Arnaud's Method in C^1 Billiards

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Abstract

We extract and generalize a central argument from [Arn13] to provide a framework for generic periodicity results in C^1 billiards. Applications of this method are demonstrated, including the existence of short periodic trajectories for generic C^1 tables.

Contents

1 Introduction

Broadly speaking, mathematical billiards are dynamical systems given by the motion of a free-moving point ("ball") in a bounded domain ("table") which reflects specularly ("bounces") off the boundary. As with many dynamical systems, questions of periodicity–i.e. about trajectories which repeat after a finite number of bounces—are among the most central and challenging in the study of billiards. Indeed, in [Bir27], where Birkhoff formally introduced the study of billiards in convex tables with smooth boundaries, he proved that all such tables admit at least two distinct periodic orbits of every period. On the other hand, it is unknown if every polygonal billiard table admits any periodic orbit, even in the case of triangles.

Billiards in strictly convex tables with C^∞ boundaries and billiards in polygons have been extensively studied, using very different methods. However, these techniques often fail for tables with less rigid boundary conditions, particularly non-convex, non-polygonal tables, and tables with less regular boundaries. Thus, there are far fewer results for such tables.

For tables with non-convex C^1 boundaries, one result is provided by M.-C. Arnaud in [Arn13]. In this paper, she shows that generically (for a topology discussed below), periodic trajectories are dense in a C^1 table. This result is particularly striking since it is unknown if non-convex C^1 tables always admit periodic orbits.

Moreover, a central argument which is sketched in [Arn13] shows that, while periodic trajectories don't always persist under C^1 perturbations, any C^1 table with a periodic orbit can be perturbed to a table where the periodic trajectory is stable. This fact provides a powerful tool for studying periodicity in C^1 billiards, particularly in showing that generic tables have a given type of periodic orbit.

The space of C^1 billiards and several primary results are introduced in §2, with further discussion in the Appendix. Then, we carefully excise the argument from [Arn13] in §3 and §4, culminating in Thm. 4.3. Applications of this argument, including the main theorem of [Arn13] and a result about short periodic trajectories, are given in §5.

2 C^1 Billiards: The Setting, and Basic Facts

In this paper we consider C^1 billiards. Formally, a C^1 billiard table is a connected, open subset of $\mathbb{R}^2$ with boundary C^1-diffeomorphic to the circle S^1. This is more or less the weakest reasonable setting for planar billiards "without pockets," as we need well-defined tangent vectors at the boundary against which billiard trajectories will satisfy the law of reflection.

Before we define our space of billiards, let us review the C^1 topology. Let M, N be C^1 manifolds with M compact, and let $C^1(M, N)$ be the set of C^1 maps from M to N.

We follow the standard convention of referring to $C^1(M, \mathbb{R})$ as $C^1(M)$.

A topology is generated on $C^1(M, N)$ from the subbasis of neighborhoods defined as follows: We pick a chart (U, ϕ) for M and (V, ψ) for N, a compact set $K \subset U$, a function $f \in C^1(M, N)$ such that $f(K) \subset V$, and an $\varepsilon > 0$. Then we take the subset of all $g \in C^1(M, N)$ such that $g(K) \subset V$ and

$$\left\|\psi \circ f(x) - \psi \circ g(x)\right\| + \left\|D_{\phi(x)}\left(\psi \circ f \circ \phi^{-1}\right) + D_{\phi(x)}\left(\psi \circ f \circ \phi^{-1}\right)\right\| < \varepsilon$$

for all $x \in K$.

For our purposes, N will always be $\mathbb{R}^n$ for some n. Since M is compact, $C^1(M, \mathbb{R}^n)$ can be given the structure of a normed vector space by taking a finite cover of charts (ϕ_i, U_i), and defining the norm

$$\|f\|_{C^1} = \max_{x \in M}\left\{\|f(x)\| + \min_{i,\, x \in U_i}\left\|D_{\phi_i(x)}\left(f \circ \phi_i^{-1}\right)\right\|\right\}.$$

One can check that this agrees with the topology above, and moreover that this space is complete, so $C^1(M, \mathbb{R}^n)$ is a Banach space.

We need two other standard facts about the C^1 topology: First, if M, N are C^k manifolds for $k \geq 1$, then $C^k(M, N)$, the set of C^k functions from M to N, is dense in $C^1(M, N)$. Second, if $\mathrm{Emb}^1(M, N)$ is the set of C^1 embeddings of M into N, then $\mathrm{Emb}^1(M, N)$ is an open subset of $C^1(M, N)$. Proofs can be found in [Hir76, §2], as well as numerous other useful results about C^1 topologies and the analogously defined C^k topologies.

We now define our space of C^1 billiard tables. From the Jordan Curve Theorem, any injective, continuous map of the circle into the plane bounds a connected open set, and thus all C^1 embeddings of the circle S^1 into the plane define a C^1 billiard table. One can observe that the behaviour of billiards within a table does not depend on the size of the table, but only on the shape. Thus it suffices to consider tables with unit length boundaries. Since the boundary of any such table is parameterized by a C^1 embedding of S^1, which we consider to have unit length, we can reparameterize the boundary by arc-length, i.e. with a unit speed parameterization, which simplifies calculations. Therefore, we will define our billiard space to be the set

$$\mathcal{B} = \left\{\beta \in \mathrm{Emb}^1\left(S^1, \mathbb{R}^2\right) \mid \forall t \in S^1, \left\|\beta'(t)\right\| = 1\right\}. \tag{2.1}$$

Recall that a *Baire space* is a topological space where countable intersections of open, dense sets are dense. Such an intersection is called a *generic* set, and is considered "large" in a Baire space because its complement has empty interior, and this remains true for countable intersections of generic sets. The Baire Category Theorem asserts that open subsets of complete metric spaces are Baire spaces.

It is straightforward to check that the subset $\mathcal{L}$ of $C^1(S^1, \mathbb{R}^2)$ consisting of unit speed loops is closed, and is thus a complete metric space since $C^1(S^1, \mathbb{R}^2)$ is complete. Since $\mathrm{Emb}^1\left(S^1, \mathbb{R}^2\right)$ is open in $C^1(S^1, \mathbb{R}^2)$, it follows that $\mathcal{B}$ is open in $\mathcal{L}$, and thus a Baire space.

In [Arn13], Arnaud works in a quotient of $\mathcal{B}$ as defined above which views billiard tables as independent of the parameterization of the boundary. In the Appendix, we discuss this space, and show how generic properties of our billiard space $\mathcal{B}$ remain generic under the quotient.

We now formalize billiards within a table. In the framework of discrete dynamical systems, billiards are studied as the dynamics of the *billiard map*, which is a function defined on the space of inward-pointing unit vectors along the boundary of the table, a space which is diffeomorphic to an open annulus. If we input an inward pointing vector v at a point b along the boundary, the billiard map takes us to the point b' where the ray in the direction of v from b next meets the boundary, and outputs the vector v' satisfying the law of reflection at b' with v. For strictly convex billiards, this map has many useful properties which aid in the analysis, in particular that it is a symplectomorphism for a symplectic structure on the annulus, and satisfies a certain important "twist" condition. This was key to Birkhoff's original work on billiards. See Birkhoff's work in [Bir27] as well as [Tab95][§1, 2] for a more recent treatment, and see [Gol01] for a more general discussion of such "symplectic twist maps."

Unfortunately, one can observe that in the setting of non-convex billiards, which we examine here, the billiard map is not even continuous in general. Instead, since we are only considering periodicity, it suffices to take the following, geometric approach to studying billiard trajectories, which is also fairly intuitive. We begin by defining some more general terminology.

Take $\beta \in \mathcal{B}$, and recall that β is an embedding of S^1 into $\mathbb{R}^2$.

Take an n-tuple points $(t_1, \ldots, t_n) \in \left(S^1\right)^n = \mathbb{T}^n$. We will think of these indices as being cyclically ordered, i.e. t_i for $i \in \mathbb{Z}/n\mathbb{Z}$. Then we say that $(t_1, \ldots, t_n)$ is a *valid n-gon* in β if for each $i = 1, \ldots, n+1$, $t_i \neq t_{i+1}$, and the line segment between $\beta(t_i)$ and $\beta(t_{i+1})$ lies completely inside the interior of the region bounded by β—only meeting the boundary at the endpoints—and moreover is not tangent to $\beta'(t_i)$ or $\beta'(t_{i+1})$. We say that $(t_1, \ldots, t_n)$ is a *glancing n-gon* in β if it satisfies the same conditions, except that the line segments are allowed to "glance" against the boundary. We say that $(t_1, \ldots, t_n)$ is an *invalid n-gon* in β if the segments still are not tangent to β at the endpoints, but may leave the interior of the table. See Fig. 1.

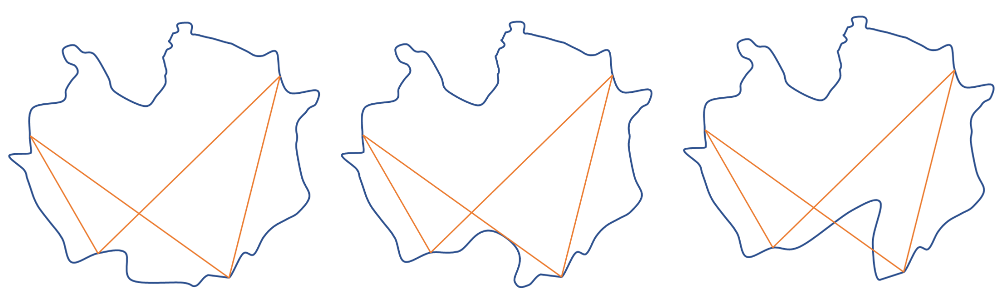
We then say that a valid n-gon $(t_1, \ldots, t_n)$ in β is a *valid n-periodic billiard orbit* in β if the segments also satisfies the law of reflection, "angle of incidence equals angle of reflection," at the boundary. The points $t_1, \ldots, t_n$ are called "bounce points" for the orbit. *Glancing n-periodic billiard orbit* and *invalid n-periodic billiard orbit* are defined correspondingly. See Fig. 1.

Remark 2.2. In the literature on non-convex billiards, there does not seem to be a consensus between whether the term "billiard orbit" includes glancing orbits, or only what we have called valid orbits. Invalid billiard orbits are always excluded. The divide is often is determined by whether one is studying billiards as flows, in which case allowing glancing orbits is more natural, or as discrete systems arising from the billiard map. While Thm. 4.3 only applies to valid orbits, a table with a glancing billiard orbit can be perturbed such that the orbit is valid, as discussed in Lemma 5.4.

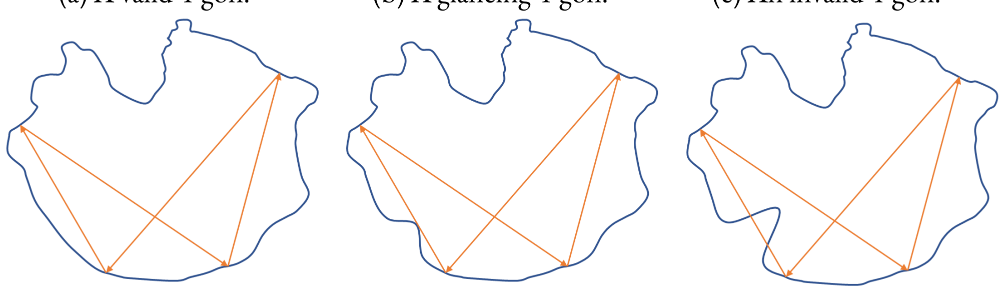
We can see that the law of reflection is equivalent to requiring that

$$\left\langle \beta'(t_i), \frac{\beta(t_i) - \beta(t_{i-1})}{\left\|\beta(t_i) - \beta(t_{i-1})\right\|} \right\rangle = \left\langle \beta'(t_i), \frac{\beta(t_{i+1}) - \beta(t_i)}{\left\|\beta(t_{i+1}) - \beta(t_i)\right\|} \right\rangle. \tag{2.3}$$

Note that, in general (2.3) also allows $\beta(t_{i-1})$, $\beta(t_i)$, and $\beta(t_{i+1})$ to be collinear in the case that $\frac{\beta(t_i)-\beta(t_{i-1})}{\left\|\beta(t_i)-\beta(t_{i-1})\right\|} = \frac{\beta(t_{i+1})-\beta(t_i)}{\left\|\beta(t_{i+1})-\beta(t_i)\right\|}$. This possible for invalid n-gons, but this can't happen for valid or glancing n-gons, as this would either cause a segment to leave the table or lie tangent to the table at a bounce point.



(a) A valid 4-gon. (b) A glancing 4-gon. (c) An invalid 4-gon.



(d) A valid 4-periodic billiard orbit. (e) A glancing 4-periodic billiard orbit. (f) An invalid 4-periodic billiard orbit.

Figure 1: Examples of valid, glancing, and invalid n-gons and billiard orbits. Observe the reflections for the billiard orbits.

From (2.3), we get a very useful property of billiard orbits, which is that they locally extremize the perimeter of the corresponding n-gon. This allows us to use critical point theory, which will be central to our analysis.

Proposition 2.4. Let $\beta \in \mathcal{B}$ be a C^1 billiard. Let $\mathbb{T}^n$ be the torus $\left(S^1\right)^n$. Define $L_n = L_n^\beta \colon \mathbb{T}^n \to \mathbb{R}$ by

$$\begin{aligned} L_n(t_1, \ldots, t_n) &= \left\|\beta(t_2) - \beta(t_1)\right\| + \cdots + \left\|\beta(t_n) - \beta(t_{n-1})\right\| + \left\|\beta(t_1) - \beta(t_n)\right\| \\ &= \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \left\|\beta(t_{i+1}) - \beta(t_i)\right\| \end{aligned} \tag{2.5}$$

Then $(t_1, \ldots, t_n) \in \mathbb{T}^n$ is a critical point of L_n if and only if, for any consecutive points t_{i-1}, t_i, t_{i+1}, either

(i) $\beta(t_{i-1}), \beta(t_i), \beta(t_{i+1})$ are collinear, or

(ii) the line from $\beta(t_{i-1})$ to $\beta(t_i)$ satisfies the law of reflection with the line from $\beta(t_i)$ to $\beta(t_{i-1})$.

In particular, if $(t_1, \ldots, t_n)$ is also a valid, glancing, or invalid n-gon in β, then it is the corresponding type of n-periodic billiard orbit.

Proof. We compute that

$$\begin{aligned} dL_n(t_1, \ldots, t_n) &= \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \left(\frac{\left\langle \beta'(t_i), \beta(t_i) - \beta(t_{i-1}) \right\rangle}{\left\|\beta(t_i) - \beta(t_{i-1})\right\|} - \frac{\left\langle \beta'(t_i), \beta(t_{i+1}) - \beta(t_i) \right\rangle}{\left\|\beta(t_{i+1}) - \beta(t_i)\right\|} \right) dx_i \\ &= \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \left\langle \beta'(t_i), \frac{\beta(t_i) - \beta(t_{i-1})}{\left\|\beta(t_i) - \beta(t_{i-1})\right\|} - \frac{\beta(t_{i+1}) - \beta(t_i)}{\left\|\beta(t_{i+1}) - \beta(t_i)\right\|} \right\rangle dx_i. \end{aligned} \tag{2.6}$$

Hence $(t_1, \ldots, t_n)$ is a critical point if and only if

$$\left\langle \beta'(t_i), \frac{\beta(t_i) - \beta(t_{i-1})}{\left\|\beta(t_i) - \beta(t_{i-1})\right\|} - \frac{\beta(t_{i+1}) - \beta(t_i)}{\left\|\beta(t_{i+1}) - \beta(t_i)\right\|} \right\rangle = 0$$

for each i. However, this is equivalent to (2.3), as desired. □

Corollary 2.7. If β is a strictly convex billiard table, then a critical point of L_n^β corresponds to an n-periodic billiard trajectory for β. Hence, such a billiard has an n-periodic trajectory, namely at the maximum of L_n.

Proof. The convexity guarantees that the trajectories stay within the table. One need only show that the maximum occurs away from points in the torus which correspond to sequences on the boundary where two adjacent points are the same (where L isn't differentiable). See [Tab05, §6], which moreover contains the proof of a famous stronger result which Birkhoff proved in [Bir27] using symplectic topology. □

3 A Morse-Theoretic Lemma

As periodic billiard trajectories are critical points of the length function for the table, a key step in our argument is showing that critical points persist locally under C^1 perturbations. In general, this is not true: consider, for example, the critical point at 0 of x^3. The requirement we need is that the critical point is *non-degenerate*, a property which thankfully is dense, even for length functions on C^1 billiards, as we will see later.

A critical point p of a C^2 function f from a manifold into $\mathbb{R}$ is non-degenerate if the Hessian matrix

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j}$$

of second derivatives is non-singular at p. This matrix is defined in coordinates, but it is straight-forward to check that the non-singularity of the Hessian is coordinate-independent.

Non-degenerate critical points play a central role in Morse Theory, where they provide topological information about the manifold on which they are defined. However, we can also intuit that non-degenerate critical points should be stable under perturbation by comparing x^2 to x^3. This one-dimensional case, though, is particularly simple because non-degenerate critical points are strict local extrema, which clearly persist under perturbation.

In [Arn13], Arnaud indicates a proof for the following result via Conley Index Theory applied to the gradient flow. A more topological argument comes from the fact that a non-degenerate critical point of $f : M \to \mathbb{R}$ is a transverse intersection of the differential $df : M \to T^*M$ with the zero-section in the cotangent bundle, and transverse intersections are stable. Results in this direction can be found in [Hir76]. However, the following proof uses only the elementary homological fact that the sphere is not contractible, in an argument similar to standard proofs of Brouwer's fixed point theorem (see i.e. [Hat01]). Observe also that a very similar proof is given for [Hir76, Thm. 1.7].

Lemma 3.1. Let M be a smooth n-dimensional manifold (compactness is not required), and let $f : M \to \mathbb{R}$ be a C^2 function with a non-degenerate critical point p. Then for any neighborhood U of p, there exists a neighborhood $\mathcal{N}$ of f in $C^1(M)$ such that every $g \in \mathcal{N}$ has at least one critical point in U.

Proof. Possibly passing to a smaller neighborhood of p, we can pick a coordinate chart sending U to a neighborhood V of $0 \in \mathbb{R}^n$, such that p maps to 0. Then in this chart, we view the differential df as a map from V to $\mathbb{R}^n$ which sends 0 to 0, and the Hessian is its differential. Since the Hessian is non-singular, we can find a neighborhood W of 0 such that the restriction of df to W is a diffeomorphism onto its image. We can change coordinates such that W contains D, the closed unit ball about 0, such that ∂D is the sphere $S = S^{n-1}$. Then, we can pick a neighborhood $\mathcal{N}$ about f in the C^1 topology such that, if we take any g in $\mathcal{N}$, then in the coordinates above $\left\|df(x) - dg(x)\right\| < \varepsilon$ for all $x \in D$.

Suppose $g \in \mathcal{N}$ does not have a critical point in U. Then, in particular, 0 does not belong to $dg(D)$. Since 0 belongs to the interior of D, and $df(0) = 0$, 0 also belongs to the interior of $df(D)$. Thus, if ε is sufficiently small, we can use bump functions to construct a continuous function $\omega : D \to df(W)$ such that ω agrees with dg near 0 and agrees with df away from 0. In particular, we can construct ω such that $\omega(D)$ does not contain 0, but ω agrees with df on the sphere S.

Then, $h : D \to S$,

$$h(x) = \frac{(df)^{-1} \circ \omega(x)}{\left\|(df)^{-1} \circ \omega(x)\right\|}$$

is a well-defined, continuous function from D to S which fixes the sphere. But this means $H : [0, 1] \times S \to S$

$$H(t, x) = h(tx)$$

is a homotopy contracting the sphere to a point, which is impossible. □

4 Arnaud's Method

We now formalize the central argument which Arnaud outlined in [Arn13] to show that generic C^1 billiards have dense periodic orbits. The argument, which we will call *Arnaud's Method,* says that near any table with a periodic orbit, we can find an open set with "similar" periodic orbits. In particular, this implies that if tables with some type of periodic orbit are dense in $\mathcal{B}$, they are in fact generic. This is especially useful for extending (often challenging) results about periodic orbits in different billiard settings to results for generic billiards in $\mathcal{B}$, as Arnaud does in [Arn13]. We will discuss this result and another application in §5.

We prove Arnaud's Method (Thm. 4.3) via the following series of lemmas. First, we show that valid n-gons are C^1 stable:

Lemma 4.1. Let $\beta \in \mathcal{B}$, and let $p = (t_1, \ldots, t_n) \in \mathbb{T}^n$ correspond to a valid n-gon for β (in particular, a billiard orbit). Then there exists a neighborhood $U \subset \mathbb{T}^n$ about p and a neighborhood $\mathcal{U} \subset \mathcal{B}$ about β such that, for any $q \in U$ and any $\alpha \in \mathcal{U}$, q corresponds to a valid n-gon for α.

Proof. We need to keep the lines between successive points from meeting the boundary anywhere else, and from being tangent to the boundary at these points. Hence, it will suffice to consider the case where $n = 2$, as for greater n we intersect the neighborhoods corresponding to each pair of successive points.

Since $\mathcal{B}$ has the C^1 topology, it is straightforward to show that the evaluation maps $E_0 : \mathcal{B} \times S^1 \to \mathbb{R}^2$ and $E_1 : \mathcal{B} \times S^1 \to \mathbb{R}^2$ given by

$$E_0(\alpha, x) = \alpha(x), \quad E_1(\alpha, x) = \alpha'(x)$$

are continuous (cf. [Hat01, p. 530]). Hence, if $D \subset \mathbb{T}^2$ is the diagonal $\{(x_1, x_2) \in \mathbb{T}^2 \mid x_1 = x_2\}$, then we have a continuous map $\Theta : \mathcal{B} \times (\mathbb{T}^2 \setminus D) \times S^1 \to \mathbb{R}$,

$$\Theta(\alpha, x_1, x_2, y) = \left(1 - \left\langle \frac{\alpha(x_2) - \alpha(x_1)}{||\alpha(x_2) - \alpha(x_1)||} \right\rangle^2, \alpha'(y)\right).$$

Observe that this gives the size of the component of $\alpha'(y)$ which is perpendicular to the line from $\alpha(x_1)$ to $\alpha(x_2)$, measuring how far the line is from being parallel to $\alpha'(y)$. Since $\beta(t_1), \beta(t_2)$ are part of a valid n-gon in β, we know that

$$\Theta(\beta, t_1, t_2, t_1),\ \Theta(\beta, t_1, t_2, t_2) > 0.$$

Thus, there exist $\varepsilon > 0$, open intervals $I_1, I_2 \subset S^1$ about t_1 and t_2, respectively, and a neighborhood $\mathcal{V} \subset \mathcal{B}$ of β such that, for all $x_1, y_1 \in I_1,\ x_2, y_2 \in I_2$, and all $\alpha \in \mathcal{V}$,

$$\Theta(\alpha, x_1, x_2, y_1),\ \Theta(\alpha, x_1, x_2, y_2) > \varepsilon.$$

Let λ be the line passing through $\alpha(x_1)$ and $\alpha(x_2)$. First, observe that the above bound on Θ keeps λ from being tangent to α at $\alpha(x_1)$ and $\alpha(x_2)$. Then, recall that α is a unit speed parameterization. Hence, we can observe that the path $\alpha(s)$ for $s \in I_i$, is bounded outside the cone of lines through $\alpha(x_i)$ which make an angle less than $\arcsin(\varepsilon)$ with λ. Therefore, within I_1 and I_2, α only intersects λ at x_1 and x_2. See Figure 2.

Thus, we only need to keep the line between our points from touching points further away on the boundary, i.e. outside of I_1 and I_2. As with Θ, we can observe that the function $\Delta : \mathcal{B} \times I_1 \times I_2 \to \mathbb{R}$ which maps α, x_1, x_2 to the distance between $\alpha\left(S^1 \setminus (I_1 \cap I_2)\right)$ and the line segment from $\alpha(x_1)$ to $\alpha(x_2)$ is continuous. Moreover, since the line from $\beta(t_1)$ to $\beta(t_2)$ doesn't meet β elsewhere, and $S^1 \setminus (I_1 \cap I_2)$ is compact,

$$\Delta(\beta, t_1, t_2) > 0.$$

Hence there exists a neighborhood $U \subset I_1 \times I_2$ about (t_1, t_2) and a neighborhood $\mathcal{U} \subset \mathcal{V}$ about β such that, for $(x_1, x_2) \in U,\ \alpha \in \mathcal{U}$

$$\Delta(\alpha, x_1, x_2) > 0.$$

This means that the line from $\alpha(x_1)$ to $\alpha(x_2)$ does not meet $\alpha\left(S^1 \setminus (I_1 \cap I_2)\right)$, and since $(x_1, x_2) \in I_1 \times I_2$, $\alpha(I_1)$ and $\alpha(I_2)$ only meet the line at $\alpha(x_1)$ and $\alpha(x_2)$.

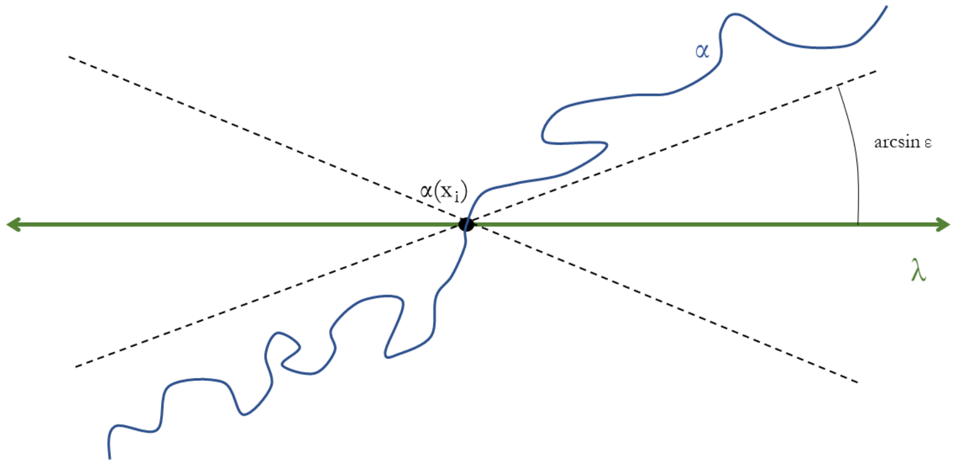


Figure 2: α is bounded outside of the cone about λ.

Finally, we might worry that the trajectory could go from lying fully inside the table to fully outside of the table. However, this can be avoided by considering the winding number of the boundary about, say, the midpoint of the trajectory (see e.g. [GP74, §2.5]), which can be written as a continuous function on a neighborhood $t_1, t_2 \in S^1$ and $\beta \in \mathcal{B}$ as per the other parts of this proof. This function must be locally constant since it is integer-valued.

□

The following lemma allows us to apply Lemma 3.1:

Lemma 4.2. In any neighborhood of a table $\beta \in \mathcal{B}$ with an n-periodic billiard orbit $(t_1, \ldots, t_n) \in \mathbb{T}^n$, there exists a table α with a periodic orbit $(s_1, \ldots, s_n)$ arbitrarily close to $(t_1, \ldots, t_n)$, such that $(s_1, \ldots, s_n)$ is a non-degenerate critical point of L^α.

Proof. From Lemma 4.1, so long as we can find a table as close as we desire to β with a non-degenerate critical point as close as we desire to $(t_1, \ldots, t_n)$, it will correspond to a valid billiard trajectory.

Since there are only finitely many points $t_1, \ldots, t_n$, a standard approximation argument allows us to find a C^2 (or smooth) map $\gamma : S^1 \to \mathbb{R}^2$ in any C^1 neighborhood of β which agrees with β up to the first derivative at $t_1, \ldots, t_n$.

Recall that

$$\left.\frac{\partial L_n^\gamma}{\partial x_i}\right|_{(t_1,\ldots,t_n)} = \left\langle \gamma'(t_i)\,, \frac{\gamma(t_i) - \gamma(t_{i-1})}{\|\gamma(t_i) - \gamma(t_{i-1})\|} - \frac{\gamma(t_{i+1}) - \gamma(t_i)}{\|\gamma(t_{i+1}) - \gamma(t_i)\|} \right\rangle.$$

Thus,

$$
\left.\frac{\partial^2 L_n^\gamma}{\partial x_i \partial x_j}\right|_{(t_1,\dots,t_n)} = \begin{cases}
\left\langle \gamma''(t_i)\,,\, \dfrac{\gamma(t_i)-\gamma(t_{i-1})}{\left\|\gamma(t_i)-\gamma(t_{i-1})\right\|} - \dfrac{\gamma(t_{i+1})-\gamma(t_i)}{\left\|\gamma(t_{i+1})-\gamma(t_i)\right\|} \right\rangle \\
\quad + \left\langle \gamma'(t_i)\,,\, \dfrac{\gamma'(t_i)}{\left\|\gamma(t_i)-\gamma(t_{i-1})\right\|} \right. \\
\qquad \left. - \dfrac{\left\langle \gamma'(t_i)\,,\, \gamma(t_i)-\gamma(t_{i-1}) \right\rangle}{\left\|\gamma(t_i)-\gamma(t_{i-1})\right\|^3} \Big(\gamma(t_i)-\gamma(t_{i-1})\Big) \right\rangle \\
\quad + \left\langle \gamma'(t_i)\,,\, \dfrac{\gamma'(t_i)}{\left\|\gamma(t_{i+1})-\gamma(t_i)\right\|} \right. \\
\qquad \left. + \dfrac{\left\langle \gamma'(t_i)\,,\gamma(t_{i+1})-\gamma(t_i) \right\rangle}{\left\|\gamma(t_{i+1})-\gamma(t_i)\right\|^3} \Big(\gamma(t_{i+1})-\gamma(t_i)\Big) \right\rangle, & i = j \\
\\
\left\langle \gamma'(t_i)\,,\, \dfrac{\left\langle \gamma'(t_j)\,,\gamma(t_i)-\gamma(t_j) \right\rangle}{\left\|\gamma(t_i)-\gamma(t_j)\right\|^3} \Big(\gamma(t_i)-\gamma(t_j)\Big) \right. \\
\quad \left. - \dfrac{\gamma'(t_j)}{\left\|\gamma(t_i)-\gamma(t_j)\right\|} \right\rangle, & j = i \pm 1 \\
\\
0, & j \neq i, i \pm 1.
\end{cases}
$$

The main observation about this formula is that it is a function of γ, γ', and γ'' at $t_1, \dots, t_n$, and the only term containing $\gamma''(t_i)$ is

$$
\left\langle \gamma''(t_i)\,,\, \frac{\gamma(t_i)-\gamma(t_{i-1})}{\left\|\gamma(t_i)-\gamma(t_{i-1})\right\|} - \frac{\gamma(t_{i+1})-\gamma(t_i)}{\left\|\gamma(t_{i+1})-\gamma(t_i)\right\|} \right\rangle
$$

in the formula for $\dfrac{\partial^2 L_n}{\partial x_i^2}$, on the diagonal of the Hessian matrix. Then we can perturb γ in the C^1 topology to $\alpha : S^1 \longrightarrow \mathbb{R}^2$ such that for each t_i, $\gamma(t_i) = \alpha(t_i)$, $\gamma'(t_i) = \alpha'(t_i)$, and $\gamma''(t_i)$ is whatever we desire. As noted previously,

$$
\frac{\gamma(t_i)-\gamma(t_{i-1})}{\left\|\gamma(t_i)-\gamma(t_{i-1})\right\|} - \frac{\gamma(t_{i+1})-\gamma(t_i)}{\left\|\gamma(t_{i+1})-\gamma(t_i)\right\|} = 0
$$

if and only if $\gamma(t_i)$ lies on the line between $\gamma(t_{i-1})$ and $\gamma(t_{i+1})$. This is not allowed for a valid billiard trajectory, since it would mean the line would be tangent to $\gamma'(t_i)$ at $\gamma(t_i)$.

Therefore, we can control the diagonal of the Hessian for α while leaving the rest the same as for γ, thereby making the Hessian non-singular so that $(t_1, \ldots, t_n)$ is a non-degenerate critical point for L_n^α.

The attentive reader may have noted that γ and α were defined as C^1 perturbations of β, but not necessarily maps in $\mathcal{B}$. Indeed, it is not necessarily possible to perform these perturbations that fixed the maps up to the first derivative at each t_i while maintaining that α has unit length and is parameterized by unit speed (one can consider what happens if α has segments which are straight lines). However, to fix this, one can conformally scale α and reparameterize by length to get a map $\hat{\alpha} \in \mathcal{B}$ with the desired dynamics. It is clear that we can force α to have a length arbitrarily close to 1, so we can force $\hat{\alpha}$ to remain as close as we like to β, and moreover, under the reparameterization to unit speed, the bounce points will shift to $(s_1, \ldots, s_n)$ as close to $(t_1, \ldots, t_n)$ as we desire.

□

We are now ready for Arnaud's Method:

Theorem 4.3 (Arnaud's Method)**.** Let $\beta \in \mathcal{B}$, and let $p = (t_1, \ldots, t_n) \in \mathbb{T}^n$ correspond to a periodic billiard trajectory in β. Then for any neighborhood $V \subset \mathbb{T}^n$ of p and any neighborhood $\mathcal{U} \subset \mathcal{B}$ of β, there exists a non-empty open set $\mathcal{G} \subset \mathcal{U}$ such that, for any $\alpha \in \mathcal{G}$, there exists $q \in V$ corresponding to a periodic billiard trajectory in α.

Proof. First, we use Lemma 4.2 to find $\hat{\beta} \in \mathcal{U}$ for which p still corresponds to a periodic billiard trajectory, but is now a non-degenerate critical point of $L_n^{\hat{\beta}}$. Then we apply Lemma 4.1 to get a neighborhood $U \subset V$ of p and $\mathcal{U}' \subset \mathcal{U}$ of $\hat{\beta}$ such that every point in U corresponds to a valid n-gon in $\mathcal{U}'$.

Next, we apply Lemma 3.1 to get a neighborhood $\mathcal{N} \subset C^1(\mathbb{T}^n)$ of $L_n^{\hat{\beta}}$ such that every $f \in \mathcal{N}$ has a critical point in U.

Observe (i.e. by (2.5) and (2.6)) that a perturbation of a table in $\mathcal{B}$ corresponds to a C^1 perturbation of the corresponding length function. Precisely, this means we can find a neighborhood $\mathcal{G} \subset \mathcal{U}'$ of $\hat{\beta}$ such that, if $\alpha \in \mathcal{G}$, $L_n^\alpha \in \mathcal{N}$.

Thus, for $\alpha \in \mathcal{G}$, L_n^α has a critical point in $q \in U$. Since q corresponds to a valid n-gon in α, this means q corresponds to a valid n-periodic billiard trajectory for α, as desired.

□

5 Applications

We conclude with applications of Arnaud's Method. We begin with Arnaud's result in [Arn13], which showed that generic C^1 tables have dense periodic orbits, which in this case means that for generic $\beta \in \mathcal{B}$, the points $t \in S^1$ which belong to periodic billiard orbits for β are dense in S^1. The argument relies on a corresponding (and much more challenging) result from rational polygonal billiards:

Theorem 5.1 ([Bos+98]). Periodic billiard orbits are dense in rational polygons.

The method is then to approximate C^1 tables by rational polygons, outlined as follows:

Theorem 5.2 ([Arn13]). Generic tables in $\mathcal{B}$ have dense periodic orbits.

Proof. Arnaud's proof is more or less as follows:

First, let $\{U_i\}$ be a countable basis for $\mathbb{R}^2$. For each U_i, we consider the sets $\mathcal{Q}_i \subset \mathcal{B}$ which consist of tables β such that either β doesn't enter U_i or there is bounce point of a periodic orbit for β in U_i. We see, then, that $\bigcap_i \mathcal{Q}_i$ is exactly the set of tables with dense periodic orbits. Our goal is to show that each $\mathcal{Q}_i$ contains an open, dense subset of $\mathcal{B}$.

$\mathcal{Q}_i$ contains every table that does not enter U_i. Hence we only need to show that $\mathcal{Q}_i$ contains an open subset of $\mathcal{B}$ near every table which does enter U_i. Let β be such a table.

Take $\varepsilon > 0$. It is fairly straightforward to show that we can find a unit-length rational polygon P (considered as a piecewise-smooth map from S^1 into $\mathbb{R}^2$) such that $\left\|P(t) - \beta(t)\right\| < \frac{\varepsilon}{4}$ for all $t \in S^1$, and $\left\|P'(t) - \beta'(t)\right\| < \frac{\varepsilon}{4}$ except at the vertices $v_1, \ldots, v_N \in S^1$. For this to be true, P' can only differ from $\beta'(t_i)$ by at most $\frac{\varepsilon}{4}$ on either side of a vertex v_i. Thus, we can "round" the corners of P arbitrarily close to the vertices to get a table $\alpha \in \mathcal{B}$ such that $\left\|\alpha(t) - \beta(t)\right\|$, $\left\|\alpha'(t) - \beta'(t)\right\| < \frac{\varepsilon}{2}$ for all $t \in S^1$, i.e. $\left\|\beta - \alpha\right\|_{C^1} < \varepsilon$, and such that P agrees with α except on a neighborhood of the vertex as small as we desire.

With ε small enough, we can require that P enters U_i. Since P is a rational polygon, it has a periodic orbit $(t_1, \ldots, t_n)$ which enters U_i. As a polygonal billiard trajectory, this orbit cannot hit a vertex. Thus, we can require that α agree with P except on neighborhoods which exclude the bounce points $t_1, \ldots, t_n$. Moreover, since the orbit was a valid trajectory in P, we can keep α close enough to P such that the rounded corners won't cross the lines of the trajectory.

Thus, we get that $(t_1, \ldots, t_n)$ is also a valid periodic orbit for α, so $\alpha \in \mathcal{Q}_i$. Since $\alpha \in \mathcal{B}$, we can apply Arnaud's Method (Thm. 4.3) to get an open neighborhood of $\mathcal{B}$ about α which is contained in $\mathcal{Q}_i$. Since we could make α as close to β in $\mathcal{B}$ as we desired, this shows that $\mathcal{Q}_i$ contains an open, dense subset of $\mathcal{B}$, as required.

□

We now provide another application, which is to show that generic C^1 billiard tables have either a 2 or a 3 periodic orbit, using a related result for C^2 billiards.

Theorem 5.3 ([BG89]). Every C^2 billiard table has a glancing or valid 2 or 3-periodic trajectory.

The proof of Thm. 5.3 given in [BG89] uses some rather intense Palais-Smale theory to approximate billiard trajectories as Lagrangian flows with potential that diverges at the boundary of the table. An updated proof with other related results can be found in [Iri14]. In order to apply Arnaud's method, we need the fact that tables with glancing 2- and 3-periodic orbits can be perturbed to have valid orbits.

Lemma 5.4. Suppose $(x_1, \ldots, x_n)$ is a glancing n-periodic orbit for $\beta \in \mathcal{B}$, with $n = 2, 3$. Then there exists $\alpha \in \mathcal{B}$ arbitrarily close to β and $(y_1, \ldots, y_n)$ arbitrarily close to $(x_1, \ldots, x_n)$ such that $(y_1, \ldots, y_n)$ is a valid n-periodic orbit for α.

Proof. Let $\Gamma_i \subset S^1$ be the set of points "glancing" against the segment σ_i between $\beta(x_i)$ and $\beta(x_{i+1})$, i.e. the points t such that σ_i is tangent to β at t. Observe that, since we do not allow σ_i to be tangent to β at x_i or x_{i+1}, Γ_i is bounded away from x_i, x_{i+1}. Then, since $\Gamma_i = \beta^{-1}(\sigma_i) \setminus \{x_i, x_{i+1}\}$ which is a compact set minus two isolated points, and is this also compact. Moreover, since $n = 2$ or 3, if there is a third point x_j, it does not lie on the segment, and thus does not belong to Γ_i. Thus, the set of all "glance points" $\Gamma = \bigcup_{i=1}^{n} \Gamma_i$ is a compact subset of S_1 which is does not contain any x_i.

Let η be the unit exterior normal map corresponding to β, i.e. the map such that $\alpha(t)$ is a unit vector pointing outwards at $\beta(t)$ which is perpendicular to $\beta'(t)$. Observe that η is continuous. Then since Γ is compact, we can take a smooth bump function ϕ and find a sufficiently small ε such that, if $\alpha : S^1 \to \mathbb{R}^2$ is given by $\alpha(t) = \beta(t) + \varepsilon\phi(t)\eta(t)$, then $\alpha(t) = \beta(t)$ around the x_i and away from Γ, and $\alpha(t)$ lies in the exterior of β around Γ.

It follows that $(x_1, \ldots, x_n)$ is still a billiard orbit for α, but we have pushed all of the glancing points away, so this orbit is valid. Moreover, we can make α as close to β in the C^1 topology as we like. However, there is no guarantee that α still has unit length and is parameterized by unit speed. Observe, though, that we can control the change in length via ε. Thus, similar to the proof of Lemma 4.2, we can scale and reparameterize to find the desired table in $\mathcal{B}$ close to β.

□

Theorem 5.5. Generic tables in $\mathcal{B}$ have valid 2 or 3-periodic orbits.

Proof. C^2 tables are dense in $\mathcal{B}$. Thm. 5.3 gives at least a glancing 2 or 3-periodic orbit for each C^2 table, and perturbing these via Lemma 5.4 gives a dense set of tables in $\mathcal{B}$ with valid 2- or 3-periodic orbits. Hence, we apply Arnaud's Method (Thm. 4.3) to these tables to produce a dense, open set in $\mathcal{B}$ with valid 2- or 3-periodic orbits. □

Appendix: Quotiented Spaces of Billiard Tables

Here we discuss the definition of the billiard space used in [Arn13], which is a quotient of the billiard space $\mathcal{B}$ defined in §2. Recall that $\mathrm{Emb}^1(S^1, \mathbb{R}^2)$ is the set of C^1 embeddings of the circle into the plane, which is an open subset of the Banach space $C^1(S^1, \mathbb{R}^2)$, and that $\mathcal{L} \subset C^1(S^1, \mathbb{R}^2)$ is the set of unit-speed loops, which is closed in $C^1(S^1, \mathbb{R}^2)$, and thus a complete metric space. In §2, we defined $\mathcal{B}$ as the intersection of $\mathrm{Emb}^1(S^1, \mathbb{R}^2)$ and $\mathcal{L}$.

In [Arn13], Arnaud defines a smaller space as follows: First, we define an equivalence relation $\sim$ on $\mathcal{L}$ by identifying any two maps which agree up to pre-composition with an isometry of S^1, i.e. if we think of S^1 as $\mathbb{R}/\mathbb{Z}$, then $\alpha \sim \beta$ if $\alpha(x) = \beta(\pm(x+\tau))$, for $\tau \in \mathbb{R}/\mathbb{Z}$. We denote the equivalence class of $\alpha \in \mathcal{L}$ by $[\alpha]$. If d is the metric $\mathcal{L}$ inherits from $C^1(S^1, \mathbb{R}^2)$, then we define $\hat{d}$ on the set of equivalence classes, $\mathcal{L}/\sim$, by

$$\hat{d}\left([\alpha], [\beta]\right) = \inf_{\alpha\in[\alpha],\beta\in[\beta]} \left\{d(\alpha, \beta)\right\}.$$

Arnaud's billiard space, which we call $\widetilde{\mathcal{B}}$, is the image of $\mathcal{B}$ under the quotient.

In general, a quotient metric such as $\hat{d}$ may not be a metric. For example, the quotient metric may not satisfy the triangle inequality. Even under a standard modification of the above formula to resolve this (see [BBI01, §3]), for some equivalence relations it is only possible to obtain a compatible pseudometric, for which the distance between distinct equivalence classes is allowed to be 0. Finally, even when $\hat{d}$ is a valid metric on the set of equivalence classes, this metric may not induce the standard quotient topology.

However, in this case $\hat{d}$ is a metric which induces the quotient topology on $\mathcal{L}/\sim$. The key observations to prove this are as follows: First, each equivalence class, as a subset of $\mathcal{L}$, is homeomorphic to the union of two circles—one for isometries which preserve the orientation of S^1, one for those which reverse it—and in particular is compact. Thus for any classes $[\alpha]$ and $[\beta]$, there are $\overline{\alpha} \in [\alpha]$ and $\overline{\beta} \in [\beta]$ such that

$$\hat{d}\left([\alpha], [\beta]\right) = d\left(\overline{\alpha}, \overline{\beta}\right).$$

But then we observe that, for any isometry ϕ of S^1,

$$d\left(\alpha \circ \phi, \beta \circ \phi\right) = d\left(\alpha, \beta\right) = \hat{d}\left([\alpha], [\beta]\right). \tag{5.6}$$

Using this fact, it is fairly straightforward to show that $\hat{d}$ is a metric on $\mathcal{L}/\sim$ which induces the quotient topology. In particular, this means that open subsets of $\mathcal{L}$ project to open subsets of $\mathcal{L}/\sim$, and thus $\widetilde{\mathcal{B}}$ is open in $\mathcal{L}/\sim$. The observations above and the fact that $\mathcal{L}$ is complete also allows us to show that $\hat{d}$ is a complete metric on $\mathcal{L}/\sim$. Thus, as an open subset of a complete metric space, $\widetilde{\mathcal{B}}$ is a Baire space. Moreover, since $\mathcal{L}/\sim$ has the quotient topology, we can see that open dense sets project to open dense sets, and it follows that generic sets project to generic sets. Therefore, the results proven in this paper for $\mathcal{B}$ apply to $\widetilde{\mathcal{B}}$.

Finally, it is not hard to repeat all of this analysis for the space where we also identify maps which agree after composition with affine isometries of $\mathbb{R}^2$. In this case, the equivalence classes are not compact (they are homeomorphic to $O(2) \times \mathbb{R}^2$), but we still have the property that every point in one equivalence class is a closest point to another, giving an analogue of (5.6). In any case, what this means is that our analysis still agrees with the principle that only the "shape" of the billiard table matters.

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