AN INTRODUCTION TO THE BURAU REPRESENTATION OF THE ARTIN BRAID GROUP

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Abstract. This paper is an introduction to homological representations of the braid group. First, we introduce the classical definition of the Artin braid group and discuss some geometric interpretations. In particular, we consider the braid group as the group of isotopy classes of self-homeomorphisms of a punctured disk. Finally, we introduce the Burau representation and prove that this representation is not faithful for $n > 5$.

Contents

1. The Artin Braid Group

Braid groups form a rich class of mathematical objects and lend themselves to a variety of interesting geometric interpretations, some of which will later be explored in this paper. In this section, we begin by presenting the braid group in terms of generators and relations as originally defined by Emil Artin in [1].

Definition 1.1. The *Artin braid group* B_n is the finitely generated group with $n-1$ generators $\sigma_1, \sigma_2, ..., \sigma_{n-1}$ satisfying the following relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } i, j \in \{1, ..., n-1\} \text{ with } |i-j| \geq 2$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } i \in \{1, ..., n-2\}$$

The relations above are referred to as the *braid relations*. An element of B_n is called a *braid*. By definition, $B_1 = \{1\}$ is trivial. The group B_2 only has a single generator σ_1 and no relations, so it is isomorphic to $\mathbb{Z}$. For $n \geq 3$, B_n is nonabelian. To see this, we first prove the following lemma:

Lemma 1.2. *Let $f : B_n \to G$ be a group homomorphism from B_n to a group G. Then the elements $\{a_i = f(\sigma_i)\}_{i=1}^{n}$ satisfy the braid relations. Conversely, if some $\{a_1, ..., a_{n-1}\} \in G$ satisfy the braid relations, then there exists a unique group homomorphism $f : B_n \to G$ such that $a_i = f(\sigma_i)$ for any $i \in \{1, ..., n-1\}$.*

Proof. The first direction only requires a quick verification. Given $f : B_n \to G$ a group homomorphism and $i, j \in \{1, ..., n-1\}$, we have

$$a_i a_j = f(\sigma_i)f(\sigma_j) = f(\sigma_i \sigma_j) = f(\sigma_j \sigma_i) = f(\sigma_j)f(\sigma_i) = a_j a_i,$$

and for $i \in \{1, ..., n-2\}$ we have

$$a_i a_{i+1} a_i = f(\sigma_i)f(\sigma_{i+1})f(\sigma_i) = f(\sigma_{i+1})f(\sigma_i)f(\sigma_{i+1}) = a_{i+1} a_i a_{i+1}.$$

For the other direction, let $\mathbb{F}_n$ denote the free group generated by $\{\sigma_1, ..., \sigma_{n-1}\}$. Suppose $\{a_1, ..., a_{n-1}\} \in G$ satisfy the braid relations. Then there exists a unique group homomorphism $\tilde{f} : \mathbb{F}_n \to G$ such that $\tilde{f}(\sigma_i) = a_i$ for all $i \in \{1, ..., n-1\}$. Note because $\{a_i\}_{i=1}^{n-1}$ satisfy the braid relations,

$$\tilde{f}(\sigma_i \sigma_j) = \tilde{f}(\sigma_i)\tilde{f}(\sigma_j) = a_i a_j = a_j a_i = \tilde{f}(\sigma_j)\tilde{f}(\sigma_i) = \tilde{f}(\sigma_j \sigma_i)$$

for all $i, j \in \{1, ..., n-1\}$. Moreover, for any $i \in \{1, ..., n-2\}$

$$\tilde{f}(\sigma_i \sigma_{i+1} \sigma_i) = a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} = \tilde{f}(\sigma_{i+1} \sigma_i \sigma_{i+1})$$

Thus $\tilde{f}$ induces a unique group homomorphism $f : B_n \to G$ such that $a_i = f(\sigma_i)$ for all $i \in \{1, ..., n-1\}$. ☐

We can now show that B_n is nonabelian for $n \geq 3$:

Lemma 1.3. *The braid group B_n is nonabelian for $n \geq 3$.*

Proof. Let S_n denote the symmetric group whose elements consist of permutations on the set $\{1, ..., n\}$. S_n is generated by $n-1$ simple transpositions $\{s_i\}_{i=1}^{n-1}$, where s_i is the permutation that just exchanges i and $i+1$. Notice that $\{s_i\}_{i=1}^{n-1}$ satisfy the braid relations. Hence by the previous lemma there exists a group homomorphism $f : B_n \to S_n$ such that $s_i = f(\sigma_i)$ for all $i \in \{1, ..., n-1\}$. Because the simple transpositions generate S_n, this homomorphism is surjective. Since S_n is nonabelian for $n \geq 3$, so is B_n. ☐

Looking at the braid relations, we see that we can define a group homomorphism $\imath : B_n \to B_{n+1}$ where $\imath(\sigma_i) = \sigma_i$ for all $i \in \{1, ..., n-1\}$. We refer to this homomorphism as the *natural inclusion*, and in the next section, we will see it is injective.

2. Geometric Braids

Braid groups have a variety of geometric interpretation interpretations, one of which we will now discuss. When we picture a braid, we imagine strands that weave around each other without intersecting. The following definition makes this notion precise:

Definition 2.1. A *geometric braid on n strings* is a subset $\beta \subset \mathbb{R}^2 \times [0,1]$ consisting of n disjoint topological intervals (called the strings of β) such that the projection $\mathbb{R}^2 \times [0,1] \to [0,1]$ maps each string homeomorphically onto $[0,1]$. Moreover, β satisfies:

$$(1)\ \beta \cap (\mathbb{R}^2 \times \{0\}) = \{(1,0,0), (2,0,0), ..., (n,0,0)\}$$
$$(2)\ \beta \cap (\mathbb{R}^2 \times \{1\}) = \{(1,0,1), (2,0,1), ..., (n,0,1)\}$$

Each string of β starts from some point $(i,0,0)$ and ends at some $(s(i),0,1)$ where $i, s(i) \in \{1,2,...,n\}$. By looking at where each string ends up, we obtain a permutation $(s(1), s(2), ..., s(n))$ of the set $\{1,2,...,n\}$, referred to as the *underlying permutation* of β. A braid with a trivial underlying permutation (i.e. $(s(1), s(2), ..., s(n)) = (1,2,...,n)$) is called *pure*.

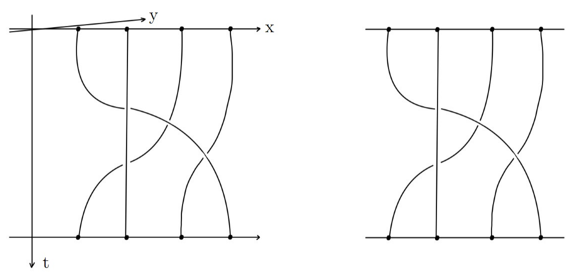

FIGURE 1. A geometric braid on four strings (left), the corresponding braid diagram (right).

On the left of Figure 1 is an example of a geometric braid on four strings. We can imagine projecting a geometric braid onto $\mathbb{R} \times [0,1]$ and indicating which string crosses over the other at any intersection. This is called a *braid diagram.* Any braid diagram d also represents some associated geometric braid β. The braid diagram of the geometric braid in Figure 1 is displayed on the right.

Looking at a geometric braid, we might ask ourselves how much we can deform the braid before we stop considering it "the same". More formally, we introduce the following definition:

Definition 2.2. Two geometric braids β_1 and β_2 are *isotopic* if there exists a continuous map $F : \beta_1 \times [0,1] \to \mathbb{R}^2 \times [0,1]$ such that:

$$F(x,0) = id_{\beta_1}$$
$$F(x,1) = id_{\beta_2},$$

and at any point $t \in [0,1], F_t : \beta_1 \to \mathbb{R}^2 \times [0,1]$ defines a geometric braid.

Intuitively, this definition is saying two geometric braids are isotopic if we can continuously deform one into the other in such a way that at any point along the deformation, we have a geometric braid. Two braid diagrams d_1, d_2 are isotopic if and only if their associated braids, β_1, β_2, are isotopic.

We can define multiplication of two geometric braids as follows: Given two geometric braids β_1, β_2, consider the associated braid diagrams d_1, d_2 respectively. The product $d_1 d_2$ is obtained by attaching d_2 to the bottom of d_1 and then compressing the diagram to fit in $\mathbb{R} \times [0,1]$. The product $d_1 d_1$ represents the product of the geometric braids, $\beta_1\beta_2$.

Let $\mathcal{B}_n$ denote the set of all geometric braids on n strings. The following lemma shows that any geometric braid $\beta \in \mathcal{B}_n$ has a two sided inverse $\beta^{-1} \in \mathcal{B}_n$, implying $\mathcal{B}_n$ is actually a group.

Lemma 2.3. *Every geometric braid $\beta \in \mathcal{B}_n$ has a two sided inverse $\beta^{-1} \in \mathcal{B}_n$.*

Proof. Start by defining braids $\sigma_i^{+1}, \sigma_i^{-1}$ for each $i \in \{1, 2, ..., n-1\}$. Let σ_i^{+1} be represented by the diagram where the only intersection is the i-th string crossing under the $(i+1)$-th string, and σ_i^{-1} be represented by the diagram where the only intersection is the i-th string crossing over the $(i+1)$-th string (see Figure 2).

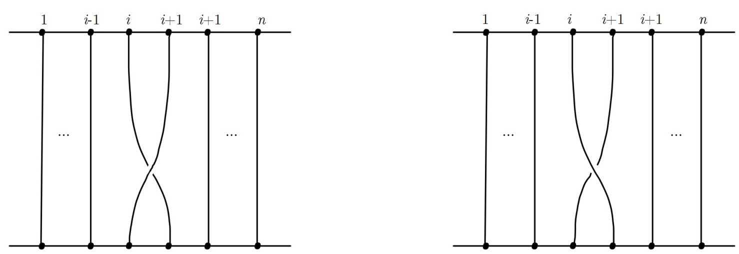

FIGURE 2. The generator σ_i^{+1} (left), the generator σ_i^{-1} (right).

Notice that $\sigma_i^{+1}\sigma_i^{-1} = \sigma_i^{-1}\sigma_i^{+1} = 1$ for any i. We claim that $\sigma_1^{+1}, ..., \sigma_{n-1}^{+1}, \sigma_1^{-1}, ..., \sigma_{n-1}^{-1}$ generate $\mathcal{B}_n$.

To see this, let $\beta \in \mathcal{B}_n$ and consider an associated braid diagram d. Then by slightly deforming neighborhoods of each crossing in β, d is isotopic to a diagram d' such that if the strings of d' have k crossings occurring at points $(x_1, s_1), ..., (x_k, s_k) \in \mathbb{R} \times [0,1]$, the second coordinates $s_1, ..., s_k \in [0,1]$ are all distinct. Then we may choose $t_0, ..., t_k \in [0,1]$ such that

$$0 = t_0 < t_1 < ... < t_k = 1$$

and the intersection $d \cap (\mathbb{R} \times [t_j, t_{j+1}])$ for $j \in \{0, ..., k-1\}$ contains exactly one crossing. Then each intersection $d \cap (\mathbb{R} \times [t_j, t_{j+1}])$ is a diagram of either σ_i^{+1} or σ_i^{-1} for some $i \in \{1, ..., n-1\}$. Hence we can write d' as the product

$$d' = \sigma_{i_1}^{\varepsilon_1}\sigma_{i_2}^{\varepsilon_2}...\sigma_{i_k}^{\varepsilon_k}$$

where $\varepsilon_1, ..., \varepsilon_k = \pm 1$ and $i_1, ..., i_k \in \{1, ..., n-1\}$. We know d' has an associated geometric braid β', and because d' is isotopic to d, β' is isotopic to β. Let β^{-1} be the geometric braid associated to the diagram $d'^{-1} = \sigma_{i_1}^{-\varepsilon_1}\sigma_{i_2}^{-\varepsilon_2}...\sigma_{i_k}^{-\varepsilon_k}$. Then β^{-1} is a two-sided inverse of β', and therefore also of β in $\mathcal{B}_n$. □

Thus, we see $\mathcal{B}_n$ forms a group. In fact, as the following theorem states, this group is actually isomorphic to the Artin braid group B_n that we defined in Section 1:

Theorem 2.4. *Let $\varepsilon = \pm 1$. Then there exists a unique isomorphism*

$$\phi_\varepsilon : B_n \to \mathcal{B}_n.$$

We will not prove this theorem in this paper. For a proof, see [7]. However, the following figure provides some geometric intuition behind the connection between the Artin braid group and geometric braids. For simplicity, we have taken the case where $n = 4$ and $\varepsilon = +1$. As Figure 3 suggests, the generators σ_i satisfy the braid relations.

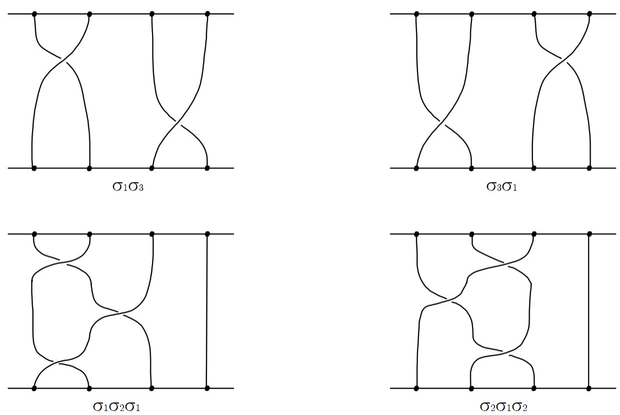

FIGURE 3. The relation $\sigma_1\sigma_3 = \sigma_3\sigma_1$ (top), the relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ (bottom).

Moreover, we can now understand why the natural inclusion $\imath : B_n \to B_{n+1}$ is injective:

Lemma 2.5. *The natural inclusion $\imath : B_n \to B_{n+1}$ is injective for all n*

Proof. Associate $\mathcal{B}_n$ with B_n via the isomorphism in Theorem 2.4. Then, given a geometric braid β, $\imath : B_n \to B_{n+1}$ adds to β a vertical string that is completely unlinked from the original strings of β, forming a braid on $n+1$ strings. Denote this resulting geometric braid on $n+1$ strings by $\imath(\beta)$. Then given two geometric braids β_1, β_2 on n strings such that $\imath(\beta_1)$ is isotopic to $\imath(\beta_2)$, restricting the isotopy to the first n strings yields an isotopy of β_1 into β_2. Hence, $\imath$ is injective. □

3. Self-Homeomorphisms and Mapping Class Groups

We now turn our attention to another geometric interpretation of braid groups. We will view braids as isotopy classes of self-homeomorphisms of a 2-dimensional disk. For the remainder of this section, let M denote an oriented topological manifold, potentially with boundary ∂M, and let $Q \subset M^\circ$ be a finite subset of the interior of M.

Definition 3.1. A *self-homeomorphism* of the pair (M, Q) is a homeomorphism $f : M \to M$ such that:

$$f(x) = x \text{ for all } x \in \partial M$$
$$f(Q) = Q$$

These conditions are simply saying that f fixes the ∂M pointwise and Q setwise.

Supposing Q consists of n points $\{q_1, q_2, ..., q_n\}$, notice any self-homeomorphism f of (M, Q) induces a (potentially trivial) permutation $\{f(q_1), f(q_2), ..., f(q_n)\}$ on Q.

Definition 3.2. Two self-homeomorphisms f_0, f_1 of (M, Q) are *isotopic* if there exist self-homeomorphisms f_t for $t \in (0, 1)$ of (M, Q) such that the map $M \times [0, 1] \to M$ sending $(x, t) \to f_t(x)$ is continuous.

Definition 3.3. The *mapping class group* $\mathfrak{M}(M, Q)$ of (M, Q) is defined as the group of isotopy classes of self-homeomorphisms of (M, Q), with composition as the group operation: for $f, g \in \mathfrak{M}(M, Q)$, $fg = f \circ g$.

Of particular interest is the case where M is a 2-disk and Q consists of n points of the interior of M, M°. In this case, the associated mapping class group $\mathfrak{M}(M, Q)$ is isomorphic to the braid group B_n. To build some theory as to why this is true, we introduce the following definitions:

Definition 3.4. A *spanning arc* on (M, Q) is a subset $\alpha \subset M$ which is homeomorphic to [0, 1] and disjoint from $Q \cup \partial M$ except at its endpoints, which lie in Q. A spanning arc is *simple* if it contains no self-intersections. Two spanning arcs α_1, α_2 are *isotopic* if there exists a self-homeomorphism $f : (M, Q) \to (M, Q)$ which sends α_1 to α_2, is isotopic to the identity on (M, Q), and is the identity on Q.

For the remainder of this paper, all spanning arcs considered are assumed to be simple.

Definition 3.5. Let $\alpha \subset M$ be a spanning arc on (M, Q). The *half-twist* τ_α associated to α is defined as follows: Pick a small neighborhood U_α of α disjoint from the other points in Q, and identify it with the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ so that the orientation of M is preserved. Then $\tau_\alpha : M \to M$ is the identity outside of U_α. Inside U_α, τ_α sends $z \in \mathbb{C}$ with $|z| \leq 1/2$ to $-z$, and sends $z \in \mathbb{C}$ with $1/2 < |z| < 1$ to $\exp(-2\pi i|z|)z$.

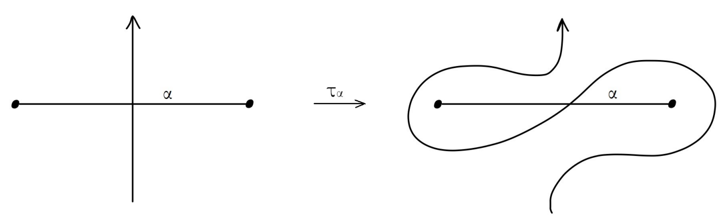

FIGURE 4. The action of τ_α on a line transversely intersecting α at one point.

Notice that τ_α, considered as an isotopy class in $\mathfrak{M}(M, Q)$, does not depend on the choice of U_α. Next, we state a few properties of half-twists that hint at the connection to the braid group B_n:

(i) Let $f : (M_1, Q_1) \to (M_2, Q_2)$ be an orientation-preserving homeomorphism. Let $\alpha \subset M_1$ be a spanning arc on (M_1, Q_1). Then $f(\alpha)$ is a spanning arc on (M_2, Q_2) and $\tau_{f(\alpha)} = f \tau_\alpha f^{-1} \in \mathfrak{M}(M_2, Q_2)$.

(ii) Let α_1, α_2 be two isotopic spanning arcs, which implies they must have the same endpoints. Then $\tau_{\alpha_1} = \tau_{\alpha_2}$. This is true because by definition, α_1, α_2 being isotopic means there has to exist a self-homeomorphism $f : (M, Q) \to (M, Q)$ which sends

α_1 to α_2, is isotopic to the identity on (M, Q), and is the identity on Q. Then (i) implies

$$\tau_{\alpha_2} = \tau_{f(\alpha_1)} = f\tau_{\alpha_1}f^{-1}$$

Because f is isotopic to the identity, $f\tau_{\alpha_1}f^{-1} = \tau_{\alpha_1}$ (where this equality is understood to be in $\mathfrak{M}(M, Q)$).

(iii) Given two disjoint spanning arcs α_1, α_2 on (M, Q), we have

$$\tau_{\alpha_1}\tau_{\alpha_2} = \tau_{\alpha_2}\tau_{\alpha_1} \in \mathfrak{M}(M, Q)$$

This equality follows from using disjoint neighborhoods of α_1, α_2 when constructing $\tau_{\alpha_1}, \tau_{\alpha_2}$.

(iv) Let α_1, α_2 be two spanning arcs on (M, Q) that share one endpoint and are otherwise disjoint. Then,

$$\tau_{\alpha_1}\tau_{\alpha_2}\tau_{\alpha_1} = \tau_{\alpha_2}\tau_{\alpha_1}\tau_{\alpha_2} \in \mathfrak{M}(M, Q).$$

To see this, first note that we have

$$\tau_{\alpha_1}(\alpha_2) = \tau_{\alpha_2}^{-1}(\alpha_1),$$

as shown in Figure 5. Then (ii) implies that

$$\tau_{\tau_{\alpha_1}}(\alpha_2) = \tau_{\tau_{\alpha_2}^{-1}}(\alpha_1).$$

From (i), we get that

$$\tau_{\alpha_1}\tau_{\alpha_2}\tau_{\alpha_1}^{-1} = \tau_{\alpha_2}^{-1}\tau_{\alpha_1}\tau_{\alpha_2}$$

which is equivalent to saying $\tau_{\alpha_1}\tau_{\alpha_2}\tau_{\alpha_1} = \tau_{\alpha_2}\tau_{\alpha_1}\tau_{\alpha_2}$.

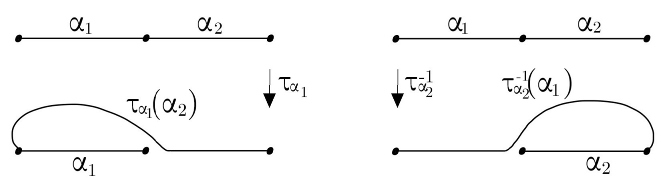

FIGURE 5. The action of τ_{α_1} on α_2 (left), the action of $\tau_{\alpha_2}^{-1}$ on α_1 (right).

We now arrive at the main result of this section: Let $Q_n \subset \mathbb{R}^2$ be the set consisting of the n points $\{(1, 0), (2, 0), ..., (n, 0)\}$. Let $D \subset \mathbb{R}^2$ be a closed disk such that $Q_n \subset D^\circ$. For every $i \in \{1, ..., n-1\}$, define

$$\alpha_i = [i, i+1] \times \{0\}$$

to be the spanning arc connecting the i^{th} and $(i+1)^{th}$ basepoints in Q_n. Each of these spanning arcs induces a half-twist

$$\tau_{\alpha_i} \in \mathfrak{M}(D, Q_n).$$

Notice (iii) and (iv) above are equivalent to the braid relations that we defined in Section 1. Hence by Lemma 1.2 there exists a group homomorphism

$$\eta : B_n \to \mathfrak{M}(D, Q_n)$$

such that $\eta(\sigma_i) = \tau_{\alpha_i}$ for all $i \in \{1, ..., n-1\}$.

In fact, we have the much stronger result:

Theorem 3.6. *The homomorphism $\eta : B_n \to \mathfrak{M}(D, Q_n)$ is an isomorphism.*

The proof of this theorem is lengthy and requires us to define yet another geometric interpretation of braid groups, thus it will not be included in this paper (see [7]). However, this theorem finally shows that we can interpret the braid group B_n as the mapping class group of a disk with n holes. With this, we are ready to move on to discussing representations of the braid group.

4. The Burau Representation

This section begins by giving a representation of B_n in terms of explicit matrices. In Section 5, we will interpret B_n as the group of isotopy classes of self-homeomorphisms of a punctured disk and obtain a representation of B_n by observing how B_n acts on the homology of certain covering spaces. It will later be seen that these two representations are actually equivalent.

Definition 4.1. A *Laurent polynomial* in t with coefficients in the field $\mathbb{Z}$ is a formal sum of the form

$$\lambda = \sum_{k \in \mathbb{Z}} n_k t^k$$

where $n_k \in \mathbb{Z}$. Let $\Lambda = \mathbb{Z}[t, t^{-1}]$ denote the ring of such Laurent polynomials.

The Burau representation, as defined below, is a linear representation of B_n consisting of $n \times n$ matrices over Λ.

Definition 4.2. Let $n \geq 2$. Define U_i as the following $n \times n$ matrix with entries in the ring $\Lambda = \mathbb{Z}[t, t^{-1}]$:

$$U_i = \begin{bmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{bmatrix}$$

where I_k denotes the $k \times k$ identity matrix.

Notice each U_i has a block diagonal form: the blocks are $(i-1) \times (i-1)$ and $(n-i-1) \times (n-i-1)$ identity matrices, and the 2×2 matrix

$$U = \begin{bmatrix} 1-t & t \\ 1 & 0 \end{bmatrix}.$$

Proposition 4.3. *Each $U_i, i \in \{1, ..., n-1\}$ is invertible.*

Proof. The Cayley-Hamilton theorem states that any 2×2 matrix M satisfies the equation

$$M^2 - \mathrm{tr}(M)M + \det(M)I_2 = 0.$$

Thus, U satisfies $U^2 - (1-t)U - tI_2 = 0$. The identity matrices also satisfy this relation, so we get that for all $i \in \{1, ..., n\}$,

$$U_i^2 - (1-t)U_i - tI_n = 0.$$

This is equivalent to

$$U_i(U_i - (1-t)I_n) = tI_n.$$

Multiplying by t^{-1} on both sides shows that U_i is invertible (over Λ) and that

$$U_i^{-1} = t^{-1}(U_i - (1-t)I_n) = \begin{bmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & t^{-1} & 1-t^{-1} & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{bmatrix}.$$

□

Proposition 4.4. *Let $n \geq 2$. The matrices U_i for all $i \in \{1, ..., n\}$ satisfy the braid relations, i.e.*

$$U_iU_j = U_jU_i \text{ for all } i,j \text{ with } |i-j| \geq 2$$

$$U_iU_{i+1}U_i = U_{i+1}U_iU_{i+1} \text{ for all } i \in \{1, ..., n-2\}.$$

Proof. Verifying both equalities only requires a bit of matrix multiplication. For the first, let $i, j \in \{1, ..., n-1\}$ such that $|i-j| \geq 2$. Assume without loss of generality that $j > i$. Then

$$U_iU_j = \begin{bmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{bmatrix} \begin{bmatrix} I_{j-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-j-1} \end{bmatrix}$$

$$= \begin{bmatrix} I_{i-2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-t & t & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{j-i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-t & t & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{n-j-2} \end{bmatrix}$$

$$= \begin{bmatrix} I_{j-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-j-1} \end{bmatrix} \begin{bmatrix} I_{i-1} & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-i-1} \end{bmatrix} = U_jU_i.$$

For the second equality, it suffices to check that

$$\begin{bmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{bmatrix}.$$

Multiplying out the left side yields

$$\begin{bmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-t & t-t^2 & t^2 \\ 1-t & t & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The right side is equal to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-t & t \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1-t & t-t^2 & t^2 \\ 1-t & t & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

□

Thus, by Lemma 1.2, there exists a group homomorphism

$$\psi_n : B_n \to \mathrm{GL}_n(\Lambda)$$

such that $\psi_n(\sigma_i) = U_i$ for every $i \in \{1, ..., n-1\}$ (where $\mathrm{GL}_n(\Lambda)$ denotes the group of invertible $n \times n$ matrices over Λ). This is called the *Burau representation* of B_n.

5. The Burau Representation is not Faithful for $n > 5$

A natural question to ask now is how "good" this representation actually is, or more formally, for which n is the Burau representation *faithful*? Namely, for which n is the homomorphism $\psi_n : B_n \to \mathrm{GL}_n(\Lambda)$ injective? It turns out that for $n \geq 5$, $\mathrm{Ker}\psi_n \neq \{1\}$. For $n \leq 3$, it can be shown that the Burau representation is faithful (see [7]), and the case where $n = 4$ is still an open problem. However, we will only prove this statement for $n > 5$ here, as the case where $n = 5$ is more complicated (see [2] for a proof). First, notice that under the natural inclusion $\imath : B_n \to B_{n+1}$, $\mathrm{Ker}\psi_n \subset \mathrm{Ker}\psi_{n+1}$. Hence, to prove that $\mathrm{Ker}\psi_n \neq \{1\}$ for all $n > 5$, it suffices to show $\mathrm{Ker}\psi_6 \neq \{1\}$. To do this, we must introduce some new definitions and concepts.

For now, let $n \geq 1$. Let $Q = \{(1, 0), (2, 0), ..., (n, 0)\} \subset \mathbb{R}^2$. Let $D \subset \mathbb{R}^2$ be a closed disk with counterclockwise orientation such that $Q \subset D^\circ$. Set $\Sigma = D - Q$, and fix a basepoint $d \in \partial D$.

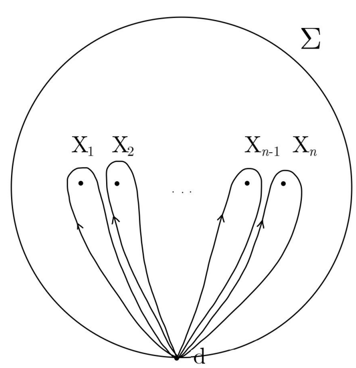

Figure 6. The surface Σ along with the loops $X_1, ..., X_n$.

Definition 5.1. Let γ be a loop in D based at d. The *total winding number* $w(\gamma)$ of γ is defined as the sum of its winding numbers around $(1,0),(2,0),...,(n,0)$.

Let $\varphi : \pi_1(\Sigma, d) \to \{t^k\}_{k\in\mathbb{Z}}$ be a group homomorphism which sends $[\gamma] \in \pi_1(\Sigma, d)$ to $t^{-w(\gamma)}$. Because $\{t^k\}_{k\in\mathbb{Z}}$ is infinite cyclic, the kernel of φ corresponds to an infinite cyclic covering space of Σ, which we will denote $\tilde{\Sigma}$. Pick some $\tilde{d} \in \partial\tilde{\Sigma}$, and note $\bigcup_{k\in\mathbb{Z}} t^k\tilde{d}$ is equal to all of the points in $\tilde{\Sigma}$ which get projected down to $d \in \Sigma$ by the covering map. Let

$$\tilde{H} = H_1(\tilde{\Sigma}, \bigcup_{k\in\mathbb{Z}} t^k\tilde{d}).$$

The group $\{t^k\}_{k\in\mathbb{Z}}$ acts on $\tilde{\Sigma}$ via deck transformations, and this in turn induces a left action of $\{t^k\}_{k\in\mathbb{Z}}$ on $\tilde{H}$.

Proposition 5.2. *$\tilde{H}$ is a free module over $\Lambda = \mathbb{Z}[t, t^{-1}]$ of rank n.*

Proof. First, $\Sigma = D - Q$ deformation retracts onto the graph $\Gamma \subset \Sigma$ consisting of one vertex d and n oriented loops, which we denote $X_1, ..., X_n$, as shown in Figure 6. Each X_i has total winding number -1, and therefore $\varphi(X_i) = t$. Also, the $[X_i]$ generate $\pi_1(\Sigma, d)$. The covering space $\tilde{\Sigma}$ deformation retracts onto a graph $\tilde{\Gamma} \subset \tilde{\Sigma}$. The vertices of $\tilde{\Gamma}$ are the lifts of d, $\{t^k\tilde{d}\}_{k\in\mathbb{Z}}$, and the edges of $\tilde{\Gamma}$ are the lifts of the X_i, $\{t^k\tilde{X}_i\}_{k\in\mathbb{Z}}$. For each $k \in \mathbb{Z}$, the edge $t^k\tilde{X}_i$ connects $t^k\tilde{d}$ to $t^{k+1}\tilde{d}$, as in Figure 7 below. So

$$\tilde{H} = H_1(\tilde{\Sigma}, \bigcup_{k\in\mathbb{Z}} t^k\tilde{d}) = H_1(\tilde{\Gamma}, \bigcup_{k\in\mathbb{Z}} t^k\tilde{d}) = \bigoplus_{i=1}^{n} \Lambda[\tilde{X}_i].$$

Therefore, $\tilde{H}$ is a free Λ-module with basis $[\tilde{X}_1], ..., [\tilde{X}_n]$. $\square$

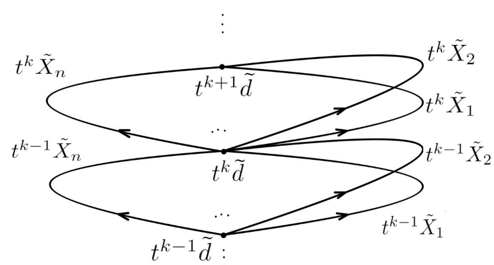

FIGURE 7. The graph $\tilde{\Gamma} \subset \tilde{\Sigma}$.

Define $\mathrm{Aut}(\tilde{H})$ as the group of automorphisms of $\tilde{H}$ that are linear in $\mathbb{Z}[t, t^{-1}]$. By definition, a self-homeomorphism f of Σ fixes d (because f fixes ∂D pointwise), and therefore induces an automorphism $f_* : \pi_1(\Sigma, d) \to \pi_1(\Sigma, d)$. Let $\mathfrak{M}_\varphi(\Sigma, d)$ denote the group of isotopy classes of self-homeomorphisms f such that $\varphi \circ f_* = \varphi$. Then any $f \in \mathfrak{M}_\varphi(\Sigma, d)$ has a unique lift to the covering space $\tilde{\Sigma}$ fixing $\tilde{d}$, denoted $\tilde{f}$. The map sending f to $\tilde{f}$ defines a group homomorphism $\mathfrak{M}_\varphi(\Sigma, d) \to \mathrm{Aut}(\tilde{H})$, which we call the *twisted homological representation* of $\mathfrak{M}_\varphi(\Sigma, d)$.

Given a loop γ in Σ based at d, if we apply some $f \in \mathfrak{M}_\varphi(\Sigma, d)$ and permute the points of Q, notice that the total winding number of γ remains the same. This is because the total winding number does not care about which particular point of Q the loop goes around. Thus, restricting D to Σ defines a homomorphism

$$\mathfrak{M}(D,Q) \to \mathfrak{M}_\varphi(\Sigma, d).$$

We compose this with the twisted homological representation $\mathfrak{M}_\varphi(\Sigma, d) \to \mathrm{Aut}(\tilde{H})$ which we defined earlier to get a homomorphism

$$\Psi_n : \mathfrak{M}(D,Q) \to \mathrm{Aut}(\tilde{H}).$$

Next, we will show $\mathrm{Ker}\Psi_n \neq \{1\}$ for $n > 5$.

Definition 5.3. Let α_1, α_2 be two spanning arcs on (D, Q). We say α_1 and α_2 are *transversal* if they do not share any endpoints and intersect transversely a finite number of times.

Let α_1, α_2 be two transversal spanning arcs on (D, Q) with some orientation, and pick arbitrary lifts $\tilde{\alpha_1}, \tilde{\alpha_2} \subset \tilde{\Sigma}$ with induced orientations. Notice $\bigcup_{k\in\mathbb{Z}} t^k\tilde{\alpha_1}$ is equal to all of the possible lifts of α_1. Let $t^k\tilde{\alpha_1} \cdot \tilde{\alpha_2}$ denote the algebraic intersection number of $t^k\tilde{\alpha_1}$ and $\tilde{\alpha_2}$.

Definition 5.4. Given two transversal spanning arcs α_1, α_2 on (D, Q), their *algebraic intersection* is defined as

$$\langle \alpha_1, \alpha_2 \rangle = \sum_{k\in\mathbb{Z}} (t^k\tilde{\alpha_1} \cdot \tilde{\alpha_2}) t^k.$$

Notice that $\tilde{\alpha_2}$ is mapped bijectively to α_2 by the covering map, and $(\bigcup_{k\in\mathbb{Z}} t^k\tilde{\alpha_1})$ is mapped bijectively onto $\alpha_1 \cap \alpha_2$, which is finite by assumption. Hence the algebraic intersection number of two transversal spanning arcs is finite. Moreover, any point $p \in \alpha_1 \cap \alpha_2 \subset \Sigma$ has a unique lift to an intersection point of $t^k\tilde{\alpha_1}$ with $\tilde{\alpha_2}$. We will denote the unique k for which this occurs by $k_p \in \mathbb{Z}$. If we let $\varepsilon_p = \pm 1$ denote the intersection sign of α_1 and α_2 at $p \in \alpha_1 \cap \alpha_2$, then

$$\langle \alpha_1, \alpha_2 \rangle = \sum_{p\in\alpha_1\cap\alpha_2} \varepsilon_p t^{k_p}.$$

Remark 5.5. For any $p, q \in \alpha_1 \cap \alpha_2$, $k_p - k_q$ equals the total winding number of the loop in Σ which starts at p and goes to q along α_1, and comes back to p along α_2.

Recall that any spanning arc α on (D, Q) corresponds to a half-twist $\tau_\alpha : (D, Q) \to (D, Q)$, and that $\tau_\alpha \in \mathfrak{M}(D, Q)$ We will denote the restriction of this half-twist to $\Sigma = D - Q$ by τ_α again. The next lemma provides a relationship between spanning arcs and how Ψ_n acts on the product of their half-twists:

Lemma 5.6. *Let α_1, α_2 be two transversal spanning arcs on (D, Q). If $\langle \alpha_1, \alpha_2 \rangle = 0$, then $\Psi_n(\tau_{\alpha_1}\tau_{\alpha_2}) = \Psi_n(\tau_{\alpha_2}\tau_{\alpha_1})$.*

Proof. Figure 4 depicts the action of τ_{α_1} on a curve crossing α_1 transversely. So, given $g \in H = H_1(\Sigma)$, τ_{α_1} inserts the "figure-eight" curve α_1' at every intersection of g with α_1.

Therefore,

$$(\tau_{\alpha_1})_*(g) = g + ([\alpha_1] \cdot g)[\alpha_1']$$

where $[\alpha_1] \cdot g$ is the intersection number of the loops $[\alpha_1]$ and g (and the sign of each crossing is determined by the orientation of D).

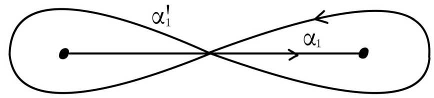

FIGURE 8. The "figure-eight" α_1'.

Note that α_1' lifts to a closed curve in $\tilde{\Sigma}$ because the total winding number of α_1' is zero. Thus $\tilde{\tau}_{\alpha_1}$ acts on $[\tilde{\gamma}] \in \tilde{H}$ by inserting lifts of $\alpha_1', \tilde{\alpha_1'}$, i.e.

$$(\tilde{\tau}_{\alpha_1})_*([\gamma]) = [\gamma] + \lambda_\gamma[\tilde{\alpha_1'}]$$

where $\lambda_\gamma \in \Lambda$ and the coefficients of λ_γ are the algebraic intersection numbers of γ with the lifts $\tilde{\alpha_1}$ of α_1. Similarly,

$$(\tilde{\tau}_{\alpha_2})_*([\gamma]) = [\gamma] + \mu_\gamma[\tilde{\alpha_2'}]$$

for $\mu_\gamma \in \Lambda$ and α_2' is defined analogously to α_1'. Because $\langle \alpha_1, \alpha_2 \rangle = 0$, any lift of α_1, and therefore also of α_1', has algebraic intersection number zero with any lift of α_2. Therefore,

$$(\tilde{\tau}_{\alpha_2})_*(\tilde{\alpha_1'}) = \tilde{\alpha_1'}$$

and similarly

$$(\tilde{\tau}_{\alpha_1})_*(\tilde{\alpha_2'}) = \tilde{\alpha_2'}.$$

Thus for any $\gamma \in \tilde{H}$,

$$(\tilde{\tau}_{\alpha_1}\tilde{\tau}_{\alpha_2})_*([\gamma]) = [\gamma] + \lambda_\gamma[\tilde{\alpha_1'}] + \mu_\gamma[\tilde{\alpha_2'}] = (\tilde{\tau}_{\alpha_2}\tilde{\tau}_{\alpha_1})_*([\gamma]).$$

Hence $(\tilde{\tau}_{\alpha_1}\tilde{\tau}_{\alpha_2})_* = (\tilde{\tau}_{\alpha_2}\tilde{\tau}_{\alpha_1})_*$. □

So, to show that $\mathrm{Ker}\Psi_n \neq \{1\}$, it suffices to construct two spanning arcs α_1, α_2 such that $\langle \alpha_1, \alpha_2 \rangle = 0$ and $\tau_{\alpha_1}\tau_{\alpha_2} \neq \tau_{\alpha_2}\tau_{\alpha_1}$ in $\mathfrak{M}(D, Q)$. We claim that the spanning arcs for $n = 6$ in Figure 9 below satisfy these properties. First, we check that $\langle \alpha_1, \alpha_2 \rangle = 0$. The points in the intersection $\alpha_1 \cap \alpha_2$ have been labeled $p_1, ..., p_4$.

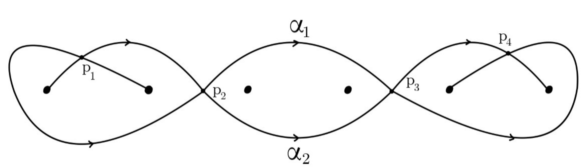

FIGURE 9. Two spanning arcs in the case $n = 6$.

Choose lifts and orientations of α_1 and α_2 such that $k_{p_1} = 0, \varepsilon_{p_1} = 0$. We let $\varepsilon_{p_i} = 1$ when α_2 is directed up at p_i and let $\varepsilon_{p_i} = -1$ when α_2 is directed down at p_i for all $i = 1, ..., 4$. Then, using Remark 5.5, we can find the k_{p_i} and ε_{p_i}, and so

$$\langle \alpha_1, \alpha_2 \rangle = 0 + (-1)(2) + (1)(4) + (-1)(2) = 0$$

It remains to show that τ_{α_1} and τ_{α_2} do not commute. To do this, we begin by defining Dehn twists:

Definition 5.7. Let c be a simple closed curve on Σ (i.e. a closed curve contained wholly within Σ° and having no self-intersections). Identify an annular neighborhood of c in Σ with $S^1 \times [0,1]$ such that the orientation on Σ is preserved and $c = S^1 \times \{1/2\}$. The *Dehn twist about* c,

$$t_c : \Sigma \to \Sigma,$$

is defined as follows: outside of $S^1 \times [0,1]$, t_c is the identity, and any $(x,s) \in S^1 \times [0,1]$ is sent to $e^{2\pi i s} \in S^1 \times [0,1]$.

Definition 5.8. Let c,d be two simple closed curves on Σ. We say that c and d are *isotopic* if there exists a self-homeomorphism $f \in \mathfrak{M}(\Sigma)$ such that f is isotopic to the identity and $f(c) = d$, and write $c \sim d$ Note that if c,d are isotopic, so are their corresponding Dehn twists t_c and t_d.

Definition 5.9. The *geometric intersection number* between two simple closed curves c,d on a surface Σ is defined as the minimal number of intersection points of simple closed curves on Σ isotopic to c and d, i.e.

$$i(c,d) = \min\{|c' \cap d'| : c' \sim c, d' \sim d\}.$$

Let us also consider how half-twists about arcs are related to Dehn twists. Let M be an oriented surface and $Q \subset M^\circ$ a finite subset of the interior of M. Let $\Sigma = M - Q$, and let α be a spanning arc on (M,Q). Define $D \subset M$ to be a closed (topological) disk such that $\alpha \subset D^\circ$, and let $c = \partial D$ be the boundary of this disk. Then the Dehn twist $t_c : \Sigma \to \Sigma$ corresponding to c is related to the half-twist $\tau_\alpha : \Sigma \to \Sigma$ about α as follows:

$$t_c = \tau_\alpha^2.$$

We now state some facts about Dehn twists, the proofs of which can be found in [4].

Fact 5.1. Let c,d be two simple closed curves in a surface Σ. Then we have

$$i(t_c(d), d) = i(c,d)^2.$$

Fact 5.2. Let c,d be two simple closed curves in a surface Σ. Then

$$t_c = t_d \text{ if and only if } c = d$$

Fact 5.3. Let c be a simple closed curve in a surface Σ. Given $f \in \mathfrak{M}(\Sigma)$, $f(c)$ is also a simple closed curve in Σ, and

$$t_{f(c)} = f t_c f^{-1}$$

The following lemma uses these facts to provide a simple condition on when two Dehn twists commute:

Lemma 5.10. *Let c,d be two simple closed curves on an oriented surface Σ. Then $t_c t_d = t_d t_c$ if and only if $i(c,d) = 0$.*

Proof. If $i(c,d) = 0$, then c,d are isotopic to simple disjoint curves c', d' respectively. We can then find disjoint annular neighborhoods around c' and d'. Because Dehn twists act as the identity outside of this annular neighborhood, we have that $t_c = t_{c'}$ and $t_d = t_{d'}$ commute.

Now suppose two Dehn twists t_c, t_d commute. Because t_d is a self-homeomorphism of Σ, by Fact 5.3,

$$t_d = t_c t_d t_c^{-1} = t_{t_c(d)}.$$

Then by Fact 5.2, $d = t_c(d)$, which implies $i(t_c(d), d) = i(d, d) = 0$. By Fact 5.1, this means $i(c, d) = 0$, so c, d are isotopic to disjoint simple closed curves. ☐

We will also use the following lemma, whose proof can be found in [5].

Lemma 5.11. *Let c, d be two simple closed curves on a surface Σ which intersect transversely at a finite number of points. Then there exist isotopic simple closed curves $c' \sim c, d' \sim d$ such that $|c' \cap d'| < |c \cap d|$ if and only if c and d have a "digon" (i.e., an embedded disk whose boundary is made up of one subarc of c and one subarc of d).*

We are finally ready to show that the half-twists $\tau_{\alpha_1}, \tau_{\alpha_2}$ corresponding to the spanning arcs in Figure 6 do not commute:

If the half twists $\tau_{\alpha_1}, \tau_{\alpha_2} \in \mathfrak{M}(D, Q)$ commute, so must their restrictions to $\Sigma = D - Q$. Let $D_1, D_2 \subset M$ be closed topological disks such that $\alpha_1 \subset D_1^\circ, \alpha_2 \subset D_2^\circ$, and D_1, D_2 meet Q only at the endpoints of α_1, α_2 respectively. Let $c(\alpha_1), c(\alpha_2)$ denote the boundaries of D_1, D_2. Then the corresponding Dehn twists also commute because

$$t_{c(\alpha_1)} = \tau_{\alpha_1}^2 : \Sigma \to \Sigma \text{ and } t_{c(\alpha_2)} = \tau_{\alpha_2}^2 : \Sigma \to \Sigma.$$

By Lemma 5.10, $c(\alpha_1), c(\alpha_2)$ must be isotopic to disjoint simple closed curves. Then Lemma 5.11 states that $c(\alpha_1), c(\alpha_2)$ must have a digon. But, as shown in Figure 10 below, $c(\alpha_1), c(\alpha_2)$ have no digons in Σ. Hence $c(\alpha_1), c(\alpha_2)$ do not commute and neither do the half-twists $\tau_{\alpha_1}, \tau_{\alpha_2}$.

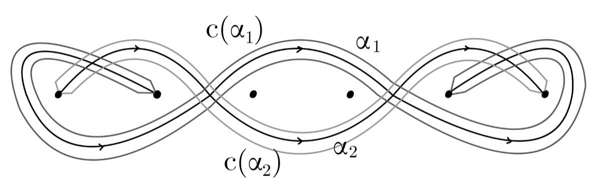

FIGURE 10. The simple closed curves $c(\alpha_1), c(\alpha_2)$.

Finally, it remains to show that the representation Ψ_n of B_n above is equivalent to the Burau representation ψ_n:

Theorem 5.12. *There exists a homomorphism $\mu : GL_n(\Lambda) \to Aut(\tilde{H})$ such that the following diagram commutes:*

$$\begin{tikzcd} B_n \arrow[r, "\eta"] \arrow[d, "\psi_n"'] & \mathfrak{M}(D, Q) \arrow[d, "\Psi_n"] \\ GL_n(\Lambda) \arrow[r, "\mu"'] & Aut(\tilde{H}) \end{tikzcd} .$$

Proof. Recall $\tilde{H} = H_1(\tilde{\Sigma}, \bigcup_{k\in\mathbb{Z}} t^k \tilde{d})$ is a free Λ-module with basis $[\tilde{X}_1], ..., [\tilde{X}_n]$. We identify $\mathrm{Aut}(\tilde{H})$ with $\mathrm{GL}_n(\Lambda)$ as follows: a matrix $(\lambda_{i,j}) \in \mathrm{GL}_n(\Lambda)$ acts on $\tilde{H}$ by sending each $[\tilde{X}_j]$ to $\sum_i \lambda_{i,j}[\tilde{X}_i]$. Define

$$\mu : \mathrm{GL}_n(\Lambda) \to \mathrm{GL}_n(\Lambda) = \mathrm{Aut}(\tilde{H})$$

by $\mu(M) = (M^T)^{-1}$.

To show the diagram in the statement of the theorem commutes, we need to check that for any $b \in B_n$,

$$\Psi_n \eta(b) = \mu\psi_n(b).$$

It suffices to check this for the generators $\{\sigma_1^{-1}, ..., \sigma_{n-1}^{-1}\}$ for B_n. Let $i \in \{1, ..., n-1\}$. The homeomorphism $\eta(\sigma_i^{-1})$ acts on D by rotating the arc connecting the points $(i, 0), (i+1, 0) \in Q$ clockwise by π. For $k \neq i, i+1$, X_k remains fixed, X_i is mapped to a loop homotopic to $X_i X_{i+1} X_i^{-1}$, and X_{i+1} is mapped to a loop homotopic to X_i, as seen in Figure 11 below.

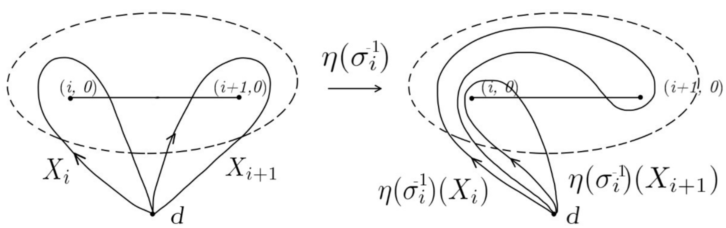

FIGURE 11. The action of $\eta(\sigma_i^{-1})$ on the loops X_i, X_{i+1} in D. Outside of the dashed neighborhood of the arc connecting $(i, 0)$ and $(i+1, 0)$, $\eta(\sigma_i^{-1})$ is the identity.

Lifting η to $\tilde{\Sigma}$ therefore keeps $\tilde{X}_k$ fixed for $k \neq i, i+1$, maps $\tilde{X}_{i+1}$ to $\tilde{X}_i$, and maps $\tilde{X}_i$ to $\tilde{X}_i(t\tilde{X}_{i+1})(t\tilde{X}_i)^{-1}$. Thus $\Psi_n\eta(\sigma_i^{-1})$ sends

$$[\tilde{X}_i] \text{ to } (1-t)[\tilde{X}_i] + t[\tilde{X}_{i+1}] \text{ and } [\tilde{X}_{i+1}] \text{ to } [\tilde{X}_i].$$

The matrix of this automorphism is

$$\begin{bmatrix} I_{n-1-i} & 0 & 0 & 0 \\ 0 & 1-t & 1 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & 0 & I_{i-1} \end{bmatrix}$$

which is precisely equal to $U_i^T = \mu\psi_n(\sigma_i^{-1})$. □

Thus, we see that the Burau representation of the Artin braid group B_n is not faithful for all $n > 5$.

6. Acknowledgments

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