INTERSECTIONS OF BROWNIAN PATHS

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Abstract. This expository paper aims to derive intersection properties of Brownian motion in the Euclidean space. Highlights of our methods include Hausdorff dimension, potential theory of Brownian motion, and ideas of percolation and trees. We address the existence or non-existence of intersection points for multiple independent Brownian motions, and the closely related question about self-intersections of finite multiplicity.

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1. Introduction

Brownian motion is a mathematical model for random continuous movements. A familiar picture of Brownian motion is provided by simple random walks. Consider a sequence of independent, identically distributed random variables $\{X_i\}_{i\in\mathbb{N}}$, with $\mathbb{P}\{X_i = 1\} = \mathbb{P}\{X_i = -1\} = 1/2$. The simple random walk started from the origin is the process $\{S_n \mid n \in \mathbb{N}\}$ given by $S_n = \sum_{i=1}^n X_i$. We can turn this discrete process into a continuous one by linear interpolation. Let

$$S(t) = S_{\lfloor t\rfloor} + (t - \lfloor t\rfloor)(S_{\lfloor t\rfloor+1} - S_{\lfloor t\rfloor}).$$

Then, if we zoom out the timeline and set $S_n^*(t) = S(nt)/\sqrt{n}$, the central limit theorem tells us that the distribution of $S_n^*(t)$ converges to the normal distribution with mean 0 and

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variance t. It can be shown that $\{S_n^*(t) \mid t \geq 0\}$ converges in distribution to Brownian motion, which we define below. By this description, we may think of Brownian motion as the macroscopic picture of microscopic fluctuations. One can also infer why we define Brownian motion to have the following properties.

Definitions 1.1. A real-valued stochastic process $\{B(t) \mid t \geq 0\}$ is a **linear Brownian motion** started in x if it satisfy the following properties:

- ◇ $B(0) = x$;
- ◇ the function $t \mapsto B(t)$ is continuous almost surely;
- ◇ $B(t)$ has **independent increments**, i.e., for all times $s_1 \leq t_1 \leq s_2 \leq t_2 \leq \cdots \leq s_n \leq t_n$, the random variables $B(t_1) - B(s_1), \cdots, B(t_n) - B(s_n)$ are independent;
- ◇ $B(t)$ has normally distributed **stationary increments**, i.e., for all times s and t, the random variables $B(t+s) - B(s)$ are normally distributed with mean 0 and variance t.

A **d-dimensional Brownian motion** started in $(x_1, \ldots, x_d)^T$ is a stochastic process whose coordinates consist of d independent linear Brownian motions started respectively in $x_1, \ldots, x_d$. A **standard Brownian motion** (in some arbitrary or specified dimension d) is a Brownian motion started from the origin.

Notation 1.2. We write $\mathbb{P}_x$ and $\mathbb{E}_x$ for conditional probability and conditional expectation, respectively, given that the Brownian motion in question starts in x.

Fact 1.3. Brownian motion exists, and in fact can be explicitly constructed

Brownian motion has a ubiquitous presence in allied fields of mathematics. It is most well known as a model for phenomena in statistical physics and financial markets, from the movement of particles in a dust cloud to stock prices, among many other applications. In mathematics itself, Brownian motion is a foundational object related to a wide range of problems in probability. The specific topic we investigate in this paper − the intersections of Brownian paths − is also far from an isolated subject. The intersection properties of Brownian motion are analogous to those of random walks, and both are closely linked to statistical physics, see [5]. As another example, the fact that Brownian motion almost surely has no self-intersections in dimensions $d \geq 4$, which we prove in Theorem 4.3, suggests that the self-avoiding random walk and the simple random walk behave similarly. There is a precise statement of this for $d \geq 5$ in [6].

We shall shortly give a brief overview of some basic properties of Brownian motion. Then, we make a detour to Hausdorff measure and Hausdorff dimension in section 2. An immediate application of this discussion allows us to determine the size of Brownian paths. Other ideas introduced here, including the capacity of a set and tree-like approximation of a set by dyadic cubes, will be important in what follows. Section 3 returns the focus to Brownian motion. The main question of interest here is the hitting of a fixed set in $\mathbb{R}^d$ by d-dimensional Brownian motion. We are able to determine whether the hitting probability is zero by a nonrandom attribute of a set – its capacity, with the powerful theorem of Kakutani 3.16. Our investigations culminate in Sections 4 and 5, where we obtain precise intersection properties of Brownian motion in different dimensions.

1.1. **Basic Properties of Brownian Motion.** Using standard facts of normal distributions, one can deduce the following two invariance properties of Brownian motion. Note that it suffices to show them for linear Brownian motions.

Proposition 1.4 (Scaling invariance). *If $\{B(t) \mid t \geq 0\}$ is a standard Brownian motion, then for any $a > 0$, the process $\{W(t) \mid t \geq 0\}$ given by $W(t) = \frac{1}{a}B(a^2 t)$ is also a standard Brownian motion*

Proposition 1.5 (Time inversion). *If $\{B(t) \mid t \geq 0\}$ is a standard Brownian motion, then the process $\{W(t) \mid t \geq 0\}$ given by*

$$W(t) = \begin{cases} 0 & \text{for } t = 0 \\ tB(1/t) & \text{for } t > 0 \end{cases}$$

is also a standard Brownian motion.

These two propositions suggest the fractal nature of Brownian motion, i.e., it shares similar geometric structures at all scales. A standard tool in the study of fractal sets is Hausdorff dimension, which we will inverstigate in the next section. Before that, we state a precise continuity condition satisfied by Brownian motions. A function $f : [0, \infty) \to \mathbb{R}$ is **locally α-Hölder continuous at x** for some $x \geq 0$ if there exists $\delta > 0$ and a **Hölder constant** $C > 0$ such that

$$|f(x) - f(y)| \leq C|x - y|^\alpha, \quad \text{for all } y \geq 0 \text{ satisfying } |y - x| < \delta.$$

The theorem below can be shown with the help of an explicit construction of Brownian motion; see 1.20 in [1].

Theorem 1.6. *For any $\alpha < 1/2$, Brownian motion is almost surely everywhere locally α-Hölder continuous.*

Hölder continuity is useful when we want to relate the Hausdorff dimension of the range of a function to that of its domain. In particular, we apply it to determine the dimension of a Brownian path in Theorem 2.17.

2. Hausdorff Dimension

While the usual integral dimension tells the size of a line apart from that of a disk, it is too crude to distinguish the size of a line from that of other curves. This observation suggests that to make sense of the size of Brownian paths, we need a more general notion of dimension. In this section, we discuss the general properties of Hausdorff dimension and provide techniques to estimate it. Among other subsequent applications, we use these techniques to determine the dimension of Brownian paths. As we will see in section 4, one can determine whether two independent Brownian motions intersect each other by analyzing the dimensions of their paths and that of the ambient space. With this purpose in mind, we restrict our discussion to subsets of Euclidean spaces.

Definition 2.1. For a set $E \subset \mathbb{R}^d$ and $\alpha \geq 0$, $\delta > 0$, let

$$\mathcal{H}^\alpha_\delta(E) \coloneqq \inf \left\{ \sum_{i=1}^\infty |E_i|^\alpha \mid E \subset \bigcup_{i=1}^\infty E_i \text{ and } |E_i| \leq \delta \text{ for all } i \right\}$$

be the *α-value* of the most efficient covering (approximately speaking) of E by sets of diameter bounded by δ, where $|\cdot|$ denotes diameter. Then the **α-Hausdorff measure** of E is

$$\mathcal{H}^\alpha(E) = \lim_{\delta \downarrow 0} \mathcal{H}^\alpha_\delta(E).$$

It can be shown that $\mathcal{H}^\alpha$ is an exterior measure.

Note that $\mathcal{H}^\alpha_\delta(E)$ increases as δ decreases, because we have less choices of coverings; hence the limit exists and is a supremum. Also observe that if $0 \le \alpha \le \beta$ and $\mathcal{H}^\alpha(E) = 0$, then $|E_i|^\beta \le |E_i|^\alpha$ for any covering $\{E_i\}_{i\in\mathbb{N}}$ of E by small sets, so $\mathcal{H}^\beta(E) = 0$. This allows us to define the Hausdorff dimension.

Definition 2.2. The **Hausdorff dimension** of a set $E \subset \mathbb{R}^d$ is

$$\dim E = \inf\{\alpha \mid \mathcal{H}^\alpha(E) = 0\} = \sup\{\alpha \mid \mathcal{H}^\alpha(E) > 0\}.$$

Intuitively, $\mathcal{H}^\alpha(E)$ measures the α-dimensional mass of E among sets of dimension α, so that it will be negligible when α is greater than the dimension of E and large if α is smaller than the "true" dimension. The following alternative formulation of Hausdorff dimension confirms this intuition.

Proposition 2.3.

$$\dim E = \inf\{\alpha \mid \mathcal{H}^\alpha(E) < \infty\} = \sup\{\alpha \mid \mathcal{H}^\alpha(E) = \infty\}$$

Proof. It suffices to show that for all $\beta > \alpha$, $\mathcal{H}^\alpha(E) < \infty$ implies $\mathcal{H}^\beta(E) = 0$. Suppose $\mathcal{H}^\alpha(E)$ equals some finite constant C. Then for any $\delta > 0$, there exists a covering $E_1, E_2, \dots$ of E where each E_i has diameter $\le \delta$ and α-value $\le C + 1$, so that $\mathcal{H}^\alpha_\delta(E) \le C + 1$. Then, for any covering $E_1, E_2, \dots$ of E with $|E_i| \le \delta$, $\sum_i |E_i|^\beta \le \delta^{\beta-\alpha} \sum_i |E_i|^\alpha$. This shows $\mathcal{H}^\beta_\delta(E) \le \delta^{\beta-\alpha}\mathcal{H}^\alpha_\delta(E) \le \delta^{\beta-\alpha}(C+1)$. Let $\delta \downarrow 0$ to get $\mathcal{H}^\beta(E) = 0$. $\square$

As one naturally expects, the Hausdorff dimension of subsets of $\mathbb{R}^d$ are no greater than d. Since $\mathbb{R}^d$ can be covered by a countable number of unit cubes and α-Hausdorff measure is countably subadditive, we can see this by showing that the unit cube Q has zero α-Hausdorff measure for all $\alpha > d$. This is immediate if we partition Q into $(1/(\delta/\sqrt{d}))^d$ cubes each with diameter $\le \delta$, and observe that $\mathcal{H}^\alpha_\delta(Q) \le \sqrt{d}^d \delta^{\alpha-d}$, which goes to 0 as δ does.

At this point, we can already give an upper bound for the dimension of the range of Brownian motion using local Hölder continuity 1.6. Suppose $f : [0,\infty) \to \mathbb{R}^d$ is a locally α-Hölder continuous function, and A is a subset of $[0,1]$. At each point $x \in [0,\infty]$, there exists a neighborhood A_x of x such that f is α-Hölder continuous in A_x with Hölder constant C_x. Then, denoting the image of A_x by B_x, we have

$$\mathcal{H}^\beta(B_x) \le C_x^\beta \mathcal{H}^{\alpha\beta}(A_x)$$

for any $\beta \ge 0$, whence $\dim(B_x) \le \frac{1}{\alpha}\dim(A_x)$. It is not hard to check that Hausdorff dimension is countably stable, i.e.,

$$\dim \bigcup_{i=1}^\infty E_i = \sup_{1\le i<\infty}\{\dim E_i\}.$$

As a result, $\dim(B) \le \frac{1}{\alpha}\dim(A)$ by taking countable unions. When f is replaced by the random function $B(t)$, we take the limit $\alpha \uparrow 1/2$ to derive the following upper bound on the dimension of Brownian paths.

Proposition 2.4. *For any fixed set $A \subset [0,\infty)$ and d-dimensional Brownian motion $B(t)$,*

$$\dim B(A) \le (2\dim A) \wedge d, \quad \textit{almost surely.}$$

This bound is sharp, as we will obtain a lower bound in Theorem 2.17 after introducing requisite techniques. In general, an α-dimensional set can have α-Hausdorff measure ranging from zero, finite nonzero, to infinity. It is an interesting fact that for $d \ge 2$, the 2-Hausdorff measure of $B(A)$ for any $A \subset [0,\infty)$ is almost surely zero. Here we show a weaker result.

Proposition 2.5. $\mathcal{H}^2(B([0,1])) < \infty$ *almost surely.*

Proof. Consider a covering of $B([0,1])$ by closed balls

$$\mathcal{B}\Big(B(\tfrac{k}{n}),\ \max_{\frac{k}{n}\le t\le \frac{k+1}{n}} \big|B(t)-B(\tfrac{k}{n})\big|\Big), \quad k=0,\dots,n-1.$$

Since Brownian motion is uniformly continuous on the unit interval, for any $\delta > 0$, we can find a large enough n such that the diameter of each ball in the above cover is less than δ. Note that by stationary increment and scaling invariance,

$$\mathbb{E}\Big[\Big(\max_{\frac{k}{n}\le t\le \frac{k+1}{n}} \big|B(t)-B(\tfrac{k}{n})\big|\Big)^2\Big] = \mathbb{E}\Big[\Big(\max_{0\le t\le \frac{1}{n}} |B(t)|\Big)^2\Big] = \frac{1}{n}\mathbb{E}\Big[\Big(\max_{0\le t\le 1} |B(t)|\Big)^2\Big].$$

Although we will not go into the details here, the right hand side expectation is finite. Therefore, the expected value of $\mathcal{H}^2_\delta$ is bounded above by

$$\mathbb{E}\Big[4\sum_{k=0}^{n-1}\Big(\max_{\frac{k}{n}\le t\le \frac{k+1}{n}} \big|B(t)-B(\tfrac{k}{n})\big|\Big)^2\Big] = 4\mathbb{E}\Big[\Big(\max_{0\le t\le 1} |B(t)|\Big)^2\Big].$$

Then by Fatou's lemma,

$$\mathbb{E}\Big[\liminf_{n\to\infty} 4\sum_{k=0}^{n-1}\Big(\max_{\frac{k}{n}\le t\le \frac{k+1}{n}} \big|B(t)-B(\tfrac{k}{n})\big|\Big)^2\Big] < \infty,$$

which shows that the liminf is almost surely finite and concludes the proof. □

2.1. **Lower bounds for the Hausdorff Dimension.** Observe from this proof and Proposition 2.3 that it is relatively easy to give an upper bound for the Hausdorff dimension of a set: it is enough to find an explicit cover with finite α-value in the limit where the diameters of the covering sets shrink to zero. To find lower bounds for the Hausdorff dimension, we use some physical intuition.

We say a measure μ on a set $E \subset \mathbb{R}^d$ is a **mass distribution** on E if $0 < \mu(E) < \infty$. Intuitively, μ spreads a positive finite mass over E. If we can distribute this mass in such a way that its local density is small, then the set E must be large in some sense.

Proposition 2.6. *(Mass distribution principle) Let $E \subset \mathbb{R}^d$ and $\alpha \ge 0$. If there exists a mass distribution μ on E and constants $C, \delta > 0$ such that*

$$\mu(V) \le C|V|^\alpha$$

for all Borel sets V with $|V| \le \delta$, then $\dim E \ge \alpha$.

Proof. For any covering $E_1, E_2, \dots$ of E with $|E_i| \le \delta$,

$$\sum_{i=1}^\infty |E_i|^\alpha \ge \sum_{i=1}^\infty \frac{\mu(E_i)}{C} \ge \frac{\mu(E)}{C} > 0.$$

The statement follows by taking the infimum over all such covers and passing to the limit $\delta \downarrow 0$. □

Next, we provide another method for obtaining lower bounds for the Hausdorff dimension.

Definition 2.7. Let $E \subset \mathbb{R}^d$ and $K : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$. The K**-energy** of a measure μ on E is

$$I_K(\mu) = \iint K(x, y)\, d\mu(x)\, d\mu(y),$$

and the K**-capacity** of E is

$$\mathrm{Cap}_K(E) = 1/[\inf\{I_K(\mu) \mid \mu \text{ a probability measure on } E\}].$$

Notation 2.8. In most of our applications, $K(x, y) = f(|x - y|)$, in which case we use the terms *f-energy* and *f-capacity.* If further $K(x, y) = |x - y|^{-\alpha}$, we use the terms *$\alpha$-energy* and *$\alpha$-capacity.* In these cases we change the subscripts in the notation accordingly.

The idea of the energy method is a simple one similar to that of the mass distribution principle. The condition that the mass in each small set is bounded by a power of its size is replaced by finiteness of an energy, which suggests that the local density is low enough to overcome the singularity of the integrand.

Theorem 2.9 (Energy method)**.** *Let $E \subset \mathbb{R}^d$, $\alpha \geq 0$, and μ be a mass distribution on E. Then, for any $\delta > 0$,*

$$\mathcal{H}^\alpha_\delta(E) \geq \frac{\mu(E)^2}{\iint_{|x-y|<\delta} \frac{d\mu(x)\, d\mu(y)}{|x-y|^\alpha}}.$$

Therefore,

- ◇ *If $I_\alpha(\mu) < \infty$, or in particular if $\mathrm{Cap}_\alpha(E) > 0$, then $\dim E \geq \alpha$.*
- ◇ *If $\mathcal{H}^\alpha(E) < \infty$, then for any mass distribution μ on E, $I_\alpha(\mu) = \infty$, so $\mathrm{Cap}_\alpha(E) = 0$.*

Proof. Given $\epsilon > 0$, choose $E_1, E_2, \ldots$ to be a pairwise disjoint covering of E by sets of diameter less than δ, such that $\sum |E_i|^\alpha \leq \mathcal{H}^\alpha_\delta(E) + \epsilon$. Then

$$\mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i) = \sum_{i=1}^{\infty} |E_i|^{\frac{\alpha}{2}} \frac{\mu(E_i)}{|E_i|^{\frac{\alpha}{2}}},$$

and

$$\iint_{|x-y|<\delta} \frac{d\mu(x)\, d\mu(y)}{|x-y|^\alpha} \geq \sum_{i=1}^{\infty} \iint_{E_i \times E_i} \frac{d\mu(x)\, d\mu(y)}{|x-y|^\alpha} \geq \sum_{i=1}^{\infty} \frac{\mu(E_i)^2}{|E_i|^\alpha}.$$

Now we apply the Cauchy-Schwarz inequality to get

$$\mu(E)^2 \leq \sum_{i=1}^{\infty} |E_i|^\alpha \sum_{i=1}^{\infty} \frac{\mu(E_i)^2}{|E_i|^\alpha} \leq (\mathcal{H}^\alpha_\delta(E) + \epsilon) \iint_{|x-y|<\delta} \frac{d\mu(x)\, d\mu(y)}{|x-y|^\alpha}.$$

The stated inequality follows by passing to the limit $\epsilon \downarrow 0$.

If $I_\alpha(\mu) < \infty$, then letting $\delta \downarrow 0$ gives $\mathcal{H}^\alpha(E) = \infty$, hence $\dim E \geq \alpha$. Note that $\mathrm{Cap}_\alpha(E) > 0$ implies that there exists a mass distribution μ on E with $\mu(E) = 1$ and $I_\alpha(\mu) < \infty$.

For the last statement, assume $I_\alpha(\mu) < \infty$. Then letting $\delta \downarrow 0$ and using dominated convergence gives $\mathcal{H}^\alpha(E) = \infty$, which shows the contrapositive of what we want. ☐

2.2. **Trees and Frostman's Lemma.** We provide a converse to the mass distribution principle 2.6, which leads to an extremely useful formulation of the Hausdorff dimension of a set in Theorem 2.16. The proof of Frostman's lemma relies on an introduction of trees – an essential tool in fractal geometry.

Definitions 2.10. A **tree** is a connected graph $T = (V, E)$, where V is a finite or countable set of vertices, and $E \subset V \times V$ is the set of ordered edges, such that the following properties hold:

- ⋄ every vertex $v \in V$ has a unique **parent** $w \in V$ with $(w, v) \in E$, except for a unique **root**, which has no parent;
- ⋄ there is a unique self-avoiding path from the root to any other vertex v, where the number of edges in this path is called the **generation** of v;
- ⋄ for every vertex $v \in V$, the number of **children** $w \in V$ with $(v, w) \in E$ is finite.

When T is finite, the **boundary** ∂T of T consists of the vertices with no children. When T is infinite, calling any infinite self-avoiding path started in the root a **ray**, the boundary ∂T is the set of rays. A **cutset** is a set Π of edges if every ray contains an edge in Π.

Notation 2.11. We denote the root by r, and for any other vertex v, denote the parent of v by $\hat{v}$. For any $v, w \in V$, let $v \wedge w$ denote the least common ancestor of v and w, i.e., the vertex in the intersection of the paths from r to v and from r to w with minimal generation. Similarly, for any two rays ξ and η, $\xi \wedge \eta$ is the vertex in the intersection of the rays with minimal generation. We also write $v \leq w$ when $v = v \wedge w$.

Definitions 2.12. For an infinite tree, assign to each edge a **capacity** with a mapping $C : E \to [0, \infty)$. A **flow** with **strength** $c > 0$ is a mapping $\theta : E \to [0, c]$ with the following properties:

- ⋄ flow is unleashed at the root and conserved at each other vertex, i.e.
$$\sum_{w:\ \hat{w}=r} \theta(r, w) = c \quad \text{and} \quad \theta(\hat{v}, v) = \sum_{w:\ \hat{w}=v} \theta(v, w), \quad \forall\, v \neq r;$$
- ⋄ for any edge $e \in E$, $\theta(e) \leq C(e)$.

We cite without proof the following well-known result of graph theory, and a useful measure extension theorem.

Theorem 2.13 (Max-flow min-cut)**.**
$$\max\Big\{ \text{strength}\,(\theta) \mid \theta \textit{ a flow with capacities } C \Big\} = \inf \Big\{ \sum_{e \in \Pi} C(e) \mid \Pi \textit{ a cutset} \Big\}.$$

Theorem 2.14 (Carathéodory's extension theorem)**.** *Let $\mathcal{C}$ be an algebra, which means $A, B \in \mathcal{C}$ implies A^c and $A \cup B$ are in $\mathcal{C}$. If $\tilde{\nu}$ defined on $\mathcal{C}$ is countably additive, then $\tilde{\nu}$ extends to a measure ν on $\sigma(\mathcal{C})$.*

Whereas the mass distribution principle gives a lower bound on dimension given a suitable mass distribution, we now construct a mass distribution given a lower bound on dimension.

Theorem 2.15 (Frostman's lemma)**.** *If $A \subset \mathbb{R}^d$ is closed with $\mathcal{H}^\alpha(A) > 0$, then there exists a Borel probability measure μ supported on A and a positive constant C such that $\mu(D) \leq C|D|^\alpha$ for all Borel sets $D \in \mathbb{R}^d$.*

Proof. Note that a bounded piece of A must have positive α-Hausdorff measure. By changing coordinates and letting $\mu \equiv 0$ on the rest of A, we may assume $A \subset [0,1]^d$. The idea of the proof is to exploit a bijection between A and the boundary of a tree, and use the max-flow min-cut theorem to construct a measure that is positive on A.

To this end, we divide the unit cube dyadically, where each of the 2^{nd} cubes at step n is further divided into 2^d subcubes. Associate a tree $T = (V, E)$ to the dyadic cubes, such that the root is associated with $[0,1]^d$ and each vertex has 2^d children. As the dyadic cubes approximate A, we remove edges ending in vertices associated with subcubes that do not intersect A. Then we get a natural bijection $\Phi : \partial T \to A$.

For edges e at generation n, we assign the capacities $C(e) = (\sqrt{d}2^{-n})^\alpha$. It is clear that if we associate to every edge e in a cutset Π of T the cube corresponding to the initial vertex of e, then Π corresponds to a covering of A. Since we can choose cutsets where the diameters of the associated cubes are arbitrarily small, for any $\delta > 0$

$$\inf\Big\{\sum_{e\in\Pi} \mid \Pi \text{ a cutset}\Big\} \geq \inf\Big\{\sum_{i=1}^\infty |A_i|^\alpha \mid A \subset \bigcup_{i=1}^\infty A_i \text{ and } |A_i| < \delta\Big\}$$

Letting $\delta \downarrow 0$, the left hand side is bounded from zero as $\mathcal{H}^\alpha(A) > 0$. By the max-flow min-cut theorem, there exits a flow $\theta : E \to [0,\infty)$ of positive strength and $\theta(e) \leq C(e)$ for all edges $e \in E$.

For each edge $e \in E$, we let $T(e) \subset \partial T$ to be the set consisting of all rays containing e. One can check that the collection $\mathcal{C}(\partial T)$ of finite disjoint unions of the sets $T(e)$ is a an algebra on ∂T, and we define $\tilde{\nu}$ on $\mathcal{C}(\partial T)$ by $\tilde{\nu}(T(e)) = \theta(e)$. Conservation of flow implies that $\tilde{\nu}$ is countably additive, so that using Carathéodory's extension theorem, $\tilde{\nu}$ extends to a measure ν on $\sigma(\mathcal{C}(\partial T))$. The desired measure on A is given by

$$\mu = \nu \circ \Phi^{-1}, \text{ which satisfies } \mu(C) = \theta(e),$$

where C is the cube associated with the initial vertex of e.

To check that each Borel set $D \subset \mathbb{R}^d$ satisfies the claimed bound, choose an integer n such that $2^{-n} < \left|D \cap [0,1]^d\right| \leq 2^{-n+1}$. Then $D \cap [0,1]^d$ can be covered by 3^d cubes of side length 2^{-n}. It follows that $\mu(D) \leq 3^d C(e) \leq 3^d d^{\alpha/2}|D|^\alpha$. The proof is completed by normalizing μ. □

Frostman's lemma has the following important corollary.

Theorem 2.16. *For any closed set $A \subset \mathbb{R}^d$,*

$$\dim A = \sup\big\{\alpha \mid \mathrm{Cap}_\alpha(A) > 0\big\}.$$

Proof. The $\geq$ direction follows from the energy method 2.9. Thus it remains to show that for any $\alpha < \dim A$, there exists a probability measure on A with finite α-energy. Choose β slightly larger than α with $\mathcal{H}^\beta(A) > 0$. Possibly replacing A by a smaller subset with positive β-Hausdorff measure, we can assume A is contained in a set of diameter less than one. By Frostman's lemma, there exists a probability measure μ on A and a positive constant C such that $\mu(D) \leq C|D|^\beta$ for all Borel sets D. Pick $x \in E$ and consider the annuli $S_k(x) = \{y \mid 2^{-k} < |x-y| \leq 2^{-k+1}\}$. Then

$$I_\alpha(\mu) \leq \int \frac{d\mu(y)}{|x-y|^\alpha} = \sum_{k=1}^\infty \int_{S_k(x)} \frac{d\mu(y)}{|x-y|^\alpha} \leq \sum_{k=1}^\infty \mu(S_k(x))2^{k\alpha} \leq C\sum_{k=1}^\infty (2^{-k+2})^\beta 2^{k\alpha} < \infty,$$

which shows the $\leq$ direction and concludes the proof. □

2.3. **First Application to Brownian motion.** We now reap the fruit of our labor in the previous subsection. Note that the following theorem implies that for $d \geq 2$, the dimension of a Brownian path is almost surely 2.

Theorem 2.17. *For any fixed set $A \subset [0, \infty)$ and d-dimensional Brownian motion $B(t)$,*

$$\dim B(A) = (2 \dim A) \wedge d, \quad \textit{almost surely.}$$

Proof. In light of Proposition 2.4, we only need the lower bound. By Theorem 2.16, for any $\alpha < \dim A \wedge d/2$, there exists a probability measure μ on A such that $I_\alpha(\mu) < \infty$. We derive a mass distribution $\tilde{\mu}$ on $B(A)$ given by $\tilde{\mu}(E) = \mu(B^{-1}(E) \cap A)$ for any Borel $E \subset \mathbb{R}^d$. Equivalently, $\tilde{\mu}$ satisfies

$$\int_{\mathbb{R}^d} f(x)\, d\tilde{\mu}(x) = \int_A f(B(t))\, d\mu(t)$$

for any measurable f. To apply the energy method, we want to show that

$$\mathbb{E} I_{2\alpha}(\tilde{\mu}) = \mathbb{E} \iint \frac{d\tilde{\mu}(x)\, d\tilde{\mu}(y)}{|x-y|^{2\alpha}} = \mathbb{E} \int_0^1 \int_0^1 \frac{d\mu(t)\, d\mu(s)}{|B(t)-B(s)|^{2\alpha}} < \infty.$$

The expected value of the integrand is

$$\mathbb{E}|B(s)-B(t)|^{-2\alpha} = \mathbb{E}\big[\big((t-s)^{1/2}|B(1)|\big)^{-2\alpha}\big] = |t-s|^{-\alpha} \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2}} \frac{2}{|z|^{2\alpha}} e^{-|z|^2/2} dz,$$

where the integral can be evaluated in polar coordinates and equals some finite constant C because $2\alpha < d$. Using Fubini's theorem,

$$\mathbb{E} I_{2\alpha}(\tilde{\mu}) = C \iint \frac{d\mu(t)\, d\mu(s)}{|t-s|^{\alpha}} \leq C I_\alpha(\mu) < \infty.$$

This means $I_{2\alpha}(\tilde{\mu}) < \infty$ almost surely, hence $\dim B(A) \geq 2\alpha$ almost surely. The lower bound follows by letting $\alpha \uparrow \dim A \wedge d/2$. ☐

The notions of energy and capacity are developed in this section as an estimate for the Hausdorff dimension of a general set, and we have just seen its power in determining the precise relation between the dimensions of the domain and range of Brownian motion. In the next section, we return our focus to further properties of Brownian motion. The central problem of interest is to estimate the probability that a Brownian path visits a certain set. It turns out that energy and capacity also play a crucial role there, as Theorem 3.16 shows.

3. Recurrence and Transience

Recall that the study of Brownian motion can be motivated by simple random walk, which is a canonical discrete-time and discrete-space Markov chain. We now investigate Brownian motion in the framework of Markov processes.

3.1. **Markov Properties of Brownian Motion.** Since Brownian motion has independent and stationary increments, it is a **time-homogeneous Markov process** in the sense that $\{B(t+s)-B(s) \mid t \geq 0\}$ is a Brownian motion independent of the process $\{B(t) \mid 0 \leq t \leq s\}$, for any fixed time s. More technically, let

$$\mathcal{F}(s) := \bigcap_{u>s} \sigma(B(t) \mid 0 \leq t \leq u).$$

Observe that $\{B(t+s') - B(s') \mid t \geq 0\}$ is independent of $\mathcal{F}(s)$ for any $s' > s$. Taking the limit $s' \downarrow s$, we find that $\{B(t+s) - B(s) \mid t \geq 0\}$ is independent of $\mathcal{F}(s)$. Brownian motion enjoys a similar property when s is replaced by a particular class of random times.

Definition 3.1. A random variable T taking values in $[0, \infty]$ is a **stopping time** with respect to $\{B(t) \mid t \geq 0\}$ if $\{T \leq t\} \in \mathcal{F}(t)$ for every $t \geq 0$. Intuitively, this means that we can determine whether $T \leq t$ just with the information of Brownian motion up to time t.

Theorem 3.2 (Strong Markov Property). *If T is an almost surely finite stopping time, then the process $\{B(T+t) - B(T) \mid t \geq 0\}$ is a standard Brownian motion independent of $\mathcal{F}(T)$.*

Because Brownian motion starts afresh at any fixed or stopping time, it is natural to formulate the distribution of d-dimensional Brownian motion in terms of the **Markov transition kernel**

$$\mathfrak{p}(t, x, y) = (2\pi t)^{-d/2} \exp\left(-\frac{|x-y|^2}{2t}\right),$$

where $\mathfrak{p}(t, x, \cdot)$ is just the density of a normal distribution with mean x and covariance matrix t times the identity. Then, for any set $A \in \mathbb{R}^d$,

$$\mathbb{P}\{B(t) \in A \mid \mathcal{F}(s)\} = \int_A \mathfrak{p}(t-s, B(s), y)\, dy. \tag{3.3}$$

3.2. **Recurrence, Transience, and the Green Kernel.** A general question one can ask about Markov processes is whether they diverge in a certain sense. The notions of recurrence and transience provide a rough answer to this question. We then introduce tools to more precisely estimate the total time that Brownian motion spends in a given set.

Definition 3.4. Let $\{X(t) \mid t \geq 0\}$ be a Markov process where t can take values in $\mathbb{R}$ or $\mathbb{N}$, and $X(t)$ can take values in $\mathbb{R}^d$ or $\mathbb{N}$. Then we say X is

- ◇ **transient** if almost surely, $\lim_{t\uparrow\infty} |X(t)| = \infty$;
- ◇ **point recurrent** if almost surely, for every x in the state space there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ with $t_n \to \infty$, such that $X(t_n) = x$ for all n;
- ◇ **neighborhood recurrent** if $X(t)$ takes values in $\mathbb{R}^d$ and, almost surely, for any x in $\mathbb{R}^d$ and $\epsilon > 0$, there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ with $t_n \to \infty$, such that $X(t_n) \in \mathcal{B}(x, \epsilon)$ for all n.

The following result is standard; for details, see sections 8.4 and 8.5 of [2], or sections 3.1 and 3.2 of [1], which gives a more rigorous treatment.

Theorem 3.5. *d-dimensional Brownian motion is*

- ◇ *point recurrent* *for $d = 1$;*
- ◇ *neighborhood recurrent, but not point recurrent* *for $d = 2$;*
- ◇ *transient* *for $d \geq 3$.*

In fact, when $d \geq 3$, let T_r be the first time the Brownian motion visits $\mathcal{B}(0, r)$. Then, for any x outside $\mathcal{B}(0, r)$,

$$\mathbb{P}_x\{T_r < \infty\} = \frac{r^{d-2}}{|x|^{d-2}}. \tag{3.6}$$

As one might expect, when Brownian motion is point recurrent or neighborhood recurrent, the total time it spends in a bounded set is infinite, whereas in the transient case this time is finite almost surely. For linear Brownian motion run up to a finite time t, it can be shown that the **occupation measure** given by

$$\mu_t(A) = \int_0^t \mathbb{1}_A(B(s))\,ds \quad \text{for } A \subset \mathbb{R} \text{ Borel}$$

is almost surely absolutely continuous with respect to the Lebesgue measure (see [1] 3.26). For the two dimensional case, we also want to estimate the time Brownian motion spends in a bounded set before some finite time. Instead of a fixed time, we are interested in a stopping time.

Definition 3.7. Let $\{B(t) \mid t \geq 0\}$ be a standard d-dimensional Brownian motion. Then a **transient Brownian motion** is the process $\{B(t) \mid 0 \leq t < T\}$ in either of the following two cases:

I. $d \geq 3$ and $T = \infty$;
II. $d \geq 2$ and T is the first exit time from a bounded region $D \subset \mathbb{R}^d$.

The following proposition is an appropriate modification of (3.3). See [1] 3.30.

Proposition 3.8. *For a transient Brownian motion, there exists a transition (sub-)density* $\mathfrak{p}^*(t,x,y)\colon [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0,1]$ *such that, for any* $t > 0$,

$$\mathbb{P}_x\big\{B(t) \in A \text{ and } t < T\big\} = \int_A \mathfrak{p}^*(t,x,y)\,dy \quad \text{for } A \subset \mathbb{R}^d \text{ Borel.} \tag{3.9}$$

The explicit forms of this transition (sub-)density are given in each case by

I. $\mathfrak{p}^*(t,x,y) = \mathfrak{p}(t,x,y)$;
II. $\mathfrak{p}^*(t,x,y) = \mathfrak{p}(t,x,y) - \mathbb{E}_x\big[\mathfrak{p}(t-T, B(T), y)\mathbb{1}\{t > T\}\big]$.

If $\{X_n \mid n \in \mathbb{N}\}$ is a transient Markov chain, then the expectation $\mathbb{E}_x\big[\sum_n \mathbb{1}_y(X_n)\big]$ estimates the number of times y is visited by the Markov chain started from x. A similar quantity is the following.

Definition 3.10. For transient Brownian motion with transition (sub-)densities as above, the **Green kernel** $G\colon \mathbb{R}^d \to \mathbb{R}^d \to [0,\infty]$ is the function

$$G(x,y) = \int_0^\infty \mathfrak{p}^*(t,x,y)\,dt$$

As the discrete analog suggests, $G(x,\cdot)$ is the density of the expected occupation measure for transient Brownian motion started in x.

Theorem 3.11. *For any measurable* $f\colon \mathbb{R}^d \to [0,\infty]$,

$$\mathbb{E}_x \int_0^T f(B(t))\,dt = \int f(y)\,G(x,y)\,dy.$$

In particular,

$$\mathbb{E}_x \int_0^T \mathbb{1}_A(B(t))\,dt = \int_A G(x,y)\,dy. \tag{3.12}$$

Proof. Using Fubini's theorem and (3.9)

$$\begin{aligned}
\mathbb{E}_x \int_0^T f(B(t))\,dt = \int_0^\infty \mathbb{E}_x\big[f(B(t))\mathbb{1}_{\{t<T\}}\big]\,dt \qquad &= \int_0^\infty \int \mathfrak{p}^*(t,x,y)\,f(y)\,dy\,dt \\
= \int \int_0^\infty \mathfrak{p}^*(t,x,y)\,dt\,f(y)\,dy \qquad &= \int G(x,y)\,f(y)\,dy.
\end{aligned}$$

□

After some calculation, we obtain the following expressions for the Green kernel. Details can be found in [1] 3.33 and 3.37.

Proposition 3.13. *For the two cases of transient Brownian motion in Definition 3.7,*

I. $G(x,y) = C|x-y|^{2-d}$ *for a constant C depending only on d;*
II. $G(x,y) = \frac{1}{\pi}\log|x-y|^{-1} + \mathbb{E}_x\left[\frac{1}{\pi}\log|B(T)-y|\right]$.

3.3. **Polar Sets and Capacities.** As we have seen above, the Green kernel is linked to how much time Brownian motion spends in a given Borel set in $\mathbb{R}^d$. Now we want to investigate a closely related question, namely to estimate the probability that a Brownian motion ever hits a given set.

Throughout this section, we assume D is a bounded region in $\mathbb{R}^d$ and $A \subset D$ is closed. It is perhaps surprising that the precise connection between the Green kernel and the hitting probability of A is provided by the notion of capacity introduced in the previous section. To begin with, we define a measure on A whose total mass is the hitting probability of A.

Definition 3.14. With notation as above, let $\{B(t) \mid 0 \le t < T\}$ be a transient Brownian motion, and $\tau = \inf\{0 < t < T \mid B(t) \in A\}$ be the first hitting time of A. The **harmonic measure** on A is the measure μ such that for any $A' \subset A$ Borel,

$$\mu(A') = \mathbb{P}_0\big\{B(\tau) \in A' \ \bigcap \ \tau < T\big\}.$$

That is, μ describes the (sub-)probability distribution of $B(\tau)$ on A. Note also that

$$\mu(A) = \mathbb{P}_0\big\{\tau < T\big\} = \mathbb{P}_0\big\{B(t) \in A \text{ for some } 0 < t < T\big\} \tag{3.15}$$

We are specifically interested in whether the hitting probability of A is positive. The set A is called **polar** for transient Brownian motion, if for all $x \in A^c$,

$$\mathbb{P}_x\big\{B(t) \in A \text{ for some } 0 < t < T\big\} = 0,$$

and nonpolar otherwise. Polarity of A for usual Brownian motion is similarly defined. It can be shown that if there exists an x in the unbounded component of A^c such that Brownian motion started in x has a positive chance to hit A, then Brownian motion started anywhere in the unbounded component of A^c has a positive chance to hit A (see [1] 3.42). Thus, for A not containing the origin, it suffices to consider standard Brownian motion and the probability in (3.15) to determine the polarity of A.

The criterion we wish to prove for the remainder of this section is:

Theorem 3.16 (Kakutani)**.** *Let Λ be a closed set in $\mathbb{R}^d$. For the* ***radial potential*** *f defined by*

$$f(\epsilon) := \begin{cases} \left|\log \epsilon^{-1}\right| & \text{for } d = 2, \\ \epsilon^{2-d} & \text{for } d \ge 3, \end{cases}$$

$$\Lambda \textit{ is polar for } d\textit{-dimensional Brownian motion} \iff \mathrm{Cap}_f(\Lambda) = 0.$$

Notation 3.17. For quantities $a(\Lambda)$ and $b(\Lambda)$ depending on Λ, write $a(\Lambda) \asymp b(\Lambda)$ if there exist constants c and C not depending on Λ, such that

$$c\, b(\Lambda) \le a(\Lambda) \le C\, b(\Lambda).$$

Comparing the radial potential to the expressions of the Green kernel given in Proposition 3.13, we find that the radial potential is a constant multiple of the Green kernel in $d \geq 3$ and also a constant multiple of the first term in the case $d = 2$. It therefore seems tempting, at least when $d \geq 3$, to prove Kakutani's theorem by proving

$$\mathbb{P}_x\big\{B(t) \in \Lambda \text{ for some } t > 0\big\} \asymp \mathrm{Cap}_G(\Lambda)$$

for some $x \in \Lambda^c$. However, we note that the left hand side probability depends on the starting point x, whereas the G-capacity of Λ is translation invariant. Moreover, by scaling invariance 1.4, the probability that $B(t)$ ever hits Λ does not change if we scale Λ up or down with respect to the starting point; whereas the G-capacity cannot in general stay constant.

Therefore, we consider the following scale-invariant version of the Green kernel, and prove Kakutani's theorem by showing a quantitative estimate.

Definition 3.18. The **Martin kernel** $M\colon D \times D \to [0,\infty]$ is given by

$$M(x,y) \coloneqq \frac{G(x,y)}{G(0,y)} \quad \text{for } x \neq y; \quad M(x,x) \coloneqq \infty.$$

Theorem 3.19. *Let $\{B(t) \mid 0 \leq t < T\}$ be a transient d-dimensional Brownian motion and $A \subset D \subset \mathbb{R}^d \backslash \{0\}$ compact. Then*

$$\frac{1}{2}\,\mathrm{Cap}_M(A) \leq \mathbb{P}_0\big\{B(t) \in A \textit{ for some } 0 < t < T\big\} \leq \mathrm{Cap}_M(A).$$

Note that the above theorem estimates the hitting probability for transient Brownian motion, but Kakutani's theorem concerns polarity for usual Brownian motion, which does not agree with the transient case in the plane. Thus to apply this estimate, we need the following fact.

Proposition 3.20. *A compact set $A \subset \mathbb{R}^2\backslash\{0\}$ is polar for planar Brownian motion if and only if it is polar for transient Brownian motion defined until first exit from every bounded domain D.*

Proof. The only if part is straightforward to see. To prove the contrapositive of the if direction, assume that A is nonpolar for planar Brownian motion. Then for some large enough time S, the probability that Brownian motion hits A before S is at least q for some $q > 0$. For any bounded D containing A,

$$\mathbb{P}\{\text{hits } A \text{ before leaving } D\} \geq \mathbb{P}\{\text{hits } A \text{ before } S \bigcap \text{ does not exit } D \text{ before } S\}.$$

By neighborhood recurrence, we can choose a large enough D such that the probability that Brownian motion exits D before S is less than $q/2$. This shows that the right hand side probability in the above display is at least $q/2$. Therefore, the transient Brownian motion defined until first exit from D is also nonpolar. □

Assuming Theorem 3.19, we prove Kakutani's theorem.

Proof. (of Theorem 3.16) By taking countable unions, it suffices to consider compact subsets A of Λ contained in a large domain D and not containing the origin, so that $G(0,y)$ is bounded away from zero and infinity. As a result, $\mathrm{Cap}_M(A)$ vanishes if and only if $\mathrm{Cap}_G(A)$ vanishes. By Theorem 3.19 and Proposition 3.20, A is polar for Brownian motion if and only if its M-capacity vanishes. Thus it remains to show that the G-capacity of A vanishes if and only if its f-capacity vanishes.

When $d \geq 3$, $G(x,y)$ is a constant multiple of $f(|x-y|)$, hence the above statement. When $d = 2$, recall that

$$G(x,y) = \frac{1}{\pi} \log |x-y|^{-1} + \mathbb{E}_x \big[\frac{1}{\pi} \log |B(T) - y| \big].$$

As A is a positive distance away from D^c, the right hand side expectation is bounded above for $x, y \in A$. Therefore, the finiteness of G-energy is dependent only on the contribution of the first term, which is a constant multiple of $f(|x-y|)$. This concludes the proof. □

Now, we give the proof of Theorem 3.19.

Proof of the upper bound in Theorem 3.19. Given the link between the harmonic measure μ and hitting probability in (3.15), we want to consider the M-energy of μ. Let τ be the first hitting time of A, and y be any point in D. By the strong Markov property,

$$\begin{aligned} \mathbb{P}_0\{|B(t)-y| < \epsilon \bigcap t \leq T\} &\geq \mathbb{P}_0\{|B(t)-y| < \epsilon \bigcap \tau \leq t \leq T\} \\ &= \mathbb{E}\,\mathbb{P}_{B(\tau)}\{|B(t)-y| < \epsilon \bigcap t \leq T\}. \end{aligned} \tag{3.21}$$

Integrating both sides of (3.21) over t and using the defining property of the Green kernel in (3.12), the left hand side yields

$$\mathbb{E}_0 \int_0^T \mathbb{1}\{|B(t)-y| < \epsilon\}\, dt = \int_{\mathcal{B}(y,\epsilon)} G(0,y')\, dy'. \tag{3.22}$$

By the definition of harmonic measure and Fubini's theorem, the right hand side becomes

$$\begin{aligned} &\int_0^\infty \mathbb{E}\,\mathbb{P}_{B(\tau)}\{|B(t)-y| < \epsilon \bigcap t \leq T\}\, dt \\ &= \int_0^\infty \int_A \mathbb{P}_x\{|B(t)-y| < \epsilon \bigcap t \leq T\}\, d\mu(x)\, dt \\ &= \int_A \mathbb{E}_x \int_0^\infty \mathbb{1}\{|B(t)-y| < \epsilon\} \mathbb{1}\{t \leq T\}\, dt\, d\mu(x) \\ &= \int_A \int_{\mathcal{B}(y,\epsilon)} G(x,y')\, dy'\, d\mu(x). \end{aligned} \tag{3.23}$$

Combining (3.21), (3.22), and (3.23), we obtain that

$$\int_{\mathcal{B}(y,\epsilon)} \int_A G(x,y')\, d\mu(x)\, dy' \leq \int_{\mathcal{B}(y,\epsilon)} G(0,y')\, dy'.$$

Divide both sides by $\mathcal{L}(\mathcal{B}(y,\epsilon))$ and take the limit $\epsilon \downarrow 0$ to get

$$\int_A G(x,y)\, d\mu(x) \leq G(0,y) \implies \int_A M(x,y) d\mu(x) \leq 1.$$

Since this holds for all $y \in D$, $I_M(\mu) \leq \mu(A)$. As $\mu/\mu(A)$ is a probability measure on A,

$$\mathrm{Cap}_M(A) \geq \frac{1}{I_M(\mu/\mu(A))} \geq \frac{\mu(A)^2}{I_M(\mu)} \geq \mu(A) = \mathbb{P}_0\big\{B(t) \in A \text{ for some } 0 < t < T\big\}.$$

□

To tackle the lower bound, we need two technical lemmas. The first one is standard, whereas the second one is proved in [1] 8.23.

Lemma 3.24 (Paley-Zygmund Inequality)**.** *If X is a nonnegative random variable with $\mathbb{E}[X^2] < \infty$, then*

$$\mathbb{P}\{X > 0\} \geq \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]}.$$

Lemma 3.25. *For $x, y \in A$, let*

$$h_\epsilon(x,y) = \int_{\mathcal{B}(y,\epsilon)} G(x,y')\,dy' \quad \textit{and} \quad h^*_\epsilon(x,y) = \sup_{|x-x'|<\epsilon} \int_{\mathcal{B}(y,\epsilon)} G(x',y')\,dy'. \tag{3.26}$$

Then there is a constant C such that for all sufficiently small ϵ,

$$h^*_\epsilon(x,y) \leq C\,G(x,y).$$

Proof of the lower bound in Theorem 3.19. Take $\epsilon > 0$ to be smaller than half the distance of A to $D^c \cup \{0\}$, and define $h_\epsilon(x,y)$ and $h^*_\epsilon(x,y)$ for x, y in A as in (3.26). Fix a probability measure ν on A, and define the random variable

$$Z_\epsilon = \int_A \int_0^T \frac{\mathbb{1}\{B(t) \in \mathcal{B}(y,\epsilon)\}}{h_\epsilon(0,y)}\,dt\,d\nu(y).$$

By (3.12) and the definition of h_ϵ, $\mathbb{E}_0 Z_\epsilon = 1$. To calculate the second moment, we use the symmetry between integrating over $s \leq t$ and over $t \leq s$, along with Markov property:

$$\begin{aligned}
\mathbb{E}_0 Z_\epsilon^2 &= 2\mathbb{E}_0 \int_0^T ds \int_s^T dt \iint \frac{\mathbb{1}\{B(s) \in \mathcal{B}(x,\epsilon)\}\mathbb{1}\{B(t) \in \mathcal{B}(y,\epsilon)\}}{h_\epsilon(0,x)\,h_\epsilon(0,y)}\,d\nu(x)\,d\nu(y) \\
&\leq 2\mathbb{E}_0 \iint \int_0^T \mathbb{1}\{B(s) \in \mathcal{B}(x,\epsilon)\}\,ds \frac{h^*_\epsilon(x,y)}{h_\epsilon(0,x)\,h_\epsilon(0,y)}\,d\nu(x)\,d\nu(y) \\
&= 2 \iint \frac{h^*_\epsilon(x,y)}{h_\epsilon(0,y)}\,d\nu(x)\,d\nu(y).
\end{aligned}$$

Note that $\lim_{\epsilon\downarrow 0} \mathcal{L}(\mathcal{B}(0,\epsilon))^{-1} h^*_\epsilon(x,y) = G(x,y)$ and $\lim_{\epsilon\downarrow 0} \mathcal{L}(\mathcal{B}(0,\epsilon))^{-1} h_\epsilon(0,y) = G(0,y)$. Since $G(0,y)$ is bounded away from zero and infinity for all $y \in A$, we apply Lemma 3.25 to get that, for some constant C' and all sufficiently small $\epsilon > 0$,

$$\frac{h^*_\epsilon(x,y)}{h_\epsilon(0,y)} \leq C' \frac{G(x,y)}{G(0,y)} = C'\,M(x,y).$$

If ν has infinite M-energy, then the lower bound trivially holds; if ν is of finite M-energy, we use dominated convergence to obtain

$$\lim_{\epsilon\downarrow 0} \mathbb{E} Z_\epsilon^2 \leq 2 \iint \frac{G(x,y)}{G(0,y)}\,d\nu(x)\,d\nu(y) = 2I_M(\nu).$$

Now, using the Paley-Zygmund inequality,

$$\mathbb{P}\{\exists\, t > 0,\ y \in A \text{ such that } B(t) \in \mathcal{B}(y,\epsilon)\} \geq \mathbb{P}\{Z_\epsilon > 0\} \geq \frac{(\mathbb{E}Z_\epsilon)^2}{\mathbb{E}Z_\epsilon^2} = (\mathbb{E}Z_\epsilon^2)^{-1}$$

By compactness of A and continuity and transience of Brownian motion, if the Brownian path visits every ϵ-neighborhood of A, then it must also intersect A. It follows that

$$\mathbb{P}\{\exists\, t > 0 \text{ such that } B(t) \in A\} \geq \lim_{\epsilon\downarrow 0} (\mathbb{E}Z_\epsilon^2)^{-1} \geq \frac{1}{2I_M(\nu)}.$$

As ν is an arbitrary probability measure on A, this concludes the proof. □

4. First Look at Intersection: Existence

Let $\{B_1(t) \mid t \geq 0\}$ and $\{B_2(t) \mid t \geq 0\}$ be two independent d-dimensional Brownian motions with arbitrary starting points. We ask whether the paths of B_1 and B_2 intersect in different dimensions. Because of point recurrence of linear Brownian motion, an intersection exists almost surely in dimension $d = 1$. The idea for the cases $d \geq 2$ is that, by independence, we view the path of B_1 as fixed, so that nonexistence of intersection is equivalent to the polarity of the path of B_1 with respect to the other Brownian motion B_2. Then, we apply Kakutani's theorem and results on Hausdorff dimension discussed earlier.

Theorem 4.1. *Almost surely,*

- ◇ *for $d \geq 4$, two independent Brownian paths in $\mathbb{R}^d$ have empty intersection, except for a possible common starting point;*
- ◇ *for $d \leq 3$, two independent Brownian paths in $\mathbb{R}^d$ have an intersection other than a possible common starting point.*

Proof. Note that in $d \geq 3$, $\mathrm{Cap}_f(A) = \mathrm{Cap}_{d-2}(A)$ for any Borel $A \subset \mathbb{R}^d$, where f is the radial potential. By Theorem 2.17 and Theorem 2.16 applied to the path of B_1,

$$2 = \dim B_1[0, \infty) = \sup\{\alpha \mid \mathrm{Cap}_\alpha(B_1[0, \infty)) > 0\}. \tag{4.2}$$

Therefore, for $d \geq 5$, as $d - 2$ is strictly larger than 2, $\mathrm{Cap}_f(B_1[0, \infty)) = 0$.

When $d = 4$, by Proposition 2.5, $\mathcal{H}^2(B_1[0, \infty)) < \infty$. Using the second form of the energy method 2.9, we get $\mathrm{Cap}_f(B_1[0, \infty)) = \mathrm{Cap}_2(B_1[0, \infty)) = 0$. By Kakutani's theorem 3.16, this implies $B_1[0, \infty)$ is polar for B_2.

To prove the statement for $d \leq 3$, observe that if two Brownian motions in $\mathbb{R}^3$ almost surely intersect, then by projection onto the first two coordinates, two Brownian paths in $\mathbb{R}^2$ must also intersect almost surely. Therefore, it suffices to consider $d = 3$. In this case, $\mathrm{Cap}_f(B_1[0, \infty)) = \mathrm{Cap}_1(B_1[0, \infty)) > 0$, by (4.2), showing that two Brownian paths in $\mathbb{R}^3$ intersect with positive probability.

To show this probability is in fact one, we first assume B_1 and B_2 start in different points. By rotational invariance and scaling invariance, the probability that B_1 and B_2 never intersect is independent of their starting positions; denote this probability by q. Then, for any given $\epsilon > 0$, we can choose a large time t such that the probability that B_1 and B_2 do not intersect before t is at most $q + \epsilon$. Using the Markov property,

$$\begin{aligned} q &\leq \mathbb{P}\{B_1(t_1) \neq B_2(t_2) \text{ for all } 0 < t_1, t_2 \leq t\}\, \mathbb{P}\{B_1(t_1) \neq B_2(t_2) \text{ for all } t_1, t_2 > t\} \\ &\leq (q + \epsilon) q \end{aligned}$$

As $q < 1$ and $\epsilon > 0$ is arbitrary, this shows $q = 0$, whence the two paths intersect almost surely. In the case that B_1 and B_2 share a starting point, we use the Markov property to conclude

$$\mathbb{P}\{B_1(t_1) \neq B_2(t_2) \text{ for all } t_1, t_2 > 0\} = \lim_{t' \downarrow 0} \mathbb{P}\{B_1(t_1) \neq B_2(t_2) \text{ for all } t_1, t_2 > t'\} = 0$$

□

In fact, when B_1 and B_2 share a starting point in $d \leq 3$, they almost surely intersect nontrivially before any $t > 0$. This is because given any $\epsilon > 0$, we can find a large enough time t_ϵ such that the probability that B_1 and B_2 intersect before t_ϵ is at least $1 - \epsilon$. As t_ϵ is finite, using scaling invariance, we see that the probability that B_1 and B_2 intersect before t is also bounded below by $1 - \epsilon$.

It is natural to ask for the existence of self-intersection of a single Brownian motion. As a moment of thought would suggest, this question is essentially equivalent to the corresponding one for independent Brownian paths. Thus the result below follows from Theorem 4.1.

Theorem 4.3. *Almost surely,*

- ◇ *for $d \geq 4$, a Brownian path in $\mathbb{R}^d$ does not intersect itself;*
- ◇ *for $d \leq 3$, a Brownian path in $\mathbb{R}^d$ intersects itself.*

Proof. Observe that in both cases, we can prove the statements by just considering Brownian motion run until time 1. For $d \geq 4$, it suffices to show that for any rational $q \in (0,1)$, there almost surely exists no times $0 \leq t_1 < q < t_2 \leq 1$ with $B(t_1) = B(t_2)$. Given q, we consider

$$B_1(t) = B(q+t) - B(q), \quad 0 \leq t \leq 1-q$$
$$B_2(t) = B(q-t) - B(q), \quad 0 \leq t \leq q.$$

By the defining properties of Brownian motion, B_1 and B_2 are independent Brownian motions. Theorem 4.1 shows that B_1 and B_2 almost surely do not intersect, hence the two sections of B before and after time q almost surely do not intersect.

When $d \leq 3$, the Brownian motions

$$B_1(t) = B\big(\frac{1}{2}+t\big) - B\big(\frac{1}{2}\big), \quad 0 \leq t \leq \frac{1}{2},$$
$$B_2(t) = B\big(\frac{1}{2}-t\big) - B\big(\frac{1}{2}\big), \quad 0 \leq t \leq \frac{1}{2},$$

almost surely intersect before time $\frac{1}{2}$, as they have the same starting point. □

5. Second Look at Intersection: Points of Finite Multiplicity

The previous section tells us that two independent Brownian paths almost surely have empty intersection in $d \geq 4$ and nonempty intersection in $d \leq 3$. It is then natural to ask for more details in the latter case: do k independent d-dimentional Brownian motions share a common point? It turns out that, although an intersection between two paths almost surely exists in both cases, the intersection behaviors in dimensions two and three are drastically different when more Brownian paths are taken into consideration.

Theorem 5.1. *Almost surely,*

- ◇ *for $d = 3$, three independent Brownian paths in $\mathbb{R}^3$ have empty intersection, except for a possible common starting point;*
- ◇ *for $d = 2$ and any interger $k \geq 2$, k independent Brownian paths in $\mathbb{R}^2$ have an intersection other than a possible common starting point.*

Given the techniques used in the previous section, we prove the first statement by considering two Brownian paths as fixed and showing their union to be polar for the third one. Specifically, we employ a similar strategy as for the $d = 4$ case in the proof of Theorem 4.1. Recall that it suffices to exhibit a cover with finite α-value to show the α-Hausdorff measure to be finite, as in the proof of Proposition 2.5. Moreover, approximation of a closed set by dyadic cubes was an element in the proof of Theorem 2.15. We also make use of these earlier introduced ideas.

Proof of Theorem 5.1 for $d = 3$. It suffices to show that three Brownian paths almost surely have empty intersection in any unit cube Q not containing the starting points. By Kakutani's theorem, we only need to show $\mathrm{Cap}_1(B_1[0,\infty) \cap B_2[0,\infty) \cap Q) = 0$. Further, using the energy method 2.9, we can show this by proving that $\mathcal{H}^1(B_1[0,\infty) \cap B_2[0,\infty) \cap Q) < \infty$.

To do this, let $\mathfrak{Q}_n$ be the collection of the n-th generation of dyadic cubes each of side length 2^{-n}, and $\mathfrak{Q}_n^*$ be the subcollection of cubes in $\mathfrak{Q}_n$ that are hit by both B_1 and B_2. Then, noting that the distance between Q and the starting points is bounded from below and using (3.6), there exists a constant C such that for any cube $Q_n \in \mathfrak{Q}_n$;

$$\mathbb{P}\{Q_n \in \mathfrak{Q}_n^*\} = \mathbb{P}\{B(t) \in Q_n \text{ for some } t > 0\}^2 \leq C2^{-2n}.$$

For every n, $\mathfrak{Q}_n^*$ is a covering of $B_1[0,\infty) \cap B_2[0,\infty) \cap Q$. The expected one-value of this covering is

$$\mathbb{E}\Big[\sum_{Q_n \in \mathfrak{Q}_n^*} |Q_n|\Big] = 2^{3n}(C2^{-2n})\sqrt{3}2^{-n} \leq C\sqrt{3}$$

This shows that, by Fatou's lemma,

$$\mathbb{E}\Big[\liminf_{n\to\infty} \sum_{Q_n \in \mathfrak{Q}_n^*} |Q_n|\Big] \leq \liminf_{n\to\infty} \mathbb{E}\Big[\sum_{Q_n \in \mathfrak{Q}_n^*} |Q_n|\Big] \leq C\sqrt{3}.$$

Therefore, almost surely, $\mathcal{H}^1(B_1[0,\infty) \cap B_2[0,\infty) \cap Q) < \infty$. □

We shall prove Theorem 5.1 for $d = 2$ through a new route. Note that it suffices to show that the intersection of k independent Brownian paths has positive Hausdorff dimension. Perhaps surprisingly, a much stronger result holds.

Theorem 5.2. *For $d = 2$ and any integer $k \geq 2$, let $\{B_1(t) \mid t \geq 0\}, \ldots, \{B_k(t) \mid t \geq 0\}$ be k independent planar Brownian motions. Then, almost surely,*

$$\dim\big(B_1[0,\infty) \cap \ldots \cap B_k[0,\infty)\big) = 2.$$

5.1. **The Percolation Method.** As the upper bound in the above theorem is immediate, we again need to find lower bounds for Hausdorff dimension. In addition to the ones discussed in section 2.1, we introduce a new method to tackle this problem. The idea is to construct a random test set Γ; if Γ has a positive chance to intersect a fixed set A, then A must be somewhat large. A slight modification of the simple construction of dyadic cubes turns out to be very useful.

Given a unit cube Q, let $\mathfrak{Q}_n$ be the collection of the n-th generation of dyadic cubes each of side length 2^{-n}. We construct a random collection $\mathfrak{Q}_1^\star$ by selecting each of the 2^d cubes in $\mathfrak{Q}_1$ independently with probability $2^{-\gamma}$ for some fixed $\gamma \in [0,d]$. Then, inductively, construct $\mathfrak{Q}_{n+1}^\star$ by selecting each of the cubes in $\mathfrak{Q}_{n+1}$ that are contained in a larger cube in the previously selected collection $\mathfrak{Q}_n^\star$. In this way, we obtain a **percolation limit set**

$$\Gamma[\gamma] := \bigcap_{n=1}^{\infty} \Gamma_n, \quad \text{where} \quad \Gamma_n := \bigcup_{Q_n \in \mathfrak{Q}_n^\star} Q_n.$$

We also define

$$\mathfrak{Q} := \bigcup_{n=1}^{\infty} \mathfrak{Q}_n \quad \text{and} \quad \mathfrak{Q}^\star := \bigcup_{n=1}^{\infty} \mathfrak{Q}_n^\star.$$

The percolation limit set $\Gamma[\gamma]$ plays the role of a random test set, where the probability for $\Gamma[\gamma]$ to hit a given set is positively related to the retention probability of dyadic cubes, hence negatively related to γ. This idea is more precisely stated in the following theorem.

Theorem 5.3 (Hawkes)**.** *For any $\gamma \in [0,d]$ and fixed closed set $A \subset Q$, the following statements hold.*

(i) If $\dim A < \gamma$, then $A \cap \Gamma[\gamma] = \emptyset$ almost surely.

(ii) If $\dim A > \gamma$*, then* $\mathbb{P}\{A \cap \Gamma[\gamma] \neq \emptyset\} > 0$*.*

Note that (i) gives a lower bound for $\dim A$ when $\mathbb{P}\{A \cap \Gamma[\gamma] \neq \emptyset\} > 0$.

Proof of (i). Suppose $\dim A < \gamma$. By the definition of Hausdorff dimension, for any $\epsilon > 0$, there exists a covering $A_1, A_2, \dots$ of A such that $\sum_i |A_i|^\gamma < \epsilon$. Since each A_i is contained in no more than 3^d dyadic cubes of smaller diameter, we may assume $A_1, A_2, \dots \in \mathfrak{Q}$. For $A_i \in \mathfrak{Q}_n$, $|A_i| = \sqrt{3}2^{-n}$, whereas $\mathbb{P}\{A_i \in \mathfrak{Q}_n^\star\} = (2^{-\gamma})^n$. Let N be the (a priori possibly infinite) number of cubes in $A_1, A_2, \dots$ that survive the percolation. Since these cubes form a covering of $A \cap \Gamma[\gamma]$,

$$\mathbb{P}\{A \cap \Gamma[\gamma] \neq \emptyset\} \leq \mathbb{P}\{N > 0\} \leq \mathbb{E}N = \sum_{i=1}^\infty \mathbb{P}\{A_i \in \mathfrak{Q}^\star\} = \frac{1}{\sqrt{3}^\gamma} \sum_{i=1}^\infty |A_i|^\gamma < \frac{\epsilon}{\sqrt{3}^\gamma}.$$

Since $\epsilon > 0$ is arbitrary, this implies that $A \cap \Gamma[\gamma]$ almost surely. □

Proof of (ii). Just like the proof for the lower bound in Theorem 3.19, this proof also relies on the Paley-Zygmund inequality 3.24 and finiteness of an energy. By Theorem 2.16, if $\dim A > \gamma$, them there exists a probability meaure μ on A with $I_\gamma(\mu) < \infty$. For every positive integer n, define the random variables

$$Z_n = \sum_{Q_n \in \mathfrak{Q}_n^\star} \frac{\mu(Q_n)}{|Q_n|^\gamma} = \sum_{Q_n \in \mathfrak{Q}_n} \mu(Q_n) 2^{n\gamma} \mathbb{1}_{\{Q_n \in \mathfrak{Q}_n^\star\}}.$$

The point of this definition is that, if $Z_n > 0$, then Γ_n must intersect A, so that by compactness, $Z_n > 0$ for all n implies $A \cap \Gamma[\gamma] \neq \emptyset$. Since $Z_{n+1} > 0$ only if $Z_n > 0$,

$$\mathbb{P}\{A \cap \Gamma[\gamma] \neq \emptyset\} \geq \mathbb{P}\{Z_n > 0 \text{ for all } n\} = \lim_{n \to \infty} \mathbb{P}\{Z_n > 0\}.$$

We want to use the Paley-Zygmund inequality to find a lower bound for $\mathbb{P}\{Z_n > 0\}$ that is independent of n. Thus, calculate the first and second moment of Z_n:

$$\begin{aligned} \mathbb{E}[Z_n] &= \sum_{Q_n \in \mathfrak{Q}_n} \mu(Q_n)\, 2^{n\gamma}\, \mathbb{P}\{Q_n \in \mathfrak{Q}_n\} = \sum_{Q_n \in \mathfrak{Q}_n} \mu(Q_n) = 1 \\ \mathbb{E}[Z_n^2] &= \sum_{Q_n \in \mathfrak{Q}_n} \sum_{R_n \in \mathfrak{Q}_n} \mu(Q_n)\mu(R_n)\, 2^{2n\gamma}\, \mathbb{P}\{Q_n \in \mathfrak{Q}_n^\star \text{ and } R_n \in \mathfrak{Q}_n^\star\}. \end{aligned}$$

Note that if $C \in \mathfrak{Q}_m^\star$ is the smallest cube containing Q_n and R_n, then the respective retentions of Q_n and R_n are not independent until after step m of the percolation. Therefore, the probability in the above display is $2^{-\gamma m} 2^{-2\gamma(n-m)}$. Also note that for any $x \in Q_n$ and $y \in R_n$,

$$|x - y| \leq \sqrt{d} 2^{-m} \implies 2^{\gamma m} \leq d^{\gamma/2} |x - y|^{-\gamma}.$$

Combining what we have above,

$$\mathbb{E}[Z_n^2] = \sum_{Q_n \in \mathfrak{Q}_n} \sum_{R_n \in \mathfrak{Q}_n} \mu(Q_n)\mu(R_n) 2^{\gamma m} \leq d^{\gamma/2} \iint \frac{d\mu(x)\, d\mu(y)}{|x - y|^\gamma} = d^{\gamma/2} I_\gamma(\mu).$$

Therefore, applying the Paley-Zygmund inequality, we have $\mathbb{P}\{Z_n > 0\} \geq 1/\big(d^{\gamma/2} I_\gamma(\mu)\big)$, where the right hand side is positive and independent of n. This gives the desired result. □

In order to use Hawkes's theorem to prove the remaining part of Theorem 5.1, we need a zero-one law, which relies on Blumenthal's well-known zero-one law.

Theorem 5.4 (Blumenthal's zero-one law). *Let $x \in \mathbb{R}^d$ and A be an event in the **germ σ-algebra** $\mathcal{F}(0) = \cap_{u>t} \sigma(B(t) \mid 0 \le t \le u)$, then $\mathbb{P}_x(A)$ is either zero or one.*

Proof. Recall from section 3.1 that any $A \in \sigma(B(t) \mid t \ge 0)$ is independent of $\mathcal{F}(0)$. As $\mathcal{F}(0) \subset \sigma(B(t) \mid t \ge 0)$, any $A \in \mathcal{F}(0)$ is independent of itself, hence has probability either zero or one. ☐

Lemma 5.5. *For any $\gamma > 0$ and independent Brownian motions $\{B_1 \mid t \ge 0\}, \ldots, \{B_k \mid t \ge 0\}$,*

$$\mathbb{P}\big\{ B_1[0,\infty) \cap \ldots \cap B_k[0,\infty) \ge \gamma \big\}$$

is either zero or one, and is independent of the starting points of $B_1, \ldots, B_k$.

Proof. As one might expect, we use Blumenthal's zero-one law in conjunction with scaling invariance. For any $s \ge 0$ and $t \in (0, \infty]$, let

$$I(s,t) = B_1(s,t) \cap \ldots \cap B_k(s,t) \quad \text{and} \quad p(t) = \mathbb{P}\big\{ \dim I(0,t) \ge \gamma \big\}.$$

Suppose first that $B_1, \ldots, B_k$ share a starting point. By monotonicity of the events,

$$\mathbb{P}\big\{ \dim I(0,t) \ge \gamma \text{ for all } t > 0 \big\} = \lim_{t \downarrow 0} p(t).$$

Since the event in question is in the germ σ-algebra, the left hand side probability is either zero or one. On the other hand, scaling invariance says that $p(t)$ is the same for all $0 < t < \infty$, hence it is either zero and one for all $0 < t < \infty$. Taking $t \to \infty$, we have either $p(\infty) = 0$ or $p(\infty) = 1$, proving the case when $B_1, \ldots, B_k$ start at the same point.

Now, suppose $p(\infty) = 0$ and pick any finite positive time s. By Markov property,

$$\begin{aligned} 0 &= \mathbb{P}\big\{ \dim I(s,\infty) \ge \gamma \big\} \\ &= \int \mathbb{P}\big\{ \dim I(s,\infty) \ge \gamma \mid B_1(s) = x_1 \cap \ldots \cap B_k(s) = x_k \big\} \prod_{i=1}^{k} \mathfrak{p}(t, 0, x_i)\, dx_1 \ldots dx_k. \end{aligned}$$

This implies that $\mathbb{P}\{\dim I(0,\infty) \ge \gamma\} = 0$ for $\mathcal{L}_{kd}$-almost every configuration of starting points. Thus, for the case where $B_1, \ldots, B_k$ have arbitrary starting points, $\mathbb{P}\{\dim I(s,\infty) \ge \gamma\} = 0$ for any $s > 0$. By the countable stability of Hausdorff dimension, $\dim I(0,\infty) \ge \gamma$ if and only if $\dim I(s,\infty) \ge \gamma$ for sufficiently small rational $s > 0$. Therefore,

$$\mathbb{P}\big\{ \dim I(0,\infty) \ge \gamma \big\} = \lim_{s \downarrow 0} \mathbb{P}\big\{ \dim I(s,\infty) \ge \gamma \big\} = 0.$$

One can argue analogously for the case $p(\infty) = 1$. ☐

Proof of Theorem 5.2, hence Theorem 5.1 for $d = 2$. We want to show that the dimension of the intersection of any k Brownian paths in any unit square is at least two. Fix $\gamma < 2$, and choose $\beta_1, \ldots, \beta_k > 0$ such that $\gamma + \beta_1 + \ldots + \beta_k < 2$. Observe that for independent percolation limit sets $\Gamma[\gamma]$ and $\Gamma[\beta_1], \ldots, \Gamma[\beta_k]$,

$$\Gamma[\gamma] \cap \bigcap_{i=1}^{k} \Gamma[\beta_i]$$

is a percolation limit set with parameter $\gamma + \beta_1 + \ldots + \beta_k$. Recall that $\dim B_1[0,\infty) = 2 > \gamma + \beta_1 + \ldots + \beta_k$. Then, by part *(ii)* of Hawkes's theorem 5.3,

$$\mathbb{P}\Big\{ B_1[0,\infty) \cap \Gamma[\gamma] \cap \bigcap_{i=1}^{k} \Gamma[\beta_i] \ne \emptyset \Big\} > 0.$$

Now, if we view $\Gamma[\beta_k]$ as the random test set, then with positive probability,

$$\mathbb{P}\Big\{B_1[0,\infty)\cap\Gamma[\gamma]\cap\bigcap_{i=1}^{k}\Gamma[\beta_i]\neq\emptyset\mid B_1[0,\infty)\cap\Gamma[\gamma]\cap\bigcap_{i=1}^{k-1}\Gamma[\beta_i]\Big\}>0.$$

By part (i) of Hawkes's theorem, this implies that

$$\mathbb{P}\Big\{\dim\big(B_1[0,\infty)\cap\Gamma[\gamma]\cap\bigcap_{i=1}^{k-1}\Gamma[\beta_i]\big)\geq\beta_k\Big\}>0. \tag{5.6}$$

Conditioning on this event, as $\beta_k>0$, the α-capacity of

$$B_1[0,\infty)\cap\Gamma[\gamma]\cap\bigcap_{i=1}^{k-1}\Gamma[\beta_i]$$

is positive for any positive $\alpha<\beta_k$. One checks that $\log\epsilon^{-1}<\epsilon^{-\alpha}$ for all sufficiently small ϵ, hence the set in the above display also has positive capacity with respect to the potential kernel in the plane. By Kakutani's theorem, it is therefore nonpolar for the independent Brownian motion $\{B_2(t)\mid t\geq 0\}$, i.e.,

$$\mathbb{P}\Big\{B_1[0,\infty)\cap B_2[0,\infty)\cap\Gamma[\gamma]\cap\bigcap_{i=1}^{k-1}\Gamma[\beta_i]\neq\emptyset\Big\}>0.$$

Repeating the above argument k times yields

$$\mathbb{P}\Big\{\bigcap_{i=1}^{k}B_i[0,\infty)\cap\Gamma[\gamma]\neq\emptyset\Big\}>0,$$

under the condition that the event in (5.6) holds. Finally, by part (i) of Hawkes's theorem,

$$\mathbb{P}\Big\{\bigcap_{i=1}^{k}B_i[0,\infty)\Big\}\geq\gamma\Big\}>0.$$

Using Lemma 5.5, this probability must in fact be one. Taking the limit $\gamma\uparrow 2$ concludes the proof. ☐

5.2. **Multiple Points of Brownian Motion.** Given the discussion in section 4, one naturally expects an analog of Theorem 5.1 for self-intersections. A point $x\in\mathbb{R}^d$ is said to have **multiplicity** k or to be a k**-multiple point** for a Brownian motion $\{B(t)\mid t\geq 0\}$ in $\mathbb{R}^d$ if there exist distinct times $t_1,\ldots,t_k$, such that $x=B(t_1)=\ldots=B(t_k)$. Now that we want to divide a Brownian path into k sections for $k\geq 3$, we cannot simply translate the intersection behavior of these sections into that of independent Brownian motions started at the same point, as we did in the proof of Theorem 4.3. Hence we need the following version of Brownian motion with nonrandom ending point.

Definition 5.7. For $x,y\in\mathbb{R}^d$, let $\{B(t)\mid t\geq 0\}$ be a d-dimensional Brownian motion started in x. A d-dimensional **Brownian bridge** with start in x and end in y is the process given by

$$X(t)=B(t)-t\big(B(1)-y\big),\quad 0\leq t\leq 1.$$

In other words, for $0\leq t\leq 1$, the distribution of $X(t)$ is the conditional distribution of $B(t)$ given $B(1)$. It can be checked that for any $0<t<1$, the distribution of a Brownian bridge run up to time t is mutually absolutely continuous with that of the corresponding

Brownian motion run up to time t; see for example Exercise 1.5 in [1]. Then, the following statements for Brownian bridges are immediate.

Corollary 5.8 (Corollary of Theorem 5.1)**.** *Let* $\{X_1(t) \mid 0 \le t \le 1\}, \dots, \{X_k(t) \mid 0 \le t \le 1\}$ *be k independent Brownian bridges with arbitary starting and ending points. Then,*

- ◇ *for $d = 3$ and $k = 3$, almost surely, three independent Brownian bridges in $\mathbb{R}^3$ have empty intersection, except for possible common starting and ending points.*
- ◇ *for $d = 2$ and any integer $k \ge 2$, with positive probability, k independent Brownian bridges in $\mathbb{R}^2$ have an intersection other than possible common starting and ending points.*

Now, we are ready to refine the result of Theorem 4.3 when $d \le 3$.

Theorem 5.9. *Almost surely,*

- ◇ *for $d = 3$, a Brownian path in $\mathbb{R}^3$ has no triple point.*
- ◇ *for $d = 2$ and any integer $k \ge 2$, a Brownian path in $\mathbb{R}^2$ has points of multiplicity at least k.*

Proof. For $d = 3$, it suffices to show that for any rationals $0 < q_1 < q_2 < q_3$ and $\epsilon < (q_3 - q_2) \wedge (q_2 - q_1)$, there almost surely exist no times $t_i \in (q_i, q_i + \epsilon)$ such that $B(t_1) = B(t_2) = B(t_3)$. Conditioning on the values of the Brownian motion at times q_i and $q_i + \epsilon$, we get three independent Brownian bridges

$$X_i(t) = B(q_i + t) - B(q_i), \quad \text{for } 0 \le t \le \epsilon \; \text{ and } \; i = 1, 2, 3.$$

By Corollary 5.8, X_1, X_2 and X_3 almost surely have empty intersection. Nonexistence of triple point follows by taking expectation over the values of the Brownian motion at times q_i and $q_i + \epsilon$.

For $d = 2$, as the corresponding statement in Corollary 5.8 only holds with positive probability, we again resort to Blumenthal's zero-one law. Consider any $\delta > 0$, rationals $0 < q_1 < \cdots < q_k < q_{k+1} = \delta$, and $\epsilon > 0$ with $q_i + \epsilon < q_{i+1}$ for $i = 1, \dots, k$. Conditioning on the values of the Brownian motion at times q_i and $q_i + \epsilon$, we get k Brownian bridges

$$X_i(t) = B(q_i + t) - B(q_i), \quad \text{for } 0 \le t \le \epsilon \; \text{ and } \; i = 1, \dots, k,$$

which intersect with positive probability. Taking an expectation, we infer that with positive probability, Brownian motion run up to time δ has a k-multiple point. By scaling invariance, this probability is the same for all $\delta > 0$. Therefore,

$$\begin{aligned} &\mathbb{P}\big\{\text{for all } \delta > 0, \exists\, 0 < t_1 < \ldots < t_k < \delta \text{ such that } B(t_1) = \cdots = B(t_k)\big\} \\ = \; &\lim_{\delta \downarrow 0} \mathbb{P}\big\{\exists\, 0 < t_1 < \ldots < t_k < \delta \text{ such that } B(t_1) = \cdots = B(t_k)\big\} > 0. \end{aligned}$$

By Blumenthal's zero-one law, this probability must be one. This shows that a Brownian motion in the plane almost surely has a k-multiple point. □

We end this paper by stating an intriguing result. The interested reader may read section 9.3 of [1] for details.

Theorem 5.10. *Almost surely, a Brownian motion in the plane has a point of uncountable multiplicity.* ■

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