SOLUTION TO THE PLATEAU PROBLEM FOR MAPS FROM THE UNIT DISK

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ABSTRACT. The Plateau problem investigates the existence of a minimal surface spanning a given closed curve. The aim of this paper is to provide a solution to the classical Plateau problem for maps from the unit disk. To do so, we construct a convergent, energy minimizing sequence of maps with the given boundary whose limit is the desired minimal surface. As an important preliminary, we introduce the theory of Sobolev spaces. The only prerequisite is basic Riemannian geometry in $\mathbb{R}^3$.

CONTENTS

1. INTRODUCTION

The Plateau problem, in the most general sense, is stated as follows:

$$\text{Given a closed curve } \Gamma, \text{find a minimal surface with boundary } \Gamma.$$

The problem was first posed by Lagrange in 1760 and was extensively studied by the Belgian physicist Plateau in the 19th century, from whom it got its name. The Plateau problem has led to significant developments in geometry and partial differential equations, such as the construction of the Lebesgue integral, the regularity theory of De Giorgi, and the development of geometric measure theory.

In this paper, we present the proof of the Plateau problem for parameterized disks in $\mathbb{R}^3$. This version of the problem was solved independently by J. Douglas [1] and T. Rado [2] in early 1930's.

This problem is formulated in the following theorem.

Theorem 1.1. *Given a piecewise C^1 Jordan curve $\Gamma \subset \mathbb{R}^3$, there exists a map $u : D \subset \mathbb{R}^2$ to $\mathbb{R}^3$ so that*

(1) $u : \partial D \to \Gamma$ is monotone[1] *and onto.*
(2) $u \in C^0(\overline{D}) \cap W^{1,2}(D)$[2] *and is C^∞ on D.*
(3) The image of u minimizes area among all maps from disks with boundary Γ.

As with other minimization problems, the most natural approach to proving this theorem would be to construct a sequence of mappings from disks such that the area goes to the infimum and use compactness to get a convergent subsequence. However, there are two obstacles to this method:

(1) The diffeomorphism group of the disk is not compact. [3] This is an issue because the area of a mapping depends only on the image. Thus, if we have a sequence $\phi_k : D \to D$ of diffeomorphisms of the disk, and if $u : D \to \mathbb{R}^3$ is a fixed map, it is possible that the sequence $u(\phi_k) : D \to \mathbb{R}^3$ does not converge to a map, even though they all have the same image. [4]
(2) The bound on the area of a map does not sufficiently restrict the map. It is possible to construct a sequence of surfaces in $\mathbb{R}^3$ with boundary $\partial D \subset \mathbb{R}^3$ such that the area approaches the area of the disk (which we know is minimal), but the closure of the surface is the entire Euclidean space. This is possible by successively thinner and longer "tentacles" emanating from the disk. An illustration can be seen in Figure 1.3.4 of [5].

Because of these reasons, minimizing area is not the way to go. We want to have a convergent subsequence approaching a well-behaved minimal surface. Therefore, instead of area, we will minimize the L^2-norm of the derivative of the map, which is the energy. Before moving on to the advantages of minimizing energy, we will make a quick digression into Sobolev spaces.

2. Sobolev Spaces

The purpose of this section is to introduce Sobolev spaces and refer to some fundamental properties that will be used in the proof of Theorem 1.1. In particular, the Rellich Compactness of Sobolev spaces will be important. For a rigorous treatment of the subject, we refer the reader to chapter 5 of [6].

Before giving the definition, we introduce the notion of weak derivatives. Essentially, Sobolev functions satisfy a weaker differentiability criterion:

Definition 2.1. Suppose $u, v \in L^1_{loc}(U)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex. We say that v is the α^{th}-weak partial derivative of u, denoted

$$D^\alpha u = v,$$

if

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v \phi dx,$$

for any compactly supported smooth function ϕ in U. Here, $D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

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[1] inverse image of every connected set is connected for $f : \partial D \to \Gamma$.
[2] Sobolev space of L^2-integrable functions with L^2-integrable weak derivatives (see §2 below).

The motivation for this definition comes from the integration by parts formula. Weak derivatives are ubiquitous in the theory of PDEs. For examples and a proof of uniqueness of the weak derivative, we refer the reader to [6].

Now we can define Sobolev spaces.

Definition 2.2. The Sobolev space $W^{k,p}(U)$ consists of all locally summable functions $u : U \to \mathbb{R}$ such that for each multiindex α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$.

As seen in the statement of Theorem 1.1, the Sobolev space that is of interest to us is $W^{1,2}(D)$. In the remainder of this section, we will restrict our attention to this space and the results shown will be special cases. Analogous statements remain true in any other Sobolev space.

The $W^{1,2}$-norm of a function $f \in W^{1,2}(D)$ is

$$|f|^2_{W^{1,2}} = \int_D |f|^2 + \int_D |\nabla f|^2 = \int_D |f|^2 + \int_D |\partial_x f|^2 + |\partial_y f|^2 \tag{2.3}$$

We set $C_0^\infty(D)$ to be the space of smooth functions with compact support on D, and $W_0^{1,2}(D) \subset W^{1,2}(D)$ the closure of $C_0^\infty(D)$ with respect to the $W^{1,2}$-norm. We note that Sobolev spaces are complete normed spaces for any $k \in \mathbb{N}$, $1 \leq k \leq \infty$.

We now state two lemmas that will play important roles in the proof of the Plateau problem.

Lemma 2.4. *(Rellich Compactness) If u_k is a sequence in $W^{1,2}(D)$ with*

$$\sup_k \int_D |u_k|_{W^{1,2}(D)} < \infty,$$

Then there is a subsequence $u_{k'}$ and $u \in W^{1,2}(D)$ so that $u_{k'} \to u$ strongly in L^2, $\nabla u_{k'} \to \nabla u$ weakly in L^2, and

$$\int_D |\nabla u|^2 \leq \liminf \int_D |\nabla u_k|^2. \tag{2.5}$$

Moreover, if each u_k is in $W_0^{1,2}$, then so is u.

For a proof, see chapter 5 of [6].

Lemma 2.6. *(Dirichlet Poincaré Inequality) There exists $C < \infty$ such that if $u \in W_0^{1,2}(D)$, then*

$$\int_D u^2 \leq C \int_D |\nabla u|^2. \tag{2.7}$$

Proof. Since $W_0^{1,2}(D)$ is the closure of $C_0^\infty(D)$, we may assume that u is smooth and has compact support in D. By the Sobolev inequality as can be seen in chapter 3 of [4], we get a constant c such that

$$\int_D u^2 \leq c \left(\int_D |\nabla u| \right)^2 \leq c\,\pi \int_D |\nabla u|.$$

To get the second inequality, we used Cauchy-Schwarz with the L^2 inner product. Note that $C = c\,\pi$. $\square$

2.1. **Weakly harmonic functions** A weakly harmonic function is a function $u \in W^{1,2}(D)$ such that for every $\phi \in C_0^\infty(D)$,

$$\int_D \langle \nabla u, \nabla \phi \rangle = 0. \tag{2.8}$$

As for weak derivative, the motivation for this definition comes from the integration by parts formula. Note that smooth harmonic functions are weakly harmonic by integration by parts.

The below regularity lemma for weakly harmonic functions is important for us to get a function that is C^∞ in the interior, as required by Theorem 1.1.

Lemma 2.9. *(Weyl's Lemma) If $v : D \to \mathbb{R}$ is in $W^{1,2}(D)$ and is weakly harmonic, then v is C^∞ on the interior of D.*

Proof. To prove Weyl's lemma we recall here the technique of convolution with an approximate identity.

Let $\psi : [0, 1] \to \mathbb{R}$ be a smooth nonnegative monotone increasing function such that it is constant on $[0, \frac{1}{3}]$, has support on $[0, \frac{2}{3}]$, and

$$2\,\pi \int_0^1 \psi(t)\, t\, dt = 0. \tag{2.10}$$

Let $\phi_t : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\phi_t(x) = t^{-2}\, \psi(|x|/t). \tag{2.11}$$

We note that ϕ_t is a nonnegative, smooth, radially symmetric function supported on D_t, which is the disk of radius t. By a change of variables, we see that it has total integral equal to one. Fix some $t < 1$. Let $v_t : D_{1-t} \to \mathbb{R}$ be the convolution of v and ϕ_t. Since ϕ_t is supported on a ball of radius t, this integral is well defined. By the convolution formula and applying a change of variables, we have that

$$v_t(y) = \int_{\mathbb{R}^2} v(y + x)\, \phi_t(x)\, dx = \int_{\mathbb{R}^2} v(z)\, \phi_t(z - y)\, dz. \tag{2.12}$$

Therefore, v_t is smooth because in the equation above, we see that ϕ_t carries the burden of differentiability, and ϕ_t is smooth by definition. Thus, the lemma is proven if we show that $v_t = v$.

To show this, we first use the mean value inequality for weakly harmonic functions. The following proposition gives the mean value inequality in its full generality.

Proposition 2.13. *(The Mean Value Inequality) If $\Sigma^k \subset \mathbb{R}^n$ is a minimal submanifold and f is a continuous function on Σ, then*

$$\begin{aligned} &r^{-k} \int_{B_r \cap \Sigma} f - s^{-k} \int_{B_s \cap \Sigma} f \\ &= \int_{(B_r - B_s)\cap\Sigma} f \frac{|x^N|^2}{|x|^{k+2}} + \frac{1}{2} \int_s^r \tau^{-k-1} \int_{B_\tau \cap \Sigma} (\tau^2 - |x|^2) \Delta_\Sigma f \, d\tau, \end{aligned} \tag{2.14}$$

where B_r is a ball of radius r centered at the origin and x^N is the projection of the coordinate vector onto the normal subspace to the submanifold.

For a proof, see chapter 1 of [4].

To apply the mean value inequality for v, we let $\Sigma = D \subset \mathbb{R}^2$. Then the right hand side of (2.14) vanishes by the weak harmonicity of v and the fact that $|x^N| = 0$. By letting $s \to 0$, we get that

$$\begin{aligned} r^{-2}\int_{B_r} v(y+x)\,dx &= \pi v(y) \\ \frac{1}{|B_r|}\int_{B_r} v(y+x)\,dx &= v(y), \end{aligned} \tag{2.15}$$

where we have used the continuity of v to get the right hand side. It is important to note that we need $r < 1 - |y|$ since v is defined in D. (2.15) is the mean value property for balls and it implies the mean value property on spheres, which is that

$$\frac{1}{|\partial B_r|}\int_{\partial B_r} v(y+x)\,dx = v(y). \tag{2.16}$$

Writing (2.16) in polar coordinates, we get that

$$\int_0^{2\pi} v(y + (r\cos\theta, r\sin\theta))\,d\theta = 2\,\pi v(y). \tag{2.17}$$

Writing (2.12) in polar coordinates, restricting to the support of ϕ_t and using the radial symmetry of ϕ_t, we get

$$\begin{aligned} v_t(y) &= \int_{r=0}^{t}\int_{\theta=0}^{2\pi} v(y+(r\cos\theta, r\sin\theta))\,\phi_t(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta \\ &= \int_{r=0}^{t} t^{-2}\psi(\frac{r}{t})\int_{\theta=0}^{2\pi} v(y+(r\cos\theta, r\sin\theta))\,r\,dr\,d\theta \\ &= 2\,\pi v(y)\,t^{-2}\int_{r=0}^{t}\psi(\frac{r}{t})\,r\,dr = v(y). \end{aligned} \tag{2.18}$$

□

Corollary 2.19. *If $v : D \to \mathbb{R}$ is a smooth harmonic function, then v is real analytic in D.*

Proof. Let u be a complex-valued function given by $u = v_x - iv_y$. Since v is smooth, so is u. We have that

$$\begin{aligned} u_x &= v_{xx} - iv_{yx} \\ u_y &= v_{xy} - iv_{yy}. \end{aligned}$$

Since v is smooth, we get that $u_x = u_y$ and therefore u is holomorphic since it satisfies the Cauchy-Riemann equations. This implies that v_x and v_y are real analytic. Thus, by integrating, we get that v is real analytic. □

2.2. **Boundary regularity of weakly harmonic functions** Now that we have interior smoothness for weakly harmonic functions, we investigate continuity on the closed disk, which is also required in Theorem 1.1. In this subsection, we prove that when boundary values are continuous, harmonic functions are continuous on the entire closed disk.

Lemma 2.20. *Suppose that $w \in C^0(\overline{D}) \cap W^{1,2}(D)$. If u is weakly harmonic on D and $u - w \in W_0^{1,2}(D)$, then $u \in C^0(\overline{D})$.*

Remark 2.21. Since $W_0^{1,2}(D)$ is the closure of smooth compactly supported functions on D, $u - w \in W_0^{1,2}(D)$ implies that u and w agree on ∂D.

Proof. By Weyl's lemma, Lemma 2.9, we know that u is continuous in the interior. We only have to show continuity on the boundary of the disk. Moreover, by the symmetry of ∂D, it is enough to show that u is continuous at $(1,0)$. To do so, we will use a *barrier* function, $b(x,y) = (1-x)$. This is a barrier function at $(1,0) \in \partial D$, because:

- b vanishes at $(1,0)$ and is positive in $\overline{D} - (1,0)$.
- b is continuous on $\overline{D}$ and harmonic in D.

Fix any $\epsilon > 0$. By the definition of continuity, there exists $\delta > 0$ such that

$$|w(z) - w(1,0)| < \epsilon \text{ for all } z \in \overline{D} \cap B_\delta(1,0).$$

Since w and b are continuous on the compact set $\overline{D}$ and since $b > 0$ on $\overline{D} - (1,0)$, we can find some $k > 0$ such that

$$k\, b(z) = k\,(1-x)(z) > 2 \sup_{\overline{D}} |w| \text{ for all } z \notin \overline{D} \cap B_\delta(1,0).$$

We define continuous harmonic functions w^+ and w^- by

$$w^\pm = w(1,0) \pm (\epsilon + k\,(1-x)).$$

On $\overline{D}$, we get that

$$w^- \le w \le w^+.$$

Since this inequality also holds on ∂D and since u, w^+, and w^- are harmonic, we get that

$$w^- \le u \le w^+ \text{ on } \overline{D}. \tag{2.22}$$

Since b is continuous and zero at $(1,0)$, we can choose $\delta_0 > 0$ and less that δ such that

$$k\,(1-x) < \epsilon \text{ on } \overline{D} \cap B_{\delta_0}(1,0).$$

Since $\epsilon > 0$ was chosen arbitrarily, we are done. ☐

3. Area vs. Energy

Now that we reviewed some necessary preliminaries about Sobolev functions, we return to our initial discussion. In section one, we saw that there are difficulties in working with a sequence constructed from area minimization, namely the noncompactness of the diffeomorphism group of the disk and the fact that a bound on area does not give sufficient control on the map.

It turns out that the answer to this issue is given by energy, i.e. the L^2-norm of the differential of the map. We will minimize energy instead of area. In this section, we will investigate the relation between energy and area. In particular, we will see that the energy minimizer also minimizes the area and yields a good parameterization.

Let $u : D \to \mathbb{R}^3$ be a map such that $u \in W^{1,2}(D)$ and $u = (u^1, u^2, u^3)$. Then the energy is defined by

$$E(u) = \frac{1}{2}\int_D |\nabla u|^2\, dx\, dy = \frac{1}{2}\int_D (|u_x|^2 + |u_y|^2)\, dx\, dy. \tag{3.1}$$

On the other hand, the area of the three dimensional image of u is given by

$$\text{Area}(u) = \int_D (|u_x|^2\, |u_y|^2 - \langle u_x, u_y\rangle^2)^{\frac{1}{2}}\, dx\, dy. \tag{3.2}$$

We have the following important proposition.

Proposition 3.3.

$$Area(u) \leq E(u), \tag{3.4}$$

with equality if and only if $\langle u_x, u_y \rangle$ *and* $|u_x|^2 - |u_y|^2$ *are zero as* L^1 *functions.*

Proof. We have that

$$(|u_x| - |u_y|)^2 \geq 0, \tag{3.5}$$

which implies

$$\frac{1}{2}\left(|u_x|^2 + |u_y|^2\right) \geq |u_x||u_y| \geq |u_x \times u_y|. \tag{3.6}$$

Thus, the inequality follows.

To get the forward direction, note that if $Area(u) = E(u)$, then

$$\begin{aligned} \frac{1}{2}\int_D (|u_x|^2 + |u_y|^2) - 2\left(|u_x|^2\,|u_y|^2 - \langle u_x, u_y\rangle^2\right)^{\frac{1}{2}}\,dx\,dy = 0 \\ \int_D \frac{1}{2}(|u_x| - |u_y|)^2 + 2\,|u_x|\,|u_y| - 2\left(|u_x|^2\,|u_y|^2 - \langle u_x, u_y\rangle^2\right)^{\frac{1}{2}}\,dx\,dy = 0 \end{aligned} \tag{3.7}$$

The last part holds if and only if $\langle u_x, u_y \rangle$ and $|u_x|^2 - |u_y|^2$ are zero as L^1 functions. □

We say that u is *almost conformal* when equality holds. We say that u is *conformal* if it is an almost conformal immersion.

There is a major benefit of working in two dimensions, which is the existence of isothermal coordinates; coordinates that make our mapping almost conformal. In particular, given any u as above that defines a surface in $\mathbb{R}^3$, there exists a diffeomorphism $\phi : D \to D$ such that $u \circ \phi : D \to \mathbb{R}^3$ is almost conformal [7]. We know that the area depends only on the image, so we get that $\text{Area}(u) = \text{Area}(u(\phi)) = E(u(\phi))$.

We define

$$X_\Gamma = \{\psi : D \to \mathbb{R}^3 | \psi \in C^0(\overline{D}) \cap W^{1,2}(D) \text{ and} \\ \psi|_{\partial D} : \partial D \to \Gamma \text{ is monotone and onto}\}.$$

Then

$$A_\Gamma = \inf_{\psi \in X_\Gamma} \text{Area}(\psi) \text{ and } E_\Gamma = \inf_{\psi \in X_\Gamma} E(\psi) \tag{3.8}$$

A crucial lemma to conclude that an energy minimizing sequence also minimizes area is the following:

Lemma 3.9. $A_\Gamma = E_\Gamma$

Proof. From the inequality 3.4 we see that $A_\Gamma \leq E_\Gamma$

We will use the existence of isothermal coordinates to prove the converse. Fix some $\epsilon > 0$. We can choose some $u \in X_\Gamma$ such that

$$\text{Area}(u) = \int_D \sqrt{\det g_{ij}} < A_\Gamma + \epsilon/2,$$

where $g_{ij} = \langle u_i, u_j \rangle$ is the metric on the surface with respect to the coordinates induced by the parameterization. We have that

$$\mathrm{Tr}g = |\nabla u|^2.$$

The existence of isothermal coordinates implies that the metric is not degenerate. Therefore, we define a perturbed map $u^s : D \to \mathbb{R}^5 = \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}$ by

$$u^s(x,y) = (u(x,y), s\,x, s\,y).$$

The pullback metric on D is

$$\tilde{g}_{ij} = g_{ij} + s^2 \delta_{ij}.$$

For use in the area calculation, note that

$$\det\tilde{g} = \det(g_{ij} + s^2\delta_{ij}) = \det g + s^2(\mathrm{Tr}g) + s^4 = \det g + s^2\,|\nabla u|^2 + s^4$$

$$\leq \left(\sqrt{\det g} + s\,|\nabla u| + s^2\right)^2. \tag{3.10}$$

We are still working in two dimensions, so by the existence of isothermal coordinates, we can find a conformal diffeomorphism $\phi : (D, \delta_{ij}) \to (D, \tilde{g})$ such that $u^s \circ \phi$ is almost conformal. Hence,

$$E(u \circ \phi) \leq E(u^s \circ \phi) = \mathrm{Area}(u^s \circ \phi) = \mathrm{Area}(u^s).$$

To prove the lemma, we will show that taking s sufficiently small, we can make the area of u^s as close as we like to u. Using (3.10), we get that

$$\begin{aligned}\mathrm{Area}(u^s) = \int_D \sqrt{\det\tilde{g}} &\leq \int_D \left(\sqrt{\det g} + s\,|\nabla u| + s^2\right) \\ &\leq \mathrm{Area}(u) + \pi\, s^2 + s\int_D |\nabla u|.\end{aligned}$$

Since $u \in W^{1,2}(D)$, $\int_D |\nabla u|$ is finite and we can choose s small enough so that $\mathrm{Area}(u^s) < \mathrm{Area}(u) + \epsilon$. Thus,

$$E_\Gamma \leq E(u \circ \phi) \leq \mathrm{Area}(u) + \epsilon < A_\Gamma + 2\epsilon$$

and we are done.

□

Corollary 3.11. *If we have a map $u \in X_\Gamma$ such that $E(u) = E_\Gamma$, then Area$(u) = A_\Gamma$*

Proof. If u is almost conformal, then we are done by Lemma 3.9. Suppose u is not almost conformal, which implies $\mathrm{Area}(u) < E(u)$. But then $A_\Gamma < E_\Gamma$, a contradiction. □

In the remainder of this paper we will prove Theorem 1.1 in two steps. First, we will show that for each parameterization of the boundary curve Γ there is an energy-minimizing map among maps that agree with the parameterization on the boundary. Then we will minimize energy over all boundary parameterizations to obtain *the* energy minimizer and solve the Plateau problem.

4. The Dirichlet Problem

The statement of the famous Dirichlet problem for harmonic maps from the disk is given as follows:

Theorem 4.1. *If $w \in C^0(\overline{D}) \cap W^{1,2}(D)$, then there is a unique energy-minimizing harmonic $u \in C^0(\overline{D}) \cap W^{1,2}(D) \cap C^\infty(D)$ with $u = w$ on ∂D.*

We will prove this theorem in the remainder of this section. To do so, we will use a direct variational approach. In particular, we will construct an energy-minimizing sequence which will converge to a weakly harmonic function that is regular by conclusions from section 2.

Proof. We define a subspace of functions in $W^{1,2}(D)$ that agree with w on the boundary, denoted $W^{1,2}_w(D)$, as

$$W^{1,2}_w(D) = \Big\{ f \in W^{1,2}(D) \,|\, (f - w) \in W^{1,2}_0(D) \Big\}.$$

Let E_w be the infimum of the energy for such functions.

$$E_w = \inf \left\{ \int_D |\nabla u|^2 \,|\, y \in W^{1,2}_w(D) \right\}.$$

We will choose a minimizing sequence of functions $u_l \in W^{1,2}_w(D)$ whose energy values satisfy

$$\int_D |\nabla u_l|^2 < E_w + \frac{1}{l}. \tag{4.2}$$

We have the following parallelogram identity:

$$\left| \nabla \frac{(u_i - u_j)}{2} \right|^2 + \left| \nabla \frac{(u_i + u_j)}{2} \right|^2 = \frac{1}{2} |\nabla u_i|^2 + \frac{1}{2} |\nabla u_j|^2.$$

Integrating this over D, we get

$$\begin{aligned} \frac{1}{4} \int_D |\nabla (u_i - u_j)|^2 + \int_D \left| \nabla \frac{(u_i - u_j)}{2} \right|^2 &= \frac{1}{2} \int_D |\nabla u_i|^2 + \frac{1}{2} \int_D |\nabla u_j|^2 \\ &< E_w + \frac{1}{2}(1/i + 1/j). \end{aligned} \tag{4.3}$$

Since $u_i, u_j \in W^{1,2}_w(D)$, $\frac{1}{2}(u_i + u_j)$ agrees with w on the boundary. Therefore, it has energy at least E_w by definition. We can subtract E_w from both sides to get

$$\frac{1}{4} \int_D |\nabla (u_i - u_j)|^2 < \frac{1}{2}(1/i + 1/j).$$

Since $u_i - w, u_j - w \in W^{1,2}_0(D)$, we have that $u_i - u_j \in W^{1,2}_0(D)$. Thus, the Dirichlet Poincaré inequality, Lemma 2.6, from section 2 implies that

$$\int_D (u_i - u_j)^2 \leq C \int_D |\nabla(u_i - u_j)|^2 \leq 2\,C\,(1/i + 1/j). \tag{4.4}$$

We can conclude that the sequence u_l is Cauchy in $W^{1,2}(D)$ and converges strongly in $W^{1,2}(D)$ to a function $v \in W^{1,2}(D)$ because Sobolev spaces are Banach spaces. By (4.2), we see that the energy of v satisfies

$$\int_D |\nabla v|^2 \leq E_w.$$

To prove the reverse inequality, we note that the sequence of functions $u_l - w \in W_0^{1,2}(D)$ converges to $v - w$. By the last part of Rellich compactness, Lemma 2.4, we can conclude that $v - w \in W_0^{1,2}(D)$ as well. Thus, $v \in W_w^{1,2}(D)$ and by the definition of E_w we get that

$$\int_D |\nabla v|^2 \geq E_w.$$

We see that $\int_D |\nabla(v)|^2 = E_w$ and therefore v is the energy-minimizing function. Uniqueness follows from the strong convergence of any minimizing sequence without having to pass to subsequences. We also get it immediately from the maximum principle when we show that v is harmonic.

To show v is weakly harmonic, we note that for any smooth ψ with compact support in D, we get

$$E(v) = \frac{1}{2}\int_D |\nabla v|^2 \geq \frac{1}{2}E(v + t\,\psi) = \frac{1}{2}\int_D \langle \nabla v + t\nabla\psi, \nabla v + t\nabla\psi\rangle,$$

where we used the fact that v is energy minimizing.

Differentiating at $t = 0$ gives

$$0 = \frac{d}{dt}E(v + t\,\psi)|_{t=0} = \int_D \langle \nabla v, \nabla\psi\rangle. \tag{4.5}$$

This holds for any $\psi \in C_c^\infty(D)$, so v is weakly harmonic by definition which implies that it is smooth in D by Lemma 2.9 and by Lemma 2.20 we get that it is continuous on $\overline{D}$. So v is our desired map. □

5. The Solution to the Plateau problem

In this section we will solve the Plateau problem by minimizing over possible boundary parameterizations. To do so we need some compactness result for parameterizations that will allow us to obtain a subsequence that converges to the minimum energy. The first step is to prove the Courant-Lebesgue lemma.

Given any point $p \in D$, for each $\rho > 0$ we define the set

$$C_\rho = \{q \in D \,|\, |p - q| = \rho.\}$$

Given a map $u : D \to \mathbb{R}^3$, we denote by $d(C_\rho)$ and $L(C_\rho)$ the diameter of and length of the image curve $u(C_\rho)$, respectively.

Lemma 5.1. *(Courant-Lebesgue lemma) If $u : D \to \mathbb{R}^3$, $u \in C^0(\overline{D}) \cap W^{1,2}(D)$ and $E(u) \leq K/2$, then, for every $\delta < 1$, there exists $\rho \in [\delta, \sqrt{\delta}]$ with*

$$(d(C_\rho))^2 \leq 2\pi\epsilon_\delta, \tag{5.2}$$

where $\epsilon_\delta = \frac{4\pi K}{-log\delta}$; in particular, $\epsilon_\delta \to 0$ as $\delta \to 0$.

Proof. It suffices to assume that u is C^1 by density and regularization arguments [6]. We will prove (5.2) for $L(C_\rho)$ which implies the inequality for $d(C_\rho)$.

Let $p(r)$ be given by

$$p(r) = r\int_{C_r} |\nabla u|^2\, ds, \tag{5.3}$$

where ds is the arclength measure on C_r. We have that

$$\int_\delta^{\sqrt{\delta}} p(r)\, d(\log r) = \int_\delta^{\sqrt{\delta}} p(r)\, \frac{dr}{r} \leq \int_D |\nabla u|^2\, dx\, dy \leq K. \tag{5.4}$$

To get the first inequality we plug in the expression for $p(r)$ given above. We can now use the mean value theorem from one-variable calculus to find some ρ between δ and $\sqrt{\delta}$ such that

$$p(\rho) \leq \frac{\int_\delta^{\sqrt{\delta}} p(r)\, d(\log r)}{\int_\delta^{\sqrt{\delta}}\, d(\log r)} \leq \frac{2K}{-\log\delta}. \tag{5.5}$$

For any $r \in [\delta, \sqrt{\delta}]$, the Cauchy-Schwarz inequality gives

$$(L(C_r))^2 \leq \Big(\int_{C_r} |\nabla u|^2\, ds\Big)^2 \leq 2\pi\, p(r). \tag{5.6}$$

Thus, we get $(L(C_r))^2 \leq \frac{4\pi K}{-\log\delta}$. ☐

There is one more obstacle we have to deal with: the noncompactness of the conformal group of D. The conformal group of D is the group of conformal diffeomorphisms $\phi : D \to D$. The noncompactness of this group is an issue because energy is conformally invariant in dimension two.

Lemma 5.7. *If $u \in W^{1,2}(D)$ and $\phi : D \to D$ is a conformal diffeomorphism, then $E(u) = E(u(\phi))$.*

Proof. Let us denote by g_{ij} the pullback metric under ϕ of the Euclidean metric on the disk. Recall that conformality of ϕ requires that $\langle \phi_x, \phi_y\rangle = 0$ and $|\phi_x|^2 = |\phi_y|^2$. Therefore, we get that

$$g_{ij} = \langle \phi_i, \phi_j\rangle = \frac{1}{2}|\nabla\phi|^2\delta_{ij}. \tag{5.8}$$

The metrics are related by a scalar factor. For simplicity, let $g_{ij} = \lambda^2 \delta_{ij}$. We have

$$
\begin{aligned}
E(u\circ\phi) = E_g(u) &= \frac{1}{2}\int_D g^{ij} u_i u_j \,(\det g_{ij})^{\frac{1}{2}}\,dx\,dy \\
&= \frac{1}{2}\int_D \lambda^2 u_i u_j\, \lambda^2\, dx\,dy = E(u).
\end{aligned}
\tag{5.9}
$$

□

The conformal group of D is the group of linear fractional transformations. [3] Given two triples of distinct points on the boundary, there is a conformal diffeomorphism of D, i.e., a linear fractional transformation that takes one triple to the other. Energy and area are invariant under a conformal reparameterization of D.

The above discussion is summarized in the following lemma:

Lemma 5.10. *If $u : D \to \mathbb{R}^3$ is in $C^0(\overline{D}) \cap W^{1,2}(D)$, $u : \partial D \to \Gamma$ is monotone and onto, then there is a linear fractional transformation $\phi : D \to D$ so that $u \circ \phi$ satisfies:*

(1) $E(u) = E(u\circ\phi)$.
(2) $u\circ\phi \in C^0(\overline{D}) \cap W^{1,2}(D)$.
(3) $u : \partial D \to \Gamma$ is monotone and onto.
(4) For each $i = 1, 2, 3$, we have $u\circ\phi(p_i) = q_i$.

Proof. □

The following lemma lists sufficient conditions for equicontinuity on the boundary.

Lemma 5.11. *For any constant K, define the family of maps $\mathcal{F} = \mathcal{F}_K$ to be all $\psi : D \to \mathbb{R}^3$ with*

(1) $\psi \in C^0(\overline{D}) \cap W^{1,2}(D)$ has $E(\psi) \le K/2$.
(2) $\psi|_{\partial D} : \partial D \to \Gamma$ is monotone and onto.
(3) For each $i = 1, 2, 3$, we have $\psi(p_i) = q_i$, where $p_i \in \partial D$ and $q_i \in \Gamma$ are fixed.

Then $\mathcal{F}$ is equicontinuous on ∂D. Hence, by the Arzela-Ascoli theorem, $\mathcal{F}$ is compact in the topology of uniform convergence on ∂D.

Proof. We will fix some $\epsilon > 0$. Without loss of generality, we can choose it to be smaller than $\min|q_i - q_j|$.

Since Γ is a simple closed curve of finite length, we can find some $d > 0$ such that if $p, q \in \Gamma$ with $0 < |p - q| < d$, then $\Gamma - \{p, q\}$ has exactly one component with diameter at most ϵ.

Fix some $\delta < 1$ such that $\sqrt{2\pi\epsilon_\delta} < d$ and such that given any $p \in \partial D$ at least two of the p_i are not in the ball of radius $\sqrt{\delta}$ about p. Recall that $\epsilon_\delta = \frac{4\pi K}{-\log\delta}$.

Given any $p \in \partial D$, by the Courant-Lebesgue lemma, Lemma 5.1, we can find some $\rho \in [\delta, \sqrt{\delta}]$ such that $d(C_\rho) < d$. The curve C_ρ divides ∂D into two components, A^1 and A^2, with the larger one containing at least two of the base points p_i. Let $\mathcal{A}^1 = \psi(A^1)$, $\mathcal{A}^2 = \psi(A^2)$, so these are the corresponding arcs on Γ. Since $d(C_\rho) < d$, the image of the endpoints of C_ρ are connected by a curve on Γ of length less than d. Thus, one of $\mathcal{A}^1$ and $\mathcal{A}^2$ has diameter less than ϵ. By our definition of ϵ, we can see that this component cannot contain two of q_i. Hence, $\mathcal{A}^1$ has diameter less than ϵ. Since $\mathcal{A}^1$ is the image of A^1, we have equicontinuity. □

Now we present the solution of the Plateau problem for maps from the unit disk.

Proof of Theorem 1.1 First we will show that $E_\Gamma < \infty$. Since Γ is piecewise C^1, we have a piecewise C^1 monotone, onto map $w : \partial D \to \Gamma$. In polar coordinates (ρ, θ), we set

$$\tilde{w}(\rho, \theta) = \eta(\rho) w(\theta),$$

where η is a smooth function with $\eta(1) = 1$ and $\eta(\rho) = 0$ for all $\rho < 1/2$. This gives a Lipschitz, and thus finite energy, map. Thus, $E_\Gamma < \infty$.

By Theorem 4.1, there exists a minimizing sequence $\{u_j\}$ of harmonic maps that are in $C^0(\overline{D}) \cap W^{1,2}(D)$, such that $u_j : \partial D \to \Gamma$ is monotone and onto with

$$E(u_j) \to E_\Gamma.$$

By Lemma 5.10 we can take each $u_j \in \mathcal{F}$.

By Rellich compactness, Lemma 2.4, there is a weakly convergent subsequence $u_k \to u \in W^{1,2}(D)$ with

$$E(u) \leq \lim\ \inf\{E(u_k)\} = E_\Gamma.$$

Lemma 5.11 implies that the boundary values of the u_j's form an equicontinuous family of functions on ∂D. Thus, the Arzela-Ascoli theorem yields a subsequence that converges uniformly to a continuous function. Since the u_j's are harmonic, so is $u_j - u_k$. By the maximum principle, we have that

$$\sup_{\partial D} |u_j - u_k| = \sup_{\partial D} |u_j - u_k|.$$

Consequently, the uniform convergence on ∂D implies uniform convergence on $\overline{D}$ which implies that u is harmonic. We conclude that

$$E(u) = E_\Gamma.$$

Uniform convergence also implies that $u|_{\partial D} : \partial D \to \Gamma$ is monotone and onto. From discussion our discussion in section 3, we conclude that $\text{Area}(u) = A_\Gamma$ and u is almost conformal.

We say that u is the *solution of the Plateau problem*.

□

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