# SHEAF COHOMOLOGY AND ALGEBRAIC DE RHAM THEOREM

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ABSTRACT. In this paper, we introduce sheaves and the cohomology of sheaves. We use sheaf-theoretic language and tools to specify the correct notion of de Rham cohomology of a smooth algebraic variety. We prove a comparison theorem between the de Rham cohomology of a smooth variety and the singular cohomology of its underlying manifold.

CONTENTS



## 1. INTRODUCTION

Methods of homological algebra were introduced to Algebraic Geometry by Serre in his fundamental paper [17]. The significance of the theory of cohomology of abstract algebraic varieties was first emphasized by A. Weil in [20], providing an idea to prove his famous conjecture. Thus, according to [5], the initial aim of such introduction is to find so called *Weil cohomology* of algebraic varieties to capture arithmetic information. In this paper, we aim to give a short but clear exposition to one such introduction: the de Rham cohomology of algebraic varieties.

We first recall the classical theory of de Rham:

**Definition 1.1.** *The* ***de Rham complex*** *is the complex of differential forms on a smooth manifold M, with the exterior derivative as the differential:*

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \Omega^3(M) \to \cdots,$$

*where $\Omega^k(M)$ is the abelian group of smooth k-forms on M. De Rham cohomology is the cohomology of de Rham complex.*

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De Rham cohomology is an important tool in the study of manifolds. The inexactness of the de Rham complex measures the extent to which the fundamental theorem of calculus fails on general manifolds. One of the most beautiful applications of de Rham cohomology is that it captures the information about the number of (higher-dimentional) *holes* in the manifold. The famous **Poincaré Lemma** tells us that for a contractable domain in a Euclidean space, all closed forms are exact, yielding trivial de Rham cohomology groups of all degrees. In other words, the nontriviality of the de Rham cohomology group indicates the uncontractability, i.e., the existance of *holes* in a manifold. For details, see [15], chapter 18. We also refer to [15], chapter 19 for a proof of the classical de Rham theorem:

**Theorem 1.2.** ***(Classical de Rham theorem, version 1)*** *Let $M$ be a smooth manifold, then we have an isomorphism between de Rham cohomology and singular cohomology with coefficients in $\mathbb{R}$:*

$$H^k_{\mathrm{dR}}(M) \cong H^k_{\mathrm{sing}}(M; \mathbb{R}).$$

We want to introduce classical de Rham cohomology into complex smooth algebraic variety $X$. Firstly, $X$ has an underlying structure of complex manifold which we denote by $X_{\mathrm{an}}$. We can compute singular cohomology and the complex analog of the de Rham cohomology (using holomorphic forms instead of smooth forms) over $X_{\mathrm{an}}$. For $X$, there is an algebraic version of differential form defined using regular functions. Correspondingly, we have a canonical algebraic de Rham complex $\Omega^\bullet_X$ which we will explain later in our paper. We want to explore the relation between the analytic structure and the algebraic structure of the same smooth variety $X$ by conjecturing that the cohomology of algebraic de Rham complex is isomorphic to the singular cohomology of the complex manifold $X_{\mathrm{an}}$. Dissapointingly, this natural conjecture is false in general:

$$H^k_{\mathrm{sing}}(X_{\mathrm{an}}; \mathbb{C}) \ncong H^k(\Omega^\bullet_X). \tag{1.3}$$

The correct definition of Algebraic de Rham cohomology involves the concept of *hypercohomology*. Grothendieck proves the following remarkable theorem in [6]:

**Theorem 1.4.** ***(Algebraic de Rham theorem)*** *Let $X$ be a smooth complex algebraic variety, and $X_{\mathrm{an}}$ be its underlying complex manifold. Then the singular cohomology of $X_{\mathrm{an}}$ with $\mathbb{C}$ coefficients agrees with the hypercohomology of the complex $\Omega^\bullet_X$ of sheaves of regular differentials on $X$ with its Zariski topology:*

$$H^k_{\mathrm{sing}}(X_{\mathrm{an}}, \mathbb{C}) \cong \mathbf{H}^k(X, \Omega^\bullet_X).$$

In this paper, we follow [12] to present a proof for the Algebraic de Rham theorem.

Section 2 and 3 are general introductions to the theory of sheaves and cohomology of sheaves. Section 4 introduces coherent sheaves and some important properties. We will also explain Serre's GAGA principle, which plays an important role in the proof of algebraic de Rham theorem. Section 5 introduces the construction of algebraic de Rham complex and the concept of hypercohomology, which is central to the formulation of algebraic de Rham cohomology. We also state Grothendieck's algebraic de Rham theorem in section 5. The last two sections sketch a proof of algebraic de Rhem theorem. The main idea is that we first reduce the theorem to affine case, i.e., if we can prove the algebraic de Rham theorem on smooth affine varieties, the we can generalize the affine case to all smooth varieties. We then put an affine variety $X$ into a projective variety $Y$, and we transfer information on $X$ to $Y$ on which we can apply some powerful tools (including Theorem 4.12). We split the affine case of the algebraic de Rham theorem into several isomorphisms, and we prove them separately, yielding our desired result.

Grothendieck's algebraic de Rham theorem has many applications in some deep theories in algebraic geometry, such as Deligne's theory of absolute Hodge classes. However, all of these applications involves concepts beyond the scope of the paper, so we refer to [12], chapter 3 for those who are interested in some of the applications.

## 2. Sheaf Theory

We first introduce the notion of sheaves. For any topological space X, let $\mathrm{Open}(X)$ be the category whose objects are open subsets of $X$ and morphisms are inclusions of open subsets.

**Definition 2.1.** *Let $\mathcal{C}$ be an arbitrary category. A $\mathcal{C}$-valued **presheaf** $\mathcal{F}$ on $X$ is a contravariant functor from the category $\mathrm{Open}(X)$ with values in the category $\mathcal{C}$.*

To be precise, a presheaf $\mathcal{F}$ contains the following data:

(1) For each open subset $U \subset X$, $\mathcal{F}$ associates an object $\mathcal{F}(U) \in \mathrm{Ob}(\mathcal{C})$;

(2) For each inclusion $V \hookrightarrow U$ of open sets in $X$, we have a restriction map:

$$\mathrm{res}_{U|V} : \mathcal{F}(U) \to \mathcal{F}(V). \tag{2.2}$$

For any $s \in \mathcal{F}(U)$, we use $s|_V$ to denote its image under $\mathrm{res}_{U|V}$.

Sometimes we also denote $\mathcal{F}(U)$ by $\Gamma(U, \mathcal{F})$. The elements in $\mathcal{F}(U)$ are called **sections** of $\mathcal{F}$ over $U$. Elements in $\mathcal{F}(X)$ are also called **global sections**.

The **restriction** $\mathcal{F}|_U$ of $\mathcal{F}$ to an open set $U \subset X$ is a presheaf on $U$ defined by $\mathcal{F}|_U(V) := \mathcal{F}(V)$ for all $V \subset U$.

A morphism $\Phi \in \mathrm{Hom}_{\mathcal{PS}_X}(\mathcal{F}, \mathcal{G})$ between two presheaves is a collection of morphisms

$$\Phi_U : \mathcal{F}(U) \to \mathcal{G}(U) \tag{2.3}$$

such that whenever we have an inclusion $V \subset U$, we get a commutative diagram:

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\Phi_U"] \arrow[d, "\mathrm{res}_{U|V}"'] & \mathcal{G}(U) \arrow[d, "\mathrm{res}_{U|V}"] \\ \mathcal{F}(V) \arrow[r, "\Phi_V"] & \mathcal{G}(V) \end{tikzcd} . \tag{2.4}$$

**Remark 2.5.** Through out our paper, we restrict our attention to sheaves with values in an additive category (abelian groups, commutative rings, modules, etc.)

The **kernel** of the morphism of presheaves $\Phi : \mathcal{F} \to \mathcal{G}$ is the sub-presheaf sub-presheaf of $\mathcal{F}$ defined as $\ker(\Phi)(U) := \ker(\Phi_U) \subset \mathcal{F}(U)$, such that for any other sheaf $\mathcal{H}$:

$$\mathrm{Hom}_{\mathcal{S}_X}(\mathcal{H}, \ker(\Phi)) = \{\Phi \in \mathrm{Hom}_{\mathcal{S}_X}(\mathcal{F}, \mathcal{G}) | \Phi \circ \phi = 0\}. \tag{2.6}$$

Let $\mathcal{F}$ be a presheaf on a topological space $X$. We may want to ask: to what extent are its sections over an open set $U$ determined by its restrictions to smaller open sets in $U$. This leads to the definition of **sheaf**.

**Definition 2.7.** *Let $U \subset X$ be open, and $\mathcal{U} = \{U_i\}_{i \in I}$ be an arbitrary open covering of $U$. A presheaf $\mathcal{F}$ is a* ***sheaf*** *if the following axioms are satisfied:*

*(Locality) The natural projection $\mathcal{F}(U) \xrightarrow{p} \prod_{i\in I} \mathcal{F}(U_i)$ is injective: If $s, t \in \mathcal{F}(U)$ satisfy $s|_{U_i} = t|_{U_i}$ for all $i \in I$, we have $s = t$.*

*(Gluing) For any family $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$, there exists $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.*

A morphism between sheaves is a morphism between the underlying presheaves.

**Remark 2.8.** The kernel of a morphism of sheaves is also a sheaf.

Intuitively, a sheaf allows us to recover *global* information from *local* information.

**Example 2.9.** We define $\mathcal{C}^0$ to be the sheaf of continuous $\mathbb{R}$- or $\mathbb{C}$-valued functions. One easily checks that it is a sheaf.

Despite the structure of sections over an open set, we also want to study more local structures, e.g. the structure *around* a single point $x$ in a topological space $X$. We therefore introduce the notion of **stalk**, which is a mathematical construction that captures the behavior of a sheaf around a single point.

**Definition 2.10.** *Let $\mathfrak{U}_p$ be the set of neighborhoods containing $p$. The* ***stalk*** *at $p$ is the direct limit $\mathcal{F}_p = \varinjlim_{U \in \mathfrak{U}_p} \mathcal{F}(U)$.*

A sheaf map $\mathcal{F} \to \mathcal{G}$ over the same topological space $X$ induces a group homomorphism at each $x \in X$: $f_x : \mathcal{F}_x \to \mathcal{G}_x$. The **support** of a sheaf map $f$ is $\mathrm{supp}(f) = \{x \in X | f_x \neq 0\}$.

We also want to introduce the notion of **fine sheaf**, since it is used in our proof of the algebraic de Rham theorem.

**Definition 2.11.** *Let $\mathcal{F}$ be a sheaf of abelian groups over a topological space $X$, and $\mathfrak{U} = \{U_i\}_{i\in I}$ be a locally finite cover of $X$. A* ***partition of unity*** *of $\mathcal{F}$ subordinate to $\mathfrak{U}$ is a collection $\{j_i : \mathcal{F} \to \mathcal{F}\}$ of sheaf maps such that $\mathrm{supp}(j_i) \subset U_i$, and for each $x \in X$, $\sum j_{i,x} = \mathrm{id}_{\mathcal{F}_x}$. A sheaf $\mathcal{F}$ of abelian group over a topological space $X$ is* ***fine*** *if for any locally finite open covering $\mathfrak{U}$, the sheaf $\mathcal{F}$ admits a partition of unity.*

For any presheaf, we have a canonical process of **sheafification** which associates a sheaf to the presheaf:

**Proposition 2.12.** *([13]) For any presheaf $\mathcal{G}$ on a space $X$, its sheafification consists of a sheaf $\mathcal{G}^{\#}$ together with a morphism $j : \mathcal{G} \to \mathcal{G}^{\#}$ with the following universal property: for any sheaf $\mathcal{F}$ we have an isomorphism*

$$\mathrm{Hom}_{\mathcal{PS}_X}(\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{S}_X}(\mathcal{G}^{\#}, \mathcal{F}).$$

*The sheaf $\mathcal{G}^{\#}$ is unique up to isomorphism.*

*We can construct $\mathcal{G}^{\#}(U)$ as the set of functions $s$ from $U$ to $\sqcup_{p\in U} \mathcal{F}_p$ such that:*

*(1) For any $p \in U$, $s(p) \in \mathcal{F}_p$.*

*(2) For any $p \in U$, there is an open neighborhood $V \subset U$ of $p$, and $t \in \mathcal{F}(V)$ such that $t_q = s(q)$ for all $q \in V$.*

Now, we can introduce another important example of a sheaf:

**Example 2.13.** The **constant sheaf** $\underline{A}$ on a topological space $X$ associated to a set $A$ is the sheaf of locally constant functions with value in $A$.

The constant sheaf can be viewed as the sheafification of the **constant presheaf** $\underline{PA}$ which assigns the value $A$ to each open set of $X$, i.e., $\Gamma(U, \underline{PA}) = A$ for any open set $U \subset X$.

## 3. Cohomology of Sheaf

We may now consider the cohomology theory of a space $X$ with coefficients in a sheaf associated to $X$. Later we will see that sheaf cohomology indeed gives a generalization of singular cohomology of a space.

**Definition 3.1.** *A sequence of sheaves on a space X:*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

*is* ***exact*** *if for all points $x \in X$ the induced sequence of stalks is exact.*

**Remark 3.2.** ([9]) We also note that the previous definition is equivalent to:
(i) For all open sets $U \subset X$ the sequence

$$0 \longrightarrow \mathcal{F}'(U) \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{F}''(U)$$

is exact.
(ii) For any $s'' \in \mathcal{F}''(U)$ we can find a covering $U = \cup_j U_j$ by open sets and $s_j \in \mathcal{F}(U_j)$ such that $s_j \mapsto s''|_{U_j}$.

Notice that the functor sending each sheaf to its value on U is left exact. Thus we can have the following definition:

**Definition 3.3.** *The* ***sheaf cohomology*** *$H^\bullet(X, \mathcal{F})$ is the right derived functor of the global section functor $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$.*

This definition is guaranteed by the fact that the category of sheaves has enough injective objects, which is implied by the famous Godement construction ([4]). Before we explain that, we first recall some basic ideas in homological algebra.

**Definition 3.4.** *An object $Q$ in an abelian category $\mathcal{C}$ is said to be* ***injective*** *if for every injective homomorphism $f : X \to Y$ and every morphism $g : X \to Q$ there exists a morphism $h : Y \to Q$ such that $h \circ f = g$.*

**Remark 3.5.** A sheaf $\mathcal{F}$ is **injective** if for any diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\varphi"] \arrow[d, "\phi"'] & \mathcal{B} \arrow[dl, dashed, "\eta"] \\ \mathcal{I} & \end{tikzcd}$$

with $\ker(\varphi) \subset \ker(\phi)$ there is a map $\eta : \mathcal{B} \to \mathcal{I}$ which makes the diagram commute.

A (right) **resolution** of $\mathcal{F}$ is a (possibly infinite) exact sequence of sheaves of the following form:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}_0 \longrightarrow \mathcal{I}_1 \longrightarrow \mathcal{I}_2 \longrightarrow \cdots . \tag{3.6}$$

The sequence above is an **injective resolution** of $\mathcal{F}$ if $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \ldots$ are injective.
In [4], Godement gives a famous construction to guarantee that every sheaf $\mathcal{F}$ can be embedded into an injective sheaf.

**Construction 3.7.** Given an open set $U \subseteq X$, define $\mathcal{G}^0\mathcal{F}$ to be the direct product $\prod_{p\in U}\mathcal{F}_p$. We can show that $\mathcal{G}^0\mathcal{F}$ is injective. Repeating this process will give an injective resulotion of sheaf $\mathcal{F}$:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}^0\mathcal{F} \longrightarrow \mathcal{G}^1\mathcal{F} \longrightarrow \mathcal{G}^2\mathcal{F} \longrightarrow \cdots \tag{3.8}$$

We call this the **Godement canonical resolution**.

Standard argument in homological algebra shows that the cohomology is independent of the choice of injective resolution, i.e., if we pick two injective resolutions $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ and $0 \to \mathcal{F} \to \mathcal{J}^\bullet$, they are quasi-isomorphic. We conclude that sheaf cohomology is well-defined.

We also introduce the useful notion of **acyclic sheaves**:

**Definition 3.9.** *Let $X$ be a topological space and $\mathfrak{U} = \{U_i\}_{i\in I}$ be an open covering of $X$. A sheaf $\mathcal{F}$ is **acyclic** over $X$ if all higher sheaf cohomology groups vanish, i.e. $H^k(X,\mathcal{F}) = 0$ for all $k > 0$. The sheaf $\mathcal{F}$ is **acyclic on the open covering** $\mathfrak{U}$ if the restriction of $\mathcal{F}$ is acyclic on all finite intersections $U_{i_0} \cap \cdots \cap U_{i_p}$ of open sets in $\mathfrak{U}$.*

The computation of sheaf cohomology can be complicated. We want to introduce **Čech cohomology**, which naturally arises from the category of (pre)sheaves as a way to compute sheaf cohomology.

**Definition 3.10.** *Let $X$ be a topological space and $\mathcal{F}$ be a (pre)sheaf defined over $X$. Let $\mathfrak{U} = \{U_i\}_{i\in I}$ be an open covering of $X$. For any set of indices $(i_0,\ldots,i_q) \in I^{q+1}$, we define $U_{i_0,\ldots,i_q} = U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_q}$. Let*

$$C^q(X,\mathfrak{U},\mathcal{F}) = \prod_{(i_0,\ldots,i_q)\in I^{q+1}} \mathcal{F}(U_{i_0,\ldots,i_q}). \tag{3.11}$$

*We define the boundary map $d : C^q(X,\mathfrak{U},\mathcal{F}) \to C^{q+1}(X,\mathfrak{U},\mathcal{F})$ by*

$$(dc)_{(i_0,\ldots,i_q)} = \sum_{v=0}^{q+1} (-1)^v \operatorname{res}(c_{i_0,\ldots,\hat{i}_v,\ldots,i_q}), \tag{3.12}$$

*where $\hat{i}_v$ means the index is deleted, so that the restriction map makes sense. It is clear that $d^2 = 0$, i.e., it is a valid boundary map. The complex $C^\bullet(X,\mathfrak{U},\mathcal{F})$ is called the **Čech complex**, which gives rise to **Čech cohomology**.*

Čech cohomology is regarded as a nice approximation of sheaf cohomology due to several reasons. First, if we restrict our attention to a sheaf $\mathcal{F}$ of abelian groups, there is a canonical homomorphism $\check{H}^p(X,\mathfrak{U},\mathcal{F}) \to H^p(X,\mathcal{F})$ from Čech cohomology to sheaf cohomology for any space $X$ and any open covering $\mathfrak{U}$, since we can show that $0 \to \mathcal{F} \to C^\bullet(X,\mathfrak{U},\mathcal{F})$ is a resolution quasi-isomorphic to an injective resolution of $\mathcal{F}$. Second, if we have a paracompact space (e.g. a manifold), the homomorphism is an isomorphism (for proof see [3]). Finally, injective resolutions are usually complicated when we compute the sheaf cohomology, and Čech cohomology may sometimes simplify the calculation.

**Theorem 3.13.** *For a paracompact space $X$, we have an isomorphism*

$$\check{H}^k(X,\mathfrak{U},\mathcal{F}) \cong H^k(X,\mathcal{F})$$

*between Čech cohomology and sheaf cohomology.*

In general, the choice of open covering matters in our computation. However, Theorem 3.13 shows that (at least) for a paracompact space, the choice of covering does not matter. We will use Čech cohomology quite frequently in our proof of the algebraic de Rham theorem.

One side of the algebraic de Rham theorem is the singular cohomology of the complex manifold. We quote the following theorem to emphasize the connection between singular cohomology and constant sheaf cohomology (for proof see [3]).

**Theorem 3.14.** *Let $X$ be a paracompact Hausdorff space which is locally contractible, and $A$ be an abelian group. We have an isomorphism*

$$H^k_{\text{sing}}(X, A) \cong H^k(X, \underline{A})$$

*between $A$-coefficient singular cohomology and constant sheaf cohomology.*

Combining this with classical de Rham theorem, version 1, we have:

**Theorem 3.15.** ***(Classical de Rham theorem, version 2)*** *Let $M$ be a smooth manifold. We have the following isomorphisms:*

$$H^k_{\text{dR}}(M) \cong H^k_{\text{sing}}(M; \mathbb{R}) \cong H^k(M, \underline{\mathbb{R}}) \cong \check{H}^k(M, \underline{\mathbb{R}}).$$

## 4. Coherent Sheaves

In this section, we introduce **coherent sheaves** and some useful properties, which plays an important role in the remaining of the paper. An important example of a coherent sheaf is the sheaf of regular functions over a variety, which is the starting point of the algebraic de Rham complex.

**Definition 4.1.** *Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves. The direct sum $\mathcal{F} \oplus \mathcal{G}$ of two sheaves is defined by letting $\Gamma(U, \mathcal{F} \oplus \mathcal{G}) = \mathcal{F}(U) \oplus \mathcal{G}(U)$ for all open set $U \subset X$.*

We can show that the direct sum of sheaves is a sheaf. We denote the direct sum of $k$ copies of $\mathcal{F}$ by $\mathcal{F}^{\oplus k}$.

**Definition 4.2.** *Let $\mathcal{R}$ be a sheaf of commutative rings over topological space $X$. A sheaf $\mathcal{F}$ is a sheaf of $\mathcal{R}$-modules if for any $U \subset X$, $\mathcal{F}(U)$ is a $\mathcal{R}(U)$-module such that the restriction $r : \mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the restriction $s : \mathcal{R}(U) \to \mathcal{R}(V)$: for any $a \in \mathcal{R}(U)$ and $b \in \mathcal{F}(U)$, we have $r(ab) = s(a)r(b)$.*

*A sheaf $\mathcal{F}$ of $\mathcal{R}$-modules is **locally free of rank** $k$ if for every $x \in X$, there is a neighborhood $U$ of $x$ on which there is a sheaf isomorphism $\mathcal{F}|_U \cong \mathcal{R}|_U^{\oplus k}$.*

A sheaf of $\mathcal{O}_M$-modules, where $\mathcal{O}_M$ is the sheaf of holomorphic functions over a complex manifold $M$, is called an **analytic sheaf**.

**Example 4.3.** Let $M$ be a complex manifold of dimension $n$. The sheaf $\Omega^k_{\text{an}}$ of holomorphic $k$-forms is an analytic sheaf locally free of rank $\binom{n}{k}$ with local frame $dz_{i_1} \wedge \cdots \wedge dz_{i_k}$ for $1 \leq i_1 \leq \cdots \leq i_k \leq n$.

Again, let $\mathcal{R}$ be a sheaf of commutative rings over topological space $X$, $\mathcal{F}$ be a sheaf of $\mathcal{R}$-modules on $X$. Let $f_1, \ldots, f_n$ be sections of $\mathcal{F}$ over an open set $U$. We define a sheaf map $\varphi : \mathcal{R}^{\oplus n} \to \mathcal{F}$ as follows: For any $r_1, \ldots, r_n \in \mathcal{R}(U)$, the map $\varphi|_U : \mathcal{R}^{\oplus n}(U) \to \mathcal{F}(U)$ is given by $(r_1, \ldots, r_n) \mapsto \sum r_i f_i$.

We define the **sheaf of relations** $\mathcal{S}(f_1, \ldots, f_n)$ among $f_1, \ldots, f_n$ as the kernel of the sheaf map $\varphi$, a subsheaf of $(\mathcal{R}|_U)^{\oplus n}$. We say that $\mathcal{F}|_U$ is **generated** by $f_1, \ldots, f_n$ if $\varphi|_U$ is surjective. The sheaf $\mathcal{F}$ is **of finite type** if for every $x \in X$

there is a neighborhood $U$ on which $\mathcal{F}|_U$ is generated by finitely many sections $f_1, \dots, f_n \in \mathcal{F}(U)$.

**Definition 4.4.** *A sheaf $\mathcal{F}$ of $\mathcal{R}$-modules on a topological space $X$ is **coherent** if $\mathcal{F}$ is of finite type; and for any open set $U \subset X$ and any collection of sections $f_1, \dots, f_n \in \mathcal{F}(U)$, $\mathcal{S}(f_1, \dots, f_n)$ is of finite type over $U$.*

**Example 4.5.** The sheaf $\mathcal{O}_M$ of holomorphic functions on a complex manifold $M$ is a coherent sheaf of $\mathcal{O}_M$-modules (see [2]).

Let $X$ be an algebraic variety and $U$ be an open subvariety. A function on $X$ is **regular** on $U$ if its restriction on $U$ is a **rational function** (functions of the form $g/h$, where $g, h$ are polynomials) with non-vanishing denominator. A function itself is called **regular** if for any $x \in X$, there exist an open set $U \ni x$ on which it is regular. Note that the regular function on algebraic varieties is the analogue of the smooth function on manifolds.

**Example 4.6.** The sheaf $\mathcal{O}_X$ of regular functions on an algebraic variety $X$ is a coherent sheaf of $\mathcal{O}_X$-modules (see [17]).

We also note that a sheaf of $\mathcal{O}_X$-modules over an algebraic variety $X$ is usually called an **algebraic sheaf**.

A sheaf $\mathcal{F}$ of $\mathcal{R}$-modules on a topological space $X$ is **locally finitely presented** if for every $x \in X$ there is a neighborhood $U$ on which there is an exact sequence of the form

$$\mathcal{R}|_U^{\oplus q} \longrightarrow \mathcal{R}|_U^{\oplus p} \longrightarrow \mathcal{F}|_U \longrightarrow 0 \ ; \tag{4.7}$$

We can show that a coherent sheaf $\mathcal{F}$ of $\mathcal{R}$-modules on a topological space $X$ is locally finitely presented. The following theorem is very useful in identifying coherent sheaves of modules:

**Theorem 4.8.** *Let $\mathcal{R}$ be a coherent sheaf of commutative rings over topological space $X$, $\mathcal{F}$ be a sheaf of $\mathcal{R}$-modules on $X$. $\mathcal{F}$ is coherent if and only if it is locally finitely presented.*

*Proof.* We have already explained the *only if* part. The *if* part is implied by the fact that the coherency is preserved by direct sums and cokernels (see [17]). □

A complex manifold is **Stein** if it is biholomorphic to a closed submanifold of $\mathbb{C}^N$. This is a very useful case, since the underlying manifold of a complex affine variety is stein. We have a very useful theorem on Stein manifolds (see [7]):

**Theorem 4.9.** ***(Cartan)*** *Coherent analytic sheaves are acyclic on Stein manifolds.*

Now we want to explain the basic idea of Serre's GAGA principle, which provides a bridge between varieties and their underlying manifolds. Specifically, we want to construct a functor from algebraic sheaves to analytic sheaves, and show that it is an equivalence of categories.

Let $X$ be a smooth quasi-projective complex variety, $U \subset X$ be a Zariski open subset. Let $X_{\text{an}}, U_{\text{an}}$ be their underlying manifolds, respectively. Let $\mathcal{O}_X, \mathcal{O}_{X_{\text{an}}}$ be the sheaf of regular functions and the sheaf of holomorphic functions on $X, X_{\text{an}}$, respectively. Let $\mathfrak{U} = \{U_i\}_{i\in I}$ be a Zariski open covering of $X$. Since Zariski open sets are open under complex topology, the analytification $\mathfrak{U}_{\text{an}} = \{U_i^{\text{an}}\}$ of the previous open covering is also an open covering of $X_{\mathrm{an}}$. Let $\mathcal{F}$ be a coherent algebraic sheaf over $X$. On every $U_i \subset \mathfrak{U}$ there is an exact sequence

$$\mathcal{O}_X|_{U_i}^{\oplus q} \longrightarrow \mathcal{O}_X|_{U_i}^{\oplus p} \longrightarrow \mathcal{F}|_{U_i} \longrightarrow 0\ . \tag{4.10}$$

The morphism $\mathcal{O}_X|_{U_i}^{\oplus q} \to \mathcal{O}_X|_{U_i}^{\oplus p}$ induces a morphism of analytic sheaves $\mathcal{O}_{X_{\mathrm{an}}}|_{U_i^{\mathrm{an}}}^{\oplus q} \to \mathcal{O}_{X_{\mathrm{an}}}|_{U_i^{\mathrm{an}}}^{\oplus p}$. Thus there is a coherent analytic sheaf $\mathcal{F}_{\mathrm{an}}$ such that the following sequece is exact:

$$\mathcal{O}_{X_{\mathrm{an}}}|_{U_i^{\mathrm{an}}}^{\oplus q} \longrightarrow \mathcal{O}_{X_{\mathrm{an}}}|_{U_i^{\mathrm{an}}}^{\oplus p} \longrightarrow \mathcal{F}^{\mathrm{an}}|_{U_i^{\mathrm{an}}} \longrightarrow 0\ . \tag{4.11}$$

Hence we obtain a functor from the category of smooth quasi-projective varieties and coherent algebraic sheaves to the category of complex manifolds and analytic sheaves. Serre's GAGA principle (see [18]) asserts that for smooth quasi-projective variety this is an equivalence of categories and we have the following theorem:

**Theorem 4.12.** *(**GAGA**) Let $X$ be a smooth quasi-projective variety. We have an isomorphism*

$$H^k(X, \mathcal{F}) \cong H^k(X_{\mathrm{an}}, \mathcal{F}_{\mathrm{an}})$$

*between sheaf cohomology of $\mathcal{F}$ over $X$ and sheaf cohomology of $\mathcal{F}_{\mathrm{an}}$ over $X_{\mathrm{an}}$.*

We also have the algebraic analogue of Theorem 4.9 (see [17]):

**Theorem 4.13.** *(**Serre's Vanishing Theorem**) A coherent algebraic sheaf $\mathcal{F}$ on affine variety $X$ is acyclic.*

## 5. Algebraic de Rham Complex and Hypercohomology

Consider a smooth complex algebraic variety $X$ (with Zariski topology) over an algebraically closed field $k$ (e.g., $\mathbb{C}$). We define the **algebraic de Rham complex** $\Omega_X^\bullet$ to be the **complex of sheaves of regular differentials**.

**Definition 5.1.** *The **complex of sheaves of regular differentials** starts from the sheaf $\mathcal{O}_X$ of regular functions and is induced by exterior derivations:*

$$\mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega_X^1 \xrightarrow{\mathrm{d}} \Omega_X^2 \xrightarrow{\mathrm{d}} \Omega_X^3 \xrightarrow{\mathrm{d}} \cdots .$$

where $\Omega_X^k = \bigwedge^k \Omega_X^1$. Every regular differential $\omega \in \Omega_X^k$ can be written into

$$\omega = \sum f_I \, \mathrm{d}\, g_{i_1} \wedge \cdots \wedge \mathrm{d}\, g_{i_k}, \tag{5.2}$$

where $f_I, g_{i_1}, \ldots, g_{i_k}$ are regular functions on $X$.

Now we introduce the important concept of hypercohomology, which is essential in our construction of the algebraic version of de Rham cohomology.

**Definition 5.3.** *Let $\mathcal{F}^\bullet$ be a complex of sheaves on a topological space $X$. The **hypercohomology** $\mathbf{H}^k(X, \mathcal{F}^\bullet)$ of $X$ with coefficients in the complex $\mathcal{F}^\bullet$ is the derived functor of the global section functor: $\mathbf{H}^k(X, \mathcal{F}^\bullet) = h^k(\Gamma(X, \mathcal{I}^\bullet))$, where $\mathcal{I}^\bullet$ is an injective resolution of $\mathcal{F}^\bullet$.*

Several results regarding hypercohomology are very useful.

**Theorem 5.4.** *A quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ of complexes of sheaves of abelian groups over a topological space $X$ induces a canonical isomorphism in hypercohomology $\mathbf{H}^k(X, \mathcal{F}^\bullet) \cong \mathbf{H}^k(X, \mathcal{G}^\bullet)$ (see [12]).*

**Theorem 5.5.** ***(Analytic de Rham Theorem)*** *Let $\Omega^k_{\mathrm{an}}$ be the sheaf of holomorphic k-forms on a complex manifold M. Then the singular cohomology of M with complex coefficients can be computed as the hypercohomology of the complex $\Omega^\bullet_{\mathrm{an}}$:*

$$H^k_{\mathrm{sing}}(M,\mathbb{C}) \cong \mathbf{H}^k(M,\Omega^\bullet_{\mathrm{an}}).$$

*Proof.* Let $\underline{\mathbb{C}}^\bullet$ be the complex of sheaves which has $\underline{\mathbb{C}}$ in degree zero and 0 otherwise. We interpret the holomorphic Poincaré Lemma as a quasi-isomorphism between the following two complexes:

$$\begin{tikzcd} 0 \arrow[r] & \underline{\mathbb{C}} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & \Omega^0_{\mathrm{an}} \arrow[r] & \Omega^1_{\mathrm{an}} \arrow[r] & \Omega^2_{\mathrm{an}} \arrow[r] & \cdots \end{tikzcd} \tag{5.6}$$

which, by Theorem 5.4, induces an isomorphism between hypercohomology:

$$\mathbf{H}^k(X,\underline{\mathbb{C}}^\bullet) \cong \mathbf{H}^k(X,\Omega^\bullet_{\mathrm{an}}). \tag{5.7}$$

The left hand side is $H^k_{\mathrm{sing}}(M,\mathbb{C})$. □

When $M$ is a Stein manifold, by Theorem 4.9, $\Omega^\bullet_{\mathrm{an}}$ is a complex of acyclic sheaves. Then we have

$$\mathbf{H}^k(M,\Omega^\bullet_{\mathrm{an}}) \cong h^k(\Omega^\bullet_{\mathrm{an}}(M)), \tag{5.8}$$

and hence the following corollary of the analytic de Rham theorem is obvious:

**Corollary 5.9.** ***(Classical de Rham Theorem, Complex version)*** *For a Stein manifold $M$, we have an isomorphism*

$$H^k(M,\mathbb{C}) \cong h^k(\Omega^\bullet_{\mathrm{an}}(M)).$$

We may also use Čech-cohomological methods to compute hypercohomology. Let $(\mathcal{F}^\bullet, d_{\mathcal{F}})$ be a complex of sheaves over a topological space $X$, and $\mathfrak{U} = \{U_i\}_{i\in I}$ an open covering of $X$. We consider the double complex $K^{p,q} = C^p(X,\mathfrak{U},\mathcal{F}^q)$ with commuting differentials $\delta$ and $d_{\mathcal{F}}$. The Čech cohomology $\check{H}(X,\mathfrak{U},\mathcal{F}^\bullet)$ of $\mathcal{F}^\bullet$ is the cohomology of the single complex

$$K^\bullet = \bigoplus_k \bigoplus_{p+q=k} C^p(X,\mathfrak{U},\mathcal{F}^q) \tag{5.10}$$

with total differential $d_K = \delta + (-1)^p d_{\mathcal{F}})$. We have the following comparison theorem (see [12]):

**Theorem 5.11.** *Let $(\mathcal{F}^\bullet, d_{\mathcal{F}})$ be a complex of sheaves of abelian groups over a topological space $X$ such that each sheaf $\mathcal{F}^q$ is acyclic on the open covering $\mathfrak{U} = \{U_i\}_{i\in I}$ of $X$, then there is an isomorphism $\check{H}(X,\mathfrak{U},\mathcal{F}^\bullet) \cong \mathbf{H}^k(X,\mathcal{F}^\bullet)$.*

Now we can state the remarkable Grothendieck's algebraic de Rham theorem:

**Theorem 5.12.** ***(Algebraic de Rham theorem)*** *Let $X$ be a smooth complex algebraic variety, and $X_{\mathrm{an}}$ be its underlying complex manifold. Then the singular cohomology of $X_{\mathrm{an}}$ with coefficients in $\mathbb{C}$ agrees with the hypercohomology of the complex $\Omega^\bullet_X$ of sheaves of regular differentials on $X$:*

$$H^k_{\mathrm{sing}}(X_{\mathrm{an}},\mathbb{C}) \cong \mathbf{H}^k(X,\Omega^\bullet_X).$$

**Remark 5.13.** By Theorem 5.5, Theorem 5.12 is equivalent to the following isomorphism:

$$\mathbf{H}^k(X_{\mathrm{an}}, \Omega^\bullet_{\mathrm{an}}) \cong \mathbf{H}^k(X, \Omega^\bullet_X). \tag{5.14}$$

The rest of the paper gives a complete proof to this theorem.

## 6. Reduction to the Affine Case

In this section, we will show that if Theorem 5.12 works for affine varieties, then it works in general.

Suppose $X$ is a complex smooth affine variety. Since $\Omega^k_X$ is a coherent algebraic sheaf, by Theorem 4.13, $\Omega^k_X$ is acyclic on $X$, and hence the algebraic de Rham theorem is reduced to the following conjectured isomorphism to be proven:

$$H^k_{\mathrm{sing}}(X_{\mathrm{an}}, \mathbb{C}) \cong h^k(\Omega^\bullet_X). \tag{6.1}$$

Now, we want to show that the affine case (assuming it is correct) implies the general case. We first notice that every algebraic variety $X$ has an **affine open covering** $\mathfrak{U} = \{U_i\}_{i\in I}$ in which every $U_i$ is an affine open set, and the intersection of two affine open sets is affine open. Since $\Omega^k_X$ is coherent, by Theorem 4.13, it is acyclic on an affine open covering. By Theorem 5.11, we have

$$\check{H}^k(X, \mathfrak{U}, \Omega^\bullet_X) \cong \mathbf{H}^k(X, \Omega^\bullet_X). \tag{6.2}$$

By Theorem 4.9 and Theorem 5.11, we have an analytic version:

$$\check{H}^k(X_{\mathrm{an}}, \mathfrak{U}_{\mathrm{an}}, \Omega^\bullet_{\mathrm{an}}) \cong \mathbf{H}^k(X_{\mathrm{an}}, \Omega^\bullet_{\mathrm{an}}). \tag{6.3}$$

Recall that the Čech cohomology $\check{H}^k(X, \mathfrak{U}, \Omega^\bullet_X)$ is the cohomology of single complex associated to the double complex $\bigoplus K^{p,q} = \bigoplus C^p(X, \mathfrak{U}, \Omega^q_X)$, which can be computed by a spectral sequence with the following $E_1$ page:

$$E^{p,q}_{1,\mathrm{alg}} = h^q_d(K^{p,\bullet}) = h^q_d(C^p(X, \mathfrak{U}, \Omega^\bullet_X)) \tag{6.4}$$

$$= h^q_d\left(\prod_{i_0<\cdots<i_p} \Omega^\bullet_X(U_{i_0,\cdots,i_p})\right) = \prod_{i_0<\cdots<i_p} h^q_d(\Omega^\bullet_X(U_{i_0,\cdots,i_p})) \tag{6.5}$$

$$= \prod_{i_0<\cdots<i_p} H^q(U_{i_0,\cdots,i_p,\mathrm{an}}, \mathbb{C}), \tag{6.6}$$

where the last equality comes from the assumed correctness of (6.1). Similarly, $\check{H}^k(X_{\mathrm{an}}, \mathfrak{U}_{\mathrm{an}}, \Omega^\bullet_{\mathrm{an}})$ can be computed by a similar spectral sequence with the following $E_1$ page:

$$E^{p,q}_{1,\mathrm{an}} = \prod_{i_0<\cdots<i_p} h^q_d(\Omega^\bullet_X(U_{i_1,\cdots,i_p})) \tag{6.7}$$

$$= \prod_{i_0<\cdots<i_p} H^q(U_{i_0,\cdots,i_p,\mathrm{an}}, \mathbb{C}), \tag{6.8}$$

where the last equality comes from Corollary 5.9. The isomorphism of $E_1$ terms commutes with the Čech differential, and hence induces an isomorphism in $E_\infty$ terms. Therefore we have

(6.9)

$$\mathbf{H}^k(X, \Omega^\bullet_X) \xrightarrow[(6.3)]{\cong} \check{H}^k(X, \mathfrak{U}, \Omega^\bullet_X) \xrightarrow{\cong} \check{H}^k(X_{\mathrm{an}}, \mathfrak{U}_{\mathrm{an}}, \Omega^\bullet_{\mathrm{an}}) \xrightarrow[(6.4)]{\cong} \mathbf{H}^k(X_{\mathrm{an}}, \Omega^\bullet_{\mathrm{an}}),$$

which is equivalent to Theorem 5.12 according to Remark 5.13. Thus the remaining work is to show the algebraic de Rham theorem for the affine case:

**Theorem 6.10.** ***(Affine Case)*** *Let $X$ be a smooth affine complex algebraic variety, and $X_{an}$ be its underlying complex manifold. Then the singular cohomology of $X_{an}$ with $\mathbb{C}$ coefficients can be computed as the cohomology of the complex $\Omega_X^\bullet$ of sheaves of regular differentials on $X$ with its Zariski topology:*

$$H^k_{\text{sing}}(X_{\text{an}}, \mathbb{C}) \cong h^k(\Omega_X^\bullet).$$

## 7. Affine Case

In the beginning, we want to sketch the proof of Theorem 6.10 following [12]. Through out this section, we assume $X$ to be a smooth affine complex variety.

We want to apply some powerful tools dealing mainly with projective varieties, such as Theorem 4.12. So it is very helpful to put $X$ into a projective variety. We apply Hironaka's resolution of singularities, which yields a smooth projective complex variety $Y$, an open embedding $f : X \hookrightarrow Y$, such that $D = Y - f(X)$ is a normal crossing divisor in $Y$, and $X \to Y - D$ is an isomorphism. Correspondingly, we have an induced embedding of complex manifolds $f^{\text{an}} : X_{\text{an}} \hookrightarrow Y_{\text{an}}$.

The idea is straight forward: we want to study the projective variety $Y$ to recover information on $X$. We *move* the sheaf of holomorphic forms on $X$ to $Y$ by considering the sheaf $\Omega^k_{Y_{\text{an}}}(*D)$ of meromorphic $k$-forms on $Y$ which restrict to homolorphic $k$-forms on $X$. Let $\mathcal{A}^k_{X_{\text{an}}}$ be the sheaf of smooth complex value $k$-forms on $X_{\text{an}}$, and then we consider its direct image on $Y_{\text{an}}$: $f^{\text{an}}_* \mathcal{A}^k_{X_{\text{an}}}$.

There is an inclusion $\Omega^\bullet_{Y_{\text{an}}}(*D_{\text{an}}) \hookrightarrow f^{\text{an}}_* \mathcal{A}^\bullet_{X_{\text{an}}}$, and by a lemma of Hodge and Atiyah (see [11]), this inclusion is a quasi-isomorphism, which induces an isomorphism in hypercohomology:

$$\mathbf{H}^k(Y_{\text{an}}, \Omega^\bullet_{Y_{\text{an}}}(*D)) \cong \mathbf{H}^k(Y_{\text{an}}, f^{\text{an}}_* \mathcal{A}^\bullet_{X_{\text{an}}}). \tag{7.1}$$

We assert that the following maps $a, b$ are isomorphisms:

$$\begin{tikzcd} \mathbf{H}^k(Y_{\text{an}}, \Omega^\bullet_{Y_{\text{an}}}(*D)) \arrow[r, "\cong", "(7.1)"'] \arrow[d, "a", "7.12"'] & \mathbf{H}^k(Y_{\text{an}}, f^{\text{an}}_* \mathcal{A}^\bullet_{X_{\text{an}}}) \arrow[d, "(7.3)", "b"'] \\ H^k_{\text{sing}}(X_{\text{an}}, \mathbb{C}) \arrow[r, "\cong?", "6.10"'] & h^k(\Omega^\bullet_X) \end{tikzcd} , \tag{7.2}$$

which yields Theorem 6.10. The isomorphism $a$ is taken care of by Lemma 7.12, while the isomorphism $b$ is separated into several isomorphisms:

$$\begin{tikzcd} \mathbf{H}^k(Y_{\text{an}}, \Omega^\bullet_{Y_{\text{an}}}(*D_{\text{an}})) \arrow[r, "\cong?", "b"'] \arrow[d, "7.17"'] & h^k(\Omega^\bullet_X) \arrow[d, equal, "4.13"] \\ \mathbf{H}^k(Y, \Omega^\bullet_Y(*D)) \arrow[r, "7.21"] & \mathbf{H}^k(X, \Omega^\bullet_X) \end{tikzcd} . \tag{7.3}$$

Now we explain the proof in detail. Let $\mathcal{M}_X$ be the sheaf of rational functions on $X$. Let $\mathcal{O}_X$ be the sheaf of regular functions as usual. The inclusion $\mathcal{O}_X \subseteq \mathcal{M}_X$ of sheaves of abelian groups induces the inclusion $\mathcal{O}_X^\times \subseteq \mathcal{M}_X^\times$ of sheaves of multiplicative abelian groups. Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open covering of $X$.

**Definition 7.4.** (**Cartier divisors**) *A Cartier divisor on $X$ is a global section of the quotient sheaf $\mathcal{M}_X^\times/\mathcal{O}_X^\times$. We have a group $\mathrm{Div}(X) = \Gamma(X, \mathcal{M}_X^\times/\mathcal{O}_X^\times)$. A $D \in \mathrm{Div}(X)$ is **effective** if for all $U_i \subset \mathfrak{U}$, every $f_i \in \Gamma(U_i, \mathcal{O}_X)$ is regular on $U_i$.*

**Definition 7.5.** (**Normal crossing divisor**) *An effective divisor $D = \sum D_i$ on $X$ is a simple normal crossing (or SNC divisor) if $D$ is reduced, each component $D_i$ is smooth, and if $D$ is defined in a neighborhood of any point by an equation in local coordinates of the form $z_1 \cdots z_k = 0$ for some $k \leq n$.*

Consider $\bar{X}$ the projective closure of $X$ defined by replacing the defining polynomial $f(z_1, \ldots, z_N) = 0$ by

$$f(\frac{Z_1}{Z_0}, \cdots, \frac{Z_N}{Z_0}) = 0 \tag{7.6}$$

in $\mathbb{C}P^N$, where $Z_0, \ldots, Z_N$ are the homogeneous coordinates on $\mathbb{C}P^N$, $z_i = Z_i/Z_0$. $\bar{X}$ is singular in general. By Hironaka's resolution of singularities, there is a surjective regular map $\pi : Y \to \bar{X}$, where $Y$ is a smooth projective variety, $\pi^{-1}(\bar{X} - X)$ is a normal-crossing divisor $D \subset Y$, and $\pi|_{Y-D}$ is an isomorphism. Thus we have $X = Y - D$ with an inclusion map $f : X \hookrightarrow Y$.

Let $\Omega^k_{Y_{\mathrm{an}}}(*D_{\mathrm{an}})$ be the sheaf of meromorphic $k$-forms on $Y_{\mathrm{an}}$ which restricts to holomorphic $k$-forms on $X_{\mathrm{an}}$ with poles only along $D_{\mathrm{an}}$. Let $\mathcal{A}^k_{X_{\mathrm{an}}}$ be the sheaf of smooth complex value $k$-forms on $X_{\mathrm{an}}$. Let $f^{\mathrm{an}} : X_{\mathrm{an}} \hookrightarrow Y_{\mathrm{an}}$. Consider the direct image of $\mathcal{A}^k_{X_{\mathrm{an}}}$ on $Y_{\mathrm{an}}$:

$$f^{\mathrm{an}}_* \mathcal{A}^k_{X_{\mathrm{an}}}(V) = \mathcal{A}^k_{X_{\mathrm{an}}}(V \cap X_{\mathrm{an}}) \tag{7.7}$$

for every open set $V \subset Y_{\mathrm{an}}$. Since any $s \in \Omega^k_{Y_{\mathrm{an}}}(*D)$ over $V$ is holomorphic and hence smooth on $V \cap X_{\mathrm{an}}$, $\Omega^k_{Y_{\mathrm{an}}}(*D_{\mathrm{an}})$ is a subsheaf of $f^{\mathrm{an}}_* \mathcal{A}^k_{X_{\mathrm{an}}}$. We have the following important lemma (see [11]):

**Lemma 7.8.** *(**Fundamental lemma of Hodge and Atiyah**) Let $M$ be a complex manifold, let $D$ be a normal-crossing divisor in $M$, and let $U = M - D$ and $j : U \hookrightarrow M$ the inclusion map. Let $\Omega^q_M(*D)$ be the sheaf of meromorphic $q$-forms on $M$ holomorphic on $U$ with poles along $D$. Let $\mathcal{A}^q_U$ the sheaf of smooth complex value $q$-forms on $U$. For each $q$, $\Omega^q_M(*D)$ is a subsheaf of $f_*\mathcal{A}^q_U$. The inclusion $\Omega^\bullet_M(*D) \hookrightarrow f_*\mathcal{A}^\bullet_U$ of complexes of sheaves is a quasi-isomorphism.*

Lemma 7.8 asserts that the inclusion map $\Omega^\bullet_{Y_{\mathrm{an}}}(*D_{\mathrm{an}}) \hookrightarrow f^{\mathrm{an}}_* \mathcal{A}^\bullet_{X_{\mathrm{an}}}$ is a quasi-isomorphism, and hence induces an isomorphism of hypercohomology

$$\mathbf{H}^k(Y_{\mathrm{an}}, \Omega^\bullet_{Y_{\mathrm{an}}}(*D_{\mathrm{an}})) \cong \mathbf{H}^k(Y_{\mathrm{an}}, f^{\mathrm{an}}_* \mathcal{A}^\bullet_{X_{\mathrm{an}}}). \tag{7.9}$$

The idea for the rest of the proof is as follow: if we can show that

$$H^k_{\mathrm{sing}}(X_{\mathrm{an}}, \mathbb{C}) \cong \mathbf{H}^k(Y_{\mathrm{an}}, f_* \mathcal{A}^\bullet_{X_{\mathrm{an}}}), \tag{7.10}$$

and

$$h^k(\Omega^\bullet_X) \cong \mathbf{H}^k(Y_{\mathrm{an}}, \Omega^\bullet_{Y_{\mathrm{an}}}(*D_{\mathrm{an}})), \tag{7.11}$$

then we are done. For (7.10), we prove the following generalization to any complex manifolds:

**Lemma 7.12.** *Let $M$ be a complex manifold, $U \subset M$ an open submanifold with $j : U \hookrightarrow M$ the inclusion map. We have*

$$\mathbf{H}^k(M, f^{\mathrm{an}}_* \mathcal{A}^\bullet_U) \cong H^k_{\mathrm{sing}}(U, \mathbb{C}),$$

*where $\mathcal{A}_U^\bullet$ denotes the sheaf of smooth $\mathbb{C}$-value $k$-forms on $U$.*

*Proof.* Let $\mathcal{A}^0$ be the sheaf of smooth $\mathbb{C}$-value functions on $M$. For any open set $V \subset M$, we have a canonical $\mathcal{A}^0(V)$-module structure on $(f_*^{\mathrm{an}}\mathcal{A}_U^k)(V) = \mathcal{A}_U^k(U \cap V)$:

$$\mathcal{A}^0(V) \times \mathcal{A}_U^k(U \cap V) \to \mathcal{A}_U^k(U \cap V)$$
$$(f, \omega) \mapsto f \cdot \omega.$$

Then $f_*^{\mathrm{an}}\mathcal{A}_U^k$ is a sheaf of $\mathcal{A}^0$-module, which is *fine*, and hence acyclic. We have

$$\mathbf{H}^k(M, f_*^{\mathrm{an}}\mathcal{A}_U^\bullet) \cong h^k((f_*^{\mathrm{an}}\mathcal{A}_U^k)(M)) \tag{7.13}$$
$$= h^k(\mathcal{A}_U^k)(U)) = H_{\mathrm{sing}}^k(U, \mathbb{C}). \tag{7.14}$$

□

Applying Lemma 7.12 directly, we take care of (7.10).

Let $\Omega_{Y_{\mathrm{an}}}^k(^*_{\leq n}D_{\mathrm{an}})$ be the sheaf of meromorphic $k$-forms on $Y_{\mathrm{an}}$ holomorphic on $X_{\mathrm{an}}$ with poles of order $\leq n$ along $D_{\mathrm{an}}$. Let $\Omega_Y^k(*D)$ and $\Omega_Y^k(^*_{\leq n}D)$ be the algebraic counterpart defined using functions regular on $Y$. Then we have

$$\Omega_{Y_{\mathrm{an}}}^k(*D_{\mathrm{an}}) = \varinjlim_n \Omega_{Y_{\mathrm{an}}}^k(*D_{\mathrm{an}}),\ \Omega_Y^k(*D) = \varinjlim_n \Omega_Y^k(^*_{\leq n}D) \tag{7.15}$$

Let $\Omega_X^k, \Omega_{X_{\mathrm{an}}}^k$ (resp. $\Omega_Y^k, \Omega_{Y_{\mathrm{an}}}^k$) be the sheaves of regular $k$-forms on $X$ (resp. $Y$) and holomorphic $k$-forms on $X_{\mathrm{an}}$ (resp. $Y_{\mathrm{an}}$). We observe that regular forms on $X = Y - D$ not defined on $D$ can have at most poles along $D$, thus we have

$$f_*\Omega_X^k = \Omega_Y^q(*D). \tag{7.16}$$

However, one notices that in general $f_*^{\mathrm{an}}\Omega_{X_{\mathrm{an}}}^k \neq \Omega_{Y_{\mathrm{an}}}^k(*D_{\mathrm{an}})$, since a holomorphic form on $X_{\mathrm{an}}$ not defined along $D_{\mathrm{an}}$ may have essential singularities.

**Proposition 7.17.** *(**Hyper-GAGA**) We have an isomorphism*

$$\mathbf{H}^\bullet(Y, \Omega_Y^\bullet(*D)) \cong \mathbf{H}^\bullet(Y_{\mathrm{an}}, \Omega_{Y_{\mathrm{an}}}^\bullet(*D_{\mathrm{an}}))$$

*Proof.* We apply Theorem 4.12 to obtain an isomorphism

$$H^p(Y, \Omega_Y^\bullet(^*_{\leq n}D)) \cong H^p(Y_{\mathrm{an}}, \Omega_{Y_{\mathrm{an}}}^\bullet(^*_{\leq n}D_{\mathrm{an}})) \tag{7.18}$$

For a compact or noetherian space, direct limit (of a system of sheafs) commutes with cohomology (for proof see [4],[10], and [11]). Taking direct limit of both sides as $n \to \infty$, we obtain

$$H^p(Y, \varinjlim_n \Omega_Y^\bullet(^*_{\leq n}D)) \cong H^p(Y_{\mathrm{an}}, \varinjlim_n \Omega_{Y_{\mathrm{an}}}^\bullet(^*_{\leq n}D_{\mathrm{an}})) \tag{7.19}$$

which, by commutativity, is

$$H^p(Y, \Omega_Y^q(*D)) \cong H^p(Y_{\mathrm{an}}, \Omega_{Y_{\mathrm{an}}}^q(*D_{\mathrm{an}})). \tag{7.20}$$

Notice that this is an isomorphism of the $E_1$ terms of the second spectral sequence of the hypercohomologies of $\Omega_Y^\bullet(*D)$ and $\Omega_{Y_{\mathrm{an}}}^\bullet(*D_{\mathrm{an}})$, respectively, which induces isomorphism of the $E_\infty$ terms, and hence we obtain the result. □

**Proposition 7.21.** *We have an isomorphism in hypercohomology*

$$\mathbf{H}^\bullet(Y, \Omega_Y^\bullet(*D)) \cong \mathbf{H}^\bullet(X, \Omega_X^\bullet).$$

*Proof.* Let $V$ be an affine open set in $Y$, hence noetherian. We thus have

$$H^p(V, \Omega^q_Y(*D)) = H^p(V, \varinjlim_n \Omega^q_Y({}^*_{\leq n}D)) \tag{7.22}$$

$$\cong \varinjlim_n H^p(V, \Omega^q_Y({}^*_{\leq n}D)) = 0. \tag{7.23}$$

The last equality follows from Theorem 4.13, since $\Omega^q_Y({}^*_{\leq n}D)$ is locally free, therefore coherent. Then $\Omega^q_Y(*D)$ is acyclic on an open covering $\mathfrak{U} = \{U_i\}_{i \in I}$ of $Y$. By Theorem 5.11,

$$\mathbf{H}^k(Y, \Omega^\bullet_Y(*D)) \cong \check{H}^k(Y, \mathfrak{U}, \Omega^\bullet_Y(*D)). \tag{7.24}$$

By (7.16), $\check{H}(Y, \mathfrak{U}, \Omega^\bullet_Y(*D))$ is the cohomology of the complex

$$K^{p,q} = C^p(Y, \mathfrak{U}, \Omega^\bullet_Y(*D)) = C^p(Y, \mathfrak{U}, f_*\Omega^q_X) \tag{7.25}$$

$$= \prod_{i_0 < \cdots < i_p} \Omega^q(U_{i_0, \cdots, i_p} \cap X). \tag{7.26}$$

Let $\mathfrak{U}|_X = \{U_i \cap X\}$ be an affine cover of $X$ induced by subspace topology. Since $\Omega^q_X$ is locally free, by Theorem 4.13, we have vanishing higher cohomology, i.e. $H^{p>0}(U_i \cap X, \Omega^q_X) = 0$. Thus $\Omega^\bullet_X$ is acyclic on $\mathfrak{U}|_X$, thus

$$\mathbf{H}^k(X, \Omega^\bullet_X) \cong \check{H}^k(\mathfrak{U}|_X, \Omega^\bullet_X). \tag{7.27}$$

The Čech cohomology $\check{H}^k(\mathfrak{U}|_X, \Omega^\bullet_X)$ can be computed as the cohomology of the following complex:

$$K^{p,q} = C^p(X, \mathfrak{U}|_X, \Omega^\bullet_X) \tag{7.28}$$

$$= \prod_{i_0 < \cdots < i_p} \Omega^q(U_{i_0, \cdots, i_p} \cap X), \tag{7.29}$$

from which we directly get our result. □

Since $\Omega^q_X$ is locally free, by Theorem 4.13, we have vanishing higher cohomology group $H^{p>0}(X, \Omega^q_X)$. Finially, we have

$$\mathbf{H}^p(X, \Omega^\bullet_X) \cong h^p(\Gamma(X, \Omega^\bullet_X)) = h^p(\Omega^\bullet_X), \tag{7.30}$$

and this takes care of the (7.11). This proves Theorem 6.10, which, by the previous section, implies Theorem 5.12, and hence we are done.

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