RANDOM MATRICES AND DETERMINANTAL POINT PROCESS

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Abstract. In this expository paper, we shall study the joint probability distribution of the eigenvalues of Gaussian Unitary Ensemble (GUE) random matrices, an important object in the theory of random matrix, from a point process perspective. We won't discuss the Wigner's semicircular law, but we will occasionally use that result. We shall assume that the reader is familiar with measure theoretic probability. Our approach is based on a combination of references [1] and [3].

Contents

1. A Brief Introduction to Point Process

We will present some essential general definitions and concepts regarding point processes in this section. We will build our study of GUE matrices within this abstract framework.

Definition 1.1. (Random Point Process)
A **random point process** on a measurable space $\mathfrak{X}$ is a random sum $\sum_{i\in I}\delta_{X_i}$ of Dirac masses. We restrict X to be a locally compact, complete and separable metric space. For instance, X can be a subset of $\mathbb{R}^d$. We endow $\mathfrak{X}$ with its Borel $\sigma-$field $B(\mathfrak{X})$.

The central object of this paper, the eigenvalues of the Gaussian unitary ensemble, is a random point process.

Definition 1.2. (Locally Finite Atomic Measure)
A **locally finite atomic measure** on $\mathfrak{X}$ is a positive measure $\mu : B(\mathfrak{X}) \to \mathbb{N}\cup\{\infty\}$ which takes integer values and for any compact subset K, $\mu(K) < \infty$. We denote $M(\mathfrak{X})$ to be the set of atomic measures on $\mathfrak{X}$. It then follows that μ can be written as $\sum_{i\in I}\delta_{x_i}$ for a countable index set I.

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We endow $M(\mathfrak{X})$ with the smallest $\sigma-$field which makes the maps $\mu \to \mu(B), B \in B(\mathfrak{X})$ measurable.Thus, we can consider a random point process on X as a measurable map from the probability space to to $M(\mathfrak{X})$. From this perspective, $\mu(B)$ is a random variable that is integer-valued.

Example 1.3. Suppose that $\mu : B(\mathfrak{X}) \to \mathbb{R} \sqcup \{+\infty\}$ is a locally finite positive Borel measure on $\mathfrak{X}$. A Poisson point process with intensity μ on $\mathfrak{X}$ is a random point process P_ν such that, for any family $(B_a)_{a\in A}$ of disjoint Borel subsets of $\mathfrak{X}, (P_\mu (B_a))_{a\in A}$ is a family of independent Poisson variables with parameters $\mu (B_a)$. Any locally finite positive Borel measure on $\mathfrak{X}$ gives rise to a Poisson point process, which is unique in law in $\mathscr{M}^{\text{atom}}(\mathfrak{X})$.

Suppose M is a random point process from $(\Omega, F, \mathbb{P}) \to \mathbb{M}(\mathfrak{X})$ on a locally compact polish space, then we can define, for $i \geq 1$, some random variables $X_i : (\Omega, F, \mathbb{P}) \to \mathfrak{X} \cup \{a\}$ such that $X_i = a$ iff $M(\mathfrak{X}) \leq \infty$ and $i > M(\mathfrak{X})$. Furthermore, we have $M = \sum_{i=1}^{M(\mathfrak{X})} \delta_{X_i}$.

These random variables enable us to define $M^n := \sum_{i_1\neq \ldots \neq i_n, 1\leq i_a \leq M(\mathfrak{X})} \delta_{X_{i_1},\ldots,X_{i_n}}$.

Remark 1.4. For explicit computations of M^n, see [2].

Definition 1.5. (Factorial Moment Measure)

The n^{th} factorial moment measure of M is the positive Borel measure μ_M^n on $\mathfrak{X}^n$ defined by $\mu_M^n(\prod_{i=1}^n B_i) := \mathbb{E}[M^n(\prod_{i=1}^n B_i)]$.

Here we present a calculation based on [1], and there are more calculations in [2].

Example 1.6. Suppose we have a Poisson point process P on $\mathfrak{X}$ with intensity μ, and some disjoint locally compact subsets $B_1, \ldots, B_n$ in $\mathfrak{X}$. We can construct the restriction of the Poisson point process P to $B = \bigcup_{a=1}^n B_a$ in the following manner. First, we take a Poisson random variable N with parameter $\mu(B)$, then we set

$$P_{|B} = \sum_{i=1}^{N} \delta_{X_i}$$

where the X_i 's are independent random variables in B with law $\frac{\mu(\cdot)}{\mu(B)}$, and are independent of N. We then have:

$$\mu_P^{\downarrow n} (B_1 \times B_2 \times \cdots \times B_n) = \mathbb{E}\left[\sum_{i_1\neq i_1 \neq \cdots \neq i_n} \left(\prod_{a=1}^n 1_{X_{i_a}\in B_a}\right)\right] = \left(\prod_{a=1}^n \frac{\mu(B_a)}{\mu(B)}\right) \mathbb{E}\left[N^{\downarrow n}\right] = \prod_{a=1}^n \mu(B_a)$$

By additivity, we conclude that $\mu_P^{\downarrow n} = \mu^{\otimes n}$. This identity encodes the independence of the restrictions of the Poisson point process P to disjoint subsets.

Definition 1.7. (Correlation Functions)

Suppose we have a random point process M on a locally compact polish space X, then there exists a reference Radon measure (locally finite Borel positive measure) λ on X such that the factorial moment measure μ_M^n is absolutely continuous with respect to $\lambda^{\otimes n}$ for all $n \geq 1$. We call $p_n(x_1, \ldots, x_n) = \frac{d(\mu_M^n)}{d\lambda^{\otimes n}}(x_1, \ldots, x_n)$ the **correlation function** of the random point process. The derivative here is the Radon-Nikodym derivative.

Remark 1.8. The existence of correlation functions in the above setting is guaranteed by the Radon-Nikodym theorem.

Definition 1.9. (Simple Point Process)
A point process χ is simple if $P(\exists x : \mathfrak{X}(\{x\}) > 1) = 0$.

Definition 1.10. (Determinantal Point Process)
A simple random point process χ is said to be a **determinantal point process** if its correlation functions $p_n(x_1, \ldots, x_n) = \det(K(x_i, x_j)_{1 \le i,j \le n})$ for all n with some adequate kernel K which does not depend on n.

We will show later in this paper that the eigenvalues of the GUE matrices form a determinantal point process.

2. The Joint Distribution of the Eigenvalues of GUE Matrix

In this section, we will introduce what a GUE random matrix is and investigate their joint probability distribution. Their joint distribution is crucial in proving that they form a determinantal point process.

Definition 2.1. (Gaussian Unitary Ensemble Matrix)
We consider a random matrix in the following form.
$(H_N)_{ii} = N_{\mathbb{R}}(0, \frac{1}{N}), (H_N)_{ij} = \overline{(H_N)_{ji}} = N_{\mathbb{R}}(0, \frac{1}{2N}) + iN_{\mathbb{R}}(0, \frac{1}{2N})$.

Let V_N be the space of $N \times N$ Hermitian matrices endowed with the Lebesgue measure dH_N. Then the distribution μ_{H_N} of a **Gaussian Unitary Ensemble** (abbreviated as GUE) **matrix** H_N is an absolutely continuous probability measure on V_n which can be written as
$\mu_{H_N} = \frac{1}{Z_{N,\mathrm{GUE},1}} \mathrm{e}^{-\frac{N}{2} \operatorname{tr} H^2} \prod_{1 \le i \le N} dH_{i,i} \prod_{1 \le i < j \le N} d \operatorname{Re}(H_{i,j}) \, d \operatorname{Im}(H_{i,j}), \quad$ with $Z_{N,\mathrm{GUE},1} = \sqrt{\frac{2^N \pi^{N^2}}{N^{N^2}}}$.

Theorem 2.2. ***(Joint Distribution of the Point Process of Eigenvalues of a*** GUE ***Matrix)***
The ordered random sequence $(X_{N,1} \le \ldots \le X_{N,N})$ *of eigenvalues of a GUE matrix* M_N *forms a random point process on* $\mathbb{R}$ *and admits the following density function in the Weyl chamber* $\mathbb{R}^n_{\ge}$ *with respect to Lebesgue measure:*
$\frac{1_{x_{N,1} \le \cdots \le x_{N,N}}}{Z_{N,\mathrm{GUE},2}} \mathrm{e}^{-\frac{N}{2} \sum_{i=1}^N (x_{N,i})^2} \prod_{1 \le i < j \le N} |x_{N,i} - x_{N,j}|^2 \prod_{1 \le i \le N} dx_{N,i}$ *in which* $Z_{N,\mathrm{GUE},2} = (2\pi)^{\frac{N}{2}} N^{-\frac{N^2}{2}} (N-1)!(N-2)! \cdots 1!$

For simplicity, we denote $\lambda_1 \le \ldots \le \lambda_N$ as the N eigenvalues corresponding to $x_{N,1} \le \cdots \le x_{N,N}$.
Suppose $X \in M_N$, then by Schur decomposition, we have $X = UDU^*$ with D being a diagonal matrix with real entries (note that the eigenvalues of Hermitian matrices are real), and U is an orthogonal matrix.

We fix some notations first. Set M_N to be the set of Gaussian unitary ensemble matrices. Let U_N be the set of all orthogonal matrices. We say that $U \in \mathscr{U}_N^{(2)}$ is **normalized** if every diagonal entry of U is strictly positive real. We say that $U \in \mathscr{U}_N^{(2)}$ is **good** if it is normalized and every entry of U is nonzero. The collection of good matrices is denoted $\mathscr{U}_N^{(2),\mathrm{g}}$. Finally, we call that $D \in \mathscr{D}_N$ is **distinct** if its entries are all distinct, denoting by $\mathscr{D}_N^{\mathrm{d}}$ the collection of distinct matrices, and by $\mathscr{D}_N^{\mathrm{ao}}$ the subset of matrices with decreasing entries, that is $\mathscr{D}_N^{\mathrm{do}} = \left\{D \in \mathscr{D}_N^{\mathrm{d}} : D_{i,i} > D_{i+1,i+1}\right\}$. We use $\mathscr{H}_N^{(2),\mathrm{dg}}$ to denote the subset of $\mathscr{H}(2)$ consisting of those matrices that possess a decomposition $X = UDU^*$ where $D \in \mathscr{D}_N^{\mathrm{d}}$ and $U \in \mathscr{U}_N^{(2),\mathrm{g}}$.

Let P_N^2 be the the law on V_N given by μ_{H_N}.

The key idea behind the proof is that the map $(U, D) \to UDU^*$ is injective and its complement is of Lebesgue measure zero. Then we essentially need a change of variable formula.

There are several lemmas regarding the sets and matrices we have defined.Eventually, we need to show that the complement is of Lebesgue measure zero.

Lemma 2.3. *$M_N \backslash M_N^{d,g}$ has null Lebesgue measure. Further, the map $(D_N^e, U_N^g) \to M_N^{d,g}$ given by $(D, U) \to UDU^*$ is bijective. The same map from $(D_N^d, U_N^g) \to M_N^{d,g}$ given by $(D, U) \to UDU^*$ is $N!$ to one.*

Proof. In order to prove the first part of the lemma, we note that for any nonvanishing polynomial function p of the entries of X, the set $\{X : p(X) = 0\}$ is closed and has zero Lebesgue measure (this fact can be checked by applying Fubini's Theorem). So it is enough to exhibit a nonvanishing polynomial p with $p(X) = 0$ if $X \in \mathscr{H}_N^{(2)} \backslash \mathscr{H}_N^{(2),\mathrm{dg}}$. Toward this end, we will show that for such X, either X has some multiple eigenvalue, or, for some k, X and the matrix $X^{(k)}$ obtained by erasing the k th row and column of X possess a common eigenvalue.

Given any N by N matrix H, for $i, j = 1, \ldots, N$ let $H^{(i,j)}$ be the $N-1$ by $N-1$ matrix obtained by deleting the i th column and j th row of H, and write $H^{(k)}$ for $H^{(k,k)}$. We begin by proving that if $X = UDU^*$ with $D \in \mathscr{D}_N^{\mathrm{d}}$, and X and $X^{(k)}$ do not have eigenvalues in common for any $k = 1, 2, \ldots, N$, then all entries of U are nonzero. Indeed, let λ be an eigenvalue of X, set $A = X - \lambda I$, and define A^{adj} as the N by N matrix with $A_{i,j}^{\mathrm{adj}} = (-1)^{i+j} \det\left(A^{(i,j)}\right)$. Using the identity $AA^{\mathrm{adj}} = \det(A) I$ one concludes that $AA^{\mathrm{adj}} = 0$. Since the eigenvalues of X are assumed distinct, the null space of A has dimension 1 , and hence all columns of A^{adj} are scalar multiple of some vector v_λ, which is then an eigenvector of X corresponding to the eigenvalue λ. Since $v_\lambda(i) = A_{i,i}^{\mathrm{adj}} = \det\left(X^{(i)} - \lambda I\right) \neq 0$ by assumption, it follows that all entries of v_λ are nonzero. On the other hand, each column of U is a nonzero scalar multiple of some v_λ, leading to the conclusion that all entries of U do not vanish. Note that the resultant of the characteristic polynomials of X and $X^{(k)}$, which can be written as a polynomial in the entries of X and $X^{(k)}$, and hence as a polynomial P_1 in the entries of X, vanishes if and only if X and $X^{(k)}$ have a common eigenvalue. Further, the discriminant of X, which is a polynomial P_2 in the entries of X, vanishes if and only if not all eigenvalues of X are distinct. Taking $p(X) = P_1(X) P_2(X)$, we obtain a nonzero polynomial p with $p(X) = 0$ if $X \in \mathscr{H}_N^{(2)} \backslash \mathscr{H}_N^{(2),\mathrm{dg}}$. The second part of the lemma is immediate since the eigenspace corresponding to each eigenvalue is of dimension 1, the eigenvectors are fixed by the normalization condition. The multiplicity arises from the possible permutations of the order of the eigenvalues.

□

Lemma 2.4. *The map $T : U_N^{vg} \to \mathbb{R}^{\frac{N(N-1)}{2}}$ defined by $T(U) = (\frac{U_{1,2}}{U_{1,1}} \ldots \frac{U_{1,N}}{U_{1,1}} \ldots \frac{U_{N,N-1}}{U_{N-1,N-1}})$ is injective with smooth inverse. The set $(T(U_N^{vg}))^c$ has Lebesgue measure zero.*

Proof. We begin with the first part. The proof is by an inductive construction. Clearly, $U_{1,1}^{-2} = 1 + \sum_{j=2}^{N} |U_{1,j}|^2 / |U_{1,1}|^2$. So suppose that $U_{i,j}$ are given for $1 \leq$

$i \leq i_0$ and $1 \leq j \leq N$. Let $v_i = (U_{i,1}, \ldots, U_{i,i_0}), i = 1, \ldots, i_0$. We can then solve the equation

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{i_0} \end{pmatrix} Z = - \begin{pmatrix} U_{1,i_0+1} + \sum_{i=i_0+2}^{N} U_{1,i} \left(\frac{U_{i_0+1,i}}{U_{i_0+1,i_0+1}} \right)^* \\ U_{2,i_0+1} + \sum_{i=i_0+2}^{N} U_{2,i} \left(\frac{U_{i_0+1,i}}{U_{i_0+1,i_0+1}} \right)^* \\ \vdots \\ U_{i_0,i_0+1} + \sum_{i=i_0+2}^{N} U_{i_0,i} \left(\frac{U_{i_0+1,i}}{U_{i_0+1,i_0+1}} \right)^* \end{pmatrix}$$

The very good condition on U ensures that the vector Z is uniquely determined by this equation, and one then sets

$$U_{i_0+1,i_0+1}^{-2} = 1 + \sum_{k=1}^{i_0} |Z_k|^2 + \sum_{i=i_0+2}^{N} \left| \frac{U_{i_0+1,i}}{U_{i_0+1,i_0+1}} \right|^2$$

and

$$U_{i_0+1,j} = Z_j^* U_{i_0+1,i_0+1}, \qquad \text{for } 1 \leq j \leq i_0.$$

All entries $U_{i_0+1,j}$ with $j > i_0 + 1$ are then determined by $T(U)$ This completes the proof of the first part.

To see the second part, let $\mathscr{Z}_N^{(2)}$ be the space of matrices whose columns are orthogonal, whose diagonal entries all equal to 1 , and all of whose minors have nonvanishing determinants. Define the action of T on $\mathscr{Z}_N^{(2)}$ as before. Then, $T\left(\mathscr{U}_N^{(2),\mathrm{vg}}\right) = T\left(\mathscr{Z}_N^{(2)}\right)$. Applying the previous constructions, we immediately obtain a polynomial type condition for a point in $\mathbb{R}^{N(N-1)}$ to not belong to the set $T\left(\mathscr{Z}_N^2\right)$.

□

Lemma 2.5. *The Lebesgue measure of $M_N \setminus M_N^{vg}$ is zero.*

Proof. We identify a subset of $\mathscr{H}_N^{(2),\mathrm{vg}}$ which we will prove to be of full Lebesgue measure. We say that a matrix $D \in \mathscr{D}_N^{\mathrm{d}}$ is strongly distinct if for any integer $r = 1, 2, \ldots, N-1$ and subsets I, J of $\{1, 2, \ldots, N\}$,

$$I = \{i_1 < \cdots < i_r\}, \quad J = \{j_1 < \cdots < j_r\}$$

with $I \neq J$, it holds that $\prod_{i \in I} D_{i,i} \neq \prod_{i \in J} D_{i,i}$. We consider the subset $\mathscr{H}_N^{(2),\mathrm{sdg}}$ of $\mathscr{H}_N^{(2),\ \mathrm{vg}}$ consisting of those matrices $X = UDU^*$ with D strongly distinct and $U \in \mathscr{U}_N^{(2),\mathrm{vg}}$ Given a positive integer r and subsets I, J as above, put

$$\left(\bigwedge^r X \right)_{IJ} := \det_{\mu,v=1}^{r} X_{i_\mu, j_v}$$

thus defining a square matrix $\Lambda^r X$ with rows and columns indexed by r-element subsets of $\{1, \ldots, N\}$. If we replace each entry of X by its complex conjugate, we replace each entry of $\Lambda^r X$ by its complex conjugate. If we replace X by its transpose, we replace $\wedge^r X$ by its transpose. Given another N by N matrix Y with complex entries, by the Cauchy-Binet Theorem (see lemma 2.2 in [1]), we have $\Lambda^r(XY) = (\Lambda^r X)(\Lambda^r Y)$. Thus, if $U \in \mathscr{U}_N^{(\beta)}$ then $\bigwedge^r U \in \mathscr{U}_{c_N^r}^{(2)}$ where $c_N^r = N!/(N-r)!r!$. We thus obtain that if $X = UDU^*$ then $\wedge^r X$ can be decomposed as $\wedge^r X = (\Lambda^r U)\left(\bigwedge^r D\right)(\Lambda^r U^*)$. In particular, if D is not strongly distinct then, for some $r, \Lambda^r X$ does not possess all eigenvalues distinct. Similarly, if D is strongly distinct but $U \notin \mathscr{U}_N^{(2),\text{ vg}}$, then some entry of $\Lambda^r U$ vanishes. Repeating the argument presented in the proof of the first part of lemma 2.3, we conclude that the Lebesgue measure of $\mathscr{H}_N^{(2)} \backslash \mathscr{H}_N^{(2),\text{sdg}}$ vanishes. This completes the proof of the lemma.

□

Now we can derive the joint distribution based on these lemmas. Let T be as defined in previous lemmas.

Proof. **(Proof of Theorem 2.2):** Define map $\hat{T} : T\left(\mathscr{U}_N^{(2),\text{vg}}\right) \times \mathbb{R}^N \to \mathscr{H}_N(2)$. We also have $\lambda \in \mathbb{R}^N$ and $z \in T\left(\mathscr{U}_N^{(2),\text{vg}}\right) D \in \mathscr{D}_N$ with $D_{i,i} = \lambda_i$ and $\hat{T}(z,\lambda) = T^{-1}(z) D T^{-1}(z)^*$ By Lemma 2.4, $\hat{T}$ is smooth, whereas by Lemma 2.3, it is $N!$-to-1 on a set of full Lebesgue measure and is locally one-to-one on a set of full Lebesgue measure. Letting $J\hat{T}$ denote the Jacobian of $\hat{T}$, we note that $J\hat{T}(z,\lambda)$ is a homogeneous polynomial in λ of degree (at most) $N(N-1)$, with coefficients that are functions of z (since derivatives of $\hat{T}(z,\lambda)$ with respect to the λ-variables do not depend on λ, while derivatives with respect to the z variables are linear in λ). Note next that $\hat{T}$ fails to be locally one-to-one when $\lambda_i = \lambda_j$ for some $i \neq j$. In particular, it follows by the implicit function theorem that $J\hat{T}$ vanishes at such points. Hence, $\Delta(\lambda) = \prod_{i<j} (\lambda_j - \lambda_i)$ is a factor of $J\hat{T}$. In fact, we have that $\Delta(\lambda)^2$ divides $J\hat{T}$. Since $\Delta(\lambda)$ is a polynomial of degree $N(N-1)/2$, it follows that $J\hat{T}(z,\lambda) = g(z)\Delta(\lambda)^2$ for some (continuous, hence measurable) function g. By Lemma 2.5, we conclude that for any function f that depends only on the eigenvalues of X, Writing for brevity $W = T^{-1}(z)$, we have $\hat{T} = WDW^*$, and $W^*W = I$. Using the notation $d\hat{T}$ for the matrix of differentials of $\hat{T}$, we have $d\hat{T} = (dW)DW^* + W(dD)W^* + WD\left(dW^*\right)$. Using the relation $d\left(W^*W\right) = \left(dW^*\right)W + W^*(dW) = 0$, we deduce that

$$W^*(d\hat{T})W = W^*(dW)D - DW^*(dW) + (dD)$$

Therefore, when $\lambda_i = \lambda_j$ for some $i \neq j$, a complex entry (above the diagonal) of $W^*(d\hat{T})W$ vanishes. This implies that, when $\lambda_i = \lambda_j$, there exists two real linear relations between the on-and-above diagonal entries of $d\hat{T}$, which implies in turn that $\left(\lambda_i - \lambda_j\right)^2$ must divide $J\hat{T}$. From here, we can deduce the joint distribution. Since $\Delta(\lambda)$ is a polynomial of degree $N(N-1)/2$, it follows that $J\hat{T}(z,\lambda) = g(z)\Delta(\lambda)^2$ for some (continuous, hence measurable) function g. By Lemma 2.5, we conclude that for any function f that depends only on the eigenvalues of X, it holds that

$$N! \int f(H) dP_N^2 = \int |g(z)| dz \int f(\lambda) |\Delta(\lambda)|^2 \prod_{i=1}^N e^{-\lambda_i^2/2} d\lambda_i.$$

Moving $N!$ to the right hand side, we obtain the density of the eigenvalues.

Up to the normalization constant $\left(\int |g(z)| dz\right)/N!$, we have proven the theorem.

The normalization constant can be explicitly computed to be $\frac{1}{Z_{N,GUE,2}}$. See section 3.1 in [4].

□

To get rid of the indicator variable in the fraction, we shall consider the unordered random sequence of eigenvalues $(x_{N,1}, \ldots, x_{N,N})$ with joint distribution

$\frac{1}{N!} \times \frac{1_{x_{N,N} \geq \ldots \geq x_{N,1}}}{Z_{N,GUE,2}} e^{-\frac{\lambda^2}{2}} |\Delta_N(\lambda)|^2 \prod dx_{N,i} \in \mathbb{R}^N = \frac{1}{Z_{N,GUE}} e^{-\frac{\lambda^2}{2}} |\Delta_N(\lambda)|^2 \prod dx_{N,i}$ for $Z_{N,GUE} = N! Z_{N,GUE,2}$.

This unordered random sequence yields a random point process $M_N = \sum_{i=1}^{N} \delta_{x_{N,i}}$ which is called the **Gaussian unitary ensemble (GUE) point process**. We shall prove that this random point process is determinantal.

Theorem 2.6. ***(GUE point process is determinantal)***
The eigenvalues of a random matrix H_N of the GUE form a determinantal point process associated to the kernels $K_N(x,y) = \sum_{i=0}^{N-1} \phi_{N,i}(x)\phi_{N,i}(y)$, where the $\phi_{N,i}$ are the orthonormal polynomials for the scaled normal law $\lambda = \lambda_N = N_{\mathbb{R}}(0, \frac{1}{N})$. Explicitly, we have $\phi_{N,i}(x) = \frac{H_i(\sqrt{N}x)}{\sqrt{i!}}$ in which $H_i(x) = (-1)^i e^{\frac{x^2}{2}} \frac{d^i}{dx^i}(e^{\frac{-x^2}{2}})$ which we call the N^{th} ***Hermite polynomial****.*

To show this theorem, we need to establish some general criterions.

Suppose ϕ_j, ψ_i are real valued, and we set $A_{i,j} = <\psi_i|\phi_j>_{L^2(X,\lambda)} = \int_X \psi_i(x)\phi_j(x)\lambda(dx)$. If A as a matrix is invertible, then we can define a kernel $K_N(x,y) = \sum_{1 \leq i,j \leq N} \psi_i(x) A^{-1}_{i,j} \phi_j(y)$. By scaling, we can assume $\det(A) = 1$.

The following general theorem holds.

Theorem 2.7. ***(Determinantal point process associated to a finite rank reproducing kernel)***
Let $(\phi_i, \psi_i)_{1 \leq iN}$ be a family of real valued functions in $L^2(X, \lambda)$ such that $\det(A) = 1$. Then the kernel K_N is a reproducing kernel, that is,
$\int_X K_N(x,x)\lambda(dx) = N, \int_X K_N(x,y)K_N(y,z)\lambda(dy) = K_N(x,z)$.
If the random variables $x_1, \ldots, x_N$ has a joint distribution of $\frac{1}{N!}\det(\phi_i(x_j))_{1 \leq i,j \leq N} \det(\psi_i(x_j))_{1 \leq i,j \leq N}$ and an associated random point process $M_N = \sum_{i=1}^{N} \delta_{x_i}$. Then the random point process M_N is determinantal on (X, λ) with kernel K_N.

Proof. The proof of the above theorem can be found in Theorem 2.3 in [1].

□

Now we return back to our specific question.

When we take $\phi_i = \psi_i$ to be the i^{th} normalized orthogonal polynomials for a probability measure λ, we can redefine the kernel $K_N(x,y) = \sum_{i=0}^{N-1} \phi_i(x)\phi_i(y)$.

A particular example is the GUE point process in which we take $\lambda_N = N_{\mathbb{R}}(0, \frac{1}{N})$.

Proof. **(of Theorem 2.6)**
First take $\lambda = N(0,1)$.

We first show that the $\phi_{N,i}(x)$ are orthonormal.

Since each $H_i(x)$ is a monic polynomial with degree i, we can integrate against $H_i(x)$ with each x^j to see that $\int_{\mathbb{R}} H_i(x)x^j\lambda(dx) = 0$ which concludes that, by linearity of integration, H_i and H_j are orthogonal.

A straightforward integration shows that $\int_{\mathbb{R}}(H_i(x))^2\lambda(dx) = i!$.

Hence, $\frac{H_i(x)}{\sqrt{i!}}$ are orthonormal with respect to $N(0,1)$.

Hence, $\phi_{N,i}(x)$ are orthonormal with respect to $N(0, \frac{1}{N})$.

To apply the above general theorem, we also need to verify that $\frac{1}{N!}\det(\phi_i(x_j))_{1 \leq i,j \leq N} \det(\psi_i(x_j))_{1 \leq i,j \leq N}$ is the joint distribution of the unordered random sequence of eigenvalues derived previously. The verification can be done by plugging in $\lambda_N(dx_i) = \frac{ie^{-2i^2x_i^2}}{\sqrt{2\pi}} dx_i$. Therefore the product of these measures amounts to $\frac{1}{Z_{N,GUE}} e^{\frac{-N}{2}\sum_{i=1}^{N} x_i^2} \prod_{1 \leq i \leq N} dx_i$.

Note also that $\Delta(x) = \det((x_{N,i})^{N-j})_{1\leq i,j\leq N}$ and normalization doesn't affect determinant, we have $\det((\phi_{N,i}(x_j)))^2 = \prod_{1\leq i<j\leq N} |x_i - x_j|^2$. Hence, all conditions of the general theorem are met, so the GUE point process is indeed determinantal. □

3. Spacings and Convergence of the Eigenvalues of GUE Matrix

In this section, we shall mainly follow [3] in giving some explicit results regarding the spacings of eigenvalues. We will prove the first theorem regarding the bulk distribution of the eigenvalues in this section, but we will only prove some partial results regarding the last two theorems in the next section with slightly different settings. For complete proofs of the last two theorems, see Theorem 3.1.2 and Theorem 3.1.4 in Chapter 3 of [3].

We suppose $\lambda_1 \leq \ldots \leq \lambda_n$ are n eigenvalues of a random GUE matrix.

From the Wigner's semicircular law, we can roughly infer that the n eigenvalues of the GUE matrix are spread out on an interval of width roughly equal to $4\sqrt{n}$ and hence the spacing between adjacent eigenvalues is expected to be of order $\frac{1}{\sqrt{n}}$. We will elaborate on the relevant results.

Theorem 3.1. ***(Gaubin-Mehta)***
For any compact set $A \subset \mathbb{R}$,
$\lim_{n\to\infty} P[\sqrt{n}\lambda_1, \ldots, \sqrt{n}\lambda_N \notin A] = 1+\sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_A \ldots \int_A \det_{i,j=1}^k K_s(x_i, x_j) \prod_{j=1}^k dx_j$
where

$$K_s(x, y) = \begin{cases} \frac{1}{\pi} & \text{if } x = y \\ \frac{1}{\pi}\frac{sin(x-y)}{x-y} & \text{if } x \neq y. \end{cases}$$

Theorem 3.2. ***(Jimbo-Miwa-Mori-Sato)***
$\lim_{n\to\infty} P[\sqrt{n}\lambda_1, \ldots, \sqrt{n}\lambda_n \notin (\frac{-t}{2}, \frac{t}{2})] = 1 - F(t)$
in which $1 - F(t) = e^{\int_0^t \frac{\sigma(x)}{x} dx}$ and σ is the solution of the differential equation $(t\sigma'')^2 + 4(t\sigma' - \sigma)(t\sigma' - \sigma + (\sigma')^2) = 0$ so that $\sigma = -\frac{t}{\pi} - \frac{t^2}{\pi^2} - \frac{t^3}{\pi^3} + O(t^4)$ as $t \to 0$. Furthermore, $F(t)$ is a probability distribution function.

The above two theorems give us descriptions about the bulk of the eigenvalues, and the following theorems will give us descriptions about the edge of the eigenvalues.

Theorem 3.3. $\lim P[n^{\frac{2}{3}}(\frac{\lambda_n}{\sqrt{n}} - 2) \leq t] = 1+\sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_t^{\infty} \ldots \int_t^{\infty} \det_{i,j=1}^k A(x_i, x_j) \prod_{j=1}^k dx_j =:$ *$F_2(t)$ in which A is the Airy kernel and $F_2(t) = e^{-\int_t^{\infty}(x-t)q(x)^2 dx}$ where q satisfies $q'' = tq + 2q^3, q(t) \sim Ai(t)$ as $t \to \infty$. We call the function $F_2(t)$ the Tracy-Widom distribution.*

The following lemma is useful for proving the above theorems.

Lemma 3.4. *For square integrable functions $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ on the real line, we have*
$\frac{1}{n!} \int \ldots \int \det_{i,j=1}^n (\sum_{k=1}^n f_k(x_i) g_k(x_j)) \prod dx_i = \det_{i,j=1}^n \int f_i(x) g_j(x) dx$

By orthogonality relation and the above lemma, we have $\int \det_{i,j=1}^n K^N(\lambda_i, \lambda_j) \prod_{i=1}^n d\lambda_i = n!$.

$K_n(x, y) = \sum_{k=0}^{n-1} \psi_k(x)\psi_k(y) = \sqrt{n}\frac{\psi_n(x)\psi_{n-1}(y) - \psi_{n-1}(x)\psi_n(y)}{x-y}$.

Set $S_n(x,y) = \frac{1}{\sqrt{n}} K_n(\frac{x}{\sqrt{n}}, \frac{y}{\sqrt{n}})$.

The proof of this theorem can be found in [1], lemma 2.2 (Cauchy-Binet formula).

The following convergence of kernels holds.

Lemma 3.5. $\lim_{n\to\infty} S_n(x,y) = \frac{1}{\pi}\frac{sin(x-y)}{x-y}$ *uniformly on each bounded subset of the* (x,y) *plane.*

For a complete proof of the above lemma, see page 117, lemma 3.5.1 in [3]. We will prove a slightly different version of this lemma in the next section.

Now we are ready to prove the statement about the bulk distribution.

Recall that the $\phi_{n,i}$ are orthogonal and that the GUE is determinantal.

Proof. **(of Theorem 3.1)**

Since $p_n(\lambda)$ the joint distribution of the eigenvalues is $\det(K_n(x_i,x_j)_{1\le i<j\le n})$, and $\{\phi_i\}$ are orthogonal, we have

$$\begin{aligned}
P(\lambda_i \in A \forall i) &= \det_{i,j=1}^{n-1} \int_A \phi_i(x)\phi_j(x)dx \\
&= \det_{i,j=1}^{n-1} (\delta_{i,j} - \int_{A^c} \psi_i(x)\psi_j(x)dx) \\
&= 1 + \sum_{k=1}^{n} (-1)^k \sum_{0\le v_1 \le \ldots \le v_k \le n-1} \det_{i,j=1}^{k} (\int_{A^c} \psi_{v_i}(x)\psi_{v_j}(x)dx) \textbf{(by expansion of the determinant)} \\
&= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{A^c} \cdots \int_{A^c} \det_{i,j=1}^{k} K^n(x_i,x_j) \prod_{i=1}^{k} dx_i \textbf{(by lemma 3.4)}
\end{aligned}$$

Then applying Lemma 3.5, we obtain,

$$\begin{aligned}
P[\sqrt{n}\lambda_1, \ldots, \sqrt{n}\lambda_n \notin A] &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{\sqrt{n}^{-1}A} \cdots \int_{\sqrt{n}^{-1}A} \det_{i,j=1}^{k} K^n(x_i,x_j) \prod_{i=1}^{k} dx_i \\
&= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{A} \cdots \int_{A} \det_{i,j=1}^{k} S^n(x_i,x_j) \prod_{i=1}^{k} dx_i
\end{aligned}$$

The $\sqrt{n}^{-1}$ disappears in the last equality because $S_n(x_i,x_j) = \frac{1}{\sqrt{n}} K_n(\frac{x}{\sqrt{n}}, \frac{y}{\sqrt{n}})$. □

4. Convergence of Determinantal Point Process and Other Generalities

Motivated by our previous discussion on the point process of the eigenvalues of GUE, we now discuss some generalities of determinantal point process and apply these relevant results back to GUE.

Definition 4.1. (Locally Uniform Convergence)

Let X be a topological space. Let M be a metric space. Let $\langle f_n \rangle$ be a sequence of mappings $f_n : X \to M$. Then f_n converges locally uniformly to $f : X \to M$ if every point of X has a neighborhood on which f_n converges uniformly to f.

Definition 4.2. (Convergence of Determinantal Point Process)

Consider a locally compact, complete, and separable metric space X, and a sequence of random point processes $(M_N), N \in \mathbb{N}$ on X. We say that M_N converges to a random point process M if the law of M_N as a random element of $M(X)$ converges to to the law of M. By definition of the σ-field on $M(X)$, this means that for any family of measurable subsets $B_1, \ldots, B_n \subset X$, we have the convergence in law $(M_N(B_1), \ldots, M_N(B_n)) \to_{N\to\infty} (M(B_1), \ldots, M(B_k))$.

Theorem 4.3. ***(Convergence of General Determinantal Point Process)***

Suppose that M is a determinantal point process on (X, λ) with locally bounded Hermitian kernel $K(x, y)$, and that $(M_N)_{N\in\mathbb{N}}$ is a sequence of determinantal point processes with Hermitian kernels $K_N(x, y)$. If $K_N(x, y) \to K(x, y)$ locally uniformly, then $M_N \to M$ as N goes to infinity.

Proof. Suppose that the random point processes M_N and M are determinantal, with Hermitian kernels K_N and K with respect to a common reference measure λ on X. We assume that $K(x, y)$ is locally bounded, and that $K_N(x, y) \to K(x, y)$ locally uniformly in x and y. Then, the correlation functions also converge locally uniformly, and therefore, the joint moments of the vectors $(M_N(B_1), \ldots, M_N(B_n))$ converge. As these moments determine the random point processes M_N and M, the above theorem is true.

□

Fix a point $x_0 \in (-2, 2)$. By Wigner's theorem (see section 2.1 in [3]), in a small interval $(x_0 - \varepsilon, x_0 + \varepsilon)$, we expect to see $N \times 2\varepsilon \times \frac{\sqrt{4-(x_0)^2}}{2\pi}$ eigenvalues of a random Hermitian matrix H_N of the GUE. Therefore, the distance between two consecutive eigenvalues in this interval is expected to be of order

$$\frac{2\pi}{\sqrt{4-(x_0)^2}N} = O\left(\frac{1}{N}\right).$$

We denote $x_{N,1} \geq x_{N,2} \geq \cdots \geq x_{N,N}$ as the N eigenvalues of H_N. The previous estimate leads one to introduce the following scaling of eigenvalues:

$$y_{N,i} = \frac{N\sqrt{4-(x_0)^2}}{2\pi}\left(x_{N,i} - x_0\right).$$

and we set $M_N^{\text{local },x_0} = \sum_{i=1}^N \delta_{y_{N,i}}$ which is a random point process. This renormalised random point process is expected to have points spaced by a distance of order 1. It contains information about the behavior of the eigenvalues of H_N in the neighborhood of a parameter x_0 in the bulk of the spectrum, that is to say with $-2 < x_0 < 2$. We have

$$M_N^{\text{local },x_0}(B) = M_N\left(x_0 + \frac{2\pi B}{N\sqrt{4-(x_0)^2}}\right)$$

By this equality, we can infer that $M_N^{\text{local, }x_0}$ is also a determinantal point process . If K_N is the Hermite kernel defined previously and

$$\bar{K}_N(a, b) = \sqrt{\frac{N}{2\pi}}\mathrm{e}^{-\frac{N\left(a^2+b^2\right)}{4}} K_N(a, b)$$

is the corresponding kernel with respect to the Lebesgue measure, then $M_N^{\text{local },x_0}$ has for kernel

$$K_N^{\text{local, } x_0}(x,y) = \frac{2\pi}{N\sqrt{4-(x_0)^2}}\bar{K}_N\left(x_0 + \frac{2\pi x}{N\sqrt{4-(x_0)^2}}, x_0 + \frac{2\pi y}{N\sqrt{4-(x_0)^2}}\right)$$

with respect to the Lebesgue measure. We denote a and b the two arguments of $\bar{K}_N$ in the above.

Now we shall determine the limit of this kernel as N goes to infinity. By the Christoffel-Darboux formula (see section 2.4 in [1] or the relevant wikipedia page),

$$\begin{aligned} K_N(a,b) &= \frac{k_{N-1}}{k_N}\frac{\phi_{N,N}(a)\phi_{N,N-1}(b)-\phi_{N,N-1}(a)\phi_{N,N}(b)}{a-b} \\ &= \frac{1}{\sqrt{N!(N-1)!}}\frac{H_N(\sqrt{N}a)H_{N-1}(\sqrt{N}b)-H_{N-1}(\sqrt{N}a)H_N(\sqrt{N}b)}{a-b}; \\ \bar{K}_N(a,b) &= \frac{1}{(N-1)!}\frac{e^{-\frac{N\left(a^2+b^2\right)}{4}}}{\sqrt{2\pi}}\frac{H_N(\sqrt{N}a)H_{N-1}(\sqrt{N}b)-H_{N-1}(\sqrt{N}a)H_N(\sqrt{N}b)}{a-b}. \end{aligned}$$

If $c = 2\cos\phi_c$, then we have seen in the previous deductions that

$$\begin{aligned} \left(\frac{N}{2\pi}\right)^{\frac{1}{4}} e^{-\frac{Nc^2}{4}}\frac{H_N(\sqrt{N}c)}{\sqrt{N!}} &= \frac{1}{\sqrt{\pi\sin\phi_c}}\cos\left(N\left(\frac{\sin 2\phi_c}{2}-\phi_c\right)+\frac{\phi_c}{2}-\frac{\pi}{4}\right)1_{N^{-1}}; \\ \left(\frac{N}{2\pi}\right)^{\frac{1}{4}} e^{-\frac{Nc^2}{4}}\frac{H_{N-1}(\sqrt{N}c)}{\sqrt{(N-1)!}} &= \frac{1}{\sqrt{\pi\sin\phi_c}}\cos\left(N\left(\frac{\sin 2\phi_c}{2}-\phi_c\right)+\frac{3\phi_c}{2}-\frac{\pi}{4}\right)1_{N^{-1}} \end{aligned}$$

where $1_{N-1} = 1 + O\left(N^{-1}\right)$. These estimates are uniform when c stays in a compact interval of $(-2,2)$. They are well behaved if we take c equal to a or b. Further, define $\phi = \arccos\frac{x_0}{2}$, $\phi_a = \arccos\frac{a}{2}$ and $\phi_b = \arccos\frac{b}{2}$. Given an angle ϕ_c, we can write

$$\alpha_c = N\left(\frac{\sin 2\phi_c}{2}-\phi_c\right)+\frac{\phi_c}{2}-\frac{\pi}{4}.$$

and we obtain,

$$K_N^{\text{local, } x_0}(x,y) = \frac{1_{N^{-1}}\cos(\alpha_a)\cos(\alpha_b+\phi_b) - 1_{N^{-1}}\cos(\alpha_a+\phi_a)\cos(\alpha_b)}{N\sin\phi\sqrt{\sin\phi_a\sin\phi_b}(a-b)}.$$

Notice that the angles ϕ, ϕ_a and ϕ_b all differ by a $O\left(\frac{1}{N}\right)$, where the constant in the $O(\cdot)$ only depends on x_0, x and y. Indeed, we have

$$\phi_a = \phi - \frac{\pi x}{2N\sin^2\phi} + O\left(\frac{1}{N^2}\right) \quad ; \quad \phi_b = \phi - \frac{\pi y}{2N\left(\sin^2\phi\right)} + O\left(\frac{1}{N^2}\right)$$

This enables us to greatly simply the calculations.

$$\begin{aligned} K_N^{\text{local },x_0}(x,y) &= \frac{1_{N^{-1}}\cos(\alpha_a)\cos(\alpha_b+\phi_b) - 1_{N^{-1}}\cos(\alpha_a+\phi_a)\cos(\alpha_b)}{N\sin^2\phi(a-b)} \\ &= \frac{\cos(\alpha_a)\cos(\alpha_b+\phi_b)-\cos(\alpha_a+\phi_a)\cos(\alpha_b)}{\pi(\sin\phi)(x-y)} + O\left(\frac{1}{N}\right) \\ &= \frac{\cos(\alpha_a-\alpha_b-\phi_b)-\cos(\phi_a+\alpha_a-\alpha_b)}{2\pi(\sin\phi)(x-y)} + O\left(\frac{1}{N}\right) \\ &= \frac{\sin\left(\alpha_a-\alpha_b+\frac{\phi_a-\phi_b}{2}\right)\sin\left(\frac{\phi_a+\phi_b}{2}\right)}{\pi(\sin\phi)(x-y)} + O\left(\frac{1}{N}\right) \\ &= \frac{\sin\left(N\left(\frac{\sin 2\phi_a}{2}-\frac{\sin 2\phi_b}{2}+\phi_b-\phi_a\right)\right)}{\pi(x-y)} + O\left(\frac{1}{N}\right) \end{aligned}$$

This is due to trignometric identities $\cos s\cos t = \frac{\cos(s+t)+\cos(s-t)}{2}$ and $\cos s - \cos t = -2\sin\left(\frac{s+t}{2}\right)\sin\left(\frac{s-t}{2}\right)$. Finally, we have

$$\begin{aligned} \frac{\sin 2\phi_a}{2} - \frac{\sin 2\phi_b}{2} + \phi_b - \phi_a &= \cos(\phi_a+\phi_b)\sin(\phi_a-\phi_b) + \phi_b - \phi_a \\ &= (\cos(\phi_a+\phi_b)-1)(\phi_a-\phi_b) + O\left(\frac{1}{N^2}\right) \\ &= 2\sin^2\left(\frac{\phi_a+\phi_b}{2}\right)(\phi_b-\phi_a) + O\left(\frac{1}{N^2}\right) \\ &= 2\sin^2\phi\,(\phi_b-\phi_a) + O\left(\frac{1}{N^2}\right) \\ &= \frac{\pi(x-y)}{N} + O\left(\frac{1}{N^2}\right) \end{aligned}$$

so we conclude that locally uniformly in x and y, $K_N^{\text{local },x_0}(x,y) \to \frac{\sin(\pi(x-y))}{\pi(x-y)}$. We have thus established by Theorem 4.3 the following convergence of kernels:

Theorem 4.4. ***(Gaudin-Mehta)***. *For any parameter x_0 in the bulk of the spectrum, as N goes to infinity, the rescaled local random point process $M_N^{local,\ x_0}$ converges towards the determinantal point process M whose kernel is the sine kernel*

$$K^{\text{sine}}(x,y) = \frac{\sin\pi(x-y)}{\pi(x-y)}$$

the reference measure being the Lebesgue measure on $\mathbb{R}$.

The above convergence is for the bulk distribution, and we can do a similar analysis for the edge distribution, that is to say in the neighborhood of $x_0 = 2$ or -2. By symmetry, it suffices to look at the right-side edge. If $x = 2 - t$ with t sufficiently small, then the density of eigenvalues at x is of order $\sqrt{t}$, so we can expect to see

$$O\left(N\int_0^t \sqrt{u}du\right) = O\left(Nt^{3/2}\right)$$

eigenvalues in the interval $(2-t,2)$. Since we want to see a $O(1)$ number of eigenvalues, we should choose $t = O\left(N^{-\frac{2}{3}}\right)$ so that $(\frac{3}{2})(\frac{-2}{3}) = -1$. We can expect the spacing of eigenvalues in the neighborhood of $x_0 = 2$ to be of order $N^{-\frac{2}{3}}$ (instead of N^{-1} in the bulk of the spectrum). We therefore rescale the eigenvalues of a matrix H_N of the GUE as follows, as we have rescaled the bulk distribution: we set $z_i = N^{\frac{2}{3}}(x_{N,i} - 2)$ and $M_N^{\text{edge}} = \sum_{i=1}^N \delta_{z_i}$ In other words,

$$M_N^{\text{edge}}(B) = M_N\left(2 + N^{-\frac{2}{3}}B\right)$$

The rescaled random point process M_N^{edge} is a determinantal point process on $\mathbb{R}$ with the following kernel:

$$\begin{aligned} K_N^{\text{edge}}(t,u) &= \frac{1}{N^{\frac{2}{3}}}\bar{K}_N\left(2 + \frac{u}{N^{\frac{2}{3}}}, 2 + \frac{u}{N^{\frac{2}{3}}}\right) \\ &= \frac{\mathrm{e}^{-\frac{N\left(x^2+y^2\right)}{4}}}{\sqrt{2\pi}(N-1)!}\frac{H_N(\sqrt{N}x)H_{N-1}(\sqrt{N}y) - H_{N-1}(\sqrt{N}x)H_N(\sqrt{N}y)}{t-u} \end{aligned}$$

where $x = 2 + tN^{-\frac{2}{3}}, y = 2 + uN^{-\frac{2}{3}}$ and the reference measure is the Lebesgue measure on $\mathbb{R}$.

Define

$$\psi_N(x) = \left(\frac{1}{2\pi}\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{x^2}{4}}\frac{H_N(x)}{\sqrt{N!}}$$

These normalised oscillator wave-functions form an orthonormal basis of $\mathscr{L}^2(\mathbb{R}, dx)$, and they satisfy the following differential equation:

$$\psi_N'(x) = -\frac{x}{2}\psi_N(x) + \left(\frac{1}{2\pi}\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{x^2}{4}}\frac{NH_{N-1}(x)}{\sqrt{N!}} = -\frac{x}{2}\psi_N(x) + \sqrt{N}\psi_{N-1}(x)$$

Therefore,

$$\begin{aligned} &\frac{\mathrm{e}^{-\frac{N\left(x^2+y^2\right)}{4}}}{\sqrt{2\pi}(N-1)!}\frac{H_N(\sqrt{N}x)H_{N-1}(\sqrt{N}y) - H_{N-1}(\sqrt{N}x)H_N(\sqrt{N}y)}{x-y} \\ &= \frac{\psi_N(\sqrt{N}x)\left(\psi_N'(\sqrt{N}y) + \frac{\sqrt{N}y}{2}\psi_N(\sqrt{N}y)\right) - \psi_N(\sqrt{N}y)\left(\psi_N'(\sqrt{N}x) + \frac{\sqrt{N}x}{2}\psi_N(\sqrt{N}y)\right)}{x-y} \\ &= \frac{\psi_N(\sqrt{N}x)\psi_N'(\sqrt{N}y) - \psi_N(\sqrt{N}y)\psi_N'(\sqrt{N}x)}{x-y} - \frac{\sqrt{N}}{2}\psi_N(\sqrt{N}x)\psi_N(\sqrt{N}y) \end{aligned}$$

Now we perform an asymptotic study on $H_N(\sqrt{N}x)$ and $\psi_N(\sqrt{N}x)$ when x is very close to 2. We fix $x = 2 + tN^{-\frac{2}{3}}$, where t is real. We have

$$\begin{aligned} N^{\frac{1}{4}}\psi_N(\sqrt{N}x) = \left(\frac{N}{2\pi}\right)^{\frac{1}{4}} \mathrm{e}^{-\frac{Nx^2}{4}}\frac{H_N(\sqrt{N}x)}{\sqrt{N!}} &= \left(\frac{N}{2\pi}\right)^{\frac{1}{4}}\frac{\sqrt{N!}}{2\mathrm{i}\pi N^{\frac{N}{2}}}\oint \mathrm{e}^{N\left(zx - \frac{x^2}{4} - \frac{z^2}{2} - \log z\right)}\frac{dz}{z} \\ &= \frac{\sqrt{N}1_{N^{-1/3}}}{2\mathrm{i}\pi}\oint \mathrm{e}^{N^{\frac{1}{3}}t(z-1) + N\left(-\frac{3}{2} + 2z - \frac{z^2}{2} - \log z\right)}\frac{dz}{z} \end{aligned}$$

where $1_{N^{-1/3}} = 1 + O\left(N^{-\frac{1}{3}}\right)$. The function $f(z) = 2z - \frac{z^2}{2} - \log z$ has a unique critical point at $z = 1$. This can be verified by taking derivative:

$$f'(z) = 2 - z - \frac{1}{z} = 0 \implies z = 1.$$

We have $f''(1)=0$, so in a neighborhood of the critical point, if $z=1+N^{-\frac{1}{3}}y$, then

$$p(z,N)=N^{\frac{1}{3}}t(z-1)+N\left(-\frac{3}{2}+2z-\frac{z^2}{2}-\log z\right)=-ty-\frac{y^3}{3}+o\left(y^3\right).$$

In this expansion, in order to make the term $-\frac{y^3}{3}$ decrease rapidly, we need to take $\arg(y)\in\left\{0,\frac{2\pi}{3},-\frac{2\pi}{3}\right\}$. Recall that for $x<2$, the contour chosen for the saddle point analysis was the unit circle.

For $x\simeq 2$, we shall deform this contour.

Around 1 , we take the union of the two segments

$$z=1+N^{-\frac{1}{3}}\mathrm{e}^{\pm\frac{2\mathrm{i}\pi}{3}}u,\quad 0\leq u\leq N^{\varepsilon}$$

with $\frac{1}{9}<\varepsilon<\frac{1}{6}$.

We join the endpoints of these two segments by the circle with center 0 and radius

$$r_N=\left|1+N^{\varepsilon-\frac{1}{3}}\mathrm{e}^{\frac{2i\pi}{3}}\right|=\sqrt{1-N^{\varepsilon-\frac{1}{3}}+N^{2\varepsilon-\frac{2}{3}}}=1-\frac{1}{2}N^{\varepsilon-\frac{1}{3}}+O\left(N^{2\varepsilon-\frac{2}{3}}\right).$$

We denote γ_1 and γ_2 the two parts of this new contour. On the second part γ_2, writing $z=r_N\mathrm{e}^{\mathrm{i}\psi}$, we have

$$\operatorname{Re}\left(-\frac{3}{2}+2z-\frac{z^2}{2}-\log z\right)=-\left(1-r_N\cos\psi\right)^2+\frac{\left(r_N\right)^2-1}{2}-\log r_N$$

so this quantity decreases with $\psi\in(0,\pi)$ and is always smaller than its value at $z=1+\mathrm{e}^{\frac{2i\pi}{3}}N^{\varepsilon-\frac{1}{3}}$, which is

$$\frac{-2N^{\varepsilon-\frac{1}{3}}+N^{2\varepsilon-\frac{2}{3}}-2\log\left(1-N^{\varepsilon-\frac{1}{3}}+N^{2\varepsilon-\frac{2}{3}}\right)}{4}=-\frac{N^{3\varepsilon-1}}{3}+o\left(N^{3\varepsilon-1}\right).$$

Therefore,

$$\log\left(\frac{1}{2\pi}\oint_{\gamma_2}\left|\mathrm{e}^{p(z,N)}\right|\frac{dz}{z}\right)\leq 2|t|N^{1/3}-\frac{N^{3\varepsilon}}{3}+o\left(N^{3\varepsilon}\right)=-\frac{N^{3\varepsilon}}{3}+o\left(N^{3\varepsilon}\right)$$

since $\varepsilon>\frac{1}{9}$. This implies that the contribution to the contour integral of γ_2 decreases as $\exp\left(-CN^{3\varepsilon}\right)$, so it will be negligible. On the other hand, a change of variables formula gives us

$$\oint_{\gamma_1}\mathrm{e}^{p(z,N)}\frac{dz}{z}=1_{N^{\varepsilon-1/3}}N^{-\frac{1}{3}}\int_{\zeta}\mathrm{e}^{ty-\frac{y^3}{3}}dy$$

where the path of integration on the right-hand side is the union of the two half-lines $\mathbb{R}_+\mathrm{e}^{\frac{2\mathrm{i}\pi}{3}}$ and $\mathbb{R}_+\mathrm{e}^{-\frac{2\mathrm{i}\pi}{3}}$. So, if $x=2+tN^{-\frac{2}{3}}$, then

$$N^{\frac{1}{4}}\psi_N(\sqrt{N}x)=N^{\frac{1}{6}}\left(\frac{1}{2\mathrm{i}\pi}\int_{\zeta}\mathrm{e}^{ty-\frac{y^3}{3}}dy\right)+O\left(N^{\varepsilon-\frac{1}{6}}\right).$$

The remainder is by construction a $o(1)$, and on the other hand, the path integral is the so-called **Airy function** $\mathrm{Ai}(t)$. (See also chapter 3 in [3].) This function satisfies the differential equation $\mathrm{Ai}''(t)-t\,\mathrm{Ai}(t)=0$ (Such differential equation is mentioned in section 3, Theorem 3.3,see also chapter 3 in [3]). It can also be redefined as the real semi-convergent integral $\mathrm{Ai}(t)=\frac{1}{\pi}\int_0^{\infty}\cos\left(ty+\frac{y^3}{3}\right)dy$, and its Fourier transform is

$$\widehat{\mathrm{Ai}}(\xi)=\mathrm{e}^{\frac{(\mathrm{i}\xi)^3}{3}}$$

We have proved above that

$$N^{\frac{1}{12}}\psi_N\left(2N^{\frac{1}{2}}+tN^{-\frac{1}{6}}\right)\to_{N\to\infty}\mathrm{Ai}(t)$$

This estimate can be made locally uniform in t, and it can be considered as a result of convergence of holomorphic functions of the variable t by complex analysis. Denote the left-hand side of the asymptotic formula above by $\theta_N(t)$. We have on the one hand

$$\begin{aligned} K_N^{\mathrm{edge}}(t,u) &= \frac{\psi_N(\sqrt{N}x)\psi_N'(\sqrt{N}y)-\psi_N(\sqrt{N}y)\psi_N'(\sqrt{N}x)}{t-u}-\frac{1}{2N^{\frac{1}{6}}}\psi_N(\sqrt{N}x)\psi_N(\sqrt{N}y) \\ &= \frac{\theta_N(t)\theta_N'(u)-\theta_N(u)\theta_N'(t)}{t-u}-\frac{1}{2N^{\frac{1}{3}}}\theta_N(t)\theta_N(u) \end{aligned}$$

and on the other hand, θ_N and all its derivatives converge locally uniformly on the complex plane towards the Airy function and its derivatives. We have therefore established the following result (see also Theorem 3.3 in section 3):

Theorem 4.5. ***(Airy kernel)***. *As N goes to infinity, the rescaled local random point process M_N^{edge} converges towards the determinantal point process whose kernel is the Airy kernel*

$$K^{Airy}(t,u)=\frac{\mathrm{Ai}(t)\,\mathrm{Ai}'(u)-\mathrm{Ai}'(t)\,\mathrm{Ai}(u)}{t-u},$$

These convergence results prove partially Theorem 3.2 and Theorem 3.3 in a slightly different context. We can see from our proof the power of the theory of point process in studying the distribution of eigenvalues of random matrices.

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