}$ is defined from the opposite category $\mathcal{C}^{op}$ to $\mathcal{D}$.

To stress that a functor is not contravariant, they are sometimes called covariant functors.

2.3. **Presheaves.**

Definition 2.8. For any category $\mathcal{C}$, a $\mathcal{D}-$valued presheaf is a contravariant functor $\mathcal{F} : C \to \mathcal{D}$. In particular, a set-valued presheaf for a category C is the functor $\mathcal{F} : \mathcal{C}^{op} \to Set$.

Definition 2.9. A presheaf category is the functor category with set-valued presheaves as objects and the natural transformations between them as morphisms.

So, for any small category, we have its presheaf category:

$$PSh(\mathcal{C}) := [\mathcal{C}^{op}, Set]$$

3. Limits and Colimits

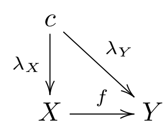
Diagrams are a key way to prove important results in category theory. We will formalize the notion of a diagram and define limits and colimits on diagrams.

Definition 3.1. A diagram in category $\mathcal{C}$ is a functor $\mathcal{D} : \mathcal{I} \to \mathcal{C}$, where $\mathcal{I}$ is any small category called the indexing category.

Definition 3.2. A constant functor on an object $c \in ob(\mathcal{C})$ from a category $\mathcal{I}$ takes all objects in $\mathcal{I}$ to c and all morphisms to the identity morphism id_c.

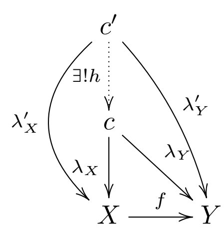
Definition 3.3. A cone with vertex c, over a diagram $\mathcal{D}$ of a category $\mathcal{C}$ is a natural transformation $\lambda : c \Rightarrow \mathcal{D}$, where the domain is the constant functor at c. The components λ_i are called the legs of the cone.

In other words, for any diagram in a category, a cone over it consists of a vertex and morphisms from the vertex to every element in the diagram, so that for every $f : X \to Y$ in the diagram, the following commutes:



Definition 3.4. The universal cone over a diagram is called the limit of the diagram.

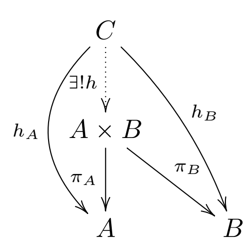
Here, universality means that for any other cone over the diagram with vertex c', there exists a unique factorization $h : c' \to c$, where c is the vertex of the limit.



Consider the following important examples of limits:

Definition 3.5. The limit over the diagram containing two objects is called the product of the two objects.

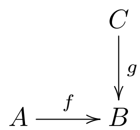
The product of objects A and B is an object $A \times B$ equipped with projection maps π_A, π_B, such that for any other C with projection maps h_A, h_B, there exists a unique factorization $h : C \to A \times B$, such that the following diagram commutes.



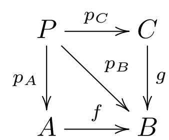
For two sets A and B, the categorical product turns out to be the Cartesian product equipped with projection maps.

Definition 3.6. The limit over a diagram with two morphisms having common codomain is defined to be pullback over the diagram.

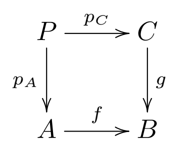
Take the following diagram:



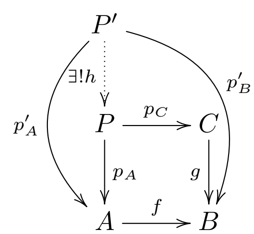
The pullback over the diagram is an object P, with morphisms to A, B, C such that the diagram commutes:



This implies that $p_B = f \circ p_A$ and $p_B = g \circ p_C$, which is equivalent to $f \circ p_A = g \circ p_C$, so the diagram that needs to commute simplifies to:

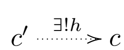


P is also such that for any other object P' with morphisms to A and C, there exists a unique factorization h from P' to P.



Definition 3.7. The limit over the empty diagram is defined as a terminal object in the category.

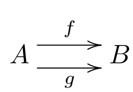
Let c be a terminal object in a category $\mathcal{C}$. Then, for any other object c' in the category, there exists a unique morphism $h : c' \to c$. Since the diagram is empty, the commuting conditions vanish.



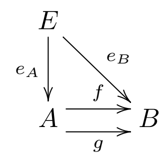
In other words, an object is called a terminal object in the category if there exists a unique morphism from any other object in the category to it.

Definition 3.8. The limit over a diagram containing two parallel morphisms is called the equaliser of the morphisms.

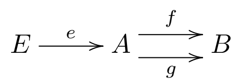
Consider the following diagram:



Then, the equalizer over it is E with morphisms e_A, e_B such that the following diagram commutes:



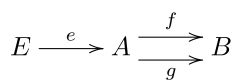
So, we have $f \circ e_A = e_B$ and $g \circ e_A = e_B$, which implies $f \circ e_A = g \circ e_A$. Take $e = e_A$ and we get that the following diagram must commute:



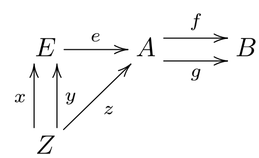
Definition 3.9. A left cancellative morphism is called a monomorphism. In other words, a morphism $f : A \to B$ is a monomorphism if for any two morphisms $x, y : C \to A$ such that $f \circ x = f \circ y$, we have $x = y$.

Proposition 3.10. *Equalisers are monomorphisms.*

Proof. Consider the following diagram:



Take morphisms $x, y : Z \to E$, such that $e \circ x = e \circ y$. Then, we have a morphism $z : Z \to A$, where $z = e \circ x = e \circ y$.



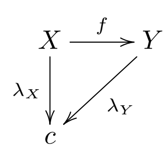
Now, $f \circ z = f \circ e \circ x = g \circ e \circ x = g \circ z$. Then, z is an equalizer of the morphisms f, g. By the universal property of E, there exists a unique morphism $h : Z \to E$, which implies that $x = y = h$.

□

There exist dual notions to cones, limits and the specific examples of limits discussed. These notions can be defined by reversing the directions of the morphisms in the diagrams.

Definition 3.11. A cocone over a diagram, or a cone under a diagram $\mathcal{D}$ with nadir c is a natural transformation $\lambda : \mathcal{D} \Rightarrow c$, where the codomain is the constant functor at c.

In other words, for any diagram in a category, a cone under it consists of a vertex and morphisms to the vertex from every element in the diagram, so that for every $f : X \to Y$ in the diagram, the following commutes:

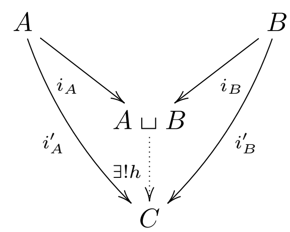


Definition 3.12. The universal cone under a diagram is called the colimit of the diagram.

Consider some important examples of colimits:

Definition 3.13. The colimit over the diagram containing two objects is called the coproduct of the two objects.

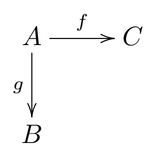
The coproduct of objects A and B is an object $A \sqcup B$ equipped with inclusion maps i_A, i_B, such that for any other C with inclusion maps i'_A, i'_B, there exists a unique factorization $h : A \sqcup B \to C$, such that the following diagram commutes.



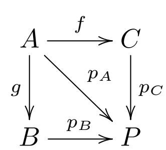
For two sets A and B, the coproduct turns out to be the disjoint union of the sets equipped with the canonical inclusion maps.

Definition 3.14. The colimit over a diagram with two morphisms having common domain is defined to be pushout over the diagram.

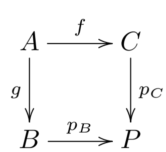
Take the following diagram:



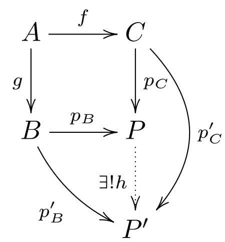
The pushout over the diagram is an object P, with morphisms from A, B, C to P such that the diagram commutes:



This implies that $p_A = p_B \circ g$ and $p_A = p_C \circ f$, which is equivalent to $p_B \circ g = p_C \circ f$, so the diagram that needs to commute simplifies to:



P is also such that for any other object P' with morphisms from B and C, there exists a unique factorization h from P to P'.



Definition 3.15. The colimit over the empty diagram is defined as an initial object in the category.

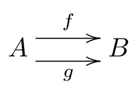
Let c be an initial object in a category $\mathcal{C}$. Then, for any other object c' in the category, there exists a unique morphism $h : c \to c'$. Since the diagram is empty, the commuting conditions are trivial.

$$c \overset{\exists! h}{\dashrightarrow} c'$$

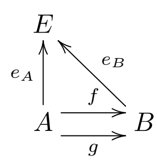
In other words, an object is called an initial object in the category if there exists a unique morphism from the object to any other object in the category.

Definition 3.16. The colimit over a diagram containing two parallel morphisms is called the coequaliser of the morphisms.

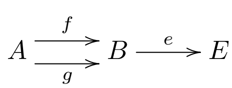
Consider the following diagram:



Then, the coequalizer over it is E with morphisms e_A, e_B such that the following diagram commutes:



So, we have $e_B \circ f = e_A$ and $e_B \circ g = e_A$, which implies $e_B \circ f = e_B \circ g$. Take $e = e_B$ and we get that the following diagram must commute:



Definition 3.17. A right cancellative morphism is called an epimorphism, that is, a morphism $f : A \to B$ is an epimorphism, if for any two morphisms $g, h : B \to C$ such that $g \circ f = h \circ f$, we have $g = h$.

Proposition 3.18. *Coequalisers are epimorphisms.*

Proof. Dual of Proposition 3.10. □

Definition 3.19. A category is complete if it contains all of its limits. Dually, a category is cocomplete if it contains all of its colimits.

4. Adjunctions & Subcategories

Definition 4.1. An adjunction is a pair of functors $\mathcal{F}, \mathcal{G} : \mathcal{A} \to \mathcal{B}$, such that there exists an isomorphism:

$$\mathcal{D}(\mathcal{F}c, d) \cong \mathcal{C}(c, \mathcal{G}d)$$

for all $c \in \mathcal{C}, d \in \mathcal{D}$, where the isomorphism is natural in both variables.
Here, naturality in $\mathcal{D}$ means that for all morphisms $k : d \to d'$ in $\mathcal{D}$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{D}(\mathcal{F}c, d) & \xrightarrow{\cong} & \mathcal{C}(c, \mathcal{G}d) \\ {\scriptstyle k_*}\downarrow & & \downarrow{\scriptstyle \mathcal{G}k_*} \\ \mathcal{D}(\mathcal{F}c, d') & \xrightarrow{\cong} & \mathcal{C}(c, \mathcal{G}d') \end{array}$$

and dually in $\mathcal{C}$, for all morphisms $h : c \to c'$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{D}(\mathcal{F}c, d) & \xrightarrow{\cong} & \mathcal{C}(c, \mathcal{G}d) \\ {\scriptstyle \mathcal{F}h_*}\downarrow & & \downarrow{\scriptstyle h_*} \\ \mathcal{D}(\mathcal{F}c', d) & \xrightarrow{\cong} & \mathcal{C}(c', \mathcal{G}d) \end{array}$$

Here, $\mathcal{F}$ is a left adjoint to $\mathcal{G}$ and $\mathcal{G}$ is a right adjoint to $\mathcal{F}$.
The corresponding morphisms $f^{\#}$ and f^{b} are called transposes or adjuncts.

$$\mathcal{F}c \xrightarrow{f^{\#}} d \rightsquigarrow c \xrightarrow{f^{b}} \mathcal{G}d$$

An important family of adjunctions are the free and forgetful functors.

$$A \underset{F}{\overset{U}{\underset{\bot}{\rightleftarrows}}} S$$

The forgetful functor U is the right adjoint to the free functor F.
Take the specific example where A is a group and S is the underlying set. Then, the forgetful functor does exactly what you would expect it to do. It forgets the overlying structure, which is the group operation and gives the underlying set for the group. The free functor returns the free group generated by the set S.

Proposition 4.2. *Given an adjunction $F \dashv G$, there exists a natural transformation $\eta : 1_C \Rightarrow GF$, called the **unit** of the adjunction. The components of the natural transformation $\eta_c : c \to GFc$ at c are defined to the transpose of the identity morphism id_{Fc}.*
*Dually, there exists a natural transformation $\epsilon : FG \to 1_D$, called the **counit** of the adjunction. The components of the natural transformation $\epsilon : FGd \to d$ at d are defined to the transpose of the identity morphism id_{Gd}.* [1]

Definition 4.3. A subcategory of a category is a sub-collection of objects and morphisms of the category such that it contains:

- the domains and codomains of all the morphisms
- the composite of all composable pairs of morphisms in the subcategory
- the identity morphism for all objects in the subcategory

In other words, a subcategory is a subcollection of objects and morphisms of a category such that it is a category.

Definition 4.4. A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is:

- **full** if the map $\mathcal{C}(x, y) \to \mathcal{D}(\mathcal{F}x, \mathcal{F}y)$ is surjective
- **faithful** if the map $\mathcal{C}(x, y) \to \mathcal{D}(\mathcal{F}x, \mathcal{F}y)$ is injective
- **fully faithful** if it is both full and faithful
- **essentially surjective** on objects if for every object $c \in \mathcal{D}$, there exists an object $d \in \mathcal{D}$, such that $\mathcal{F}c \cong d$

Fullness and faithfullness are local conditions and may not be surjective or injective on morphisms respectively but we can define corresponding global conditions.

Definition 4.5. A faithful functor that is injective on objects is called an embedding and a fully faithful functor that is injective on objects is called a full embedding.

In case of an embedding, the functor will be injective on morphisms.

Definition 4.6. Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ define a full embedding of $\mathcal{A}$ in $\mathcal{B}$. Then $\mathcal{A}$ is a full subcategory of $\mathcal{B}$.

Definition 4.7. A reflective subcategory $\mathcal{D}$ of a category $\mathcal{C}$ is a full subcategory such that its inclusion admits a left adjoint, also called the reflector or the localisation.

$$\mathcal{D} \underset{i}{\overset{L}{\underset{\hookrightarrow}{\overset{\leftarrow}{\perp}}}} \mathcal{C}$$

Proposition 4.8. *Consider an adjunction*

$$A \underset{G}{\overset{F}{\underset{\leftarrow}{\overset{\rightarrow}{\perp}}}} B$$

with counit $\epsilon : FG \Rightarrow id_B$ and unit $\eta : id_A \Rightarrow GF$. G is fully faithful if and only if every component of ϵ is an isomorphism.
Dually, F is fully faithful if and only if every component of η is an isomorphism.

Proof. We only prove the first statement and the dual case will follow just by reversing all the morphisms.
$(\Rightarrow)$
We need to show that there exists a morphism $\epsilon_b^{-1} : b \to FGb$ for all $b \in B$, such that:

$$\epsilon_b \circ \epsilon_b^{-1} = \epsilon_b^{-1} \circ \epsilon_b = id$$

Suppose not. Since $B(x, y) \to A(Gx, Gy)$ is bijective, there cannot exist a map of the form $G\epsilon_b^{-1} : Gb \to GFGb$. But, for $Gb \in A$, we have the component of the unit $\eta_{Gb} : Gb \to GFGb$. Contradiction.
$(\Leftarrow)$
In order to prove that $B(x, y) \to A(Gx, Gy)$ is bijective, we show that we have the composite isomorphism:

$$B(x, y) \to B(FGx, y) \to A(Gx, Gy)$$

The first isomorphism is given by pre-composition of the maps $f : x \to y$ with ϵ_x. So, we obtain a map $f \circ \epsilon_x : FGx \to x \to y$. For any map $g : FGx \to y$, we have a map $g \circ \epsilon_x^{-1} : x \to FGx \to y$. We know $\epsilon_x^{-1} : x \to FGx$ exists for all $x \in B$ because every component of ϵ is an isomorphism. So, we obtain an isomorphism between $B(x, y)$ and $B(FGx, y)$.

The second isomorphism is supplied by the adjunction. For any map $f^{\#} : FGx \to y$, we have its unique transpose $f^{b} : Gx \to Gy$ in A and vice versa. □

Theorem 4.9. *Right adjoints preserve limits.*

Proof. Consider a diagram $\mathcal{K} : \mathcal{I} \to \mathcal{D}$ over a category $\mathcal{D}$ with the limit cone $\lambda : \lim \mathcal{K} \Rightarrow \mathcal{K}$ in $\mathcal{D}$. Then applying the right adjoint functor $\mathcal{G} : \mathcal{D} \to \mathcal{C}$, we get a cone over the diagram $\mathcal{GK} : \mathcal{I} \to \mathcal{C}$ in the category $\mathcal{C}$. We need to prove that this cone is the limit over the diagram.

Consider another cone $\mu : c \Rightarrow \mathcal{GK}$. Then applying $\mathcal{F}$ to this diagram, we get a cone over the diagram $\mathcal{K}$ with $\mathcal{F}c$ as the vertex. By universality of $\lim \mathcal{K}$, there exists a unique morphism $\zeta : \mathcal{F}c \to \lim \mathcal{K}$.

The map ζ has a transpose $\zeta^{b} : c \to \mathcal{G}\lim \mathcal{K}$, which is a factorization of μ through $\mathcal{G}\lambda$. It is unique because for any other factorization $c \to \mathcal{G}\lim \mathcal{K}$, we will have a transpose $\zeta^{\#}$ in $\mathcal{C}$. The transpose $\zeta^{\#}$ is unique by universality of the limit cone and is always equal to ζ. Then its unique transpose ζ^{b} will equip the cone $\mathcal{G}\lim \mathcal{K}$ with the universal property of a limit. □

Dually, we will have:

Theorem 4.10. *Left adjoints preserve colimits.*

Proposition 4.11. *If $\mathcal{D} \hookrightarrow \mathcal{C}$ is a reflective subcategory, then $\mathcal{D}$ has all the colimits that $\mathcal{C}$ admits, created by applying the reflector functor to the colimit in $\mathcal{C}$.*

Proof. Let $i : \mathcal{D} \hookrightarrow \mathcal{C}$ be the full embedding whose left adjoint is the reflector functor L and consider a diagram $F : \mathcal{J} \to \mathcal{D}$.

Applying the inclusion functor, we obtain a diagram iF in $\mathcal{C}$. Take the colimit cone $\lambda : iF \Rightarrow c$ over the diagram iF in $\mathcal{C}$. Applying the reflector functor, we get a diagram LiF in $\mathcal{D}$.

Since L is a left adjoint, it preserves colimits. So, we obtain a colimit cone $L\lambda : LiF \Rightarrow Lc$ over LiF. Since i is fully faithful, we know that the counit supplies an isomorphism: $Li \cong 1_{\mathcal{D}}$. We then obtain a colimit cone $L\lambda$ over the original diagram F. □

5. Completeness of Cat

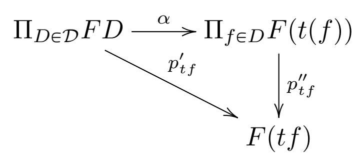
Theorem 5.1. *A category is complete if and only if every family of objects has a product and every pair of parallel arrows has an equalizer.* [2]

Proof.
($\Rightarrow$)
A complete category has all of its limits. Since products and equalizers are specific types of limits, so a complete category would have all the products and equalizers.
($\Leftarrow$)
Consider a category $\mathcal{D}$ to be an arbitrary indexing category for the diagram $F : \mathcal{D} \to \mathcal{C}$. We construct the products:

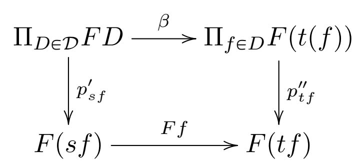
$$(\Pi_{D\in\mathcal{D}} FD,\, p'_D) \text{ and } (\Pi_{f\in D} F(t(f)),\, p''_{tf})$$

where $t(f)$ is the target of a morphism f in D.

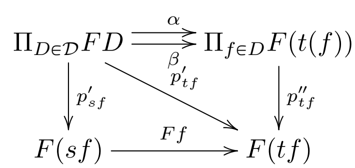
Take α to be the unique factorization $p''_{tf} \circ \alpha = p'_{tf}$



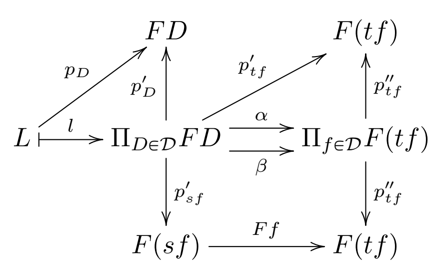
and β to be the unique factorization $p''_{tf} \circ \beta = Ff \circ p'_{sf}$



Combining the two diagrams, we get:

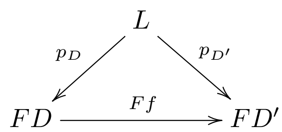


We define (L, l) to be the equalizer of the morphisms α and β and $p_D : L \to FD$ defined as $p_D = p'_D \circ l$.



Claim: L is a cone over the diagram $\mathcal{D}$.

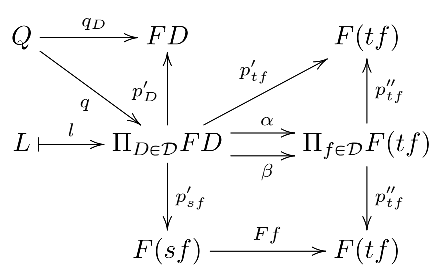
Proof. For any morphism $f : D \to D'$ in $\mathcal{D}$, we need to show that the following diagram commutes:



So, we need to prove that $p_{D'} = Ff \circ p_D$.

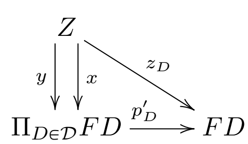
$$p_{D'} = p'_{D'} \circ l = p''_{tf} \circ \alpha \circ l = p''_{tf} \circ \beta \circ l = Ff \circ p'_D \circ l = Ff \circ p_D$$ □

Now, we need to show that L is a universal cone over the diagram D.
Take (Q, q_D) another cone over the diagram D, with $q : Q \to \Pi_{D\in\mathcal{D}}FD$, a unique morphism such that $q_D = p'_D \circ q$.



Proposition 5.2. *p'_D and p''_{tf} are monomorphisms.*

Proof. Take morphisms $x, y : Z \to \Pi_{D\in\mathcal{D}}FD$, such that $p'_D \circ x = p'_D \circ y$. Then, with a morphism $z_D : Z \to FD = p'_D \circ x = p'_D \circ y$, the following diagram commutes:



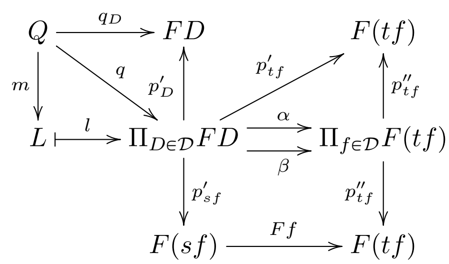
So, Z is also a product, which means there exists a unique morphism
$h : Z \to \Pi_{D\in\mathcal{D}}FD \implies h = x = y$.
We can prove similarly for p''_{tf}.

□

Claim: (Q, q) defines an equalizer over α and β.

Proof. For any $f : D \to D'$,
$p''_{tf} \circ \alpha \circ q = p'_{D'} \circ q = q_{D'} = Ff \circ q_D = Ff \circ p'_D \circ q = p''_{tf} \circ \beta \circ q$
$\implies \alpha \circ q = \beta \circ q$. □

Since (Q, q) is an equalizer over α, β, then there exists a unique factorization $m : Q \to L$, such that $q = l \circ m$.



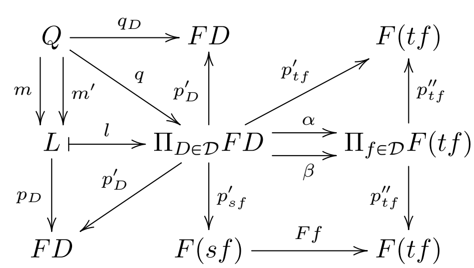
Claim: m is the factorization required to establish that (L, p_D) is the limit over the diagram D.

Proof. In other words, we need to show that $p_D \circ m = q_D$.
$p_D \circ m = p'_D \circ l \circ m = p'_D \circ q = q_D$

□

Claim: m is the unique factorization required to establish that (L, p_D) is the limit over the diagram D.

Proof. Take another factorization $m' : Q \to L$, such that $p_D \circ m' = q_D$ for all $D \in \mathcal{D}$.



Since, l is a monomorphism, it suffices to show that $l \circ m = l \circ m'$, which is equivalent to showing that $p'_D \circ l \circ m = p'_D \circ l \circ m'$.

$$p'_D \circ l \circ m = p'_D \circ q = q_D = p_D \circ m' = p'_D \circ l \circ m'$$ □

□

Dually, we will have:

Theorem 5.3. *A category is cocomplete if and only if every family of objects has a coproduct and every pair of parallel arrows has an coequalizer.*

Theorem 5.4. *Set is complete and cocomplete.*

Proof. To prove (co)completeness of *Set*, we will construct explicit (co)products and (co)equalizers.

The categorical products of sets are defined to be the Cartesian products and equalizers for any two morphisms $f, g : A \to B$ is E, a subset of A, such that for all $a \in E, fa = ga$.

The coproducts for sets are defined to be the disjoint unions of sets and the coequalizer for any two morphisms $f, g : A \to B$ is $B/(\sim)$, where $\sim$ is an equivalence relation, defined as: $\forall a \in A, f(a) \sim g(a)$. □

Theorem 5.5. *Cat is complete.*

Proof. It suffices to construct products and equalizers.

Definition 5.6. For any two categories $\mathcal{C}$ and $\mathcal{D}$, we have $\mathcal{C} \times \mathcal{D}$ such that:

- its objects are of the form (c, d), where $c \in ob(\mathcal{C})$ and $d \in ob(\mathcal{D})$
- its morphisms are of the form $(f, g) : (c, d) \to (c', d')$ for all the morphisms $f : c \to c'$ in $\mathcal{C}$ and $g : d \to d'$ in $\mathcal{D}$.
- its identities and compositions are defined component-wise.

n-ary versions of this product give us the required product and the equalizer for two morphisms $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D}$ is a subcategory of $\mathcal{C}$ with objects c such that $\mathcal{F}c = \mathcal{G}c$ and morphisms f such that $\mathcal{F}f = \mathcal{G}f$ □

6. Simplicial Sets

Let $\mathbf{\Delta}$ be the category whose objects are finite, non-empty, totally-ordered sets

$$[n] = \{0, 1, ..., n\}$$

and morphisms are order-preserving maps between the sets.

Definition 6.1. A simplicial object in a category $\mathcal{C}$ is a functor $F : \mathbf{\Delta^{op}} \to \mathcal{C}$. In particular, a simplicial set is a contravariant functor from $\mathbf{\Delta}$ to **Set**.

We use the notation **sSet** for the functor category $Set^{\Delta^{op}}$ of simplicial sets, where the morphisms are natural transformations between simplicial sets.

It is easy to show that all the morphisms in the category $\mathbf{\Delta}$ can be created by the composites of the coface and codegeneracy morphisms, denoted by δ_i and σ_i respectively, as shown. For each $n \geqslant 0$, we have $n + 1$ injections $\delta_i : [n-1] \to [n]$, defined for $0 \leqslant i \leqslant n$ as:

$$\delta_i[k] = \begin{cases} k & k < i \\ k+1 & k \geqslant i \end{cases}$$

and $n + 1$ surjections $\sigma_i : [n + 1] \to [n]$ defined as:

$$\sigma_i[k] = \begin{cases} k & k \leqslant i \\ k-1 & k > i \end{cases}$$

Clearly, the coface maps skips the i-th element in the image and the codegeneracy maps duplicates the i-th element in the image. These maps satisfy the following relations:

$$\delta_i\delta_j = \delta_j\delta_{i-1} \quad i < j \tag{6.2}$$
$$\sigma_j\sigma_i = \sigma_i\sigma_{j+1} \quad i \leqslant j$$

$$\sigma_j\delta_i = \begin{cases} 1 & i = j, j+1 \\ \delta_i\sigma_{j-1} & i < j \\ \delta_{i-1}\sigma_j & i > j+1 \end{cases}$$

Dually, the face and degeneracy morphisms in the $\mathbf{\Delta^{op}}$ category endow simplicial sets with face and degeneracy morphisms satisfying the required conditions. For a simplicial set X, we get:

$$d_i = X\delta_i : X_n \to X_{n-1} \quad 0 \leqslant i \leqslant n$$
$$s_i = X\sigma_i : X_n \to X_{n+1} \quad 0 \leqslant i \leqslant n$$

satisfying the relations dual to 6.2.

We obtain an equivalent definition for simplicial sets:

Definition 6.3. A simplicial set is a sequence of sets X_n with face and degeneracy morphisms $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ satisfying the following relations:

$$d_id_j = d_{j-1}d_i \quad i < j \tag{6.4}$$
$$s_js_i = s_{j+1}s_i \quad i \leqslant j$$

$$d_is_j = \begin{cases} 1 & i = j, j+1 \\ s_{j-1}d_i & i < j \\ s_jd_{i-1} & i > j+1 \end{cases}$$

For any simplicial set X, elements of the set X_n are called its $n-$simplices.

Proposition 6.5. *The presheaf category for any category D is complete and cocomplete.*

Proof. Take the presheaf category for a small category D and a diagram $F : I \to Set^{D^{op}}$. Then, for a fixed object $d \in D$, consider the functor $F_d : I \to Set$. We then take the (co)limit over this diagram which exists because Set is (co)complete. In other words, we take the level-wise (co)limits. □

Proposition 6.6. ***sSet*** *is complete and cocomplete.*

Proof. This follows directly from the above proposition since **sSet** is a presheaf category. □

7. Cocompleteness of *Cat*

We will show that there exists the adjunction

$$Cat \underset{N}{\overset{\Pi}{\underset{\perp}{\rightleftarrows}}} \mathbf{sSet}$$

where the inclusion is a full embedding of Cat into **sSet**. Since **sSet** is cocomplete, then by Proposition 4.11, Cat is also cocomplete.

Definition 7.1. Define the nerve of any small category $\mathcal{C}$ to be a simplicial set $N\mathcal{C}$ containing:
$N\mathcal{C}_0 = ob(\mathcal{C})$
$N\mathcal{C}_1 = mor(\mathcal{C})$
$N\mathcal{C}_2 = \{$pairs of composable arrows in $\mathcal{C}\}$
...
$N\mathcal{C}_n = \{$n-strings of composable arrows in $\mathcal{C}\}$

The degeneracy maps $s_i : N\mathcal{C}_n \to N\mathcal{C}_{n+1}$ takes the n-string of composable arrows $c_1 \to c_2 \to ... \to c_n$ and obtains a string of $n+1$ composable arrows by inserting the identity of c_i at the i-th spot.

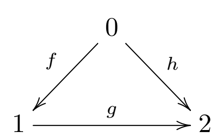
The face maps $d_i : N\mathcal{C}_n \to N\mathcal{C}_{n-1}$ leaves out the first and last arrows for when $i = 0$ and $i = n$ respectively and composes the i-th and $(i-1)$-th arrows when $0 < i < n$ to obtain a string of $n-1$ arrows. [3]

Definition 7.2. For any simplicial set S, we define Π to be the functor $\Pi : \mathbf{sSet} \to Cat$. Define:

- $ob(\Pi S) = S_0$
- $mor(\Pi S) = S_1$ modulo relations generated by S_2

The degeneracy morphisms $s_0 : S_0 \to S_1$ supply the identity morphisms and the face morphisms $d_1, d_0 : S_1 \to S_0$ assign domain and codomain to every arrow.

Once we obtain all the morphisms from S_1, we modulo by the relations generated by S_2. $h = gf$ if we have a 2-simplex in $s \in S_2$ such that $sd_0 = g$, $sd_1 = h$ and $sd_2 = f$.



We can completely define a category by its objects, morphisms, composition and identities so in this way we obtain the category ΠS for a simplicial set S.

Claim: Π is a left adjoint to N and N is right adjoint to Π.

Proof. We need to show that for any morphism $k : C \to C'$, the following diagram commutes:

$$\begin{array}{ccc} Cat(\Pi X, C) & \xrightarrow{\cong} & \mathbf{sSet}(X, NC) \\ \downarrow k_* & & \downarrow Nk_* \\ Cat(\Pi X, C') & \xrightarrow{\cong} & \mathbf{sSet}(X, NC') \end{array}$$

For any morphism $f^\# : \Pi X \to C$, we have the composite $k \circ f^\# : \Pi X \to C'$. The transpose of this map is $(k \circ f^\#)^b : X \to NC'$, which can be given by the composite $Nk_* \circ f^b : X \to NC'$. f^b takes X to NC since it is the transpose of $f^\#$.

Then, we get an induced map Nk_* from k which takes NC to NC' by acting on each set of the sequence in NC and mapping it to respective sets in NC'.

Dually, we need to show that for any morphism $h : X' \to X$, the following diagram commutes:

$$\begin{array}{ccc} Cat(\Pi X', C) & \xrightarrow{\cong} & \mathbf{sSet}(X', NC) \\ \downarrow{\scriptstyle \Pi h^*} & & \downarrow{\scriptstyle h^*} \\ Cat(\Pi X, C) & \xrightarrow{\cong} & \mathbf{sSet}(X, NC) \end{array}$$

For any morphism $f^\# : \Pi X \to C$, we have the composite $f^\# \circ \Pi h : \Pi X' \to C$. The transpose of this map is $(f^\# \circ \Pi h)^b : X' \to NC$, which can be given by the composite $f^b \circ h : X' \to NC$. By definition, h takes X' to X. f^b is the transpose of $f^\#$ and a map of simplicial sets which takes X to NC by acting on each set of the sequence in X and mapping it to respective sets in NC. □

Claim: Every component ΠNC of the counit of the adjunction is an isomorphism.

Proof. Take the category C. Applying the nerve functor, we obtain NC, defined as explained above. Applying Π, we obtain the category ΠNC, which has:

- $ob(\Pi NC) = NC_0 = ob(C)$
- $mor(\Pi NC) = NC_1 = mor(C)$
- composition given by the objects in NC_2, which is the set of pairs of composable arrows in C.
- By definition of nerve, we know that $s_0 : NC_0 \to NC_1$ is obtained by inserting the identity morphism of $c_0 \in NC_0$ at the 0-th spot. So, it is exactly the identity morphism for c_0. We also know that the degeneracy morphisms for NC supply the identity morphisms for ΠNC so we get the same identities as in C.

Hence, we have shown that $\Pi NC \cong C$ for any category C. □

Since every component of the counit is an isomorphism, we have shown that N is fully faithful by Proposition 4.8.
It is clear by construction that the inclusion N is injective. So, N is a full embedding of Cat into **sSet**, proving that Cat is a reflective subcategory of **sSet**.

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8. Bibliography

In writing this paper, the author consulted Emily Riehl's 'Category Theory in Context', Francis Borceux's 'Handbook of Categorical Algebra' and Riehl's 'A Leisurely Introduction to Simplicial Sets' notes.

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