WISHART PROCESSES AND FINANCIAL APPLICATIONS

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Abstract. Following a brief discussion on stochastic calculus preliminaries we introduce Wishart processes and derive the stochastic differential equations for these processes and their eigenvalues and eigenvectors. Some simulations of the eigenvalue processes are presented and financial applications of Wishart processes are discussed.

Contents

1. Introduction

The real world operates on randomness from the behavior of atoms to the daily weather. Many believe the financial markets to also operate largely on randomness with a common model for stock prices being a Geometric Brownian Motion. Financial derivatives (such as a stock option) whose value depends on the underlying security (such as a stock) have been modeled stochastically as early as the 1960s with the famous Black-Scholes model. This model, however, has its drawbacks one being that it assumes the underlying stock's volatility to be constant–a consequence of using GBM to model the stock's price. This has empirically shown to not be true with a famous example being the volatility smile of options of different strike prices for the same underlying stock. As a result, it is natural to attempt to model the volatility itself with several stochastic approaches being taken. An asset is rarely traded by itself instead being traded with other highly correlated assets (and sometimes uncorrelated assets depending on the strategy) to hedge risk. It thus makes sense to study the pairwise correlation processes of a group of assets and perhaps consequently try to model their covariance matrix stochastically. The field of Random Matrix Theory largely originated in the 1950s when Eugene Wigner attempted to describe atomic energy levels using the eigenvalues of certain random matrices. Wishart matrices are a certain type of random matrix and their structure

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(described in more detail in the later sections) results in their usefulness in statistics and sample covariance matrices. It thus seems reasonable to further explore Wishart processes when attempting to model the relationship between assets and deciding how to appropriately price them.

2. Stochastic Calculus Preliminaries

We draw most of these preliminaries from [1].

Definition 2.1. A stochastic process B_t is a Brownian motion with drift μ and variance σ^2 starting at the origin if

(1) $B_0 = 0$
(2) B_t is a continuous function of t
(3) B_t has independent increments i.e. if $s_1 < s_2 \leq t_1 < t_2$ then the random variables $B_{s_2} - B_{s_1}$ and $B_{t_2} - B_{t_1}$ are independent of one another
(4) For $t > s$, $B_t - B_s \sim N(\mu(t-s), \sigma^2(t-s))$

Remark 2.2. A standard Brownian motion is a Brownian motion with drift 0 and variance 1. For the rest of this paper we will be working with standard Brownian motions.

Definition 2.3. The covariation of two processes X and Y is

$$[X,Y]_t = \lim_{\|P\|\to 0} \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}})$$

where P is a partition $[t_0, t_1, \ldots, t_n]$ of $[0,t]$ and $\|P\|$ is the mesh of the partition.

Remark 2.4. The quadratic variation of a process X_t is $[X,X]_t$

Lemma 2.5. *Let B be a standard Brownian motion. Then $[B,B]_t = t$*

Proof. We show that $E([B,B]_t) = t$ and $\mathrm{Var}([B,B]_t) = 0$

$$E(\sum_{i=1}^{n}(B_{t_i} - B_{t_{i-1}})^2) = \sum_{i=1}^{n}(\mathrm{Var}(B_{t_i} - B_{t_{i-1}}) + (E(B_{t_i} - B_{t_{i-1}}))^2) = \sum_{i=1}^{n} t_i - t_{i-1} = t$$

so $E([B,B]_t) = t$. For a given $\|P\|$,

$$\begin{aligned}
\mathrm{Var}(\sum_{i=1}^{n}(B_{t_i} - B_{t_{i-1}})^2) &= \sum_{i=1}^{n}(E((B_{t_i} - B_{t_{i-1}})^4)) - (E((B_{t_i} - B_{t_{i-1}})^2))^2) \\
&= \sum_{i=1}^{n} 3(t_i - t_{i-1})^2 - (t_i - t_{i-1})^2 \\
&= \sum_{i=1}^{n} 2(t_i - t_{i-1})(t_i - t_{i-1}) \\
&\leq \sum_{i=1}^{n} 2\|P\|(t_i - t_{i-1}) = 2\|P\|t
\end{aligned}$$

which approaches 0 as $\|P\|$ approaches 0. Since variance is nonnegative, $\mathrm{Var}([B,B]_t) = 0$. □

Lemma 2.6. *Let X, Y be two independent standard Brownian motions. Then $[X, Y]_t = 0$.*

Proof. We will show that $E([X, Y]_t) = 0$ and $\mathrm{Var}([X, Y]_t) = 0$.

$$E(\sum_{i=1}^{n}(X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}})) = \sum_{i=1}^{n} E(X_{t_i} - X_{t_{i-1}})E(Y_{t_i} - Y_{t_{i-1}}) = 0$$

so $E([X, Y]_t) = 0$. For two independent variables A, B each with mean 0, $\mathrm{Var}(AB) = E(A^2B^2) - (E(AB))^2 = E(A^2)E(B^2) = \mathrm{Var}(A)\mathrm{Var}(B)$. So for a given $\|P\|$

$$\begin{aligned}\mathrm{Var}(\sum_{i=1}^{n}(X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}})) &= \sum_{i=1}^{n} \mathrm{Var}(X_{t_i} - X_{t_{i-1}})\mathrm{Var}(Y_{t_i} - Y_{t_{i-1}}) \\ &= \sum_{i=1}^{n}(t_i - t_{i-1})(t_i - t_{i-1}) \\ &\leq \sum_{i=1}^{n} \|P\|(t_i - t_{i-1}) = \|P\|t\end{aligned}$$

which approaches 0 as $\|P\|$ approaches 0. Since variance is nonnegative, $\mathrm{Var}([X, Y]_t) = 0$.

□

Definition 2.7. A sequence X_t is a simple process if there exist times $0 = t_0 < t_1 < \ldots < t_n < \infty$ and random variables Y_j for $j = 0, 1, \ldots, n$ such that for $t \in [t_j, t_{j+1})$ $X_t = Y_j$

Definition 2.8. Let X_t be a simple process as in the above definition. We then define

$$\int_0^t X_s \, dB_s = \sum_{i=0}^{n-1} Y_i(B_{t_{i+1}} - B_{t_i})$$

Theorem 2.9. *Suppose X_t is a continuous, bounded process. Then there exists a sequence of simple, bounded processes $X_t^{(n)}$ such that*

$$\lim_{n\to\infty} \int_0^t E((\|X_s - X_s^{(n)}\|)^2) \, ds = 0$$

for all t.

Remark 2.10. From the above theorem, we can expand the stochastic integral from simple processes to continuous bounded processes. For a continuous, bounded process X_t we choose a sequence of simple, bounded processes $X_t^{(n)}$ satisfying the above theorem and then define

$$\int_0^t X_s \, dB_s = \lim_{n\to\infty} \int_0^t X_s^{(n)} \, dB_s$$

Theorem 2.11. *Ito's Formula: Let B_t be a standard Brownian motion and f be a C^2 function. Then*

$$f(B_t) = f(B_0) + \int_0^t f'(B_s) \, dB_t + \frac{1}{2}\int_0^t f''(B_s) \, ds$$

Remark 2.12. We write Ito's formula in terms of stochastic integrals because these are better defined (Brownian motions are nowhere differentiable) compared to the shorthand differential form $df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt$. We do however, use the shorthand differential form throughout the paper in place of the well defined stochastic integrals.

Theorem 2.13 (Multivariate Ito's Formula)**.** *Let X, Y be two stochastic processes and f be a C^2 function. Then,*

$$\begin{aligned} df(X,Y) = &\frac{\partial f(X,Y)}{\partial X}dX + \frac{\partial f(X,Y)}{\partial Y}dY \\ &+ \frac{1}{2}\left(\frac{\partial^2 f(X,Y)}{\partial X^2}d[X,X] + 2\frac{\partial^2 f(X,Y)}{\partial X \partial Y}d[X,Y] + \frac{\partial^2 f(X,Y)}{\partial Y^2}d[Y,Y]\right) \end{aligned}$$

Remark 2.14. Let X, Y be two stochastic processes. Then $d(XY) = XdY + (dX)Y + d[X,Y]$. This follows from the above theorem. This order is important when dealing with matrix stochastic processes as we will be later in this paper. When writing $dXdY$ we in fact mean $d[X,Y]$. Similarly, $dXYdZ$ for matrices means $dXYdZ_{ij} = \sum_{k,l} dX_{ik}Y_{kl}dZ_{lj} = \sum_{k,l} Y_{kl}dX_{ik}dZ_{lj} = \sum_{k,l} Y_{kl}d[X_{ik}Z_{lj}]$

3. Wishart Processes

Definition 3.1. A Brownian matrix of dimension $n \times p$ is a $n \times p$ matrix process whose entries are independent Brownian motions.

Definition 3.2. X_t is a Wishart process of dimension p, index n, and initial state X_0 if $X_t = B_t^\top B_t$ where B_t is a Brownian matrix of dimension $n \times p$ and $X_0 = D^\top D$ for some $n \times p$ matrix D.

Theorem 3.3. *Let X_t be a Wishart process of dimension p and index n. Then X_t satisfies the stochastic differential equation*

$$dX_t = \sqrt{X_t}dW_t + dW_t^\top\sqrt{X_t} + nI_p dt$$

where W_t is a $p*p$ Brownian matrix and I_p is the $p*p$ identity matrix. Furthermore, $\sqrt{X_t}$ is the symmetric matrix C such that $C^2 = X$

Proof. We can diagonalize $X = QDQ^{-1}$ where Q is the matrix whose columns are the eigenvectors of X and D is the diagonal matrix whose diagonal entries are the corresponding eigenvalues of X. We then have $\sqrt{X} = QD^{1/2}Q^{-1}$ and consequently $\sqrt{X}^{-1} = QD^{-1/2}Q^{-1}$. Thus

$$\begin{aligned} \sqrt{X}^{-1}B^\top dB = (QD^{-1/2}Q^{-1})(XB^{-1})dB &= QD^{-1/2}Q^{-1}QDQ^{-1}B^{-1}dB \\ &= \sqrt{X}B^{-1}dB = dW \end{aligned}$$

So we have $B^\top dB = \sqrt{X}dW$ and $(dB)^\top B = (B^\top dB)^\top = (\sqrt{X}dW)^\top = dW^\top\sqrt{X}^\top = dW^\top\sqrt{X}$ since $\sqrt{X}$ is symmetric. We then have by Remark 2.14

$$dX = d(B^\top B) = B^\top dB + dB^\top B + d[B^\top, B] = \sqrt{X}dW + dW^\top\sqrt{X} + d[B^\top, B]$$

Once we show $[B^\top, B]_t = ntI_p$ we are done. We have

$$[B^\top, B]_t = \lim_{\|P\|\to 0} \sum_{i=1}^{n} (B_{t_i}^\top - B_{t_{i-1}}^\top)(B_{t_i} - B_{t_{i-1}})$$

which is a $p * p$ matrix whose entry in the i row and j column is

$$\sum_{k=1}^{n} \lim_{\|P\|\to 0} \sum_{i=1}^{N} (b_{ki}(t_i) - b_{ki}(t_{i-1}))(b_{kj}(t_i) - b_{kj}(t_{i-1})) = \sum_{k=1}^{n} [b_{ki}, b_{kj}]_t$$

where $b_{ki}(t)$ is the entry in the k row and i column of $B(t)$. Since $[b_{ki}, b_{kj}]_t = t$ only if $i = j$ and 0 otherwise (by Lemmas 2.5 and 2.6), $d[B^\top, B]_t$ is a $p * p$ matrix with nonzero entries only along the diagonal which are nt so $[B^\top, B]_t = ntI_p$. □

Remark 3.4. A Wishart process X_t of dimension $p = 1$ and index n satisfies the stochastic differential equation

$$dX_t = 2\sqrt{X_t} dW_t + ndt$$

which is a squared Bessel process with index n.

Theorem 3.5. *Let X_t be a Wishart process of dimension p and index n. Let $\lambda_i(t)$ be the p eigenvalues of X_t. Suppose the p $\lambda_i(0)$'s are distinct. Then the $\lambda_i(t)$'s satisfy the stochastic differential equation*

$$d\lambda_i(t) = 2\sqrt{\lambda_i(t)} dW_i(t) + ndt + \sum_{k\neq i} \frac{\lambda_i(t) + \lambda_k(t)}{\lambda_i(t) - \lambda_k(t)} dt$$

where the W_i are independent Brownian motions.

Proof. We follow the proof in [2]. We can choose orthogonal Q_t to diagonalize X_t such that $Q_t^\top X_t Q_t = diag(\lambda_i(t)) = \Lambda_t$ where $\lambda_i(t)$ are the eigenvalues of X_t and the columns of Q_t are the corresponding eigenvectors. Thus by Remark 2.14,

$$d\Lambda = Q^\top X dQ + d(Q^\top X)Q + d[Q^\top X, Q] = Q^\top X dQ + d(Q^\top X)Q + [d(Q^\top X), dQ]$$

Since $d(Q^\top X) = Q^\top dX + dQ^\top X + d[Q^\top, X]$

$$d\Lambda = Q^\top X dQ + Q^\top dX Q + dQ^\top X Q + dQ^\top dX Q + Q^\top dX dQ + dQ^\top X dQ$$

We now introduce the following matrices

$$dA = Q^\top dQ + \frac{1}{2} dQ^\top dQ, \quad d\Gamma = \frac{1}{2}(dA)(dA)$$
$$d\Phi = Q^\top dX Q dA, \quad \text{and} \quad d\mu = (dA)^\top \Lambda (dA)$$

and see that

$$d\Phi^\top = (QdA)^\top (Q^\top dX)^\top = dA^\top Q^\top dX^\top Q = dA^\top Q^\top dX Q.$$

Note that we can write $dQ = QdA + \frac{1}{2} QdAdA$ and thus $dQ = Q(dA + d\Gamma)$ Note for $d\Phi^\top$ we used the fact that since X is always symmetric, dX is always symmetric

and thus $(dX)^\top = dX$. We have that A is always skew symmetric. To show this, since $A_0 = 0$, we merely need to show dA is always skew symmetric.

$$\begin{aligned} dA_{ij} + dA_{ji} &= \sum_{k=1}^{p} Q_{ki} dQ_{kj} + \frac{1}{2} dQ_{ki} dQ_{kj} + Q_{kj} dQ_{ki} + \frac{1}{2} dQ_{kj} dQ_{ki} \\ &= \sum_{k=1}^{p} (Q_{ki} + dQ_{ki})(Q_{kj} + dQ_{kj}) - \sum_{k=1}^{p} Q_{ki} Q_{kj} = 0 \end{aligned}$$

since Q is always orthogonal (so $Q_t + dQ_t$ is also orthogonal) and any two columns in an orthogonal matrix are orthogonal. Since dA is always skew symmetric, $d\Gamma$ is always symmetric. We rewrite $d\Lambda$ as

$$\begin{aligned} d\Lambda = Q^\top XQ(dA + d\Gamma) + Q^\top dXQ + (dA^\top + d\Gamma^\top) Q^\top XQ + (dA^\top + d\Gamma^\top) Q^\top dXQ \\ + Q^\top dXQ(dA + d\Gamma) + (dA^\top + d\Gamma^\top) Q^\top XQ(dA + d\Gamma) \end{aligned}$$

Since $Q^\top XQ = \Lambda$

$$\begin{aligned} d\Lambda &= \Lambda dA + \Lambda d\Gamma + Q^\top dXQ + dA^\top \Lambda + d\Gamma^\top \Lambda + dA^\top Q^\top dXQ \\ &\quad + Q^\top dXQdA + dA^\top \Lambda dA \\ &= Q^\top dXQ + (dA^\top \Lambda + \Lambda dA) + (d\Gamma^\top \Lambda + \Lambda d\Gamma) + d\Phi^\top + d\Phi + d\mu \end{aligned}$$

Let $q_{ij}, da_{ij}, d\gamma_{ij}, d\phi_{ij}, d\mu_{ij}, \lambda_{ij}$ denote the entry in the i row and j column in $Q, dA, d\Gamma, d\Phi, d\mu, \Lambda$, respectively and as in the theorem statement λ_i are the eigenvalues of X. Since Λ is diagonal with $\Lambda_{ii} = \lambda_i$ we have $d\Lambda_{ii} = d\lambda_i$. Looking at diagonal entries

$$\begin{aligned} d\Lambda_{ii} &= \sum_{k=1}^{p} \sum_{l=1}^{p} q_{ki} dx_{kl} q_{li} + \sum_{k=1}^{p} da_{ki} \lambda_{ki} + \sum_{k=1}^{p} \lambda_{ik} da_{ki} + \sum_{k=1}^{p} d\gamma_{ki} \lambda_{ki} + \sum_{k=1}^{p} \lambda_{ik} d\gamma_{ki} \\ &\quad + d\phi_{ii} + d\phi_{ii} + d\mu_{ii} \\ &= \sum_{k=1}^{p} \sum_{l=1}^{p} q_{ki} dx_{kl} q_{li} + da_{ii} \lambda_{ii} + \lambda_{ii} da_{ii} + d\gamma_{ii} \lambda_{ii} + \lambda_{ii} d\gamma_{ii} + 2d\phi_{ii} + d\mu_{ii} \end{aligned}$$

Since dA is skew symmetric, $da_{ii} = 0$ so

$$\begin{aligned} d\lambda_i &= \sum_{k=1}^{p} \sum_{l=1}^{p} q_{ki} dx_{kl} q_{li} + 2\lambda_i d\gamma_{ii} + 2d\phi_{ii} + d\phi_{ii} \\ &= 2\sqrt{\lambda_i} dW_i + ndt + 2\lambda_i d\gamma_{ii} + 2d\phi_{ii} + d\phi_{ii} \end{aligned}$$

Looking at the off-diagonal entries,

$$
\begin{aligned}
d\Lambda_{ij} = 0 &= \sum_{k=1}^{p}\sum_{l=1}^{p} q_{ki}dx_{kl}q_{lj} + \sum_{k=1}^{p} da_{ki}\lambda_{kj} + \sum_{k=1}^{p}\lambda_{ik}da_{kj} + \sum_{k=1}^{p} d\gamma_{ki}\lambda_{kj} \\
&+ \sum_{k=1}^{p}\lambda_{ik}d\gamma_{kj} + d\phi_{ji} + d\phi_{ij} + d\mu_{ij} \\
&= \sum_{k=1}^{p}\sum_{l=1}^{p} q_{ki}dx_{kl}q_{lj} + da_{ji}\lambda_{jj} + \lambda_{ii}da_{ij} \\
&+ d\gamma_{ji}\lambda_{jj} + \lambda_{ii}d\gamma_{ij} + d\phi_{ji} + d\phi_{ij} + d\mu_{ij}
\end{aligned}
$$

Since dA is skew symmetric and $d\Gamma$ is symmetric, $da_{ji} = -da_{ij}$ and $d\gamma_{ij} = d\gamma_{ji}$ so

$$
0 = \sum_{k=1}^{p}\sum_{l=1}^{p} q_{ki}dx_{kl}q_{lj} + (\lambda_i - \lambda_j)da_{ij} + (\lambda_i + \lambda_j)d\gamma_{ij} + d\phi_{ij} + d\phi_{ji} + d\mu_{ij}
$$

and thus,

$$
(\lambda_j - \lambda_i)da_{ij} = \sum_{k=1}^{p}\sum_{l=1}^{p} q_{ki}dx_{kl}q_{lj} + (\lambda_i + \lambda_j)d\gamma_{ij} + d\phi_{ij} + d\phi_{ji} + d\mu_{ij}
$$

which gives

$$
d[a_{ij}, a_{ij}] = \frac{\lambda_i + \lambda_j}{(\lambda_j - \lambda_i)^2}dt, \quad \text{and} \quad da_{ij} = \sqrt{\frac{\lambda_i + \lambda_j}{(\lambda_j - \lambda_i)^2}}dW_{ij}
$$

Thus,

$$
\begin{aligned}
d\gamma_{ii} &= \frac{1}{2}\sum_{k=1}^{p} d[a_{ik}, a_{ki}] = -\frac{1}{2}\sum_{k=1}^{p} d[a_{ik}, a_{ik}] = -\frac{1}{2}\sum_{k\neq i}\frac{\lambda_i + \lambda_k}{(\lambda_k - \lambda_i)^2}dt \\
d\mu_{ii} &= \sum_{k=1}^{p}\sum_{j=1}^{p}\lambda_{jk}d[a_{ji}, a_{ki}] = \sum_{k=1}^{p} d[a_{ki}, a_{ki}]\lambda_k = \sum_{k\neq i}\frac{\lambda_i\lambda_k + \lambda_k^2}{(\lambda_k - \lambda_i)^2}dt
\end{aligned}
$$

We also have from the equation derived from taking off-diagonal entries that

$$
(\lambda_i - \lambda_k)d[a_{ki}, x_{kl}] = (\lambda_i + \lambda_j)(q_{ki}q_{lj} + q_{li}q_{kj})dt
$$

and

$$
d\phi_{ii} = \sum_{k,l,r=1}^{p} q_{ki}q_{lr}d[x_{kl}, a_{ri}] = \sum_{k\neq i}\frac{\lambda_i + \lambda_k}{\lambda_i - \lambda_k}dt.
$$

Thus,

$$
\begin{aligned}
2\lambda_i d\gamma_{ii} + 2d\phi_{ii} + d\phi ii &= -\sum_{k\neq i}\frac{\lambda_i^2 + \lambda_i\lambda_k}{(\lambda_k - \lambda_i)^2}dt + \sum_{k\neq i}\frac{2\lambda_i + 2\lambda_k}{\lambda_i - \lambda_k}dt + \sum_{k\neq i}\frac{\lambda_i\lambda_k + \lambda_k^2}{(\lambda_k - \lambda_i)^2}dt \\
&= \sum_{k\neq i}\frac{\lambda_i^2 - \lambda_k^2}{(\lambda_i - \lambda_k)^2}dt \\
&= \sum_{k\neq i}\frac{\lambda_i + \lambda_k}{\lambda_i - \lambda_k}dt
\end{aligned}
$$

which finally gives

$$d\lambda_i = 2\sqrt{\lambda_i}dW_i + ndt + \sum_{k\neq i} \frac{\lambda_i + \lambda_k}{\lambda_i - \lambda_k} dt$$

□

Remark 3.6. Note that the matrices $d\Gamma, d\Phi, d\mu$ are all diagonal. Furthermore, the eigenvalues behave like squared bessel processes of index n with the additional repulsion force displayed in the last $\sum_{k\neq i} \frac{\lambda_i(t)+\lambda_k(t)}{\lambda_i(t)-\lambda_k(t)} dt$ term. In fact, the eigenvalues never collide in finite time [2]. We have that since $X_t = B_t^\top B_t$, X_t is always positive semidefinite and thus has nonnegative eigenvalues at all times. Furthermore, the eigenvalues remain positive [3].

Theorem 3.7. *Let $X(t)$ be a Wishart process of dimension p and index n and the p $\lambda_i(0)'s$ are distinct. Let $Q(t)$ be the matrix with columns the eigenvectors of $X(t)$. Let $q_{ij}(t)$ denote the entry in the i row and j column of $Q(t)$. Then $q_{ij}(t)$ satisfies the stochastic differential equation*

$$dq_{ij}(t) = \sum_{k\neq j} q_{ik}(t)\sqrt{\frac{\lambda_k(t)+\lambda_j(t)}{(\lambda_k(t)-\lambda_j(t))^2}} dW_{kj}(t) - \frac{1}{2} q_{ij}(t) \sum_{k\neq j} \frac{\lambda_k(t)+\lambda_j(t)}{(\lambda_k(t)-\lambda_j(t))^2} dt$$

where the $dW_{kj}(t)'s$ are independent Brownian motions.

Proof. As in Theorem 3.5, we can diagonalize $X_t = Q_t \Lambda_t Q_t^\top$ where Λ_t is diagonal with entries the eigenvalues of X_t and Q_t orthogonal with columns the corresponding eigenvectors. We can write $\Lambda_t = Q_t^\top X_t Q_t$. From the definition of dA and the expressions for da_{ij}^2 and da_{ij} in Theorem 3.5's proof we have

$$\begin{aligned} dq_{ij} &= \sum_{k=1}^{p} q_{ik} da_{kj} + \sum_{k=1}^{p}\sum_{l=1}^{p} \frac{1}{2} q_{il} da_{lk} da_{kj} = \sum_{k\neq j} q_{ik} da_{kj} + \frac{1}{2}\sum_{k=1}^{p} q_{ik} da_{jk} da_{kj} \\ &= \sum_{k\neq j} q_{ik} da_{kj} - \frac{1}{2}\sum_{k\neq j} q_{ik} da_{kj}^2 \\ &= \sum_{k\neq j} q_{ik}\sqrt{\frac{\lambda_k+\lambda_j}{(\lambda_k-\lambda_j)^2}} dW_{kj} - \frac{1}{2} q_{ij} \sum_{k\neq j} \frac{\lambda_k+\lambda_j}{(\lambda_k-\lambda_j)^2} dt \end{aligned}$$

□

4. Simulations

Using the stochastic differential equation in Theorem 3.5, we simulate in the appendix the eigenvalues of a Wishart process of dimension $p = 10$ and index $n = 20$ from $t = 0$ to $t = 100$. We clearly have to discretize this process and choose time steps $\frac{100-0}{N} = \frac{100}{N}$ for various values of N. Thus we generate $dW_i \sim N(0, 100/N)$ at each step. We use R to run these simulations and plot the processes. We use N=20000 for the simulations below. The first two simulations are the eigenvalue processes with the initial B_0 matrix having iid entries drawn from a normal distribution with mean 10 and standard deviation 10 (random choice to have initial eigenvalues spaced out from one another). The next two simulations are the eigenvalue processes with the initial B_0 matrix having iid entries drawn from a normal distribution with mean 0

and standard deviation 1 (initial eigenvalues are very close to each other). The last two simulations are the eigenvalue processes again with the initial B_0 matrix having iid entries drawn from a normal distribution with mean 0 and standard deviation 1 (initial eigenvalues are very close to each other). However, the plots show the eigenvalue processes from t=(100/20000)*300 to t=(100/20000)*600. This is not only presented to show a close-up of the processes during a certain time window when it is hard to distinguish them in the 3rd and 4th plots but also to again show the eigenvalue processes for a matrix with eigenvalues spaced apart.

Clearly the actual eigenvalue processes will never collide since when getting the eigenvalues for $X(t)$ at each time step, R returns the eigenvalues in order. However, when simulating these eigenvalue processes, there is a chance of collision due to the discretization of this process. Decreasing the time step greatly reduces the chance of this happening and large jumps when two eigenvalues come very close to each other. Even with extremely small time steps there is still a chance of collision due to the random dW term which is different for each eigenvalue and R rounding errors. When N was increased to 50000 and 100000 (simulations not pictured) there were no visible collisions. In the last two plots, we can also see much more clearly the non-colliding behavior of the eigenvalues.

5. Financial Applications

The structure of Wishart processes (the transpose of a Brownian matrix multiplied by the same Brownian matrix) result in the natural investigation at attempting to model stochastic covariance matrices with Wishart processes. Many processes in finance are frequently modeled by stochastic processes due to the randomness of the market. While the famous Black-Scholes model for pricing options does assume the underlying stock follows a geometric Brownian motion, it also assumes the volatility of the underlying stock is constant which has repeatedly been shown to not be the case–leading to attempts to model volatilities stochastically. In [4], Fonseca attempts to model the $n \times 1$ vector asset S_t with the stochastic differential equation below

$$dS_t = diag(S_t)(r\mathbf{1}dt + \sqrt{X_t}W_t)$$

where X_t is a covariance matrix following a Wishart process and $\mathbf{1}$ is the $n \times 1$ vector consisting of only 1s. This model is better at pricing options than Black Scholes as it reproduces some of the nonconstant volatility observed in real market conditions such as the volatility skews and smiles. For the single asset, Fonseca models the asset S_t with the below stochastic differential equation

$$dS_t = S_t(rdt + Tr(\sqrt{X_t}dW_t))$$

where X_t is again Wishart compared to the traditional geometric Brownian motion $dS_t = S_t(\mu dt + \sigma dW_t)$ (where the volatility is constant) used to model assets.

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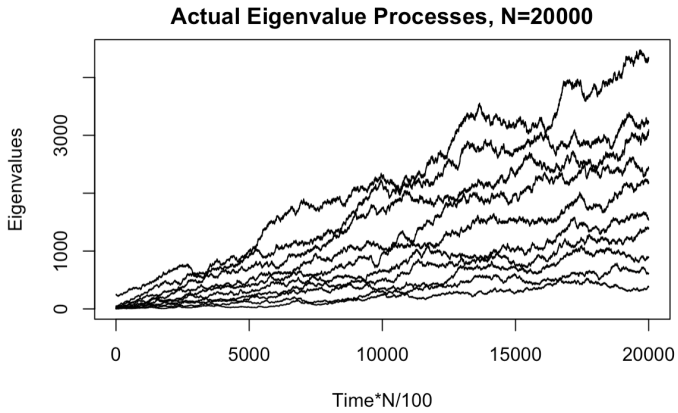
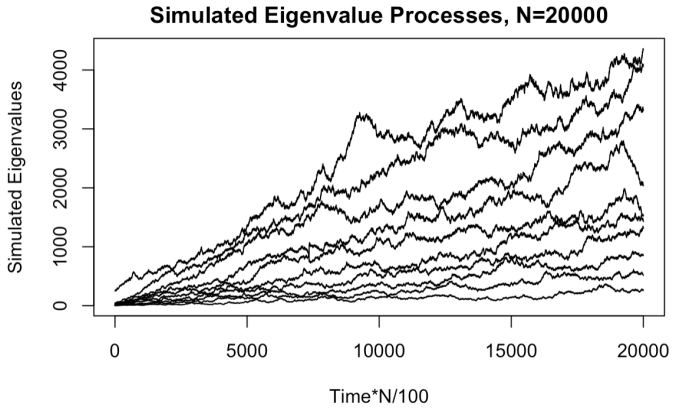
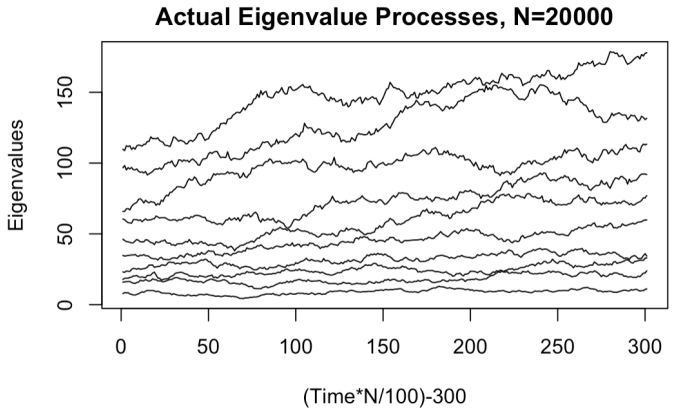
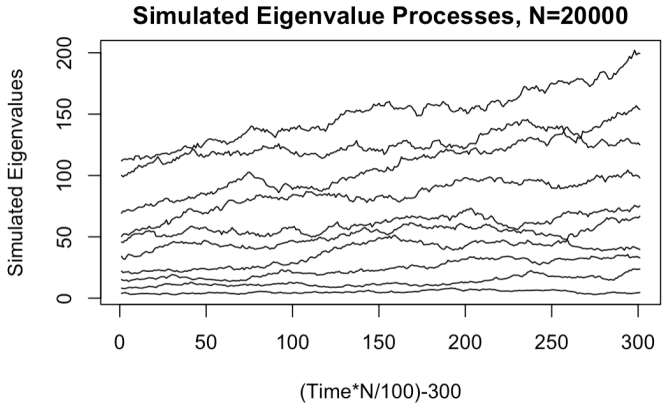
6. Appendix


Actual Eigenvalue Processes, N=20000
Eigenvalues
0
1000
3000
0
5000
10000
15000
20000
Time*N/100


Simulated Eigenvalue Processes, N=20000
Simulated Eigenvalues
0
1000
2000
3000
4000
0
5000
10000
15000
20000
Time*N/100


Actual Eigenvalue Processes, N=20000
Eigenvalues
0
50
100
150
0
50
100
150
200
250
300
(Time*N/100)-300


Simulated Eigenvalue Processes, N=20000
Simulated Eigenvalues
0
50
100
150
200
0
50
100
150
200
250
300
(Time*N/100)-300