A DETAILED EXPOSITION FOR THE PSEUDO-ARC

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Abstract. The pseudo-arc is an important topological space in continuum theory. The purpose of this paper is to provide an in-depth and straightforward exposition of the pseudo-arc with many pictures. We cover two constructions of the pseudo-arc, some of its properties, and its place in the complete classification of compact homogeneous subsets of the plane.

Contents

1. Introduction

The pseudo-arc is one of the crown jewels of continuum theory, a branch of general topology begun in the late 1800s. In order to understand the place of the pseudo-arc, we provide a brief introduction to continuum theory, starting with the definition of a *continuum*.

Definition 1.1. A *continuum* K is a compact, connected metric space. If $L \subset K$ is also a continuum, we say L is a *subcontinuum* of K; if $L \neq K$, then it is *proper*. If K has more than one point, we say it is *nondegenerate*.

The definition of continuum was first formed by Cantor in 1883; however, since connectedness and compactness had not been formalized yet, Cantor's definition of a continuum looked a bit different. Cantor defined a continuum K to be a perfect subset of $\mathbb{R}^n$ such that, for all $a, b \in K$, $\epsilon > 0$, there exists a finite sequence of points $a = p_0, p_1, \ldots, p_n = b$ and $|p_i - p_{i+1}| < \epsilon$ for each i. The finite sequence of points rules out sets like the Cantor set from being what Cantor considered a continuum [13]. As topology evolved, continuum theory evolved with it. Once connectedness and compactness had become well-established in the 1920s, continuum gained its modern definition.

The following are examples of continua.

Date: September 3rd, 2021.

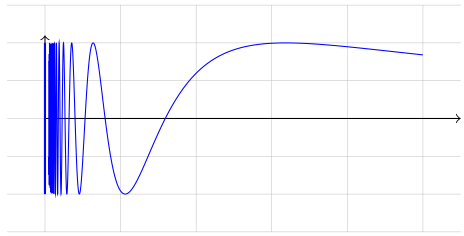

FIGURE 1.1. The topologist's sine curve, a connected but not path-connected subset of $\mathbb{R}^2$, is a continuum.

Examples 1.2. (a) The unit interval $[0,1]$ is the simplest example of a nondegenerate continuum. For the rest of the paper, we define $I \coloneqq [0,1]$.
(b) An *arc*, defined as a topological space homeomorphic to I.
(c) The circle, S^1.
(d) An *n-cell*, defined as the set $\prod_{i=1}^{n}[0,1] \subset \mathbb{R}^n$.
(e) The *Hilbert cube*, defined as the set $\prod_{i=1}^{\infty}[0,1]$ with metric defined such that

$$d(x,y) = \sum_{i=1}^{\infty} \frac{\min\{|x_i - y_i|, 1\}}{2^i}.$$

Note that this is equivalent to the subspace topology induced by the product topology on $\mathbb{R}^{\mathbb{N}}$ (See Definition 2.2.1). [2, Thm. 7.1]
(f) The *topologist's sine curve*, defined as $\{0\} \times [-1,1] \cup \{(x, \sin(1/x)) \mid x \in (0,1]\}$. See Figure 1.1.

Continua can be thought of as a generalization of the notion of a *curve*. Curves can be found in all areas of mathematics, so it makes sense to study their topological properties. We define a few of these properties which will show up in Section 3. We also give some questions that naturally arise about these properties.

Definition 1.3. A continuum K is *planar* if it can be embedded in the plane.[1]

In this paper, we will be mainly dealing with planar continua. The focus of continuum theory is not on planar continua necessarily, but planar continua tend to be the easier to work with and visualize than other continua, as well as being complicated enough on their own. See Nadler's textbook [24] for a general introduction to continuum theory.

Definition 1.4. A topological space K is *homogeneous* if, for all $x, y \in K$, there exists a homeomorphism $f\colon K \to K$ such that $f(x) = y$.

The circle and the Hilbert cube are the only continua in Examples 1.2 that are homogeneous. In fact, every n-sphere S^n is homogeneous: for any points x, y, consider a rotation of the n-sphere such that x is sent to y. The proof that the Hilbert cube is homogeneous is more subtle; it seems impossible that points with all coordinates either 0 or 1 can be sent into points in the middle of the cube by a homeomorphism.[2] See [2] for the proof.

[1] i.e. there exists a function $f\colon K \to \mathbb{R}^2$ such that $f\colon K \to f(K)$ is a homeomorphism.

[2] For an intuitive explanation for why this is not a problem, consider the Hilbert cube as a subspace of the product space $\prod_{n=1}^{\infty}\mathbb{R}$. As we explain in Definition 2.2.1, the Hilbert cube has empty interior, and therefore it is the boundary of itself.

If we restrict our continua to the plane, the following question arises.

Question 1.5. Is every nondegenerate planar continuum that is homogenous necessarily homeomorphic to the circle?

An answer in the affirmative would provide a complete classification of homogeneous planar continua. We show in Section 3 that the pseudo-arc, a continuum not homeomorphic to the circle, is actually homogeneous. However, in Section 4 we cover a result which utilizes the pseudo-arc to classify homogeneous planar continua completely.

An interesting property of the arc is that it is homeomorphic to each of its subcontinua: any subcontinuum of I is also a closed interval. The following question then arises.

Question 1.6. Is every continuum which is homeomorphic to each of its subcontinua an arc?

Although this condition appears strong, it turns out that the pseudo-arc also satisfies this property. This property gives some explanation for the name "pseudo-arc."

In working with continua which separate the plane, Schoenflies made the assumption that continua which separate the plane are the union of two proper subcontinua. This assumption seems intuitive, but it is worth examining closely. The following definitions were introduced to ask questions regarding a continuum's ability to be *decomposed*.

Definition 1.7. A continuum K is *decomposable* if there exist subcontinua $L, L' \subset K$ such that $K = L \cup L'$. If K is not decomposable, we say it is *indecomposable*. In the case that every subcontinuum of K is also indecomposable, we say K is *hereditarily indecomposable*.

A very simple example of a hereditarily indecomposable continuum is a degenerate continuum, consisting of only a point. Every continuum in Examples 1.2 is decomposable. Intuitively, it may even seem like all nondegenerate continua are decomposable, leading to the following question.

Question 1.8. Does there exist a nondegenerate indecomposable continuum? If so, does there exist a hereditarily indecomposable continuum?

It turns out there do exist hereditarily indecomposable continua, and one example is none other than the pseudo-arc.

In Section 2, we give two constructions of the pseudo-arc: one by the intersection of decreasing chained continua, and the other by the inverse limit of functions $f_n \colon I \to I$.

In Section 3, we give detailed proofs that the pseudo-arc is hereditarily indecomposable, homogeneous, and homeomorphic to each of its nondegenerate subcontinua, answering Questions 1.5, 1.6, and 1.8.

In Section 4, we expand on Question 1.5 by giving a few examples of continua similar to the pseudo-arc, then state a result which classifies planar, homogeneous continua completely. This result allows us to deduce a complete classification of homogeneous, compact subsets of the plane up to homeomorphism.

We conclude in Section 5 by giving a few unrelated results involving the pseudo-arc, as well as an open problem concerning planar continua.

2. Two Canonical Constructions of the Pseudo-Arc

We will discuss two of the most common constructions of the pseudo-arc. The first is done by taking the intersection of decreasing chained continua, and the second is done by taking an inverse limit over an inverse system of functions $f_n\colon I \to I$. It will follow from a result in the following section that these constructions are homeomorphic.

2.1. Intersection of Decreasing Chained Continua.

The construction of the pseudo-arc by intersecting nested chained continua was first done by Knaster [16] in 1922, then later by Bing [5] [4] and Moise [21]. This construction uses a method of obtaining new continua similar to the following theorem.

Theorem 2.1.1. *The intersection of nonempty nested continua is also a nonempty continuum.*

Proof. The intersection of compact sets is compact, and the intersection of nested connected sets is connected. By Cantor's intersection theorem, the resulting continuum must also be nonempty. □

We will begin by laying out some definitions vital to this construction of the pseudo-arc, the first of which being the notion of a *chain*.

Definition 2.1.2. [5] A *chain* is a finite sequence of open subsets of $\mathbb{R}^2$ $\mathscr{C} = [C_1, \dots, C_n]$ such that

$$C_i \cap C_j \neq \emptyset \iff j \in \{i-1, i, i+1\},$$

i.e. two links intersect if, and only if, they are adjacent. The open sets $C_1, \dots, C_n$ are called *links* of $\mathscr{C}$. We call C_1 and C_n *end links* of $\mathscr{C}$ and $C_2, \dots, C_{n-1}$ *interior links* of $\mathscr{C}$. We denote

$$\mathscr{C}^* = \bigcup_{i=1}^{n} C_i.$$

We call $\mathscr{C}(i,j) = [C_i, \dots, C_j]$ a *subchain* of $\mathscr{C}$. Note that we do not require $i < j$, since $[C_i, C_{i-1}, \dots, C_{j+1}, C_j]$ can also be regarded as a chain within the bigger chain $\mathscr{C}$. We can casually refer to the sum of subchains of $\mathscr{C}$ in the sense that, if $1 \le i < j < k \le n$, then $\mathscr{C}(i,j) + \mathscr{C}(j,k) = \mathscr{C}(i,k)$. We say a chain $\mathscr{C}$ *goes from p to q* if the first link of $\mathscr{C}$ contains p and the last contains q.

Remark 2.1. It is not necessary to require links of chains be subsets of $\mathbb{R}^2$. Chains can be defined in any metric space, and the resulting construction of the pseudo-arc will be homeomorphic, as proven in Section 3. However, because the constructions are homeomorphic, it is easiest to assume links are all open subsets of $\mathbb{R}^2$.

Intuitively, a chain $\mathscr{C}$ is nothing but a string of overlapping open sets, and $\mathscr{C}^*$ looks like a curve with some thickness determined by the diameter of the links (See Figure 2.1.1). It is worth remarking that the links of a chain need not be connected, though it is not immediately obvious why this is useful. The pseudo-arc can be constructed using only chains with connected links, but generalizing to include chains with disconnected links as well turns out to be not only equivalent but useful in proving the properties of the continuum.

The importance of thinking of chains as sequences of open sets rather than as the union of the sets lies in the following definitions.

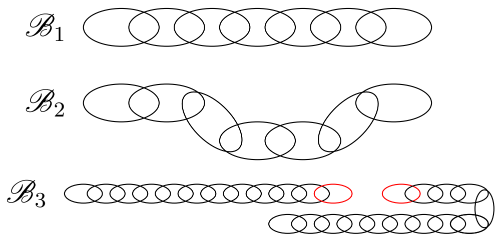

FIGURE 2.1.1. Above are three representations of chains as strings of ovals. Each oval represents an open subset of $\mathbb{R}^2$ which contains all points enclosed by the oval but not points on the outline. The red ovals of $\mathscr{B}_3$ represent one link which is not connected.

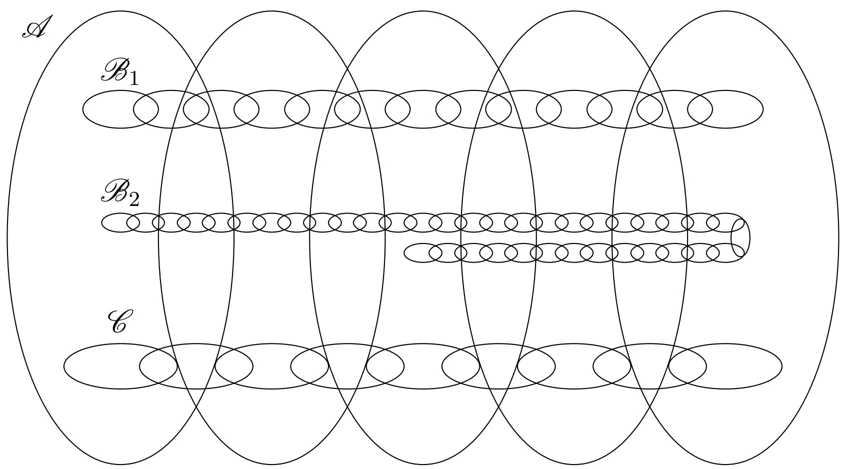

FIGURE 2.1.2. Chains $\mathscr{B}_1$ and $\mathscr{B}_2$ are refinements of $\mathscr{A}$ because for every link of each, one of the links of $\mathscr{A}$ contains it completely. This is not the case for $\mathscr{C}$ because links C_2, C_4, C_6, and C_8 each intersect two chains nontrivially.

Definition 2.1.3. [21, Def. 2][5] Suppose $\mathscr{A} = [A_1, \ldots, A_n]$ and $\mathscr{B} = [B_1, \ldots, B_m]$ are chains such that every link of $\mathscr{B}$ is a subset of some link of $\mathscr{A}$. In this case, we say $\mathscr{B}$ is a *refinement* of $\mathscr{A}$. In the case that every link of $\mathscr{A}$ is exactly the union of links of $\mathscr{B}$, we say $\mathscr{A}$ is a *consolidation* of $\mathscr{B}$.

Saying $\mathscr{B}$ is a refinement of $\mathscr{A}$ is simply saying $\mathscr{B}$ is contained in $\mathscr{A}$ in some useful way. Figure 2.1.2 depicts a chain $\mathscr{A}$ with refinements $\mathscr{B}_1$ and $\mathscr{B}_2$, as well as an example of a non-refinement $\mathscr{C}$.

Definition 2.1.4. [5] Suppose $\mathscr{A} = [A_1, \ldots, A_n]$ is a chain and $\mathscr{B} = [B_1, \ldots, B_m]$ is a refinement of $\mathscr{A}$. We say $\mathscr{B}$ is *crooked* in $\mathscr{A}$ if every subchain $\mathscr{B}(i, j)$ satisfies the following condition:

Suppose h, k are such that B_i intersects A_h and B_j intersects A_k. If $|k-h| > 2$, then there exist r, s such that B_r is a subset of a link adjacent to A_k, B_s is a subset of a link adjacent to A_h, and $\mathscr{B}(i,j) = \mathscr{B}(i,r) + \mathscr{B}(r,s) + \mathscr{B}(s,j)$. [3]

See Figure 2.1.3 for examples of chains $\mathscr{B}^n$ crooked in $\mathscr{A}^n$ when $\mathscr{A}$ has n links, $n = 3, 4, 5$.

The crookedness of $\mathscr{B}$ in $\mathscr{A}$ can be thought of as capturing how complicated $\mathscr{B}$ is in relation to $\mathscr{A}$. Put simply, the condition that needs to be satisfied can be summed up as "between every two links there is a tangle" (as long as the two links are far enough away from each other with respect to $\mathscr{A}$). For each subchain $\mathscr{B}(i,j)$, following the sequence $B_i, B_{i+1}, \dots, B_{j-1}, B_j$ link by link leads us first within one link of the far end link of $\mathscr{A}(h,k)$, then leads us within one link of the starting end link of $\mathscr{A}(h,k)$ before finally reaching the end link intersecting B_j.

Before constructing the pseudo-arc, we will outline the proof for a lemma which is somewhat intuitive.

Lemma 2.1.5. [5, Thm. 7] *Given a chain $\mathscr{A}^n = [A_1, \dots, A_n]$, there exists a chain $\mathscr{B}^n$ such that $\mathscr{B}^n$ is crooked in $\mathscr{A}$ and the closure of each link of $\mathscr{B}^n$ is a compact subset of a link of $\mathscr{A}^n$.*

Outline of Proof. We can construct $\mathscr{B}^n$ inductively given $\mathscr{B}^{n-2}$ and $\mathscr{B}^{n-1}$. For $n = 1, 2, 3$, every possible refinement (which intersects both end links of $\mathscr{A}^n$) is crooked, since any A_h^n, A_k^n must satisfy $|h-k| \leq 2$. See 2.1.3 for visual representations of possible chains $\mathscr{B}^n$ for $n = 3, 4, 5$.

We can represent chains as essentially lying in straight lines as depicted in Figure 2.1.4. Chains $\mathscr{B}^4$ and $\mathscr{B}^5$ demonstrate the following inductive construction of $\mathscr{B}^n$. Given $\mathscr{A}$ with n links, first construct $\mathscr{C}_1 = \mathscr{B}^{n-1}$ for $\mathscr{A}(1, n-1)$, then construct $\mathscr{C}_3 = \mathscr{B}^{n-1}$ for $\mathscr{A}(2, n)$. Finally, connect these two chains with $\mathscr{C}_2 = \mathscr{B}^{n-2}$ for $\mathscr{A}(n-1, 2)$. We then let $\mathscr{B}^n = \mathscr{C}_1 + \mathscr{C}_2 + \mathscr{C}_3$.

In Figure 2.1.3, $\mathscr{B}^5$ demonstrates this idea very well. In red are chains resembling $\mathscr{B}^4$ crooked in $\mathscr{C}(1,4)$ and $\mathscr{C}(2,5)$. In blue is a chain resembling $\mathscr{B}^3$ crooked in $\mathscr{C}(2,4)$. To see that this construction produces a chain crooked in $\mathscr{A}^n$, refer to Figure 2.1.4. Take two arbitrary links, then finish the proof by casework depending on which subchain each belongs to.

This construction does not handle the case that some links of $\mathscr{A}^n$ are not connected. However, a similar construction can be made using disconnected links in $\mathscr{B}^n$. $\square$

Remark 2.1.6. The notation for a chain $\mathscr{B}^n$ will generally not be related to number of links in a chain. For the rest of the paper, the index of a chain will only refer to its place in a sequence of chains $\{\mathscr{B}^n\}_{n\in\mathbb{N}}$.

Definition 2.1.7. (Construction of Pseudo-Arc) [5] Suppose $\{\mathscr{D}^n\}_{n\in\mathbb{N}}$ is a sequence of nested chains such that

(a) $\mathscr{D}^{n+1}$ is crooked in $\mathscr{D}^n$,
(b) each link of $\mathscr{D}^{n+1}$ is a compact subset of some link of $\mathscr{D}^n$, and

[3] The definitions for chain, refinement, and crooked are cherry-picked from Bing [5] [4] and Moise [21], authors of the earliest constructions (in English) of the pseudo-arc using chained continua. Moise uses disjoint open links chained by intersections of boundary points, while Bing uses the definition of chain given in this paper. Moise also gives a slightly different, less intuitive but equivalent definition of crookedness.

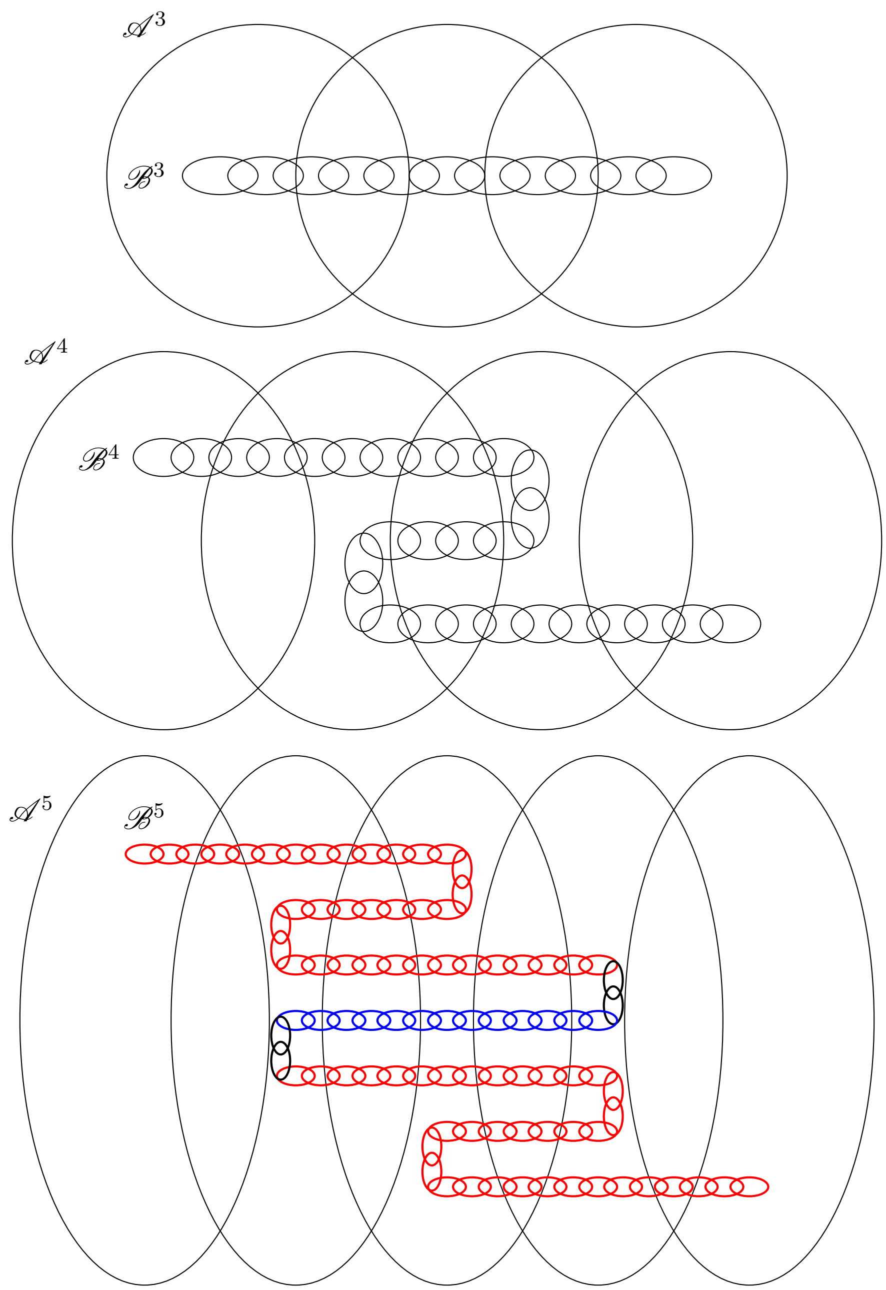

FIGURE 2.1.3. Above are examples of chains $\mathscr{B}^n$ crooked in $\mathscr{A}^n$ for $n = 3, 4, 5$. To see this, choose any two links on $\mathscr{B}^n$, check how far away the links of $\mathscr{A}^n$ containing them are from each other, and then find the required subchain decomposition of $\mathscr{B}^n$, if necessary. In red are subchains of $\mathscr{B}^5$ resembling $\mathscr{B}^4$ and in blue is a subchain resembling $\mathscr{B}^3$.

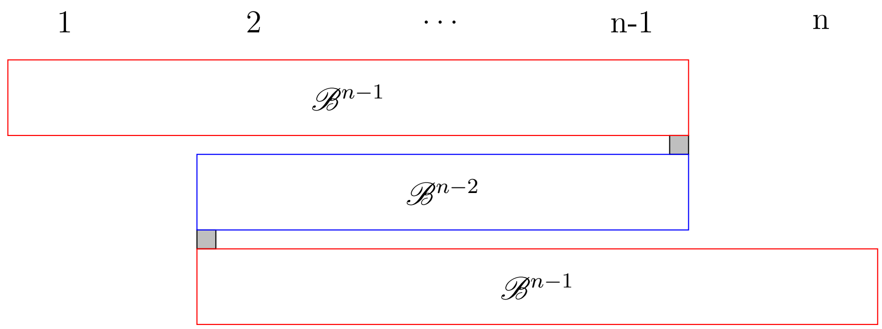

FIGURE 2.1.4. A visual representation of our construction for $\mathscr{B}^n$ in terms of $\mathscr{B}^{n-2}$ and $\mathscr{B}^{n-1}$.

(c) the diameter of each link of $\mathscr{D}^n$ is at most $\frac{1}{n}$.

Then,

$$\mathcal{M} = \bigcap_{n=1}^{\infty} \overline{(\mathscr{D}^n)^*} = \bigcap_{n=1}^{\infty} \overline{\bigcup_{D_i^n \in \mathscr{D}^n} D_i^n}.$$

is a *pseudo-arc*. We refer to $\{\mathscr{D}^n\}_{n\in\mathbb{N}}$ as a *witness*[4] that $\mathcal{M}$ is a pseudo-arc (or more simply, $\{\mathscr{D}^n\}_{n\in\mathbb{N}}$ is a witness for $\mathcal{M}$). Every pseudo-arc must have at least one witness, by definition.

Remark 2.1.8. Because our $\{\mathscr{D}^n\}_{n\in\mathbb{N}}$ is not unique, what we are defining here is not "the pseudo-arc" but a family of continua we call pseudo-arcs. In the next subsection, we will define this same family using inverse limits. We find in Section 3 that both families are the same, and the continua in this family are all homeomorphic. Then, it will make sense to refer to each one as "the pseudo-arc."

Theorem 2.1.9. *There exists a pseudo-arc with witness $\{\mathscr{D}^n\}_{n\in\mathbb{N}}$ such that, for all $n \in \mathbb{N}$, every link of $\mathscr{D}^n$ is connected.*

Proof. Lemma 2.1.5 allows us to construct a collection $\{\mathscr{D}_n\}_{n\in\mathbb{N}}$ inductively satisfying (a) and (b) such that each chain $\mathscr{D}_n$ has connected links. We can make this collection a witness by also requiring (c) when we construct $\mathscr{D}^{n+1}$. □

Theorem 2.1.10. *Every pseudo-arc $\mathcal{M}$ is a nondegenerate continuum.*

Proof. Suppose $\{\mathscr{D}^n\}_{n\in\mathbb{N}}$ is a sequence of chains satisfying the properties given in Definition 2.1.7 such that

$$\mathcal{M} = \bigcap_{n=1}^{\infty} \overline{(\mathscr{D}^n)^*}.$$

If each $\mathscr{D}^n$ is assumed to have connected links, $\mathcal{M}$ is a continuum by Theorem 2.1.1. However, even if each $\mathscr{D}^n$ is not assumed to have connected links (and even if the union of the links for some chain is not connected), $\mathcal{M}$ is still a continuum.

Assume $p, q \in \mathcal{M}$ are in distinct connected components $P, Q \subset \mathcal{M}$, respectively. Because we must have that $\mathcal{M}$ is compact, P and Q are compact, so there is $n \in \mathbb{N}$

[4]This is not a standard definition but was defined by the author so as to better refer to the collection which is used to construct a pseudo-arc. This sequence of chains will often be used when proving properties of a given pseudo-arc.

such that $\operatorname{dist}(P, Q) > \frac{1}{n}$. We reach a contradiction since $\mathscr{D}^n$ would be required to have a link which intersects P and Q.

To show that $\mathcal{M}$ is nondegenerate (contains more than one point), observe that for all $n \in \mathbb{N}$, every link of $\mathscr{D}^n$ has nonempty intersection with $\mathcal{M}$. □

2.2. **Inverse Limit.** The second construction uses an inverse limit to apply the same ideas of crookedness. We will first provide a clear description of what an inverse limit of spaces is in order to better explain its application in the construction. To bypass the introduction to inverse limits and get straight to the application of inverse limits to continuum theory, skip to Theorem 2.2.7. We begin by laying out definitions we will need in order to define an inverse limit.

Definition 2.2.1. [23, p.114] Recall that a *topology* $\mathcal{T}$ on a set X is a collection of subsets of X called *open* sets satisfying the following:

(a) $\emptyset$ and X are open,
(b) the arbitrary union of open sets is open, and
(c) the finite intersection of open sets is open.

Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be an arbitrary collection of topological spaces. The *product topology* is the topology on $\prod_{\lambda\in\Lambda} X_\lambda$ generated by arbitrary unions of the collection

$$\mathcal{B} = \left\{ \prod_{\lambda\in\Lambda} V_\lambda \mid V_\lambda \text{ are each open in } X_\lambda \text{ and } V_\lambda = X_\lambda \text{ for all but finite } \lambda \right\} \cup \{\emptyset\}.$$

This topology seems less natural than the more naive *box topology* generated by the arbitrary product of any sets open in each X_λ. The box topology is surprisingly much more pathological than we would like: component-wise continuity is not equivalent to continuity. As it turns out, the product topology is the "smallest" topology we can place on the product space such that this implication holds [23, p.114].[5]

A major difference between open sets in the product and box topologies is that open sets in the product topology are always huge. If we consider $\prod_{\lambda\in\Lambda} \mathbb{R}$ for some infinite Λ, the interior of $\prod_{\lambda\in\Lambda}[0,1]$ is naturally $\prod_{\lambda\in\Lambda}(0,1)$ in the box topology, but in the product topology the interior is empty; not a single open set fits in the product of closed intervals!

Definition 2.2.2. [14, p.76] A *directed set* is a set D with a reflexive and transitive binary relation $\leq$ such that, for all $x, y \in D$, there exists $z \in D$ such that $x \leq z$ and $y \leq z$. In other words, not every two elements of a directed set are comparable through $\leq$, but every two elements are comparable to a third which is an upper bound to both.

Example 2.2.3. One of the simplest directed sets is $(\mathbb{N}, \leq)$.

Example 2.2.4. For a nonempty set X, $(\mathcal{P}(X), \subseteq)$ (where $\mathcal{P}(X)$ is the power set of X) is another good example which demonstrates the "directed" nature of such a structure.

Definition 2.2.5. Let $(D, \leq)$ be a directed set, $\{X_i\}_{i\in D}$ a collection of topological spaces, and $\{f_{ij} \colon X_j \to X_i\}_{i\leq j\in D}$ a collection of continuous maps such that

(a) $f_{ii} = \mathrm{id}_{X_i}$ for all $i \in D$, and

[5] From a category theoretical point of view, we also find that the product topology is the most natural topology we can place on the cartesian product. See Leinster ([17], Example 5.1.4).

(b) $f_{ij} \circ f_{jk} = f_{ik}$ for all $i \le j \le k \in D$.

We call (X_i, f_{ij}, J) an *inverse system.*

In the special case that $(D, \le) = (\mathbb{N}, \le)$, observe that the condition of composition implies

$$f_{ij} = f_{i,i+1} \circ f_{i+1,i+2} \circ \cdots \circ f_{j-2,j-1} \circ f_{j-1,j}.$$

Thus, we can instead just define the adjacent functions $\{f_i \coloneqq f_{i,i+1}\}_{i \in \mathbb{N}}$ and generate the others by composition.[6] In this case we can represent the inverse system by

$$\cdots \xrightarrow{f_5} X_5 \xrightarrow{f_4} X_4 \xrightarrow{f_3} X_3 \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1,$$

which captures what is meant by "inverse system."

Definition 2.2.6. Suppose (X_i, f_{ij}, D) is an inverse system. We define the *inverse limit* of the system

$$X_\infty = \varprojlim(X_i, f_{ij}, D) = \left\{ (a_i)_{i \in D} \in \prod_{i \in D} X_i \;\middle|\; a_i = f_{ij}(a_j) \text{ for all } i \le j \in D \right\}.$$

We then give X_∞ the subspace topology induced by the product topology on $\prod_{i \in D} X_i$.

We now return solely to the implementation of inverse limits in continuum theory. The following theorem demonstrates a useful method of obtaining new continua.

Theorem 2.2.7. *If (X_i, f_{ij}, D) is an inverse system such that each X_i is a continuum, each f_{ij} is continuous, and D is a countable directed set, then $\varprojlim(X_i, f_{ij}, D)$ is a nonempty continuum.*

We will first prove the theorem for the case that $(D, \le) = (\mathbb{N}, \le)$ in the following lemma.

Lemma 2.2.8. [14, Thm. 9] *If $(X_n, f_n, \mathbb{N})$ is an inverse system such that each X_n is a continuum and each f_n is continuous, then $\varprojlim_{n \in \mathbb{N}} X_n$ is a nonempty continuum.*

Proof. For all $n \in \mathbb{N}$, let

$$G_n = \left\{ (a_n)_{n \in \mathbb{N}} \in \prod_{i=1}^{\infty} X_i \;\middle|\; a_i = f_i(a_{i+1}) \text{ for all } i \in \mathbb{N} \text{ such that } i \le n \right\}.$$

We see $\varprojlim_{n \in \mathbb{N}} X_n = \bigcap_{n=1}^{\infty} G_n$. Thus, by Theorem 2.1.1 it suffices to show that $\{G_n\}_{n \in \mathbb{N}}$ are nonempty, nested continua.

Let $(a_i)_{i \in \mathbb{N}} \in G_{n+1}$. Then, $a_i = f_i(a_{i+1})$ for all $i \le n+1$, and thus $i \le n$. Therefore, $\{G_n\}_{n \in \mathbb{N}}$ are nested.

Consider the homeomorphism $\phi \colon G_n \to \prod_{i=n+1}^{\infty} X_i$ such that

$$\phi((a_i)_{i=1}^{\infty}) = (a_i)_{i=n+1}^{\infty}.$$

To see that ϕ is a homeomorphism, note that components $a_1, \ldots, a_n$ of each $(a_i)_{i \in \mathbb{N}} \in G_n$ are determined completely by a_{n+1}. By Tychonoff's theorem[7], $\prod_{i=n+1}^{\infty} X_i$ is

[6]The reason we cannot do this for an arbitrary directed set is that we lack the concept of "adjacency."

[7]This relies on the axiom of choice. For an alternate proof which does not rely on the axiom of choice, see Ingram and Mahavier [14, Thm. 6]

compact, and therefore so is G_n. Furthermore, the countable product of connected spaces is connected, so G_n is connected. Thus, by Theorem 2.1.1,

$$\varprojlim_{n\in\mathbb{N}} X_n = \bigcap_{n=1}^{\infty} G_n$$

is a nonempty continuum. □

The proof for Theorem 2.2.7 is similar to that for the Lemma we have just proved. However, because we only have that $D = \{j_1, j_2, \dots\}$ is a countable directed set, we have to construct a countable sequence $\{k_n\}_{n\in\mathbb{N}}$ such that $j_1, \dots, j_n \leq k_n$ for each $n \in \mathbb{N}$. Then, we can use sets G_{k_n} and the proof is essentially the same.

We will first prove a quick theorem which demonstrates how Theorem 2.2.7 can be used to obtain new continua.

Theorem 2.2.9. [14, Thm. 9] *Recall that we let $I = [0,1]$. Consider the inverse system $(I, \phi_n, \mathbb{N})$, where each $\phi_n\colon I \to I$ is a homeomorphism. Then, $I_\infty = \varprojlim(I, \phi_n, \mathbb{N})$ is an arc, i.e. homeomorphic to I.*

Proof. Consider the projection π_1, where $\pi_1((x_n)_{n=1}^{\infty}) = x_1$. Because each ϕ_n is bijective, $(x_n)_{n=1}^{\infty}$ is determined completely by x_1, so $\pi_1|_{I_\infty}$ is bijective. A result from point-set topology states that a bijective, continuous function from a compact space to a Hausdorff space is a homeomorphism [23, Thm. 26.6]. We know $\pi_1|_{I_\infty}$ is continuous, I_∞ is compact, and I is Hausdorff, so we conclude I and I_∞ are homeomorphic. □

We now construct the pseudo-arc by an inverse limit of the inverse system of maps $f\colon I \to I$. We begin by defining a new but analogous concept of crookedness for maps from I to I.

Definition 2.2.10. [11] Let $g\colon I \to I$ be continuous, $a, b, c, d \in \mathbb{Q}$ such that $0 < a < b < c < d < 1$, and $[u,t] \subset I$ such that $g([u,t]) = [a,d]$. We say $[u,t]$ is *mapped crookedly onto* the increasing rational quadruple (a,b,c,d) by g if there exist three points $x < y < z \in [u,t]$ such that either

(a) $g(x) = g(z) = b$ and $g(y) = c$ or
(b) $g(x) = g(z) = c$ and $g(y) = b$.

Essentially, $[u,t]$ is mapped crookedly onto the increasing quadruple of rational numbers (a,b,c,d) if there is a large enough "backtrack" at some point in the function. Compare with Definition 2.1.4. See Figure 2.2.1 for the graph of such a function.

Definition 2.2.11. (Construction of Pseudo-Arc) [11] Suppose $\{f_n\colon I \to I\}_{n\in\mathbb{N}}$ is a collection of surjective maps such that, for every increasing quadruple of rationals (a,b,c,d), there exists $N \in \mathbb{N}$ such that, for all $j > i \geq N$ and $[u,t] \subset I$ satisfying $f_i \circ \cdots \circ f_j([u,t]) = [a,d]$, $[u,t]$ is mapped crookedly onto (a,b,c,d) by $f_i \circ \cdots \circ f_j$.

Then,

$$\mathcal{N} = \varprojlim(I, f_n, \mathbb{N})$$

is a *pseudo-arc.*

Remark 2.2.12. Because our inverse system $(I, f_n, \mathbb{N})$ is not unique, just like in Subsection 2.1, what we are defining here is not "the pseudo-arc" but a family of continua we call pseudo-arcs. We find in Section 3 that both families are the same,

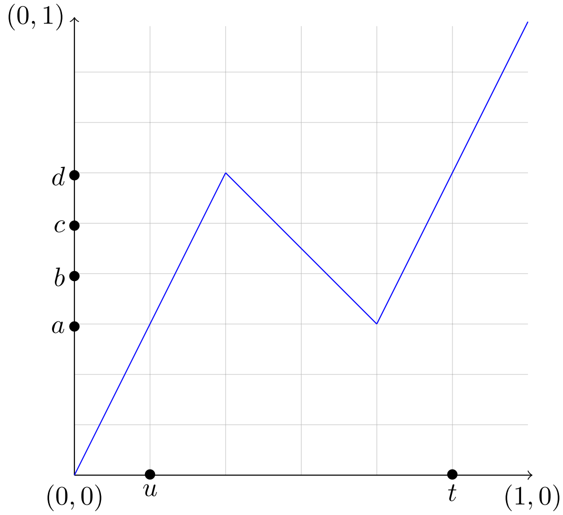

FIGURE 2.2.1. Above is the graph of a function g which maps $[u, t]$ crookedly onto (a, b, c, d).

and the continua in this family are all homeomorphic. Then, it will make sense to refer to each one as "the pseudo-arc."

Theorem 2.2.13. *Every inverse limit constructed pseudo-arc $\mathcal{N}$ is a nondegenerate continuum.*

Proof. By Theorem 2.2.7, $\mathcal{N}$ is a nonempty continuum. To see that $\mathcal{N}$ is nondegenerate, consider that each f_n is surjective. □

To show there exists a pseudo-arc using this construction, we state without proof the following Lemma proven by Henderson [11, Lem. 1].

Lemma 2.2.14. [11, Lem. 1] *There exists a surjective map $f: I \to I$ such that for every increasing quadruple of rationals (a, b, c, d), there exists $N \in \mathbb{N}$ such that, for all $n \geq N$ and $[u, t] \subset I$ satisfying $f^n([u, t]) = [a, d]$, $[u, t]$ is mapped crookedly onto (a, b, c, d) by f^n.*

In other words, for every increasing rational quadruple, after some number N, the nth iteration of f maps any interval whose image is $[a, d]$ crookedly onto (a, b, c, d).

The function from Lemma 2.2.14, which Henderson constructs more directly in his paper, "may be described roughly as starting with $g(x) = x^2$ and notching its graph with an infinite set of non-intersecting v's which accumulate at $(1, 1)$," in Henderson's words. See Figure 2.2.2 for Henderson's depiction of this function.

Theorem 2.2.15. *There exists a pseudo-arc which is the inverse limit of an inverse system $(I, f_n, \mathbb{N})$ such that all f_n are the same map.*

Proof. Using the f from Lemma 2.2.14, we have a pseudo-arc

$$\mathcal{N} = \varprojlim(I, f, \mathbb{N}).$$

□

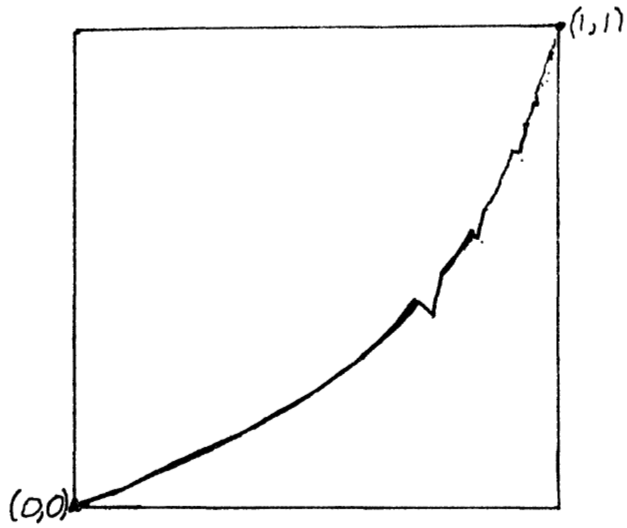

FIGURE 2.2.2. A picture of the graph for the function f drawn by Henderson for his paper [11].

3. Properties of the Pseudo-Arc

When we refer to pseudo-arcs in this section we are referring in particular to continua resulting from the chain construction in Subsection 2.1. Although both constructions are homeomorphic, we reserve the name "pseudo-arc" for chain-constructed continua until we have proven so.

The first property of a pseudo-arc we mention is the property Knaster was searching for in a continuum when he first constructed a pseudo-arc: hereditary indecomposability.

Theorem 3.1. [5, Thm. 10] *Pseudo-arcs are hereditarily indecomposable.*

Proof. Consider a pseudo-arc $\mathcal{M}$ with witness $\{\mathscr{D}^n\}_{n\in\mathbb{N}}$.

Let $\mathcal{M}'$ be a nondegenerate subcontinuum of $\mathcal{M}$. Assume for a contradiction that $\mathcal{M}'$ is decomposable. Thus, there exist proper subcontinua $\mathcal{K}$ and $\mathcal{H}$ of $\mathcal{M}'$ such that $\mathcal{K}\cup\mathcal{H}=\mathcal{M}'$. Let $p\in\mathcal{M}'\setminus\mathcal{K}$ and $q\in\mathcal{M}'\setminus\mathcal{H}$, which is possible since the subcontinua are proper. Because their union is $\mathcal{M}'$, we then have that $p\in\mathcal{H}$ and $q\in\mathcal{K}$.

Because $\mathcal{K}$ is compact, there exists $j_1\in\mathbb{N}$ such that the ball of radius $\frac{2}{j_1}$ centered at p does not intersect $\mathcal{K}$. Similarly, there exists $j_2\in\mathbb{N}$ such that the ball of radius $\frac{2}{j_2}$ does not intersect $\mathcal{H}$. By taking $j=\max\{j_1,j_2\}$, we have j such that

$$\operatorname{dist}(p,\mathcal{K})\geq\frac{2}{j}\quad\text{and}\quad\operatorname{dist}(q,\mathcal{H})\geq\frac{2}{j}.$$

See Figure 3.1.

Consider $\mathscr{D}^j=[D_1^j,\ldots,D_n^j]$ and $\mathscr{D}^{j+1}=[D_1^{j+1},\ldots,D_m^{j+1}]$. Because $p,q\in\mathcal{M}'$, there exist subchains $\mathscr{D}^j(h,k)$ and $\mathscr{D}^{j+1}(u,v)$ such that p and q belong to opposite end links of these subchains. Without loss of generality, we let $h<k$.

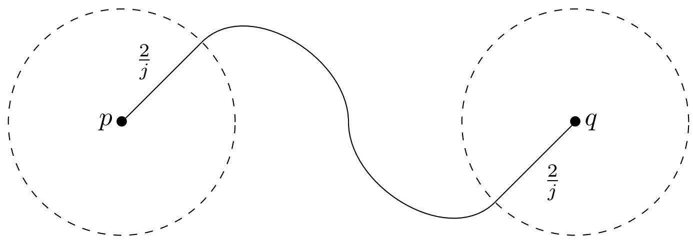

Figure 3.1. A simplified depiction of $\mathcal{M}$ with open balls of radius $\frac{2}{j}$ which separate p from $\mathcal{K}$ and q from $\mathcal{H}$.

Because $\mathcal{M}'$ is connected, each link of $\mathscr{D}^j(h,k)$ and $\mathscr{D}^{j+1}(u,v)$ has nonempty intersection with $\mathcal{M}'$. To see this for $\mathscr{D}^j(h,k)$, assume for some c such that $h < c < k$ we have $D^j_c \cap \mathcal{M}' = \emptyset$. We then have that $\mathcal{M} \subset \left(\bigcup_{i=1}^{c-1} D^j_i\right) \cup \left(\bigcup_{i=c+1}^{n} D^j_i\right)$. By taking the intersections of these sets with $\mathcal{M}'$, we have a separation for $\mathcal{M}'$, contradicting connectedness of $\mathcal{M}'$.

Without loss of generality, suppose $p \in D^j_h$ and $p \in D^{j+1}_u$. Because links in $\mathscr{D}_j$ have diameter at most $\frac{1}{j}$, $D^j_h \cup D^j_{h+1}$ is contained in the ball of radius $\frac{2}{j}$ around p. Thus, $(D^j_h \cup D^j_{h+1}) \cap \mathcal{K} = \emptyset$. Similarly, we also have that $(D^j_k \cup D^j_{k-1}) \cap \mathcal{H} = \emptyset$.

Because $\mathscr{D}^{j+1}$ is crooked in $\mathscr{D}^j$, there exist r, s such that $D^{j+1}_r \subset D^j_{k-1}$, $D^{j+1}_s \subset D^j_{h+1}$, and $\mathscr{D}^{j+1}(u,v) = \mathscr{D}^{j+1}(u,r) + \mathscr{D}^{j+1}(r,s) + \mathscr{D}^{j+1}(s,v)$. Because $D^{j+1}_r \subset D^j_{k-1}$, we have $D^{j+1}_r \cap \mathcal{H} = \emptyset$. Similarly, we have $D^{j+1}_s \cap \mathcal{K} = \emptyset$. Note also that $D^{j+1}_v \subset D^j_k \cup D^j_{k-1}$ since $\mathscr{D}^{j+1}$ is a refinement of $\mathscr{D}^j$. Thus, $D^{j+1}_v \cap \mathcal{H} = \emptyset$. Because each link has nonempty intersection with $\mathcal{M}'$, we have parts of $\mathcal{K}$ contained in both D^{j+1}_r and D^{j+1}_v, but not in D^{j+1}_s.

Because we have that $r < s < v$ or $v < s < r$, by a similar proof to that which showed links must each have nonempty intersection with $\mathcal{M}'$ or else $\mathcal{M}'$ would not be connected, we reach a contradiction and conclude $\mathcal{M}'$ is indecomposable. Thus, we conclude $\mathcal{M}$ is hereditarily indecomposable. $\square$

As it turns out, we can completely characterize pseudo-arcs by introducing another property present in pseudo-arcs.

Definition 3.2. [4, 6] A continuum M is called *chainable* (or *snake-like*) if, for all $\epsilon > 0$, there exists an ϵ-chain (a chain with links of diameter at most ϵ) which covers M.

Examples 3.3. Since the interval $I \times \{0\} \subset \mathbb{R}^2$ is chainable, all arcs are chainable. Also, by construction, pseudo-arcs are chainable.

Non-Example 3.4. A good example of a continuum which is not chainable is a *triod*. A triod is a continuum T such that, for some subcontinuum $S \subset T$, $T \setminus S$ has three connected components. See Figure 3.2.

The following lemma allows us to relate chainability to crookedness by finding a collection that resembles a witness for a pseudo-arc. Recall from Definition 2.1.2 that, for a chain $\mathscr{C}$, $\mathscr{C}^* = \bigcup_{i=1}^n C_i$.

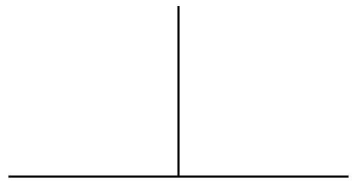

FIGURE 3.2. The simplest example of a triod.

Lemma 3.5. *A continuum M is chainable if, and only if, there exists a nested sequence of chains $\{\mathscr{C}^n\}_{n\in\mathbb{N}}$ such that, for all $n \in \mathbb{N}$,*

(a) *$\mathscr{C}^n$ is a $1/n$-chain (a chain with links of diameter at most $\frac{1}{n}$),*
(b) *$\mathscr{C}^n$ covers M,*
(c) *no link of $\mathscr{C}^n$ has empty intersection with M, and*
(d) *the closure of each link of $\mathscr{C}^{n+1}$ is a compact subset of a link of $\mathscr{C}^n$.*

Proof. Suppose M is chainable. Then, there exists a sequence of chains $\{\mathscr{B}^n\}_{n\in\mathbb{N}}$ satisfying (a) and (b). Because M is connected, a link of $\mathscr{B}^n$ with empty intersection with M cannot have links before and after it with nonempty intersection with M.

To see why, assume for a contradiction that, for $i < j < k$, B_j^n has empty intersection with M, but B_i^n and B_k^n have nonempty intersection with M. Then, the union of all links of $\mathscr{B}^n$ before the jth link and the union of all links of $\mathscr{B}^n$ after the jth link separate M into two open sets, contradicting connectedness.

Thus, we may construct a new sequence $\{\mathscr{D}^n\}_{n\in\mathbb{N}}$ by removing the links from the ends of each chain which have empty intersection with M.

To obtain (d), consider that, since it is compact, M cannot get arbitrarily close to the borders of each link, so, given $\mathscr{D}^1$, we can find a chain with links of suitably small diameter to guarantee that the closure of the union of links of the chain is a compact subset of $(\mathscr{D}^1)^*$. We can then take a subsequence $\{\mathscr{D}^{n_k}\}_{k\in\mathbb{N}}$ of these suitably small chains. By dividing the links of each $\mathscr{D}^{n_k+1}$ into smaller links, we can make it a refinement of $\mathscr{D}^{n_k}$. We let our collection $\{\mathscr{C}^n\}_{n\in\mathbb{N}}$ be defined such that each $\mathscr{C}^k$ is the properly divided $\mathscr{D}^{n_k}$. Note that this implies our sequence of chains $\{\mathscr{C}^n\}_{n\in\mathbb{N}}$ is indeed nested.

Suppose we have such a collection $\{\mathscr{C}^n\}_{n\in\mathbb{N}}$. For all $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. Then, $\mathscr{C}^n$ is a ϵ-chain covering M. $\square$

We know that pseudo-arcs are chainable and hereditarily indecomposable, but as it turns out, this characterizes pseudo-arcs completely. Given a chainable and hereditarily indecomposable continuum, we can use hereditary indecomposability to strengthen the collection from Lemma 3.5 to find a witness that our continuum is in fact a pseudo-arc. We now prove a lemma which helps us prove this characterization.

Lemma 3.6. [4, Thm. 1] *Suppose M is a hereditarily indecomposable chainable continuum. Given a sequence of chains $\{\mathscr{C}^n\}_{n\in\mathbb{N}}$ satisfying conditions (a)-(d) of Lemma 3.5, there exists a subsequence $\{\mathscr{C}^{n_t}\}_{t\in\mathbb{N}}$ such that $\mathscr{C}^{n_{t+1}}$ is crooked in $\mathscr{C}^{n_t}$.*

Proof. We will prove there is some $n \in \mathbb{N}$ such that $\mathscr{C}^n$ is crooked in $\mathscr{C}^1$. The rest of the sequence then follows.

Assume for a contradiction there does not exist $n \in \mathbb{N}$ such that $\mathscr{C}^n$ is crooked in $\mathscr{C}^1$. Define $\eta(h,k)$ to be the set of all $n \in \mathbb{N}$ such that $\mathscr{C}^n$ does not have a tangle between every two links which intersect C_h^1 and C_k^1. More precisely, $\eta(h,k)$ is the set of all $n \in \mathbb{N}$ such that there is some subchain $\mathscr{C}^n(i_n, j_n)$ satisfying $C_i^n \cap C_h^1 \neq \emptyset$ and $C_j^n \cap C_k^1 \neq \emptyset$ but there do not exist r, s such that $i < r < s < j$ or $j < s < r < i$, $C_r^n \subset C_{k-1}^1$, and $C_s^n \subset C_{h+1}^1$. If, for all h, k such that $k - h > 2$ we had that $\eta(h, k)$ were finite, we could take

$$z = \max\Bigg(\bigcup_{k-h>2} \eta(h,k)\Bigg).$$

We then have that $\mathscr{C}^{z+1}$ is crooked in $\mathscr{C}^1$, a contradiction of our assumption. Thus, we conclude there exist h, k with $k - h > 2$ such that $\eta(h, k)$ is infinite.

For each $m \in \eta(h, k)$, let r_m be such that $C_{r_m}^m$ is the first link (i.e. closest in sequence to i_m) of $\mathscr{C}^m(i_m, j_m)$ which is a subset of C_{k-1}^1. Then, define

$$W_m = \mathscr{C}^m(i_m, r_m)^* \text{ and } V_m = \mathscr{C}^m(r_m, j_m)^*.$$

We see $W_m \cap C_k^1 = \emptyset$ since $C_{r_m}^m$ is a subset of C_{k-1}^1 and no link between C_{i_m} and C_{r_m} is a subset of C_{k-1}. See Figure 3.3 for a visual representation of these sets.

Assume for a contradiction $V_m \cap C_h^1 \neq \emptyset$. Thus, there exists a link in $\mathscr{C}^m(r_m, j_m)$ with nonempty intersection with C_h^1. However, this implies there is some link $C_{s_m}^m \subset C_{h+1}^1$. Since r_m, s_m satisfy the conditions above which imply $m \notin \eta(h, k)$, we reach a contradiction. Thus, we have that for all $m \in \eta(h, k)$, $V_m \cap C_h^1 = \emptyset$.

We now show that the existence of the infinite set $\eta(h, k)$ implies we can construct two proper subcontinua of M such that a point of C_h^1 is contained in only one and a point of C_k^1 is contained in only the other.

Denote $\eta(h, k) = \{m_1, m_2, \dots\}$ where $m_t < m_{t+1}$. Consider

$$\mathcal{W} = \bigcap_{t=1}^{\infty} \overline{W_{m_t}} \text{ and } \mathcal{V} = \bigcap_{t=1}^{\infty} \overline{V_{m_t}}.$$

By Theorem 2.1.1, $\mathcal{W}$ and $\mathcal{V}$ are continua. Furthermore, since

$$\mathcal{W} \cup \mathcal{V} = \bigcap_{t=1}^{\infty} (\overline{W_{m_t}} \cup \overline{V_{m_t}}) = \bigcap_{t=1}^{\infty} \overline{\mathscr{C}^{m_t}(i_{m_t}, j_{m_t})^*}$$

and each $\mathscr{C}^{m_t}(i_{m_t}, j_{m_t})^*$ is connected, we have that $\mathcal{W} \cup \mathcal{V}$ is a subcontinuum of M. Note that $\mathcal{W} \cap C_k^1 = \emptyset$ and $\mathcal{V} \cap C_h^1 = \emptyset$. However, since $\mathscr{C}^{m_t}(i_{m_t}, j_{m_t})^*$ has nonempty intersection with C_h^1 and C_k^1 for all m_t, $\mathcal{W} \cup \mathcal{V}$ must contain points $x \in C_h^1$ and $y \in C_k^1$. We see $x \notin \mathcal{V}$ and $y \notin \mathcal{W}$, so $\mathcal{W}$ and $\mathcal{V}$ are proper subcontinua of $\mathcal{W} \cup \mathcal{V}$.

Thus, we contradict M being hereditarily indecomposable and conclude there exists $n \in \mathbb{N}$ such that $\mathscr{C}^n$ is crooked in $\mathscr{C}^1$. We can then construct an infinite set $\{n_1, n_2, \dots\} \subset \mathbb{N}$ with $n_t < n_{t+1}$ such that $\mathscr{C}_{n_1} = \mathscr{C}_1$ and $\mathscr{C}^{n_{t+1}}$ is crooked in $\mathscr{C}^{n_t}$. □

Theorem 3.7. [4, Thm. 1] *A continuum $\mathcal{M}$ is a pseudo-arc if, and only if, it is hereditarily indecomposable and chainable.*

Proof. Theorem 3.1 and Examples 3.3 together yield the forward direction.

Given a hereditarily indecomposable, chainable continuum M, by Remark 3.5, there exists a collection $\{\mathscr{C}^n\}_{n\in\mathbb{N}}$ such that, for all $n \in \mathbb{N}$,

(a) $\mathscr{C}^n$ is a $1/n$-chain,
(b) $\mathscr{C}^n$ covers M, and
(c) no link of $\mathscr{C}^n$ has empty intersection with M, and

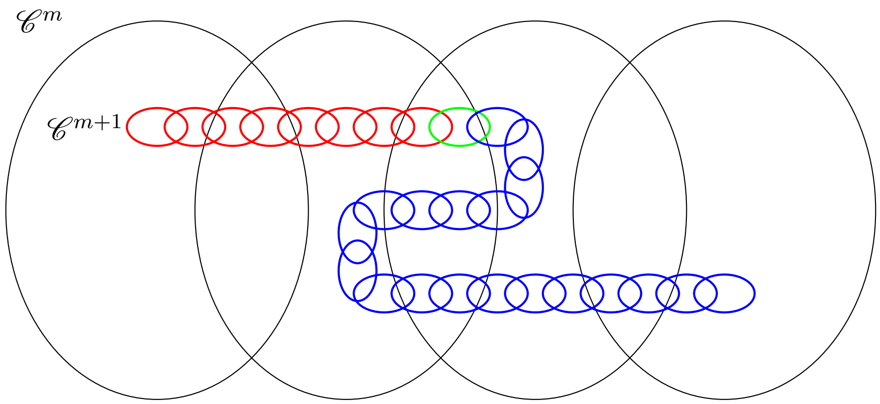

FIGURE 3.3. Above are possible $\mathscr{C}^m$ and $\mathscr{C}^{m+1}$, though $\mathscr{C}^{m+1}$ is not necessarily crooked in $\mathscr{C}^m$. In the picture, r_m is the green link, W_m is the union of the red links and the green link, and V_m is the union of the blue links and the green link.

(d) the closure of each link of $\mathscr{C}^{n+1}$ is a compact subset of a link of $\mathscr{C}^n$.

By Lemma 3.6, there exists a subsequence $\{\mathscr{C}^{n_t}\}_{t\in\mathbb{N}}$ such that $\mathscr{C}^{n_{t+1}}$ is crooked in $\mathscr{C}^{n_t}$. Then, $\{\mathscr{C}^{n_t}\}_{t\in\mathbb{N}}$ is a witness that M is a pseudo-arc. □

It is always a wonderful day when we discover a way to describe a class of objects perfectly by well-defined properties. However, we do not stop here! We have yet to show that all of these pseudo-arcs are but one, and so we shall set out to do just this.

We begin by stating without proof a lemma from Bing [5] which allows us to reallocate links of a chain crooked in another to form a chain which begins and ends where we want it to. Recall that a consolidation $\mathscr{C}$ of a chain $\mathscr{B}$ is a chain such that every link of $\mathscr{C}$ is the union of some links of $\mathscr{B}$ and that a chain goes from p to q if the first link contains p and the last link contains q.

Lemma 3.8. [5, Thm. 4] *Suppose a chain $\mathscr{B}$ is crooked in a chain $\mathscr{A}$, $p, q \in \mathscr{B}^*$, and the maximal subchain[8] of $\mathscr{B}$ from p to q has links which intersect the first and last links of $\mathscr{A}$. Then, there exists a chain $\mathscr{C}$ such that*

(a) $\mathscr{C}$ is a consolidation of $\mathscr{B}$,
(b) each link of $\mathscr{C}$ is contained in two adjacent links of $\mathscr{A}$,
(c) B_i is a subset of the first link of $\mathscr{C}$ and B_j is a subset of the last link of $\mathscr{C}$, and
(d) $\mathscr{C}$ goes from p to q.

This lemma is more natural if it is considered as the piecing together of two chains. Given that a chain $\mathscr{B}$ is crooked in $\mathscr{A}$, if we fix a link of $\mathscr{B}$, we can construct a chain $\mathscr{C}$ which is a consolidation of $\mathscr{B}$ such that the first link of $\mathscr{C}$ contains as a subset this fixed link. In Figure 3.4, we give a way to visualize this. For the lemma, the idea is to put the chain which starts at p and the one that starts at q together in a nice way. As shown in the figure, we must dispose of any notion

[8]We take the maximal subchain since it is possible that a point could be in two adjacent links.

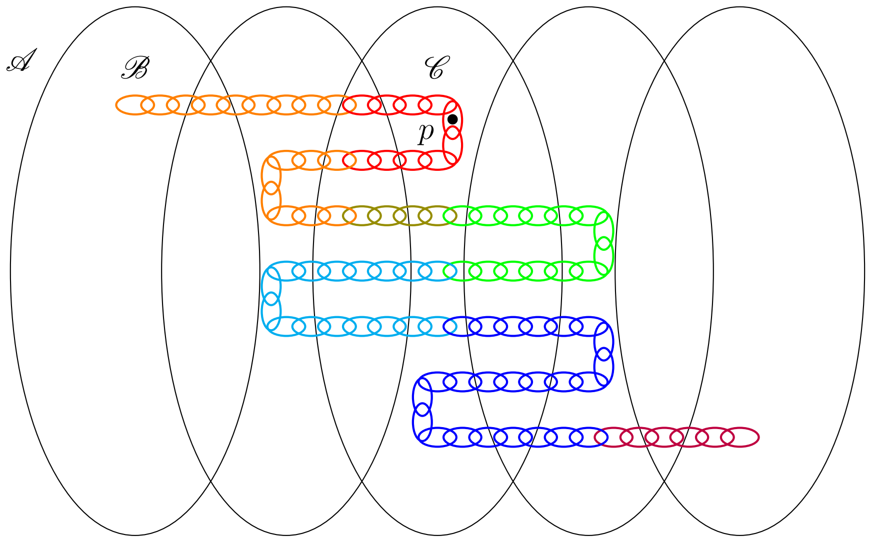

FIGURE 3.4. Given a chain $\mathscr{B}$ crooked in the chain $\mathscr{A}$, we have fixed the link containing p and showed using colors the way to consolidate $\mathscr{B}$ into a new chain $\mathscr{C}$ so that each link of $\mathscr{C}$ is a subset of two adjacent links and the first link of $\mathscr{C}$ contains the link of $\mathscr{B}$ containing p. Same-colored links denote whole links of $\mathscr{C}$. Thus, $\mathscr{C}$ has 7 links which, in sequence, are red, orange, olive, green, cyan, blue, and purple.

of connected links we may still be holding onto, as this lemma does not guarantee in the slightest that our new chain $\mathscr{C}$ has connected links.

The next lemma simply applies the above lemma to a witness for a pseudo-arc, then uses Lemma 3.6 to create a very useful witness of the same pseudo-arc.

Lemma 3.9. *Suppose $\mathcal{M}$ is a pseudo-arc with witness $\{\mathscr{D}^n\}_{n\in\mathbb{N}}$ and $p, q \in \mathcal{M}$ are such that, for all $n \in \mathbb{N}$, the maximal subchain of $\mathscr{D}^{n+1}$ from p to q has links which intersect the first and last links of $\mathscr{D}^n$. Then, there exists a witness $\{\mathscr{E}^n\}_{n\in\mathbb{N}}$ of $\mathcal{M}$ such that each chain $\mathscr{E}^n$ goes from p to q.*

Proof. We will first construct a sequence of chains $\{\mathscr{C}^n\}_{n\in\mathbb{N}}$. For all $n \in \mathbb{N}$, by Lemma 3.8, there exists $\mathscr{C}^n$ such that

(a) $\mathscr{C}^n$ is a consolidation of $\mathscr{D}^n$,
(b) each link of $\mathscr{C}^n$ is contained in two adjacent links of $\mathscr{D}^n$,
(c) $D^n_{i_n}$ is a subset of the first link of $\mathscr{C}^n$ and $D^n_{j_n}$ is a subset of the last link of $\mathscr{C}^n$, and
(d) $\mathscr{C}^n$ goes from p to q.

The method we used to obtain (d) of Lemma 3.5 allows us to assume without loss of generality that $\{\mathscr{C}^n\}$ also satisfies the condition that the closure of each link of $\mathscr{C}^{n+1}$ is a compact subset of a link of $\mathscr{C}^n$.

Although every chain of $\{\mathscr{C}^n\}_{n\in\mathbb{N}}$ goes from p to q, by applying Lemma 3.8, we have lost the property that each $\mathscr{C}^{n+1}$ is crooked in $\mathscr{C}^n$. However, what we now have is a sequence $\{\mathscr{C}^n\}_{n\in\mathbb{N}}$ such that

(a) $\mathscr{C}^n$ is a $1/n$-chain,
(b) $\mathscr{C}^n$ covers $\mathcal{M}$,
(c) no link of $\mathscr{C}^n$ has empty intersection with $\mathcal{M}$, and
(d) the closure of each link of $\mathscr{C}^{n+1}$ is a compact subset of a link of $\mathscr{C}^n$.

Therefore, we may apply Lemma 3.6 to yield a subsequence $\{\mathscr{C}^{n_k}\}_{k\in\mathbb{N}}$ such that each $\mathscr{C}^{n_{k+1}}$ is crooked in $\mathscr{C}^{n_k}$. By defining each $\mathscr{D}^k \coloneqq \mathscr{C}^{n_k}$, we have our witness $\{\mathscr{D}^n\}_{n\in\mathbb{N}}$. □

We now outline the proof for a lemma which not only proves pseudo-arcs are homeomorphic to one another, but will also be very useful in proving properties of the pseudo-arc.

Lemma 3.10. [5, Thm. 12] *Suppose $\mathcal{M}$ and $\mathcal{M}'$ are pseudo-arcs with witnesses $\{\mathscr{D}^n\}_{n\in\mathbb{N}}$ and $\{\mathscr{D}'^n\}_{n\in\mathbb{N}}$, respectively. Suppose further there exist $p, q \in \mathcal{M}$ and $p', q' \in \mathcal{M}'$ such that $\mathscr{D}^n$ goes from p to q and $\mathscr{D}'^n$ goes from p' to q' for all $n \in \mathbb{N}$.*

Then, there exists a homeomorphism $\phi\colon \mathcal{M} \to \mathcal{M}'$ such that $\phi(p) = p'$ and $\phi(q) = q'$.

Outline of Proof. We would like to identify points of $\mathcal{M}$ with sequences of indices whose links that point resides in; a point p in $\mathcal{M}$ can be determined by $(a_1, a_2, \dots) \in \mathbb{N}^{\mathbb{N}}$, where each a_n is the index of the first link of chain $\mathscr{D}^n$ which that point lies in.

To do this, we have to create new sequences by consolidating chains from both in a way so that the nth chain of both collection has the same number of links. We can further restrict these new sequences so that whenever a link of $\mathcal{D}^n$ intersects a link of $\mathscr{D}^{n+1}$, similar links in $\mathscr{D}'^n$ are reasonably close. More precisely, if the ith link of $\mathscr{D}^n$ intersects the jth link of $\mathscr{D}^{n+1}$, then the ith link of $\mathscr{D}'^n$ should be at most $\frac{1}{n}$ away from the jth link of $\mathscr{D}'^{n+1}$. Using these chains we can identify points of $\mathcal{M}$ with sequences of indices $(a_1, a_2, \dots)$.

We define a function $\phi\colon \mathcal{M} \to \mathcal{M}'$ by sending p, determined by $(a_1, a_2, \dots)$ in $\mathcal{M}$, to the point determined by $(a_1, a_2, \dots)$ in $\mathcal{M}'$. This works quite nicely once we have the condition of intersecting links in one implying reasonably close links in the other, and we end up with a homeomorphism.

Because p and q are in the first and last links of every chain, and the same is true for p' and q', we have that $\phi(p) = p'$ and $\phi(q) = q'$. □

We now wish to find a witness for every pseudo-arc such that each chain goes from one point p to another point q. As it turns out, we can do this, but only if we add the additional condition that p and q are in different *composants*.

Definition 3.11. [4] For a continuum M, the *composant* of a point p is the union of all proper subcontinua of M containing p.

Remarks 3.12. (a) The definition of composant seems to resemble that of connected component. It seems like the composant of p is the largest proper subcontinuum of M which contains p. However, this is not the case. Not only is the composant of p not necessarily a proper subcontinuum, but the composant is not necessarily even a continuum at all! To see the former, consider the composant of any point in the triod in Figure 3.2. For the latter, consider that the composant of 0 in I is $[0, 1)$.

(b) Composants do not in general partition a continuum. This can be seen by considering I and the composants of 0 and $\frac{1}{2}$. Although each is in the other's composant, their composants are different! However, the additional condition that our continuum is indecomposable allows us to define equivalence classes which are precisely the composants.

Lemma 3.13. *Let M be a nondegenerate indecomposable continuum. Then,*

(a) M is partitioned by its composants and
(b) M has uncountably many composants.

Proof. For (a), we will define an equivalence relation on M. For $p, q \in M$, define

$$p \sim q \iff \text{there exists a subcontinuum } S \subset M \text{ such that } p, q \in S.$$

Reflexivity and symmetry of $\sim$ are trivial. For transitivity, let $p, q, r \in M$ such that $p \sim q \sim r$. Let subcontinua $S, T \subset M$ such that $p, q \in S$ and $q, r \in T$. Because $q \in S \cap T$, $S \cup T$ is connected and is therefore a subcontinuum of M. If $S \cup T$ was a proper subcontinuum, then $p \sim r$. Assume for a contradiction that $S \cup T$ is not a proper subcontinuum of M. Because M is indecomposable, either S or T must not be a proper subcontinuum of M. We a reach a contradiction, so conclude $\sim$ is transitive and is thus an equivalence relation.

Because the equivalence classes formed using $\sim$ are precisely the composants of M, we conclude M is partitioned by its composants.

For (b), It turns out that composants of indecomposable continua are first category[9]. For more details, see [24, Prop 11.14]. □

We now prove that all pseudo-arcs are homeomorphic using the lemmas we have proven.

Theorem 3.14. [4, Thm. 1] *Given two pseudo-arcs $\mathcal{M}, \mathcal{M}'$, if $p, q \in \mathcal{M}$ are in different composants of $\mathcal{M}$ and $p', q' \in \mathcal{M}'$ are in different composants of $\mathcal{M}'$, there exists a homeomorphism $\phi\colon \mathcal{M} \to \mathcal{M}'$ such that $\phi(p) = p'$ and $\phi(q) = q'$.*

Proof. Let $\{\mathscr{D}^n\}_{n\in\mathbb{N}}$ be a witness for $\mathcal{M}$. We will first show that, since p and q are in different composants of $\mathcal{M}$, for all $n \in \mathbb{N}$, the maximal subchain of $\mathscr{D}^{n+1}$ from p to q has links which intersect the first and last links of $\mathscr{D}^n$.

For all $n \in \mathbb{N}$, define $\mathscr{A}^n$ to be the maximal subchain of $\mathscr{D}^n$ that goes from p to q. Let

$$\mathcal{K} = \bigcap_{n=1}^{\infty} (\mathscr{A}^n)^*.$$

We see $\mathcal{K}$ is a subcontinuum of $\mathcal{M}$ containing p and q. Since p and q are in different composants, we conclude $\mathcal{K} = \mathcal{M}$. If it were the case that some $\mathscr{A}^{n+1}$ did not have links intersecting both end links of $\mathscr{D}^n$, we would have that $\mathscr{D}^n$ has an end link with empty intersection with $\mathcal{M}$. We reach a contradiction by definition of a witness for a pseudo-arc.

By Lemma 3.9, there exists a witness $\{\mathscr{E}^n\}_{n\in\mathbb{N}}$ of $\mathcal{M}$ such that each chain $\mathscr{E}^n$ goes from p to q. Similarly, there exists a witness $\{\mathscr{E}'^n\}_{n\in\mathbb{N}}$ of $\mathcal{M}'$ such that each chain $\mathscr{E}'^n$ goes from p' to q'.

[9] i.e. The countable union of nowhere dense sets (sets A such that every open set U contains an open set $V \subset U$ such that $V \cap A = \emptyset$).

By Lemma 3.10, there exists a homeomorphism $\phi\colon \mathcal{M} \to \mathcal{M}'$ such that $\phi(p) = p'$ and $\phi(q) = q'$. □

Because a hereditarily indecomposable continuum has uncountably many composants (which means it has at least two), we can then deduce the following corollary.

Corollary 3.15. *The pseudo-arc is unique up to homeomorphism.*

This corollary means we can finally refer to the pseudo-arc as just that: "*the* pseudo-arc." For the rest of the paper, we refer to the pseudo-arc by $\mathcal{M}$. We can quickly deduce the same for all nondegerate, hereditarily indecomposable, chainable continua.

Corollary 3.16. *All nondegenerate, hereditarily indecomposable, chainable continua are homeomorphic.*

We state without proof a theorem from Ingram and Mahavier [14] which allows us to unify our constructions of the pseudo-arc.

Theorem 3.17. [14, Thm.s 96, 102] *A continuum M is chainable if, and only if, there exists an inverse system $(I, f_n, \mathbb{N})$ such that each $f_n\colon I \to I$ and M is homeomorphic to $\varprojlim(I, f_n, \mathbb{N})$.*

Corollary 3.18. *The construction of the pseudo-arc $\mathcal{M}$ by the intersection of nested chains is homeomorphic to the construction of the pseudo-arc $\mathcal{N}$ as the inverse limit of an inverse sequence of functions $f_n\colon I \to I$.*

Proof. By Theorem 3.17, $\mathcal{N}$ is chainable. The crookedness of the maps whose inverse limit is $\mathcal{N}$ provide a sufficient condition that $\mathcal{N}$ is hereditarily indecomposable. Since both continua are hereditarily indecomposable, by Corollary 3.16, $\mathcal{M}$ and $\mathcal{N}$ are homeomorphic. □

Along with these nice statements, our theorem also allows us to deduce the following corollaries.

Corollary 3.19. [4] *The pseudo-arc is homeomorphic to each of its nondegenerate subcontinua.*

Proof. A subcontinuum $\mathcal{S}$ of the pseudo-arc $\mathcal{M}$ must be hereditarily indecomposable, since any subcontinuum of $\mathcal{S}$ is also a subcontinuum of $\mathcal{M}$. Furthermore, $\mathcal{S}$ is chainable, since every chain which covers $\mathcal{M}$ also covers $\mathcal{S}$. By Corollary 3.16, $\mathcal{S}$ is homeomorphic to $\mathcal{M}$. □

Corollary 3.20. [4] *The pseudo-arc is homogeneous.*

Proof. Let $p, q \in \mathcal{M}$. Let $r \in \mathcal{M}$ such that r is not in the same composant as p or q. We can apply Theorem 3.14 to $\mathcal{M}$ with itself to yield a homeomorphism $\phi\colon \mathcal{M} \to \mathcal{M}$ such that $\phi(p) = q$ and $\phi(r) = r$. □

Thus, we have answered both questions we asked in the introduction.

4. Classifying Homogeneous Planar Continua

In this section we briefly touch on an interesting question in continuum theory which was answered recently and involves the pseudo-arc.

Many problems in continuum theory involve planar continua and, often, planar continua which divide the plane. In fact, now that we have found an example of a homogeneous continuum not homeomorphic to the circle, a new question arises.

Question 4.1. Is every homogeneous planar continuum homeomorphic to the circle or the pseudo-arc?

In the case of an arc, we gain homogeneity by connecting the two end points, forming a circle. Thus, it seems natural to try to do the same for the pseudo-arc, hopefully preserving its homogeneity.

To do this, we define *circular chains*.

Definition 4.2. [4, Ex. 2] A *circular chain* $\mathscr{P}$ is a chain such that the first and last links intersect.

A *subchain* $\mathscr{P}(i,j)$ of a circular chain $\mathscr{P}$ is a chain that starts at P_i and ends at P_j. More precisely, if $i<j$, we define $\mathscr{P}(i,j) = [P_i, P_{i+1}, \dots, P_{j-1}, P_j]$, and if $j<i$, we define $\mathscr{P}(i,j) = [P_i, P_{i+1}, \dots, P_n, O_1, \dots, P_{j-1}, P_j]$. Thus, $\mathscr{P}(i,j)$ and $\mathscr{P}(j,i)$ are now completely different chains in a sense other than just orientation. Note that a subchain is *never* a circular chain because we do not allow $i=j$.

Definition 4.3. We say a circular chain $\mathscr{Q}$ is a *refinement* of a circular chain $\mathscr{P}$ if every link of $\mathscr{Q}$ is a subset of some link of $\mathscr{P}$.

Definition 4.4. [4, Ex. 2] We say a circular chain $\mathscr{Q}$ which is a refinement of a circular chain $\mathscr{P}$ is *crooked* in $\mathscr{P}$ if, for all subchains $\mathscr{P}(h,k)$ and $\mathscr{Q}(i,j)$ such that $\mathscr{Q}(i,j)$ is a refinement of $\mathscr{P}(h,k)$, $\mathscr{Q}(i,j)$ is crooked in $\mathscr{P}(h,k)$ (in the usual sense defined in Definition 2.1.4).

In [10], Handel constructs a map in a way which directly resembles the crookedness of a circular chain. See Figure 4.1.

Definition 4.5. [4, Ex. 2] Suppose $\{\mathscr{P}^n\}_{n\in\mathbb{N}}$ is a sequence of nested circular chains such that

(a) $\mathscr{P}^{n+1}$ is crooked in $\mathscr{P}^n$,
(b) each link of $\mathscr{P}^{n+1}$ is a compact subset of some link of $\mathscr{P}^n$,
(c) each link of $\mathscr{P}^n$ is the interior of a circle of radius $\frac{1}{n}$,[10]
(d) $(\mathscr{P}^n)^*$ is homeomorphic to an annulus, and
(e) each complementary domain[11] of $\mathscr{P}^{n+1}$ contains a complementary domain of $\mathscr{P}^n$.[12]

[10] This condition was in Bing's first construction of the pseudo-circle [4, Ex. 2] and is much stronger than the conditions put on witnesses of pseudo-arcs. We discuss the necessity of the stronger condition in Remark 4.7.

[11] A complementary domain of a closed set K is an open set U such that the boundary of U is a subset of K [22, p.291]. Equivalently, complementary domains of a closed set are the connected components of the complement of K.

[12] This condition can be thought of as forcing $\mathscr{P}^{n+1}$ to go all the way around $\mathscr{P}^n$. Without (e), we could have a situation where $\mathscr{P}^{n+1}$ is squeezed and filtered along $\mathscr{P}^n$ in the shape of a C. See Figure 4.2.

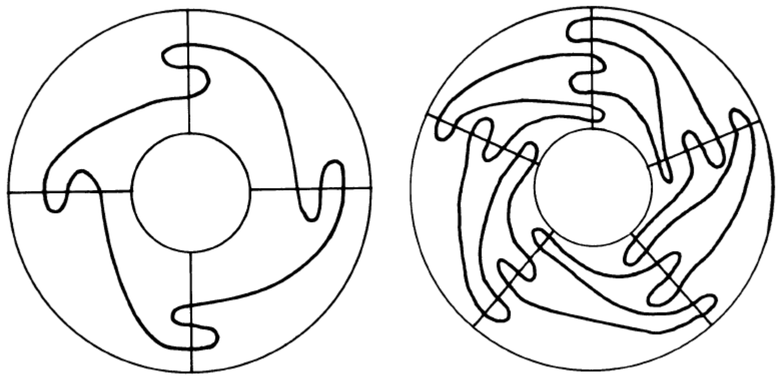

FIGURE 4.1. Above are drawings by Handel of two functions which he describes in [10]. These bold curves follow the same pattern of circular chains crooked in circular chains of 4 and 5 links.

Then,

$$\mathcal{C} = \bigcap_{n=1}^{\infty} \overline{(\mathscr{P}^n)^*} = \bigcap_{n=1}^{\infty} \overline{\bigcup_{P_i^n \in \mathscr{P}^n} P_i^n}.$$

is a *pseudo-circle*. Again, we refer to $\{\mathscr{P}^n\}_{n\in\mathbb{N}}$ as a *witness* that $\mathcal{C}$ is a pseudo-circle. Every pseudo-circle must have at least one witness, by definition.

We may again refer to each pseudo-circle as *the* pseudo-circle due to the following theorem proved by Fearnley [7].

Theorem 4.6. [7, Thm. 6.3] *The pseudo-circle is unique up to homeomorphism.*

Remark 4.7. Pseudo-circles can be defined more broadly: Rogers [25, 26] typically considers pseudo-circles to be all hereditarily indecomposable, circularly chainable, unchainable continua. Unfortunately, not all pseudo-circles (in Rogers' general sense) are homeomorphic to the one constructed by Bing which we construct here. In fact, in [26], Rogers constructs uncountably many topologically distinct (general) pseudo-circles. It turns out that if a (general) pseudo-circle is planar, it is homeomorphic to Bing's pseudo-circle, giving us a result similar to Theorem 4.6.

Theorem 4.8. [7, Thm. 6.3] *All planar hereditarily indecomposable, circularly chainable, unchainable continua are homeomorphic.*

For our purposes, we do not need to go to deeper than this and will continue to treat the pseudo-circle as a space homeomorphic to the one constructed by Bing.

The pseudo-circle looks a lot like the pseudo-arc; in fact, every subcontinuum of the pseudo-circle *is* a pseudo-arc. Naturally, we would like to discover what properties the two continua have in common. In particular, we hope the pseudo-circle is also a hereditarily indecomposable, homogeneous continua. When Bing first introduced the continuum in [4], he proved the former but left the latter as a question.

Theorem 4.9. [4, Ex. 2] *The pseudo-circle is hereditarily indecomposable.*

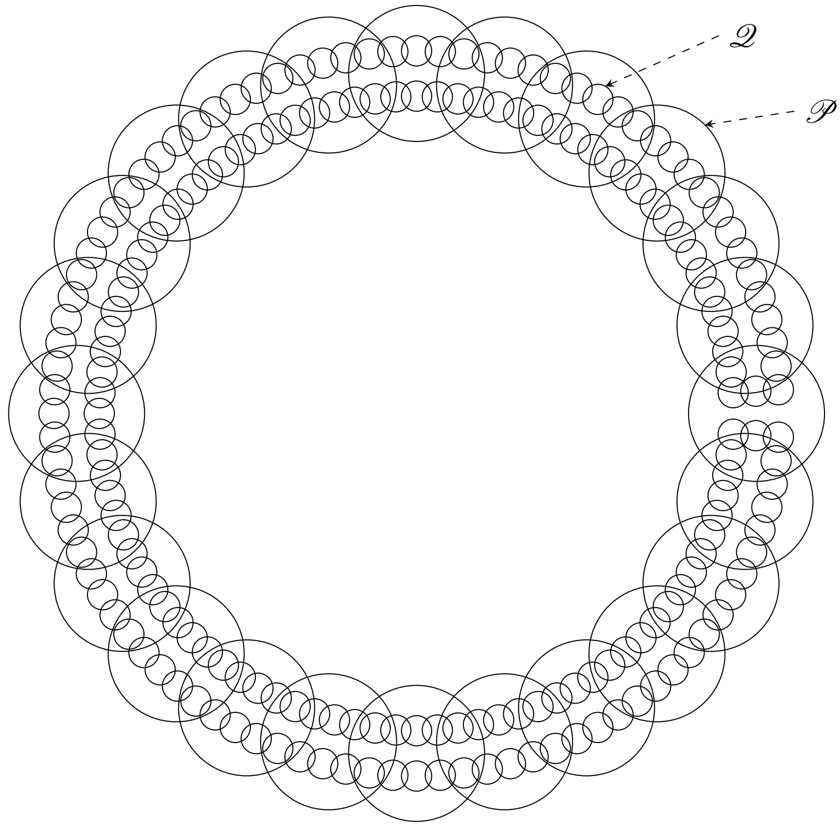

FIGURE 4.2. An example of circular chains $\mathscr{P}$ and $\mathscr{Q}$ such that $\mathscr{Q}$ is a refinement of $\mathscr{P}$. Treating $\mathscr{P}$ as $\mathscr{P}^n$ and $\mathscr{Q}$ as $\mathscr{P}^{n+1}$, these chains do not satisfy condition (e) of Definition 4.5: The complements of $\mathscr{P}^*$ and $\mathscr{Q}^*$ each have two connected components. Since the bounded connected component of $(\mathscr{Q}^*)^C$ is a subset of $\mathscr{P}^*$, it does not contain a connected component of $(\mathscr{P}^*)^C$.

Rogers [25] later proved that, quite surprisingly, the answer to Bing's question is *no*.

Theorem 4.10. [25, Thm. 1] *The pseudo-circle is* not *homogeneous.*

It seems as though the relationship between the pseudo-circle and the pseudo-arc is analogous to that between the circle and the arc. However, whereas the circle is homogeneous and the arc is not, the pseudo-arc *is* homogeneous and the pseudo-circle is *not*. This strange reversal of properties demonstrates how difficult homogeneous planar continua are to find.

In [3], Bing and Jones constructed a planar continuum $\mathcal{Q}$ such that there is a continuous function $f\colon \mathcal{Q} \to [0,1]$ so that, for all $x \in [0,1]$, $f^{-1}(\{x\})$ is a pseudo-arc. They called this continuum a *circle of pseudo-arcs*. We do not give a detailed construction of this space here. The continuum $\mathcal{Q}$ which they constructed is unique up to homeomorphism and homogeneous, providing a negative answer to Question 4.1.

In 2016, Hoehn and Oversteegen were able to use this continuum to completely classify homogeneous planar continua.

Theorem 4.11. [12, Thm. 2] *Up to homeomorphism, the only nondegenerate homogeneous planar continua are*

(a) the circle,
(b) the pseudo-arc, and
(c) the circle of pseudo-arcs.

With the help of a theorem from [1], we can deduce a strong classification of homogeneous , compact, planar sets.

Theorem 4.12. [1] *Every homogeneous compact $Y \subset \mathbb{R}^2$ is homeomorphic to $X \times Z$ for a homogeneous continuum X and a totally disconnected compact set Z. Thus, Z is either*

(a) a finite set or
(b) a Cantor set.

From Theorem 4.11 and Theorem 4.12, the following is immediate.

Theorem 4.13. [12, Thm. 3] *Up to homeomorphism, the only homogeneous compact subsets of $\mathbb{R}^2$ are*

(a) finite sets,
(b) Cantor sets, and
(c) $X \times Z$, for X one of (a)-(c) in Theorem 4.11 and Z one of (a)-(b) in Theorem 4.12.

With these powerful results, the pseudo-arc's importance in mathematics is clear.

5. Final Remarks

The pseudo-arc is much more than just a nondegenerate, hereditarily indecomposable, homogeneous continuum homeomorphic to each of its nondegenerate subcontinua. Unfortunately, the necessary time constraints of the REU program do not allow the author the amount of time necessary to cover every interesting detail of the pseudo-arc. In this section we give a few other results relating to the pseudo-arc.

We first introduce a result proven by Mazurkiewicz [20]. For a proof in English, see Bing [4, p.46].

Definition 5.1. [4, p.46] Given a metric space (X, d), let 2^X be the set of all nonempty continua which are subsets of X. We define the *Hausdorff metric* $\rho\colon 2^X \times 2^X \to \mathbb{R}$, $A, B \in 2^X$ by

$$\rho(A, B) = \max\{\sup_{a\in A} \operatorname{dist}(a, B), \sup_{b\in B} \operatorname{dist}(b, A)\},$$

where $\operatorname{dist}(a, B) \coloneqq \inf_{b\in B} d(a, b)$. We call the resulting metric space 2^X the *hyperspace* of X.

Theorem 5.2. [20] *Pseudo-arcs are residual in the hyperspace 2^X of any metric space X.*

In other words, most continua are pseudo-arcs.

This theorem is quite reminiscent of one pertaining to $C[0,1]$, the space of continuous functions from $[0,1]$ to $\mathbb{R}$ endowed with the uniform topology[13]. At first, it seems like a nowhere differentiable but continuous function must be quite a pathological example, and even if one exists, there cannot be too many. Then, it turns out that nowhere differentiable functions are residual in $C[0,1]$: *most* continuous functions are nowhere differentiable!

In our case, at first it seems like indecomposable continua must be few and far between. In fact, in the first decade of 1900, Schoenflies made the assumption that every curve which is the boundary of two disjoint open sets in the plane is decomposable; as the pseudo-circle shows, this assumption was incorrect [13]. With Theorem 5.2, we know our intuition failed us: *hereditarily indecomposable* continua are everywhere.

Another interesting place where the pseudo-arc makes an appearance is in dynamics. For example, in [10], Handel constructs a smooth diffeomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ with the pseudo-circle as an attractor (a set which the points in the domain tend towards when f is applied iteratively). In [15], Kennedy and Yorke construct another smooth function with an invariant set[14] made of uncountably many pseudo-circles.

To conclude the paper, we would like to state an open problem in continuum theory in relation to the pseudo-arc.

Definition 5.3. A space A has the *fixed point property* if every continuous map $f\colon A \to A$ has a fixed point, i.e. there exists $x \in A$ such that $f(x) = x$.

Fixed point theorems are highly sought after: knowing that some continuous map has a point which is sent to itself can be very useful. Regarding planar continua, it is natural to ask which have the fixed point property and which do not. We can quickly notice that arcs have the fixed point property, but the circle does not. The following conjecture arises.

Conjecture 5.4. [8] *Does every nonseparating planar continuum have the fixed point property?*

By nonseparating we mean the complement is connected. If this conjecture were answered in the affirmative, it would be a strong generalization of the Brouwer fixed point theorem.

The pseudo-arc seems like a strong contender to be a counterexample to the conjecture. However, Hamilton [9] proved otherwise.

Theorem 5.5. [9, Cor] *The pseudo-arc has the fixed point property.*

Conjecture 5.4 remains open.

Acknowledgments

My mentor is a chainable, hereditarily indecomposable continuum. This example is interesting because it demonstrates that not all chainable, hereditarily indecomposable continua are nondegenerate.

[13]The uniform topology is generated by the metric $d(f,g) \coloneqq \min\{1, \sup_{x\in[0,1]} |f(x) - g(x)|$

[14]An invariant set is a set S such that, for all $x \in S$, the trajectory of x never leaves S.

My mentor Jonathan DeWitt was an incredible help to me while I was working on this paper. Jon gave me the topic as an option at the very beginning of the program, and it made for an extremely interesting project. I could not have wished for a better mentor.

I would also like to thank Professor Peter May for organizing this REU, as it gave me a great opportunity to dive deep into advanced mathematics and experience writing a paper first-hand.

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