A GLIMPSE INTO HOMOLOGICAL MIRROR SYMMETRY

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ABSTRACT. This is an expository paper about homological mirror symmetry (HMS), a surprising and profound equivalence between the symplectic geometry and the complex geometry. We will introduce both sides of the mirror symmetry: A-model and B-model, including the physical background and the mathematical notions. In the end, we will give some further discussions about HMS.

CONTENTS

1. INTRODUCTION

The math phenomenon of the mirror symmetry was first discovered by string physicists in 1980s. Physical string theory is a kind of Theory of Everything, which tries to unify the quantum field theory (QFT) and the general relativity (GR), namely the quantum gravity, and explain every fundamental particle and force in our world. There are many types of string theories, of which two are called type IIA string theory and type IIB string theory. These two types of string theory are equivalent in physics and predict the same physical effects. However, the type IIA string theory only relies on the symplectic structure of a certain Calabi-Yau threefold, while the type IIB string theory is determined by the complex structure of another Calabi-Yau threefold. The pair of the Calabi-Yau manifolds above are called mirror manifolds and the phenomenon above is called mirror symmetry. Physicists put forward the following mirror symmetry conjecture in string theory:

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Conjecture 1.1. *(Mirror Symmetry Conjecture in the String Theory) Type IIA string theory compactified on a Calabi-Yau threefold is dual to type IIB string theory compactified on the mirror Calabi-Yau threefold.*

The duality here more or less means that, for any theory of the first type, there exists some isomorphic theory of the second type; in particular all physical predictions of the two theories are the same.

The non-trivial equivalence of physics implies a surprising correspondence between symplectic geometry and complex geometry, which attracted the interests of mathematicians around 1990. However, the modern physics tools, such as path integral in QFT, lie just beyond the current scope of mathematics. The lack of rigorous mathematical formulation kept mathematicians from tapping into the essence of the phenomenon along the physical road. Therefore, following the inspiration of physicists, mathematicians developed their own approaches and the first breakthrough was the homological mirror symmetry conjecture (HMS) proposed by M. Kontsevich in his 1994 ICM talk [1].

Conjecture 1.2 (4.1)**.** *(Homological Mirror Symmetry) The derived Fukaya category of a Calabi-Yau manifold is equivalent to the bounded derived category of the coherent sheaves of its mirror Calabi-Yau manifold as triangulated categories. That is*

$$\mathrm{D}\, Fuk(X) \cong \mathrm{D}^{\mathrm{b}}\, Coh(X^{\vee}),$$

where $X^{\vee}$ is the mirror Calabi-Yau manifold of X.

This rather abstract proposal took some time to be appreciated by either mathematicians or physicists. The Fukaya category is determined by the symplectic structure of the Calabi-Yau manifold X, while the coherent sheaves base on the complex structure of the Calabi-Yau manifold $X^{\vee}$. The correspondence between symplectic geometry and complex geometry might originate from the perspective of homological algebra and category theory, which, inspired by string theory, goes beyond the traditional differential and algebraic geometry. Today mirror symmetry has developed into an attractive and challenging subject in pure mathematics and provided a powerful insight into algebraic geometry, symplectic geometry, homological algebra and noncummutative geometry.

The main purpose of this expository paper is to understand the relevant notions of this conjecture, including the mathematical setup and the physical motivations. In Section 2 and Section 3, we give a systematic introduction to the A-model (symplectic side) and the B-model (complex side), which represent two kinds of topology field theory associated to $N = (2, 2)$ superconformal field theory. In Section 4, we give some further discussions about the homological mirror symmetry conjecture. For those who only focus on the mathematical contents, we recommend them to skip the introduction about the physical string theory A-model and B-model in Section 2 and Section 3.

We will focus on the mathematical and physical ideas about the homological rather than technical details. We assume that readers are familiar with the basic notions of symplectic geometry (symplectic manifold, Hamiltonian vector field etc.), algebraic geometry (coherent sheaf etc.) and homological algebra (derived category, Ext group and triangulated category etc.) for this paper.

2. A-Model: Floer Cohomology and Fukaya Category

2.1. **Closed and Open String A-model.** The physical background behind the mirror symmetry is the non-linear sigma model and the Dirichlet branes (D-branes). In the string theory, every particle is viewed as a string rather than a point and there are two kinds of strings: closed strings and open strings. A closed string is homeomorphic to a circle, while an open string is homeomorphic to a closed interval. In the superstring theory, our universe has dimension 10, which could be described by $M^{1,3} \times X$, where $M^{1,3}$ is the spacetime manifold in the general relativity, a pseudo-Riemann manifold with signature (+,-,-,-), and X is a Calabi-Yau threefold, which stands for the hidden 6 real dimensions of our universe. A Calabi-Yau threefold is by definition a compact Kähler manifold of complex dimension 3 with a nowhere vanishing holomorphic 3-form, which is called a Calabi-Yau form. We will focus on the Calabi-Yau part in our case. The trajectory of a string in the universe is called a world sheet. A world sheet can be represented by a smooth map $\phi : \Sigma \longrightarrow X$, where Σ is a Riemann surface (with boundary in the case of open string) standing for the world sheet and X is the Calabi-Yau manifold above, which is called the target manifold.

2.1.1. *Closed String A-model.* Firstly, we concentrate on closed strings, meaning that Σ does not have boundaries. Let T_X be the complex tangent bundle on X, which can be canonically decomposed as $T_X = T_X^{(1,0)} \oplus T_X^{(0,1)}$ via the complex structure of X. We denote the canonical and anti-canonical bundles on Σ by K and $\bar{K}$. We also need a choice of square roots of K and $\bar{K}$ to discuss fermions in this physical system. (A square root of a vector bundle E is by definition a vector bundle L satisfying $L^{\otimes 2} = E$.) A non-linear[1] sigma model is a 2-dimensional quantum field theory defined on the world sheet Σ. In QFT, the observables are determined via the path integral, which only relies on the action of the physical system. The non-linear action is given by

$$\begin{aligned} S = \int_\Sigma dz \wedge d\bar{z}\Big(&\frac{1}{2} g_{ij}\partial_z\phi^i\partial_{\bar{z}}\phi^j + \frac{i}{2}B_{ij}\partial_z\phi^i\partial_{\bar{z}}\phi^j \\ &+ i\psi_-^{\bar{i}} D_z \psi_-^i g_{i\bar{i}} + i\psi_+^{\bar{i}} D_{\bar{z}}\psi_+^i g_{\bar{i}i} + R_{i\bar{i}j\bar{j}}\psi_+^i\psi_+^{\bar{i}}\psi_-^j\psi_-^{\bar{j}}\Big), \end{aligned} \tag{2.1}$$

where g_{ij} is the Kähler metric, B_{ij} is a real (1,1) form, $R_{i\bar{i}j\bar{j}}$ s the Riemann tensor on X, D_z is the ∂ operator on $\bar{K}^{1/2} \otimes \phi^* T_X^{(1,0)}$, arising by pulling back the holomorphic part of the Levi–Civita connection on T_X, D_z is the $\bar{\partial}$ operator on $K^{1/2} \otimes \phi^* T_X^{(1,0)}$ and fermions fields ψ are sections of the following bundles:

$$\begin{aligned} \psi_+^i &\in \Gamma\left(K^{1/2} \otimes \phi^* T_X^{(1,0)}\right), \quad \psi_+^{\bar{i}} \in \Gamma\left(K^{1/2} \otimes \phi^* T_X^{(0,1)}\right). \\ \psi_-^i &\in \Gamma\left(\bar{K}^{1/2} \otimes \phi^* T_X^{(1,0)}\right), \quad \psi_-^{\bar{i}} \in \Gamma\left(\bar{K}^{1/2} \otimes \phi^* T_X^{(0,1)}\right). \end{aligned}$$

Now we introduce the supersymmetry (SUSY) transformations, which are in the core of the supersymmetry[2] theory. The SUSY transformations transform a bosonic field to a fermionic field and viceversa, and they are generated by four infinitesimal fermionic parameters α_+, $\widetilde{\alpha}_+$, α_-, $\widetilde{\alpha}_-$. The first two are anti-holomorphic sections of $\bar{K}^{-1/2}$ and the latter two are holomorphic sections of $K^{-1/2}$. There are four

[1]The "non-linear" here means that the target space isn't a linear space, but a non-linear manifold.

[2]The "supersymmetry" here means that every boson has its dual fermion and viceversa.

parameters and two pairs of them have two kinds of chirality respectively, so we call this theory "$N = (2, 2)$ *supersymmetry*."

By twisting SUSY transformation in 2 ways, we can get the A-model and the B-model. The twisting here means modifying the bundles on which the fermions fields $\psi^i_\pm, \psi^{\bar{i}}_\pm$ take value. In A-model, let

$$\begin{aligned} \psi^i_+ &\in \Gamma\left(\phi^* T_X^{(1,0)}\right), & \psi^{\bar{i}}_+ &\in \Gamma\left(K \otimes \phi^* T_X^{(0,1)}\right) \\ \psi^i_- &\in \Gamma\left(\bar{K} \otimes \phi^* T_X^{(1,0)}\right), & \psi^{\bar{i}}_- &\in \Gamma\left(\phi^* T_X^{(0,1)}\right) \end{aligned}$$

and set the fermionic parameters $\widetilde{\alpha}_- = \alpha_+ = 0,\ \alpha_- = \widetilde{\alpha}_+ = \alpha$. Thus the SUSY transformation in the A-model only depends on one parameter α. The SUSY transformation laws (See [16]) and the action (2.1) are rather complicated, but we can use the generator of SUSY transformation, called the BRST operator Q, to simplify the transformation laws and the action in the following ways:

$$\delta\varphi = -i\alpha\{Q, \varphi\},$$

$$S = \int_\Sigma i\{Q, V\} - 2\pi i \int_\Sigma \phi^*(B + iJ), \tag{2.2}$$

where φ is any field, δ refers to the SUSY transformation, $\{\ ,\ \}$ is the anti-commutator,

$$V = 2\pi g_{i\bar{j}}\left(\psi^{\bar{j}}_z \bar{\partial}\phi^i + \partial\phi^{\bar{j}}\psi^i_{\bar{z}}\right),$$

and $B + iJ \in H^2(X, \mathbb{C})$ is the complexified Kähler form. The most important property of Q is $Q^2 = 0$, up to the equation of motion, which means that the BRST operator Q has a differential. Given any field [3] or operator W, we say that W is Q-closed, or BRST invariant, if $\{Q, W\} = 0$, and we say that W is Q-exact if there is another operator W' satisfying $W = \{Q, W'\}$. In the string theory, the physical observables must be the products of Q-closed local operators. Every local operator can be represented as the following form

$$W_a = a_{I_1\cdots I_p}\chi^{I_1}\cdots\chi^{I_p},$$

where χ is a section of $\phi^* T_X$ and $\chi^i = \psi^i_+,\ \chi^{\bar{i}} = \psi^{\bar{i}}_-$. The capital I_q indices refer to real indices. Every local operator W_a has a corresponding p-form

$$a = a_{I_1\cdots I_p} d\phi^{I_1}\cdots d\phi^{I_p}.$$

By computing the variation of the operator W_a, one can find $\{Q, W_a\} = -W_{da}$, which implies a remarkable conclusion: The Q-cohomology is equivalent to de Rham cohomology on the target manifold X. It is consistent with the A-model being independent of the complex structure of X. Because of the correspondence between local operators and exterior differential forms, there is a natural grading of local operators called "ghost number" in physics, which means that if a is a p-form, then W_a is of ghost number p.

In the string theory, a shift in the action by a Q-exact operator $S \longrightarrow S + \int_\Sigma\{Q, S'\}$ won't change all correlation functions, which means that every physical observable will remain invariant. It is clear from (2.2) that only V depends on the complex structure of X, but a deformation $V \longrightarrow V + \delta V$ leads to $S \longrightarrow S + \int_\Sigma\{Q, \delta V\}$. Therefore physically, the closed string A-model only depends on the Kähler structure $B + iJ$, more precisely, the cohomology class of $B + iJ$.

[3]In QFT, every quantized field is an operator, which is often called a field operator.

2.1.2. *Open String A-model.* Now we consider the open string A-model, which means that the world sheet Σ has boundaries. In the open strings case, unlike closed strings, a deformation of the Kähler form by an exact form dA will change the action (2.2). Therefore, we should add an extra boundary term in the action to show the contribution of this freedom:

$$S_{\partial\Sigma} = -2\pi i \int_{\partial\Sigma} \phi^*(A). \tag{2.3}$$

In the open string case, there is another important restrictive condition called Dirichlet-branes (D-branes)[4]. The D-branes are the allowed regions of the endpoints of an open string, which means that the boundaries of an open string world sheet Σ must lie on the D-branes. The D-branes in the A-model and the B-model, also called A-branes and B-branes respectively, both have categorical structures, where the objects are D-branes and the morphisms are open strings stretching from one brane to another. The homological mirror symmetry is the equivalence between the categories of A-branes and B-branes.

Classically, an A-brane must be an equivalence class of Lagrangian submanifolds with flat $U(1)$-bundles modulo Hamiltonian deformations. [5] The connection on the $U(1)$-bundle is precisely the 1-form A in (2.3). Firstly, let us focus on the physical perspective of this categorical structure. To compute the space of open strings, let the two corresponding A-branes L_1, L_2 intersect transversely. According to the Q-invariance of the topological field theory, the open string world sheet map must be a constant map, which means that the open string is just an intersection of two A-branes. Let the intersections of L_1 and L_2 be $p_1, ..., p_N$, and to each point p_α there is a corresponding vertex operator W_{p_α} that creates an open string on p_α and the hom-set can be regarded as a complex vector space generated by these vertex operators. Each vertex operator W_{p_α} has a ghost number $\mu(p_\alpha)$, which leads to the grading of the hom-set. Apart from that, we should also consider the first order tunneling correction to our theory, that is, an open string state at one point is able to tunnel to another open string state at another point. The process of tunneling can be represented by a holomorphic[6] curve $D \longrightarrow X$, where D is a disk such that ∂D lies on the A-branes L_1 and L_2. Such a map is called a worldsheet instanton[7]. An instanton of the tunneling process is shown in the Figure 1. These instantons produce some corrections to the BRST operator Q:

$$\{Q, W_{p_a}\} = \sum_b n_{ab} W_{p_b}, \tag{2.4}$$

where n_{ab} is the number of points in a zero-dimensional instanton moduli space. Since the BRST operator Q has ghost number 1 and is nilpotent, the hom-set has the structure of Q-cohomology. The Q-cohomology coincides with the Floer cohomology introduced in Section 2.2 and the BRST operator Q is related to the

[4]The "Dirichlet" here refers to Dirichlet boundary condition in the partial differential equation theory, which is similar to the restrictive condition of D-branes.

[5]The precise definition of the A-branes need some quantum corrections that it must satisfy the tadpole cancellation property and has trivial Maslov class. More details on [16].

[6]In fact, a pseudoholomorphic curve is enough, which means that we do not need the complex structure to do the tunneling correction, a symplectic structure is enough.

[7]The name comes from that the process happens at some "instant" (though not really) within infinite interval of time.

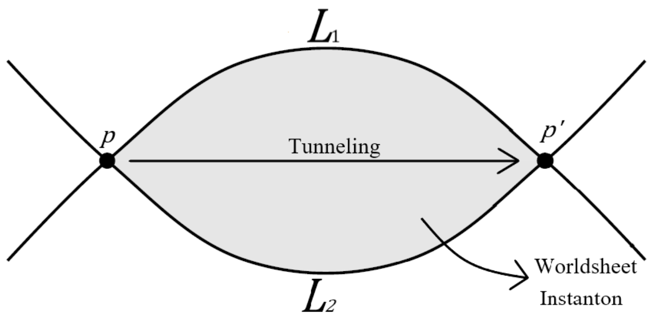

FIGURE 1. Instanton Tunneling

Floer differential. Finally, we define the category of the A-branes, which is a Fukaya category. We will introduce its rigorous mathematical definition in Section 2.4.

In the end of this section, we will discuss more about the ghost number of an A-brane L, denoted by $\mu(L)$. Using the definition of the ghost number of open string vertex operators, we could naturally define the difference between two A-branes by letting the ghost number of elements from $\mathrm{Hom}^i(L_1, L_2)$ be $i + \mu(L_2) - \mu(L_1)$. However, we can not give a natural definition of the absolute ghost number of an A-brane. This ambiguity of the grading of A-branes plays an important role when we discuss the definition of B-branes.

2.2. **Lagrangian Floer Cohomology.** Now we introduce the mathematical definition of the Lagrangian Floer cohomology. Let (M, ω) be a symplectic manifold (See [13] for basic definitions). Let L_0 and L_1 be two compact connected Lagrangian submanifolds, which stand for two A-branes in physics. Firstly, we assume that L_0 and L_1 intersect transversely. Therefore there are only finite intersections of two Lagrangian manifolds. If the transversality condition fails, then we should do Hamiltonian perturbation (see Section 2.2.3) to achieve it. To make Floer cohomology well-defined, we need to work on the Novikov field rather than classical fields.

Definition 2.5. The Novikov field Λ over a base field $\mathbb{K}$ is

$$\Lambda = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{K}, \lambda_i \in \mathbb{R}, \lim_{i \to \infty} \lambda_i = +\infty \right\},$$

where T is a formal parameter.

From now on, we work on the base field $\mathbb{K} = \mathbb{F}_2$, a field consisting of order two, unless otherwise stated. We write $\chi(L_0, L_1) = L_0 \cap L_1$, which is the set of generators of the Floer complex defined as follows.

Definition 2.6. The Floer complex of L_0 and L_1 is a free Λ-module generated by the intersections $\mathcal{X}(L_0, L_1)$, that is

$$CF(L_0, L_1) = \bigoplus_{p \in \mathcal{X}(L_0, L_1)} \Lambda \cdot p.$$

To make the Floer complex an honest chain complex, we also need to define the Floer differential and the $\mathbb{Z}$-grading of the Floer complex, which are rather complicated but the cores of the Floer cohomology theory.

2.2.1. *Floer differential.* The Floer differential is defined by counting the pseudoholomorphic curves bounded by L_0 and L_1, which represent the worldsheet instantons in physics. To be more specific, let J be a ω-compatible almost complex structure[8] on M, which means an almost complex structure such that $g_J(\cdot,\cdot) := \omega(\cdot, J\cdot)$ is a Riemann metric. We consider the space of pseudoholomorphic curves

$$u : \Sigma = \mathbb{R} \times [0,1] \longrightarrow M,$$

which means that the curve u satisfies the complex linear condition $J \circ du = du \circ J_0$, where J_0 is the standard almost complex structure of $\Sigma \subsetneq \mathbb{C}$. The complex linear condition is equivalent to the Cauchy-Riemann equation $\bar{\partial}_J u = 0$, i.e.

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = 0, \tag{2.7}$$

where (s,t) are the real coordinates of $\Sigma \subseteq \mathbb{C}$.

Definitions 2.8. Let $\widehat{\mathcal{M}}(p,q;[u],J)$ be the moduli space of pseudoholomorphic strips $u : \mathbb{R} \times [0,1] \longrightarrow (M,J)$ in the homotopy class $[u] \in \pi_2(M, L_0 \cup L_1)$ satisfying the Lagrangian boundary conditions

$$\begin{cases} u(s,0) \in L_0 \text{ and } u(s,1) \in L_1 \quad \forall s \in \mathbb{R}, \\ \lim\limits_{s\to+\infty} u(s,t) = p, \quad \lim\limits_{s\to-\infty} u(s,t) = q, \end{cases} \tag{2.9}$$

and the finite energy (or say symplectic area) condition

$$E(u) = \omega([u]) = \int u^*\omega = \iint \left|\frac{\partial u}{\partial s}\right|^2 dsdt < \infty. \tag{2.10}$$

Let $\mathbb{R}$ act on the $\widehat{\mathcal{M}}(p,q;[u],J)$ by translation reparametrization, i.e., $a \in \mathbb{R}$ acts by

$$(a \cdot u)(s,t) := u(s-a,t).$$

Then we denote the orbit space of this action by $\mathcal{M}(p,q;[u],J)$.

Remark 2.11. According to the Riemann mapping theorem, the strip Σ is biholomorphic to $D^2 \setminus \{\pm 1\}$, the closed unit disc minus two points on its boundary. The smooth map u can extend to the closed disc and ± 1 correspond to p, q. This interpretation will be useful when we define the multi-products of intersections of two Lagrangian submanifolds in Section 2.3.2.

Remark 2.12. Let $D_{\bar{\partial}_J,u}$ be the linearization of $\bar{\partial}_J$ at a given strip u. The linearized operator $D_{\bar{\partial}_J,u}$ is a $\bar{\partial}_J$-type first-order differential operator, which means that $D_{\bar{\partial}_J,u}$ maps a section of u^*TM satisfying the Lagrangian boundary conditions[9] to a u^*TM-valued $(0,1)$-form on Σ, namely a section of bundle $u^*TM \otimes \bigwedge^{0,1} T\Sigma$. The linearized operator $D_{\bar{\partial}_J,u}$ is also a Fredholm operator, whose Fredholm index is by definition

$$\mathrm{ind}_{\mathbb{R}}\, D_{\bar{\partial}_J,u} = \dim \mathrm{Ker}\, D_{\bar{\partial}_J,u} - \dim \mathrm{Coker}\, D_{\bar{\partial}_J,u}.$$

[8]According to a classical result, the space of all ω-compatible almost complex structures $\mathcal{J}(M,\omega)$ is non-empty and contractible, and all of these almost complex structures are isomorphic to each other, more details on [13] and [14].

[9]The Lagrangian boundary conditions here are that the values of the section at lines $t = 0$ and $t = 1$ are in bundles u^*TL_0 and u^*TL_1 respectively.

The Fredholm index coincides with the Maslov Index $\mathrm{Ind}([u])$ defined below, which is more computable and geometrical. For precise definition and more details about the linearization $D_{\bar{\partial}_J,u}$, please refer to D. McDuff and D. Salamon's book [14, 3.1.].

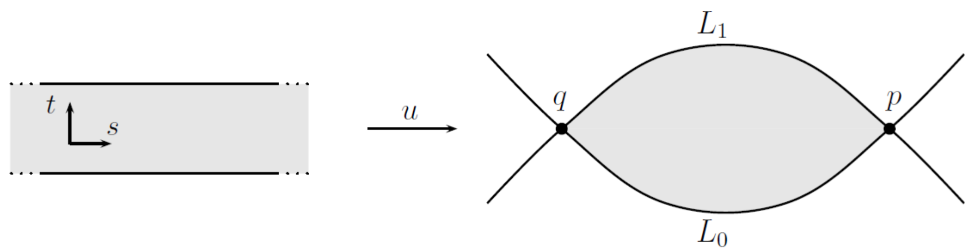

FIGURE 2. A pseudoholomorphic strip contributing to the Floer differential on $CF(L_0, L_1)$

Now we consider the structure of the moduli space $\widehat{\mathcal{M}}(p,q;[u],J)$. If every strip in $\widehat{\mathcal{M}}(p,q;[u],J)$ is regular with respect to the linearized operator $D_{\bar{\partial}_J,u}$, i.e., $D_{\bar{\partial}_J,u}$ is surjective at each point of $D_{\bar{\partial}_J,u}$, then the dimension of $\widehat{\mathcal{M}}(p,q;[u],J)$ is just the Fredholm index of $D_{\bar{\partial}_J,u}$. We can use the Hamiltonian perturbation to guarantee this condition.

Before we give the definition of the Floer differential, we introduce the Maslov index, which plays an important role in the Floer theory. Let $LGr(n)$ be the Lagrangian Grassmann (See [13, 2.3]) consisting of all Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_0)$, where ω_0 is the standard symplectic form of $\mathbb{R}^{2n}$. We have $LGr(n) \cong U(n)/O(n)$. We consider the square of the determinant map

$$det^2 : U(n)/O(n) \longrightarrow S^1.$$

The induced homomorphism det^2_* is an isomorphism between fundamental groups. The Maslov index of a loop in $LGr(n)$ is then defined to be the integer corresponding to the loop via det^2_*.

For any pseudoholomorphic strip u, the pullback bundle u^*TM is a trivial symplectic bundle because Σ is contractible. Every fiber is canonically isomorphic to each other, therefore $u|^*_{t=0}TL_0$ and $u|^*_{t=1}TL_1$ can be viewed as two paths in $LGr(n)$ (oriented with s going from $+\infty$ to $-\infty$), denoted by l_0 and l_1 respectively. We use canonical short paths to concatenate l_0 and l_1 into a loop. Given two transverse subspaces λ_0 and λ_1 of $(\mathbb{R}, \omega_0)$ and identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$. There is a symplectomorphism $A \in Sp(2n, \mathbb{R})$, which maps λ_0 to $\mathbb{R}^n \subseteq \mathbb{C}^n$ and λ_1 to $(i\mathbb{R})^n \subseteq \mathbb{C}^n$. The Lagrangian subspaces family $\lambda_t = A^{-1}((e^{-i\pi t/2}\mathbb{R})^n), t \in [0,1]$ is defined to be the canonical short path connecting λ_0 and λ_1 in $LGr(n)$. The Maslov index of strip u is defined as follows:

Definition 2.13. Given $p, q \in L_0 \cap L_1$. Let λ_p and λ_q be the canonical paths connecting T_pL_0 and T_pL_1, T_qL_0 and T_qL_1 respectively. Given $u : \Sigma \longrightarrow M$ a pseudoholomorphic[10] strip and let l_0 and l_1 be the paths in $LGr(n)$ corresponding

[10]It is easy to see that the pseudoholomorphic condition isn't necessary to define Maslov index. In fact, the Maslov index is a map from $\pi_2(M, L)$ to $\mathbb{Z}$, where L is any union of Lagrangian submanifolds.

to $u|_{t=0}^* TL_0$ and $u|_{t=1}^* TL_1$. The Maslov index $ind(u)$ is defined to be the Maslov index of the loop consisting of $-l_0, \lambda_p, l_1, \lambda_q$.

It is easy to prove that the Maslov index only depends on the relative homotopy class of u, therefore we can define the Maslov index of $[u]$. As we have mentioned in Remark 2.12, the Maslov index of a pseudoholomorphic strip u coincides with the Fredholm index of $D_{\bar{\partial}_J,u}$, and the dimension of the moduli space $\widehat{\mathcal{M}}(p,q;[u],J)$ is just the Maslov index of $[u]$. If the Maslov Index of $[u]$ is 1, then we have

$$\dim \mathcal{M}(p,q;[u],J) = \dim \widehat{\mathcal{M}}(p,q;[u],J) - 1 = \mathrm{ind}([u]) - 1 = 0,$$

which means that $\mathcal{M}(p,q;[u],J)$ is a compact 0-manifold consisting of finite points. Now we can give the definition of the Floer differential by counting pseudoholomorphic strips.

Definition 2.14. The Floer differential $\partial : CF(L_0,L_1) \longrightarrow CF(L_0,L_1)$ is the Λ-linear map defined by

$$\partial(p) = \sum_{\substack{q \in \mathcal{X}(L_0,L_1) \\ [u]:\mathrm{ind}([u])=1}} (\#\mathcal{M}(p,q;[u],J)) T^{\omega([u])} q, \tag{2.15}$$

where $\#\mathcal{M}(p,q;[u],J) \in \mathbb{F}_2$ is the number of points in $\mathcal{M}(p,q;[u],J)$ modulo 2, $[u]$ runs over the homotopy classes of Maslov index 1 in $\pi_2(M, L_0 \cup L_1)$, and $\omega([u]) = \int u^*\omega$ is the symplectic area [11]of those strips.

Remark 2.16. The sum on the right hand of (2.15) has infinite terms in general. However given an energy upper bound E_0, according to the Gromov's compactness theorem, there are only finite homotopy classes $[u]$ such that the moduli space $\mathcal{M}(p,q;[u],J)$ is non-empty and $\omega([u])$ is below E_0. Therefore the coefficients on the right hand of (2.15) is in the Novikov field Λ.

Remark 2.17. In general, the definition of the Floer differential and the Floer cohomology needs a Hamiltonian perturbation to ensure the transversality and the regular condition of $D_{\bar{\partial}_J,u}$. In this case, the Floer differential counts perturbed pseudoholomorphic strips between the perturbed intersections.

Remark 2.18. If one wants to work on $\mathbb{Z}$ rather than $\mathbb{F}_2$, then the spin structures on L_0 and L_1 are needed. The spin structures induce a canonical orientation of the moduli space $\mathcal{M}(p,q;[u],J)$, which equip the points of $\mathcal{M}(p,q;[u],J)$ with signs. Then $\#\mathcal{M}(p,q;[u],J) \in \mathbb{Z}$ refers to signed count of points in the moduli space. We omit the technical construction of spin structures here and refer readers to [3] for details.

The Floer differential has a similar form with the BRST operator Q in (2.4). The physical meaning of implementing a Floer differential can then be roughly interpreted as making an open string at one point tunnel to other points. After the definition of the Floer differential ∂, we will use the Gromov's compactness theorem to prove the most important theorem of the Floer cohomology theory: $\partial^2 = 0$, which is the core of the cohomology structure of Floer complex.

Theorem 2.19. *Assume that $\omega \cdot \pi_2(M, L_0) = 0$ and $\omega \cdot \pi_2(M, L_1) = 0$, where the dot product means the integral. Then the Floer differential satisfies $\partial^2 = 0$.*

[11]By Stokes theorem and the Lagrangian boundary condition, the symplectic area, also called Energy, only depends on the homotopy class of u.

Proof. We assume that L_0 and L_1 intersect transversely and satisfy the regular condition such that $\mathcal{M}(p,q;[u],J)$ is a manifold of dimension $\mathrm{ind}([u])-1$, since we can use Hamiltonian perturbation to guarantee this assumption. We consider the boundaries of the compactified moduli space $\overline{\mathcal{M}}(p,q;[u],J)$, especially when $\dim \mathcal{M}(p,q;[u],J) = \mathrm{ind}([u])-1 = 1$. We need to use the following Gromov's compactness theorem to govern the limits of pseudoholomorphic strips.

Lemma 2.20. *(Gromov's compactness theorem) Any sequence of pseudoholomorphic curves with uniformly bounded energy has a subsequence that converges to nodal configurations up to reparametrization. In the case of pseudoholomorphic strips satisfying Lagrangian boundary conditions, there are three types of configurations:*

(1) broken strip (the upper case of Figure 3): energy concentrates at either end $s \longrightarrow \pm\infty$, which means that there is a sequence $a_n \longrightarrow \pm\infty$ such that the translated strips $u_n(s-a_n,t)$ converge to a non-constant limit strip.

(2) disc bubble (the lower left case of Figure 3): energy concentrates at a point on the boundary of the strips, where suitable rescalings of u_n converge to a pseudoholomorphic disc in M with boundary entirely contained in either L_0 and L_1.

(3) sphere bubble (the lower right case of Figure 3): energy concentrates at an interior point of the strip, where suitable rescalings of u_n converge to a pseudoholomorphic sphere in M.

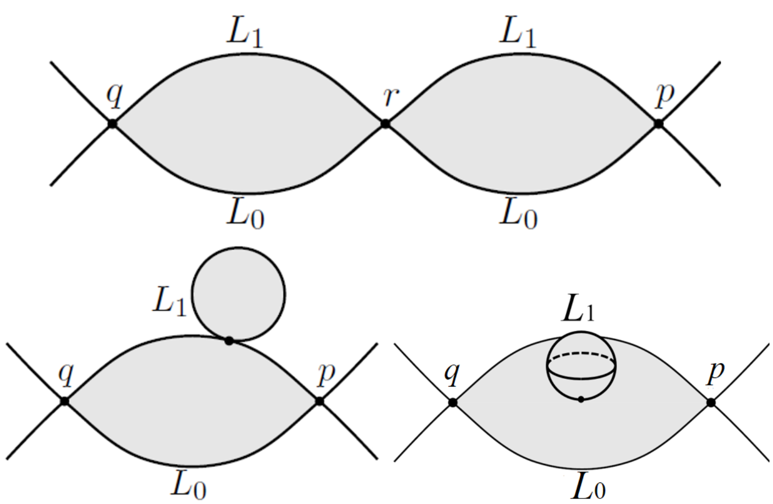

FIGURE 3. Possible limits of pseudoholomorphic strips: broken strip (upper), disc bubble (lower left) and sphere bubble (lower right).

The Broken strip is the key to prove that the Floer differential squares to zero, and the conditions $\omega \cdot \pi_2(M, L_i) = 0$ for $k = 0, 1$ eliminate the cases of disc bubbles and sphere bubbles. We can move the base point of the sphere bubbles to the boundary along a path on the strips. Therefore both the disc bubbles and the sphere bubbles can be regard as elements in $\pi_2(M, L_i)$. The symplectic area, or say energy, of the bubbles must be zero. All pseudoholomorphic strips with energy zero are constant strips, i.e. points. Therefore all bubbles reduce to points, which are equivalent to the trivial cases without bubbles.

Consider any homotopy class $[u]$ of Maslov Index 2. According to Gromov's compactness theorem, the boundaries of the compactified moduli space $\overline{\mathcal{M}}(p, q; [u], J)$ consist of broken strips connecting p to q in the class $[u]$. The components of the broken strips correspond to the moduli spaces $\mathcal{M}(p, r; [u'], J)$ and $\mathcal{M}(r, q; [u''], J)$, where r is any generator of the the Floer complex and the relevant homotopy classes satisfy $[u'] + [u''] = [u]$. Since the Maslov index is additive under the addition of homotopy classes, according to the dimension formula dim $\mathcal{M}(p, q; [u], J) = \mathrm{ind}([u]) - 1$, the Maslov index of non-constant strip must be at least one. Thus, the only possibility is a broken strip consisting of two components of Maslov index 1. Besides, the gluing theorem states that every broken strip is locally the limits of a unique family of index 2 strips. Therefore, the boundaries of the compactified moduli space is precisely the case of broken strips, and each broken strip consists of 2 strips of Maslov index 1. In conclusion, we have

$$\partial\overline{\mathcal{M}}(p, q; [u], J) = \coprod_{\substack{r\in\mathcal{X}(L_0,L_1)\\ [u']+[u'']=[u]\\ \mathrm{ind}([u'])=\mathrm{ind}([u''])=1}} \left(\mathcal{M}\left(p, r; [u'], J\right) \times \mathcal{M}\left(r, q; [u''], J\right)\right)$$

Since the total number of the boundary points of a compact 1-manifold is always zero modulo 2, we have

$$\partial^2 p = \sum_{\substack{r\in\mathcal{X}(L_0,L_1)\\ [u']:\mathrm{ind}([u'])=1}} (\#\mathcal{M}(p, r; [u'], J))T^{\omega([u'])}\partial r$$

$$= \sum_{\substack{r\in\mathcal{X}(L_0,L_1)\\ [u']:\mathrm{ind}([u'])=1}} \sum_{\substack{q\in\mathcal{X}(L_0,L_1)\\ [u'']:\mathrm{ind}([u''])=1}} (\#\mathcal{M}(p, r; [u'], J))(\#\mathcal{M}(r, q; [u''], J))T^{\omega([u'])+\omega([u''])}q.$$

$$= \sum_{\substack{q\in\mathcal{X}(L_0,L_1)\\ [u]:\mathrm{ind}([u])=2}} \left(\sum_{\substack{r\in\mathcal{X}(L_0,L_1)\\ [u']+[u'']=[u]\\ \mathrm{ind}([u'])=\mathrm{ind}([u''])=1}} (\#\mathcal{M}(p, r; [u'], J))(\#\mathcal{M}(r, q; [u''], J)) \right) T^{\omega([u])}q.$$

$$= 0.$$

□

Remark 2.21. The Gromov's compactness theorem above is a useful tool to set up the algebraic structure and this "boundary method" is widely used in proof of the A_∞ structure of the Fukaya category. One can see the power of this method in Section 2.3.

Remark 2.22. Given certain spin structures, it is easy to generalize this proof to the case of $\mathbb{Z}$-grading just by replacing the total unsigned number of the boundary points of the compactified moduli space with the signed number of the boundary points.

2.2.2. *$\mathbb{Z}$-grading of Floer complex.* In order to give a $\mathbb{Z}$-grading of the generators of the Floer complex, we need some extra data of the Lagrangian manifolds L_0 and L_1: a choice of the graded lift of the canonical section of the Lagrangian submanifold. A Lagrangian submanifold with such a choice is called a graded Lagrangian submanifold. There are two extra conditions to ensure the existence of such lifts:

(1) The first Chern class of M must be 2-torsion, namely $2c_1(TM) = 0$.

(2) The Maslov class[12] of L_i: $\mu_{L_i} \in H^1(L_i, \mathbb{Z})$ must be zero, for $i = 0, 1$.

Remark 2.23. If one is satisfied with the $\mathbb{F}_2$-grading, then we do not need the two conditions above, the only requirement is that the Lagrangian manifolds L_0 and L_1 are oriented, in which case the degree of a generator p of the Floer complex is determined by comparing the orientation of L_0 and L_1 at p: if the orientation at T_pL_0 can be continuously extended to the orientation at T_pL_1 along the canonical short path in the Lagrangian Grassmann, then the degree of p is zero, otherwise the degree of p is one.

We introduce some relevant notions before we define the $\mathbb{Z}$-grading of the Floer complex. We define $LGr(TM)$ to be the Lagrangian Grassmann of M, namely the fiber of $LGr(TM)$ is the Lagrangian Grassmann of T_pM. To construct the grading structure of Lagrangian submanifolds, we wish to construct a map $\Omega : LGr(TM) \longrightarrow S^1$, which is fiberwisely equivalent to det^2, i.e., we have the commutative diagram

$$\begin{tikzcd} LGr(n) \arrow[rr, "\cong"] \arrow[dr, "\Omega|_{\text{fiber}}"'] & & U(n)/O(n) \arrow[dl, "det^2"] \\ & S^1 & \end{tikzcd} \tag{2.24}$$

However, the isomorphism $LGr(n) \cong U(n)/O(n)$ is not canonical, thus we need an extra condition $2c_1(TM) = 0$ to ensure the existence of such map. In fact, according to the relation between the first Chern class of the tangent bundle and the canonical bundle, we have

$$c_1((\bigwedge_{\mathbb{C}}^n T^*M)^{\otimes 2}) = 2c_1(\bigwedge_{\mathbb{C}}^n T^*M) = -2c_1(TM) = 0,$$

which implies that the complex line bundle $(\bigwedge_{\mathbb{C}}^n T^*M)^{\otimes 2}$ is trivial. Let Θ be a nowhere vanishing section of $(\bigwedge_{\mathbb{C}}^n T^*M)^{\otimes 2}$. For any Lagrangian subspace $L_p \in LGr(T_pM)$, we define

$$\Omega(L_p) = \frac{\Theta(v_1, ..., v_n, v_1, ..., v_n)}{|\Theta(v_1, ..., v_n, v_1, ..., v_n)|},$$

where $v_1, ..., v_n$ is a set of real basis of L_p. It is easy to see Ω does not depend on the choice of basis. The map Ω is just a fiberwise det^2 map. It is easy to see the square of the Calabi-Yau form is a nowhere vanishing section of $(\bigwedge_{\mathbb{C}}^n T^*M)^{\otimes 2}$, therefore the condition that the first Chern class is 2-torsion is a kind of “weak” Calabi-Yau condition. We define the canonical section s_L of $LGr(TM)$ over a Lagrangian submanifold L by $s_L(p) = T_pL \in LGr(T_pM)$, and call the composite map $\varphi_L = \Omega \circ s_L : L \longrightarrow S^1$ the phase square map. A grading of the Lagrangian submanifold L is defined by a choice of the lifting of the phase square map through the standard covering $e^{2\pi it} : \mathbb{R} \longrightarrow S^1$, namely a choice of $\widetilde{\varphi}_L$ in the commutative

[12] The Maslov class should not be confused with the Maslov index we defined before. The Maslov index determines the dimension of the moduli space of a certain homotopy class, while the Maslov class describes the obstruction of lifting a canonical section of Lagrangian Grassmann bundle to its universal covering.

diagram

$$\begin{tikzcd}
\widetilde{LGr}(TM) \arrow[rr, "\widetilde{\Omega}"'] \arrow[d] & & \mathbb{R} \arrow[d, "e^{2\pi i t}"] \\
LGr(TM) \arrow[rr, "\Omega"'] \arrow[d] & & S^1 \\
M & L \arrow[l, hook'] \arrow[ul, "s_L"] \arrow[ur, "\varphi_L"'] \arrow[uul, dashed, "\widetilde{s}_L"] \arrow[uur, dashed, "\widetilde{\varphi}_L"'] &
\end{tikzcd} . \tag{2.25}$$

We define a graded Lagrangian submanifold by $\widetilde{L} = (L, \widetilde{\varphi}_L)$. Let $\widetilde{LGr}(TM) \longrightarrow LGr(TM)$ be the pullback covering space of $\mathbb{R} \longrightarrow S^1$ through Ω. By the isomorphism $det^2_* : \pi_1(U(n)/O(n)) \cong \pi_1(S^1)$ and the fact that Ω is fiberwisely equivalent to det^2, the pullback bundle $\widetilde{LGr}(TM)$ is a fiberwise universal covering of $LGr(TM)$. Let

$$\widetilde{\Omega} : \widetilde{LGr}(TM) \longrightarrow \mathbb{R}$$

be the unique lifting of Ω via the covering maps in diagram 2.25. According to the commutative diagram 2.25 and the property of the pullback covering space, there is a 1-1 correspondence between the lifting section $\widetilde{s}_L$ of the canonical section s_L and the lifting phase square map $\widetilde{\varphi}_L$ via the condition $\widetilde{\varphi}_L = \widetilde{\Omega} \circ \widetilde{s}_L$. Therefore, a grading of the Lagrangian submanifold L can be equivalently defined as a choice of the lifting canonical section $\widetilde{\varphi}_L$.

In order to understand when such a lifting exists, we define the following Maslov class.

Definition 2.26. The Maslov class μ_L of a Lagrangian submanifold L is defined to be the first $\mathbb{Z}$-coefficient cohomology class of L corresponding to the homotopy class of the phase square map φ_L, namely $\mu_L = [\varphi_L] \in [L, S^1] \cong Hom(\pi_1(L), \mathbb{Z}) \cong H^1(L, \mathbb{Z})$.

The Maslov class is the obstruction of lifting φ_L through the standard covering $\mathbb{R} \longrightarrow \mathbb{Z}$, which means that the existence of such a lifting is equivalent to $[\varphi_L] = 0$. Therefore, under the condition: $2c_1(TM) = 0$ and $\mu_L = 0$, we can equip the Lagrangian submanifold L with a grading. It is easy to see that for each connected component of L, the choices of the grading are in one-to-one correspondence to $\mathbb{Z}$.

Definition 2.27. The $\mathbb{Z}$-grading of the Floer complex between two graded Lagrangian submanifolds $\widetilde{L}_0$ and $\widetilde{L}_1$ is defined as follows. Given $p \in \widetilde{L}_0 \cap \widetilde{L}_1$, also as an element of $CF(\widetilde{L}_0, \widetilde{L}_1)$, choose a path from $\widetilde{s}_{L_0}(p)$ to $\widetilde{s}_{L_1}(p)$ in $\widetilde{LGr}(T_pM)$ and then project it to $LGr(T_pM)$. The homotopy class of the projected path does not rely on the choice of the path in the fiberwise universal covering space $\widetilde{LGr}(T_pM)$. Concatenate this projected path with the canonical short path from T_pL_1 to T_pL_0 to get a loop in $LGr(T_pM)$. Then the degree of p is defined to be the Maslov index of the loop.

It is easy to check that any strip u connecting p to q satisfies

$$\operatorname{ind}([u]) = \deg(q) - \deg(p),$$

namely the Maslov index of a strip is precisely the change of the degrees of two intersections. In particular, the Floer differential ∂ is then of degree 1. In conclusion, we have established the chain complex structure of the Floer complex $CF(\widetilde{L}_0, \widetilde{L}_1)$, and naturally we have

Definition 2.28. The Floer cohomology $HF(\widetilde{L_0}, \widetilde{L_1})$ between graded Lagrangian submanifolds $\widetilde{L_0}$ and $\widetilde{L_1}$ is defined to be $H^*(CF(\widetilde{L_0}, \widetilde{L_1}), \partial)$.

2.2.3. *Hamiltonian perturbation.* Here we give a brief introduction of the Hamiltonian perturbation, which we have mentioned in the definition of the Floer cohomology. If one wants to define the Floer complex $CF(L_0, L_1)$ when L_0 and L_1 do not intersect transversely, especially when $L_0 = L_1$, then the Hamiltonian perturbation is necessary. Let $H_t \in C^\infty(M \times [0,1])$ be a time-dependent Hamiltonian, X_{H_t} be the Hamiltonian vector field generated by H_t and ϕ_t be the one-parameter flow, also called Hamiltonian isotopy, generated by X_{H_t}. Let $J_t \in C^\infty([0,1], \mathcal{J}(M,\omega))$ be a generic smooth family of ω-compatible almost complex structures. The choices of H_t and J_t, which are called the Floer data, should guarantee the regular condition (the linearization operator $D_{\bar{\partial}_J, u}$ is surjective at any strips u) and that L_0 and $\phi_1^{-1} L_1$ intersect transversely. Then Cauchy-Riemann equation (2.7) must be replaced by the following equation

$$\frac{\partial \widetilde{u}}{\partial s} + \widetilde{J}_t(\widetilde{u}) \frac{\partial \widetilde{u}}{\partial t} = 0,$$

where $\widetilde{J} = (\phi_t)_*^{-1} J_t$ and $\widetilde{u}(s,t) = (\phi_t)^{-1}(u(s,t))$. Then the Floer complex with perturbation is defined as

$$CF(L_0, L_1; H_t, J_t) := CF(L_0, (\phi_t)^{-1}(L_1); J_t).$$

There is also a theorem saying that the Floer complex is independent of the perturbation (H_t, J_t) up to chain homotopy equivalence. Please refer to D. Auroux's note [10] for the proof of the theorem.

Theorem 2.29. *Given two Lagrangian submanifolds L_0 and L_1. Then the Floer complex $(CF(L_0, L_1), \partial)$ does not depend on the choice of the time-dependent almost-complex structure and the Hamiltonian isotopy, up to chain homotopy equivalence.*

2.3. **A_∞-Category and Fukaya Category.** In this section we introduce the Fukaya category, which encodes the higher algebraic structures on the Floer cohomology theory. We always assume that the first Chern class of the symplectic manifold M is 2-torsion and the Maslov classes of the involved Lagrangian manifolds vanish unless stated otherwise. Then the Floer complexes have $\mathbb{Z}$-gradings.

2.3.1. *A_∞-category.* Firstly, we introduce a special category called the A_∞-category, which is not an honest category but the best description of the algebraic structure of the Floer cohomology theory. The "A" here refer to associativity, and the "∞" indicates that the associativity is relaxed up to higher homotopies without bound on the degree of the homotopies.

Definition 2.30. An A_∞-category $\mathcal{A}$ consists of a collection of objects $\mathrm{Ob}(\mathcal{A})$ and $\mathbb{Z}$-graded vector spaces $\mathrm{Hom}_{\mathcal{A}}(A,B)$ as hom-sets for any pair of objects $A,B\in\mathrm{Ob}(\mathcal{A})$ such that for every $k>0$ and every set of objects $A_0,...,A_k\in\mathrm{Ob}(\mathcal{A})$, there are multi-linear maps m_k of degree $2-k$

$$m_k:\mathrm{Hom}_{\mathcal{A}}(A_{k-1},A_k)\otimes...\otimes\mathrm{Hom}_{\mathcal{A}}(A_0,A_1)\longrightarrow\mathrm{Hom}_{\mathcal{A}}(A_0,A_k)[2-k]$$

satisfying the A_∞-relations

$$\sum_{\ell=1}^{k}\sum_{j=0}^{k-\ell}(-1)^\sigma m_{k+1-\ell}\left(f_k,\dots,f_{j+\ell+1},m_\ell\left(f_{j+\ell},\dots,f_{j+1}\right),f_j,\dots,f_1\right)=0, \tag{2.31}$$

where $\sigma=j+\deg(f_1)+...\deg(f_j)$, $f_i\in\mathrm{Hom}_{\mathcal{A}}(A_{i-1},A_i)$ for $i=1,...,k$.

Remark 2.32. When $k=1$, (2.31) becomes $m_1^2=0$, which means that m_1 is a differential of the hom-set. The case $k=2$ corresponds to the following Leibniz rule

$$m_1(m_2(p_2,p_1))=\pm m_2((m_1(p_2)),p_1)\pm m_2(p_2,(m_1p_1)).$$

It implies that m_1 is a derivation with respect to the composition m_2 and therefore m_2 also induces a composition map on the homology of the hom-sets. For $k=3$, the A_∞-relation implies that the composition m_2 is not associative, but up to an explicit homotopy given by m_3:

$$\begin{aligned}&\pm m_2(f_3,m_2(f_2,f_1))\pm m_2(m_2(f_3,f_2),f_1)=\pm m_1(m_3(f_3,f_2,f_1))\\&\pm m_3(m_1(f_3),f_2,f_1)\pm m_3(f_3,m_1(f_2),f_1)\pm m_3(f_3,f_2,m_1(f_1)).\end{aligned}$$

We define the homotopy category $\mathrm{H}^0\,\mathcal{A}$ of an A_∞-category by

$$\mathrm{Ob}(\mathrm{H}^0\,\mathcal{A})=\mathrm{Ob}(\mathcal{A}),$$

and

$$\mathrm{Hom}_{\mathrm{H}^0\,\mathcal{A}}(A,B)=\mathrm{H}^0(\mathrm{Hom}_{\mathcal{A}}(A,B),m_1),$$

As a consequence of the A_∞-relation, the homotopy category $\mathrm{H}^0\,\mathcal{A}$ is an honest category, probably without identities. Although we do not require that the A_∞-category have identities, we often expect that the homotopy category $\mathrm{H}^0\,\mathcal{A}$ has identities. An A_∞ category whose homotopy category has identities is called cohomologically unital or c-unital.

An A_∞ functor between A_∞-categories is by definition a functor satisfying additional condition with respect to the A_∞-structure maps. We will not give the technical details of the definition of the A_∞-functor and refer readers to [19]. An A_∞ functor between A_∞-categories is called a quasi-equivalence if it induces an equivalence between the homotopy categories.

2.3.2. *Multi-product of the Floer complexes.* To get the A_∞ structure of the Floer cohomology theory, we need to define the multi-products of the Floer complexes. For $i=0,1,...,k$, let L_i be a Lagrangian submanifold and we assume that the transversality and the regular conditions hold via an appropriate Hamiltonian perturbation. Choose certain spin structures if one wants to work on $\mathbb{Z}$-coefficients. Consider intersections $p_i\in L_{i-1}\cap L_i$ $(i=1,...,k)$, $q\in L_0\cap L_k$ and choose $k+1$ prior fixed points $z_0,z_1,...,z_k$ on the boundary of D^2. As what we have done before, let $\widehat{\mathcal{M}}(p_1,...,p_k,q;[u],J)$ be the moduli space of pseudoholomorphic curves $u:D^2\longrightarrow M$ of finite energy in the homotopy class $[u]\in\pi_2(M,\bigcup_{i=0}^k L_i)$, mapping arcs from z_i to z_{i+1} to L_i and the boundary points $z_1,...,z_k,z_0$ to $p_1,...p_k,q$

respectively. The case when $k = 2$ is showed in Figure 2.3.2. The definition of

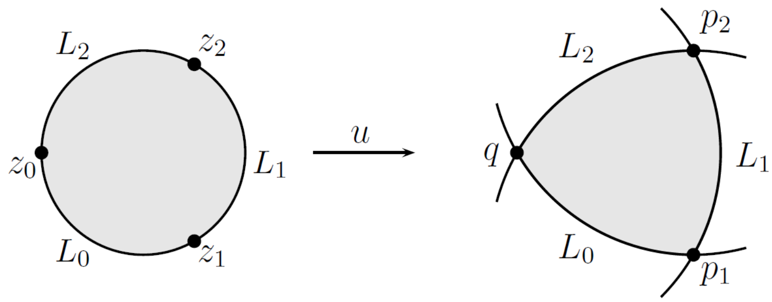

FIGURE 4. A pseudoholomorphic disc contributing to the product map

the Maslov index in this situation is similar to Definition 2.13. We pull back the tangent bundle of M through the curve, which turns out to be a trivial bundle on D^2. The tangent spaces of L_i, for $i = 0, ..., k$, together with appropriate canonical paths at the intersections, form a loop on the Lagrangian Grassmann $LGr(n)$, then the Maslov index of $[u]$ is the integer corresponding to the loop through det_*^2. One can calculate the Maslov index of u in terms of the degrees of the intersections:

$$\operatorname{ind}([u]) = \deg(q) - \sum_{i=1}^{k} \deg(p_i). \tag{2.33}$$

The dimension of the moduli space is given by the Maslov index of $[u]$ as before. Let $\mathcal{M}(p_1, ..., p_k, q; [u], J)$ be the moduli space of the pseudoholomorphic curves $u : D^2 \longrightarrow M$ (where the boundary points $z_0, ...z_k$ are not fixed a prior) satisfying the Lagrangian boundary condition above and has finite energy, up to reparametrization by $Aut(D^2)$. Since every automorphism of D^2 has the form $e^{i\theta} \cdot \frac{\alpha - z}{1 - \bar{\alpha} z}$, where $\theta \in \mathbb{R}$ standing for rotation, and $\alpha \in (D^2)^\circ \subseteq \mathbb{C}$. Therefore, there are three real degrees of freedom of $Aut(D^2)$. By counting degrees of freedom, we get the dimension of the moduli space $\mathcal{M}(p_1, ..., p_k, q; [u], J)$:

$$\begin{aligned} \dim \mathcal{M}(p_1, ..., p_k, q; [u], J) &= \dim \widehat{\mathcal{M}}(p_1, ..., p_k, q; [u], J) + (k+1) - 3 \\ &= \operatorname{ind}([u]) + k - 2. \end{aligned} \tag{2.34}$$

We define multi-products m_k in the following.

Definition 2.35. The multi-products operation

$$m_k : CF(L_{k-1}, L_k) \otimes ... \otimes CF(L_0, L_1) \longrightarrow CF(L_0, L_k)$$

is Λ-linear maps defined by

$$m_k\left(p_k, \ldots, p_1\right) = \sum_{\substack{q \in \mathcal{X}(L_0, L_k) \\ [u]:\operatorname{ind}([u]) = 2-k}} \left(\#\mathcal{M}\left(p_1, \ldots, p_k, q; [u], J\right)\right) T^{\omega([u])} q. \tag{2.36}$$

According to the index formula (2.33), the multi-product m_k is of degree $2-k$. It is obvious from the definition that the first order product m_1 is the Floer differential ∂ and the higher order products can be regard as the generalization of it. Just as the nilpotent property of the Floer differential, the multi-products satisfy the high order A_∞-relations as one can imagine.

Theorem 2.37. *If $\omega \cdot \pi_2(M, L_i)$ for $i = 0, ..., k$, then the operators m_k satisfy the A_∞-relations*

$$\sum_{\ell=1}^{k} \sum_{j=0}^{k-\ell} (-1)^\sigma m_{k+1-\ell}\left(p_k, \ldots, p_{j+\ell+1}, m_\ell\left(p_{j+\ell}, \ldots, p_{j+1}\right), p_j, \ldots, p_1\right) = 0, \tag{2.38}$$

where $\sigma = j + \deg(p_1) + ... \deg(p_j)$.

Proof. The method of this proof is similar to the nilpotent property 2.19, thus we sketch it. Consider the compactified 1-dimensional moduli space $\overline{\mathcal{M}}(p_1, \ldots, p_k, q; [u], J)$, where $\operatorname{ind}([u])$ is $3-k$. The conditions $\omega \cdot \pi_2(M, L_i)$, $i = 0, ...k$ eliminate the disc and sphere bubbles and the only possibilities are broken discs[13]. In higher order cases, the number of the boundary points z_i on each component can be both more than 1. For example, Figure 5 shows two possible broken discs when $k = 3$, and each component has two boundary points. Because of the dimension formula (2.34), the Maslov index of the pseudoholomorphic discs with k boundary points is no less than $2-k$. According to this index restriction and the additional property of the Maslov indexes, the boundaries of the moduli space precisely consist of two-component broken discs. Again, we use the fact $\#\partial\overline{\mathcal{M}}(r, q; [u], J)) = 0 \mod 2$ and interpret each terms, then we get the A_∞ relations. $\square$

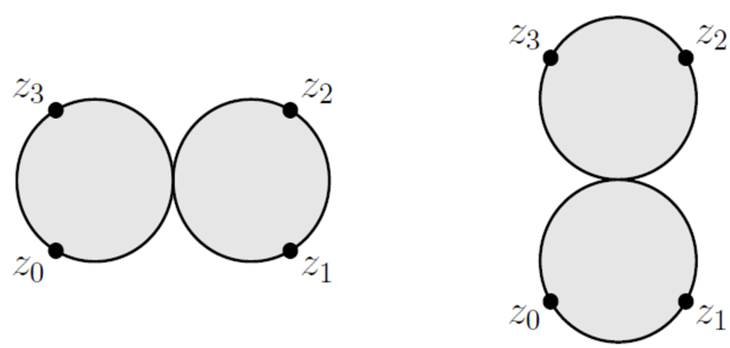

FIGURE 5. Two possible limits of a pseudoholomorphic disc when $k = 3$

Remark 2.39. In general, we should do Hamiltonian perturbation to modify the definition of the multi-products. More specifically, for every Lagrangian submanifold pair (L_i, L_j) for $i, j = 0, ..., k$, we choose a suitable time-dependent Hamiltonian $H_t^{(ij)} \in C^\infty([0,1] \times M, \mathbb{R})$ and time-dependent almost complex structure $J_t^{(ij)} \in C^\infty([0,1], \mathcal{J}(M, \omega))$. Besides, there is an extra requirement that $J_t^{(ij)}$ and $H_t^{(ij)}$ must be "compatible". Then the multi-product m_k is a map

$$CF(L_1, L_2; H_t^{(k-1\,k)}, J_t^{(k-1\,k)}) \otimes \cdots \otimes CF(L_0, L_1; H_t^{(01)}, J_t^{(01)}) \longrightarrow CF(L_0, L_2; H_t^{(0k)}, J_t^{(0k)})$$

on the perturbed Floer complex. Please refer to D. Auroux's note [10] for more details about the perturbation.

[13]The previous broken strip case can be viewed as a special broken disc, on which there is only one of the boundary points $z_i, i = 0, ..., k$.

2.3.3. *Fukaya category.* Finally, given the definition of the multi-products of the Floer complexes, we give the definition of the Fukaya category, which is an A_∞-category.

Definition 2.40. Let (M,ω) be a symplectic manifold with $2c_1(TM)=0$. The object of the Fukaya category are $Fuk(M)$ compact, closed Lagrangian submanifolds L such that $\omega\cdot\pi_2(M,L)=0$ and with vanishing Maslov classes, together with the choice of a graded lift of the phase square map φ_L. We denote the objects by $\widetilde{L}=(L,\varphi_L)$. (A choice of spin structure is needed if one wants to work over a field that isn't of characteristic 2.)

For every pair of objects (L_i,L_j), we choose a time-dependent Hamiltonian perturbation $H_t^{(ij)}\in C^\infty([0,1]\times M,\mathbb{R})$ and $J_t^{(ij)}\in C^\infty([0,1],\mathcal{J}(M,\omega))$ to achieve the transversality and the regular condition for all moduli spaces used in the definition of multi-product. The hom-set of the Fukaya category between L_i and L_j is then the perturbed Floer complex $CF(L_i,L_j;H_t^{(ij)},J_t^{(ij)})$. The multi-products m_k of the Fukaya category are given by Definition 2.35.

Remark 2.41. The actual chain-level details of the Fukaya category depend on the choices of the perturbation. However, the Fukaya category with different choices of the perturbation are quasi-equivalent to each other as A_∞ categories.

Remark 2.42. There are many definitions of the Fukaya category, relying on the desired generality and the implement details, while the common features are the same. The definition above is a relatively simple one, which is close to the original definition of K. Fukaya. But in homological mirror symmetry, people use a richer version of the Fukaya category. The object L should be equipped with a unitary local system, namely a flat vector bundle over the $\mathbb{C}$-coefficients Novikov field $\mathcal{E}\longrightarrow L$ with unitary holonomy. The Floer complex is defined by

$$CF\left(\left(L_i,\mathcal{E}_i\right),\left(L_j,\mathcal{E}_j\right)\right)=\bigoplus_{p\in\mathcal{X}(L_i,L_j)}\operatorname{Hom}\left(\mathcal{E}_{i|p},\mathcal{E}_{j|p}\right)$$

Consider objects $(L_0,\mathcal{E}_0),...,(L_k,\mathcal{E}_k)$ and intersections $p_1,...,p_k,q$. Set $p_0=p_{k+1}=q$. Let $\gamma_i\in\operatorname{Hom}(\mathcal{E}_i|_{p_i},\mathcal{E}_i|_{p_{i+1}})$ be the isomorphism induced by the parallel transport along the portion of the boundary of $[u]$ on L_i. Given elements of $\rho_i\in\operatorname{Hom}(\mathcal{E}_{i-1}|_{p_i},\mathcal{E}_i|_{p_i})(i=1,...,k)$. The composition of all these linear maps gives an element $\eta_{[u],\rho_k,...,\rho_1}:=\gamma_k\circ\rho_k\circ...\circ\gamma_1\circ\rho_1\circ\gamma_0\in\operatorname{Hom}(\mathcal{E}_0|_q,\mathcal{E}_k|_q)$. Then the multi-products are defined by

$$m_k\left(\rho_k,\ldots,\rho_1\right)=\sum_{\substack{q\in\mathcal{X}(L_0,L_k)\\ [u]:\operatorname{ind}([u])=2-k}}\left(\#\mathcal{M}\left(p_1,\ldots,p_k,q;[u],J\right)\right)T^{\omega([u])}\eta_{[u],\rho_k,\ldots,\rho_1}.$$

2.4. **Twisted Fukaya Category and Derived Fukaya Category.** To get the category of A-branes, there is one step remain: triangulation. The triangulation physically corresponds to the stability of the D-branes, which is essential in connecting with "the real world", namely the untwisted theory[14]. However, there is no natural triangulated structure in the Fukaya category. Therefore we need to "twist" it to get a triangulated A_∞-category, called twisted Fukaya category.

[14]To be more precise, D-branes in topological field theory should correspond to a BPS state in the untwisted theory. This BPS condition imposes a further condition on D-branes, namely "stability".

Roughly speaking, a triangulated category is a category which admits a shift functor and a collection of exact triangles, by which one can take mapping cones of any morphisms. For rigor definition of the triangulated category, please refer to C. A. Weibel's book [15].

Definition 2.43. Let $\mathcal{A}$ be an A_∞-category, the twisted A_∞-category $\operatorname{Tw}\mathcal{A}$ is defined as follows. The objects of $\operatorname{Tw}\mathcal{A}$ are pairs $(\bigoplus_i A_i[k_i], \delta_A)$, where $A_i \in \mathcal{A}$, $k_i \in \mathbb{Z}$ for $i = 1, ..., n$, $A_i[k_i]$ is a formal shift of A_i and

$$\delta_A \in \bigoplus_{i,j} \operatorname{Hom}_{\mathcal{A}}(A_i, A_j)[1 + k_j - k_i]$$

is a strictly upper triangular matrix, namely $(\delta_A)_{ij} = 0$ if $i \geq j$, satisfying the Maurer-Cartan equation[15]

$$\sum_{i=0}^{\infty} m_i(\delta^{\otimes i}) = 0,$$

where the multi-products of the morphism matrix are defined by the rules of matrix multiplication, namely

$$(m_d(\delta_1, ..., \delta_d))_{ij} := \sum_{i_1,...,i_{d-1}} m_d((\delta_1)_{ii_1}, ..., (\delta_d)_{i_{d-1}j}).$$

Let

$$\operatorname{Hom}_{\operatorname{Tw}\mathcal{A}}((\bigoplus_i A_i[k_i], \delta_A), (\bigoplus_j B_j[l_j], \delta_B)) := \bigoplus_{i,j} \operatorname{Hom}_{\mathcal{A}}(A_i, B_j)[l_j - k_i].$$

We also define

$$\operatorname{Hom}^p_{\operatorname{Tw}\mathcal{A}}((\bigoplus_i A_i[k_i], \delta_A), (\bigoplus_j B_j[l_j], \delta_B)) = \bigoplus_{i,j} \operatorname{Hom}_{\mathcal{A}}(A_i, B_j)[p + l_j - k_i].$$

The multi-products of $\operatorname{Tw}\mathcal{A}$ is defined by "inserting δ in all possible ways"

$$m_k^{\operatorname{Tw}\mathcal{A}}(f_d, ..., f_1) = \sum_{j_0,...,j_d \geq 0} m_{d+j_1+...+j_k}(\delta_d^{\otimes j_d}, f_d, \delta_{d-1}^{\otimes j_{d-1}}, f_{d-1}, ..., f_1, \delta_0^{\otimes j_0}),$$

where $f_i \in \operatorname{Hom}_{\operatorname{Tw}\mathcal{A}}((A_{i-1}, \delta_{i-1}), (A_i, \delta_i))$, $(A_i, \delta_i) \in \operatorname{Ob}(\mathcal{A})$.

One can check that the twisted category $\operatorname{Tw}\mathcal{A}$ is an A_∞-category. Besides, it is also a triangulated category via the following mapping cone:

$$\operatorname{cone}(f) = \left(A[1] \oplus B, \begin{pmatrix} \delta_A & 0 \\ f & \delta_B \end{pmatrix}\right),$$

where $f \in \operatorname{Hom}_{\operatorname{Tw}\mathcal{A}}((A, \delta_A), (B, \delta_B))$ and $m_1^{\operatorname{Tw}\mathcal{A}}(f) = 0$. In fact, the category $\operatorname{Tw}\mathcal{A}$ should be thought of as the smallest triangulated A_∞-category containing $\mathcal{A}$ as a full subcategory.

Definition 2.44. Let $\mathcal{A}$ be an A_∞-category. The derived A_∞-category $\operatorname{D}\mathcal{A}$ is the homotopy category of the twisted A_∞-category $\operatorname{Tw}\mathcal{A}$, namely

$$\operatorname{D}\mathcal{A} := \operatorname{H}^0 \operatorname{Tw}\mathcal{A}.$$

One can show the derived A_∞-category is an honest triangulated category. Ultimately, we get the category of A-model: the derived Fukaya category $\operatorname{D}Fuk(X)$, for any Calabi-Yau manifold X.

[15]The Maurer-Cartan equation is a finite sum because of the strictly upper triangular condition.

3. B-Model: Derived Category of Coherent Sheaves

3.1. Closed and Open String B-Model.

3.1.1. *Closed string B-model.* Now we consider the other side of the mirror symmetry: the B-model. We have mentioned in Section 2.1 that the difference between the A-model and the B-model is just the different kinds of twisting, namely the choice of the bundles where the fermionic fields $\psi_\pm^i, \psi_\pm^{\bar{i}}$ take value. In B-model, let

$$\begin{aligned} \psi_+^i &\in \Gamma\left(K \otimes \phi^* T_X^{(1,0)}\right), \quad \psi_+^{\bar{i}} \in \Gamma\left(\phi^* T_X^{(0,1)}\right) \\ \psi_-^i &\in \Gamma\left(\bar{K} \otimes \phi^* T_X^{(1,0)}\right), \quad \psi_-^{\bar{i}} \in \Gamma\left(\phi^* T_X^{(0,1)}\right) \end{aligned}$$

and set the fermionic parameters $\widetilde{\alpha}_\pm = \alpha, \alpha_\pm = 0$. Set $\eta^{\bar{j}} = \psi_+^{\bar{j}} + \psi_-^{\bar{j}}$ and $\theta_j = g_{j\bar{k}}(\psi_+^{\bar{k}} - \psi_-^{\bar{k}})$. We define a $\phi^* T_X^{(1,0)}$-valued 1-form ρ^i whose $(1,0)$-form part is given by ψ_+^i and $(0,1)$-form part is given by ψ_-^i. As in the A-model, let Q be the generator of the SUSY transformations, which is also called the BRST operator. The BRST operator Q in the B-model is also nilpotent, i.e. $Q^2 = 0$, up to equation of motion. The closed string action can be written in the following form:

$$S = i \int_\Sigma (\{Q, V\} + U),$$

where

$$V = g_{j\bar{k}} \left(\rho_z^j \bar{\partial} \phi^{\bar{k}} + \rho_{\bar{z}}^j \partial \phi^{\bar{k}}\right)$$

and

$$U = \int_\Sigma \left(-\theta_j D\rho^j - \frac{i}{2} R_{j\bar{j}k\bar{k}} \rho^j \wedge \rho^k \eta^{\bar{j}} \theta_l g^{l\bar{k}}\right).$$

The physical local observables are again given by the products of the BRST invariant fields, and every BRST invariant field, namely Q-closed, has the form

$$W_A = \eta^{\bar{k}_1} \ldots \eta^{\bar{k}_q} A_{\bar{k}_1 \ldots \bar{k}_q}^{j_1 \cdots j_p} \theta_{j_1} \ldots \theta_{j_p} \tag{3.1}$$

where

$$A = d\bar{z}^{\bar{k}_1} \ldots d\bar{z}^{\bar{k}_q} A_{\bar{k}_1 \ldots \bar{k}_q}^{j_1 \ldots j_p} \frac{\partial}{\partial z^{j_1}} \cdots \frac{\partial}{\partial z^{j_p}}$$

is a $(0,q)$-form on X valued in $\bigwedge^p T_X$. Analogously to the A-model, we find that

$$\{Q, W_A\} = -W_{\bar{\partial} A}.$$

Therefore, in the B-model, the Q-cohomology is the $\bigwedge^p T_X^{(1,0)}$-valued Dolbeault cohomology on the target manifold $H^{0,q}(X, \bigwedge^p T_X^{(1,0)})$, which implies that the B-model only relies on the complex structure of the target manifold X.

3.1.2. *Open string B-model.* In the case of open strings, as in the A-model, we should consider the worldsheet Σ with boundaries on D-branes. In the B-model, the D-branes are holomorphic submanifolds of the target Calabi-Yau threefold X. We denote the D-branes of dimension p by Dp-branes[16], where p can only be $0, 2, 4, 6$. For simplicity, let us look at the D6-branes, namely the branes filling the whole target manifold. The space-filling condition requires

$$\theta_j = g_{j\bar{k}}(\psi_+^{\bar{k}} - \psi_-^{\bar{k}}) = 0.$$

[16]More precisely, a Dp-brane is of dimension p in the direction of the Calabi-Yau direction and any number of dimensions in the uncompactified direction of the space-time.

Since every BRST invariant local operator has the form 3.1, the BRST invariant local operator only depends on $\eta^{\bar{j}}$, which corresponds to $d\bar{z}^{\bar{j}}$. Thus, in the case of D6-brane, the BRST invariant local operators must be $(0,q)$-forms, possibly valued in some bundles. Therefore, the objects of the category of D6-branes are given by holomorphic bundles $E \longrightarrow X$. Given two D6-branes $E_1 \longrightarrow X$ and $E_2 \longrightarrow X$, let $\mathrm{Hom}(E_1, E_2)$ be the set of bundle morphisms, which is also a vector bundle, called the hom-bundle. We define the local operator representing an open string state, or say a morphism of the B-branes, stretching from $E_1 \longrightarrow X$ to $E_2 \longrightarrow X$ to be W_A, where A is a $\mathrm{Hom}(E_1, E_2)$-valued $(0,q)$-form on X. Since the change of the action by a Q-exact operator will not cause any observable physical effects, the actual hom-set of the B-branes is the Dolbeault cohomology group

$$H^{0,q}_{\bar{\partial}}(X, \mathrm{Hom}(E_1, E_2)).$$

If we want to discuss the B-branes of higher codimensions, namely the branes no longer fill the whole target space X, then we need to generalize the notion of vector bundles to coherent sheaves. One of the best advantages of the coherent sheaves is that they are able to support only on a submanifold of the target space, which perfectly matches the B-branes of higher codimensions. Roughly speaking, the category of coherent sheaves is the minimal, full abelian category arising as the "completion" of the category of locally-free sheaves by adding all kernels and cokernels. For precise definitions of the coherent sheaves, please refer to R. Hartshorne's book [2]. Apart from that, we denote the locally free coherent sheaves corresponding to E_1 and E_2 by $\mathscr{E}_1$ and $\mathscr{E}_2$ respectively. There is a classical relation called Čech-Dolbeault isomorphism

$$H^{0,q}_{\bar{\partial}}\left(X, \mathrm{Hom}\left(E_1, E_2\right)\right) \cong \check{H}^q\left(X, \mathscr{H}\mathrm{om}\left(\mathscr{E}_1, \mathscr{E}_2\right)\right), \tag{3.2}$$

where $\mathscr{H}\,\mathrm{om}\left(\mathscr{E}_1, \mathscr{E}_2\right)$ is the locally free sheave corresponding to the hom-bundles and $\check{H}^q$ is the qth Čech cohomology group. Given an $\mathscr{O}_X$-module $\mathscr{F}$, the Čech cohomology groups are isomorphic to the sheaf cohomology groups

$$H^q(X, \mathscr{F}) \cong \check{H}^q(X, \mathscr{F}), \tag{3.3}$$

where the sheaf cohomology is by definition the right derived functor of the global section functor $H^q(X, \mathscr{F}) = R^q\Gamma(X, \mathscr{F})$. Since $\Gamma(X, -) = \mathrm{Hom}(\mathscr{O}_X, -)$, we have

$$H^q(X, \mathscr{H}\mathrm{om}(\mathscr{E}_1, \mathscr{E}_2)) = R^q\Gamma(X, \mathscr{H}\,\mathrm{om}(\mathscr{E}_1, \mathscr{E}_2)) \cong R^q\,\mathrm{Hom}(\mathscr{E}_1, \mathscr{E}_2) = \mathrm{Ext}^{\mathrm{q}}(\mathscr{E}_1, \mathscr{E}_2) \tag{3.4}$$

The last equality is the definition of Ext groups. Therefore, combine (3.2), (3.3) , (3.4) and we have

$$H^{0,q}_{\bar{\partial}}\left(X, \mathrm{Hom}\left(E_1, E_2\right)\right) \cong \mathrm{Ext}^{\mathrm{q}}(\mathscr{E}_1, \mathscr{E}_2).$$

Therefore, the hom-set of the category of the B-branes is the Ext group of two coherent sheaves. In conclusion, the "naive" definition of the category of B-branes is that the objects are coherent sheaves and the hom-sets are Ext groups. The "naive" here means that there aren't enough B-branes matching the A-branes and the mirror symmetry will fail in this naive definition. Therefore, we need to revise the category of B-branes to the derived category of coherent sheaves.

3.2. **Derived Category of Coherent Sheaves.** We have mentioned in Section 2.1 that there exists some ambiguity when we define the ghost number of an open string. However, in the naive definition of the category of B-branes, one can define the ghost number of an open string from $\mathrm{Ext}^{\mathrm{q}}(\mathscr{E}_1, \mathscr{E}_2)$ by q without ambiguity, which breaks the duality of the A-model and the B-model. Therefore, it is natural to define the objects of the B-branes to be the direct sums of coherent sheaves

$$\mathscr{E} = \bigoplus_{n \in \mathbb{Z}} \mathscr{E}^n,$$

where $\mathscr{E}^n$ is the component of ghost number n. The direct sums can be viewed as chain complexes with trivial differential $\{\mathscr{E}^\bullet, 0\}$, where

$$\mathscr{E}^\bullet = (... \longrightarrow \mathscr{E}^{n-1} \longrightarrow \mathscr{E}^n \longrightarrow \mathscr{E}^{n+1} ...).$$

We can then recover the ambiguity of the ghost number of open strings in B-branes by letting the ghost number of elements of $\mathrm{Ext}^{\mathrm{k}}(\mathscr{E}^n, \mathscr{E}^{n-k+q})$ be q. To get the true definition of the category of B-branes, we consider the case of D6-branes for simplicity. We should consider the deformation of the trivial case to get enough B-branes. The deformation here is to deform the trivial chain complex into non-trivial ones. These non-trivial differentials are called tachyons, particles that signify instability in the string theory. Let us look at the open strings of ghost number 1, namely elements from $\mathrm{Ext}^0(\mathscr{E}^n, \mathscr{E}^{n+1}) \cong \mathrm{Hom}(\mathscr{E}^n, \mathscr{E}^{n+1})$. We define $d = \sum_n d_n \in \mathrm{Hom}(\mathscr{E}, \mathscr{E})$, where $d_n \in \mathrm{Hom}(\mathscr{E}^n, \mathscr{E}^{n+1})$ and the corresponding form is holomorphic. Then we use d to deform the action of the non-linear sigma model by:

$$\delta S = \oint_{\partial\Sigma} \left(\psi_+^i + \psi_-^i\right) \partial_i d$$

The deformation of the action also requires a deformation of the BRST operator as well:

$$Q = Q_0 + d,$$

where Q_0 is the BRST operator before the deformation. We want to keep the nilpotence property $Q^2 = 0$ under the deformation, which is equivalent to

$$\{Q_0, d\} + d^2 = 0.$$

Recall that the BRST operator Q_0 corresponds to $\overline{\partial}$ operator. The term $\{Q_0, d\}$ is zero since the corresponding form of d is holomorphic. Therefore, we get $d^2 = 0$, which makes the object $\mathscr{E}^\bullet$ an non-trivial chain complex. Apart from that, one can show that a map of degree zero $f : \mathscr{E}^\bullet \longrightarrow \mathscr{F}^\bullet$ between two objects is a chain map if and only if it is Q-closed, and two chain maps are chain homotopy if and only if they differ by a Q-exact term. One can imagine that the actual category of the B-branes should be the chain homotopy category of the coherent sheaves $K(Coh(X))$, where the objects are chain complexes of the coherent sheaves and the morphisms are chain homotopy classes of chain maps. However, we also need the renormalization group flow to connect different configurations of the B-branes. The renormalization group flow is a flow in the "space of physical theories", which can make an unstable physical system, namely a physical system with non-trivial tachyons, into a more stable one. Two theories related by renormalization group flow are said to be of the same universality class. There is a physical conjecture that the universality class corresponds to the equivalence class of quasi-isomorphisms.

We say a chain map $f : \mathscr{E}^\bullet \longrightarrow \mathscr{F}^\bullet$ is a quasi-isomorphism if the induced homomorphism between cohomology groups is an isomorphism. In conclusion, the category of B-branes is precisely the (bounded) derived category of the coherent sheaves. The most important fact about the derived category is that it has the structure of the triangulated category. Please refer to C. A. Weibel's book [15] for more details about the derived category. One can also convince himself about the definition of the category of B-branes via the fact

$$\mathrm{Ext}^{\mathrm{q}}(\mathscr{E}_1, \mathscr{E}_2) \cong \mathrm{Hom}_{\mathrm{D^b}\, Coh(X)}(\mathscr{E}_1, \mathscr{E}_2[q]).$$

4. Homological Mirror Symmetry Conjecture

We have set up the categories of both A-branes and B-branes, where the objects are D-branes and the morphisms are open strings. The comparisons between A-branes and B-branes can be concluded in the following table.

	A-branes	B-branes
Geometry	Symplectic	Complex/Algebraic
Category	Fukaya category	Derived category
D-branes	Lagrangian submanifolds	Complexes of coherent sheaves
Open strings	Floer cohomologys	Ext groups

Table 1. The features of A-branes and B-branes

The following homological mirror symmetry conjecture proposed by M. Kontsevich is a remarkable attempt to explain the mathematical essence behind the duality of two types of string theory.

Conjecture 4.1 (M. Kontsevich [1]). *(Homological Mirror Symmetry) The derived Fukaya category of a Calabi-Yau manifold is equivalent to the bounded derived category of the coherent sheaves of its mirror Calabi-Yau manifold as triangulated categories. That is*

$$\mathrm{D}\, Fuk(X) \cong \mathrm{D^b}\, Coh(X^\vee),$$

where $X^\vee$ is the mirror Calabi-Yau manifold of X.

4.0.1. *Karoubi completion.* With the development of the mirror symmetry, we need a little revision of the original statements by considering the Karoubi completeness. The category $\mathrm{D^b}\, Coh(X)$ is Karoubi complete, while $\mathrm{Tw}\, Fuk(X)$ does not have this property.

Definition 4.2. A category $\mathcal{A}$ is called Karoubi complete if every idempotent splits, i.e., for every $p \in Hom_{\mathcal{A}}(A, A)$ such that $p^2 = p$, there is a pair of morphisms $s : A \longrightarrow B$ and $i : B \longrightarrow A$ so that $s \circ i = \mathrm{Id}_B$ and $i \circ s = p$. The B in this definition is called the direct image of p. An A_∞-category is called Karoubi complete if the homotopy category is Karoubi complete.

For categories that are not Karoubi complete, we have the operation of Karoubi completion.

Definition 4.3. Given a triangulated category $\mathcal{A}$. A functor $F : \mathcal{A} \longrightarrow \mathcal{S}$ is a Karoubi completion of $\mathcal{A}$ if F is fully faithful and for every $s \in \mathcal{S}$, there is an idempotent $p \in \operatorname{Hom}_{\mathcal{A}}(A, A)$ such that s is isomorphic to the direct image of $F(p)$. A functor $F : \mathcal{A} \longrightarrow \mathcal{S}$ of triangulated A_∞-categories is a Karoubi completion of $\mathcal{A}$ if the induced functor between the homotopy categories is a Karoubi completion. We also say that $\mathcal{S}$ is the Karoubi completion of $\mathcal{A}$ if there is a functor $F : \mathcal{A} \longrightarrow \mathcal{S}$ that is a Karoubi completion of $\mathcal{A}$. We denote the Karoubi completion of $\mathcal{A}$ by $\Pi\mathcal{A}$.

We also have the following proposition about the existence-uniqueness of the Karoubi completion. In fact, P. Seidel gives the explicit construction of the Karoubi completion in his book [3].

Proposition 4.4. *Every triangulated A_∞-category $\mathcal{A}$ admits a Karoubi completion, and any pair of Karoubi completion of $\mathcal{C}$ are quasi-equivalent.*

Therefore, the category of A-branes should be revised to be the Karoubi completion of the twisted Fukaya category $\Pi \operatorname{Tw} Fuk(X)$.

4.0.2. *Dg enhancement.* We need a category of B-model corresponding to $\Pi \operatorname{Tw} Fuk(X)$ in A-model, which is called the dg enhancement of the derived category of the coherent sheaves. The dg here means the differential graded category, which is by definition the chain complex enrichment of a category.

Definition 4.5. A category $\mathcal{A}$ is a differential graded (dg) category if every hom-set is a $\mathbb{Z}$-graded chain complex satisfying:
(1) The Leibniz rule about the differential and the composition.

$$d_{\mathcal{A}}(g \circ f) = (d_{\mathcal{A}} g) \circ f + (-1)^{i+j} g \circ (d_{\mathcal{A}} f),$$

where $d_{\mathcal{A}}$ is the differential of the hom-set, $f \in \operatorname{Hom}^i_{\mathcal{A}}(A, B)$ and $g \in \operatorname{Hom}^j_{\mathcal{A}}(B, C)$.
(2) The identities map is of degree 0.

A dg category is precisely a unital A_∞-category satisfying $m_i = 0$ for $i > 2$. Like A_∞-category, we can also define the homotopy category by taking the 0th cohomology of the hom-sets. Given a category $\mathcal{A}$, we say that a dg category $\mathcal{C}$ is a dg enhancement of $\mathcal{A}$ if $\mathrm{H}^0 \mathcal{C} = \mathcal{A}$. There are many kinds of construction of the dg enhancement and V. Lunts and D. Orlov proved a beautiful result that the dg enhancement of $\mathrm{D^b}(X)$ is unique up to quasi-equivalence in their paper [20]. Therefore the category of B-branes is revised to be the dg enhancement of the bound derived category of coherent sheaves, denoted by $\mathrm{D^b_{dg}}(X)$. Ultimately, we give the following revised homological mirror symmetry conjecture as the end of this paper.

Conjecture 4.6. *(Revised Homological Mirror Symmetry) Let X and $X^\vee$ be a pair of Calabi-Yau manifolds. There is a quasi-equivalence of A_∞-category*

$$\Pi \operatorname{Tw} Fuk(X) \cong \mathrm{D^b_{dg}}(X^\vee).$$

Remark 4.7. Mathematically, one can think that the definition of the mirror manifolds is given by HMS. However, it is difficult to construct the mirror manifolds via this homological algebraic definition. In fact, Strominger, Yau and Zealow put forward the SYZ symmetry conjecture in their paper [5], which provides a more constructable geometrical definition of the mirror manifolds. Roughly speaking, the SYZ symmetry states that one can get the mirror manifolds by dualizing the special Lagrangian fibrations. See [5] for more details about the SYZ symmetry.

Remark 4.8. Because of the youth and mathematical difficulty of HMS, there are only several established examples, such as elliptic curves, projective lines, and quadratic surfaces etc., among which the elliptic curves are the simplest examples. The construction of the mirror manifolds in the case of elliptic curves is achieved by exchanging the lattice parameter and the symplectic area. We refer the readers to [6] and [7] for elliptic curves, [12] for projective lines and [4] for quadratic surfaces.

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