WISHART PROCESSES AND FINANCIAL APPLICATIONS

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ABSTRACT. Following a brief discussion on stochastic calculus preliminaries we introduce Wishart processes and derive the stochastic differential equations for these processes and their eigenvalues and eigenvectors. Some simulations of the eigenvalue processes are presented and financial applications of Wishart processes are discussed.

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1. INTRODUCTION

The real world operates on randomness from the behavior of atoms to the daily weather. Many believe the financial markets to also operate largely on randomness with a common model for stock prices being a Geometric Brownian Motion. Financial derivatives (such as a stock option) whose value depends on the underlying security (such as a stock) have been modeled stochastically as early as the 1960s with the famous Black-Scholes model. This model, however, has its drawbacks one being that it assumes the underlying stock’s volatility to be constant—a consequence of using GBM to model the stock’s price. This has empirically shown to not be true with a famous example being the volatility smile of options of different strike prices for the same underlying stock. As a result, it is natural to attempt to model the volatility itself with several stochastic approaches being taken. An asset is rarely traded by itself instead being traded with other highly correlated assets (and sometimes uncorrelated assets depending on the strategy) to hedge risk. It thus makes sense to study the pairwise correlation processes of a group of assets and perhaps consequently try to model their covariance matrix stochastically. The field of Random Matrix Theory largely originated in the 1950s when Eugene Wigner attempted to describe atomic energy levels using the eigenvalues of certain random matrices. Wishart matrices are a certain type of random matrix and their structure
(described in more detail in the later sections) results in their usefulness in statistics and sample covariance matrices. It thus seems reasonable to further explore Wishart processes when attempting to model the relationship between assets and deciding how to appropriately price them.

2. Stochastic Calculus Preliminaries

We draw most of these preliminaries from [1].

**Definition 2.1.** A stochastic process \( B_t \) is a Brownian motion with drift \( \mu \) and variance \( \sigma^2 \) starting at the origin if

1. \( B_0 = 0 \)
2. \( B_t \) is a continuous function of \( t \)
3. \( B_t \) has independent increments i.e. if \( s_1 < s_2 \leq t_1 < t_2 \) then the random variables \( B_{s_2} - B_{s_1} \) and \( B_{t_2} - B_{t_1} \) are independent of one another
4. For \( t > s \), \( B_t - B_s \sim N(\mu(t - s), \sigma^2(t - s)) \)

**Remark 2.2.** A standard Brownian motion is a Brownian motion with drift 0 and variance 1. For the rest of this paper we will be working with standard Brownian motions.

**Definition 2.3.** The covariation of two processes \( X \) and \( Y \) is

\[
[X, Y]_t = \lim_{\|P\| \to 0} \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}})
\]

where \( P \) is a partition \( [t_0, t_1, \ldots, t_n] \) of \( [0, t] \) and \( \|P\| \) is the mesh of the partition.

**Remark 2.4.** The quadratic variation of a process \( X_t \) is \( [X, X]_t \)

**Lemma 2.5.** Let \( B \) be a standard Brownian motion. Then \( [B, B]_t = t \)

**Proof.** We show that \( E([B, B]_t) = t \) and \( \text{Var}([B, B]_t) = 0 \)

\[
E(\sum_{i=1}^{n} (B_{t_i} - B_{t_{i-1}})^2) = \sum_{i=1}^{n} (\text{Var}(B_{t_i} - B_{t_{i-1}}) + (E(B_{t_i} - B_{t_{i-1}}))^2) = \sum_{i=1}^{n} t_i - t_{i-1} = t
\]

so \( E([B, B]_t) = t \). For a given \( \|P\| \),

\[
\text{Var}(\sum_{i=1}^{n} (B_{t_i} - B_{t_{i-1}})^2) = \sum_{i=1}^{n} (E((B_{t_i} - B_{t_{i-1}})^4)) - (E((B_{t_i} - B_{t_{i-1}})^2))^2
\]

\[
= \sum_{i=1}^{n} 3(t_i - t_{i-1})^2 - (t_i - t_{i-1})^2
\]

\[
= \sum_{i=1}^{n} 2(t_i - t_{i-1})(t_i - t_{i-1})
\]

\[
\leq \sum_{i=1}^{n} 2\|P\|(t_i - t_{i-1}) = 2\|P\|t
\]

which approaches 0 as \( \|P\| \) approaches 0. Since variance is nonnegative, \( \text{Var}([B, B]_t) = 0 \). □
Lemma 2.6. Let $X, Y$ be two independent standard Brownian motions. Then $[X, Y]_t = 0$.

Proof. We will show that $E([X, Y]_t) = 0$ and $\text{Var}([X, Y]_t) = 0$.

$$E(\sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}})) = \sum_{i=1}^{n} E(X_{t_i} - X_{t_{i-1}})E(Y_{t_i} - Y_{t_{i-1}}) = 0$$

so $E([X, Y]_t) = 0$. For two independent variables $A, B$ each with mean 0, $\text{Var}(AB) = E(A^2B^2) - (E(AB))^2 = E(A^2)E(B^2) = \text{Var}(A)\text{Var}(B)$. So for a given $\|P\|$

$$\text{Var}(\sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}})) = \sum_{i=1}^{n} \text{Var}(X_{t_i} - X_{t_{i-1}})\text{Var}(Y_{t_i} - Y_{t_{i-1}}) = \sum_{i=1}^{n} (t_i - t_{i-1})(t_i - t_{i-1}) \leq \sum_{i=1}^{n} \|P\|(t_i - t_{i-1}) = \|P\|t$$

which approaches 0 as $\|P\|$ approaches 0. Since variance is nonnegative, $\text{Var}([X, Y]_t) = 0$.

□

Definition 2.7. A sequence $X_t$ is a simple process if there exist times $0 = t_0 < t_1 < \ldots < t_n < \infty$ and random variables $Y_j$ for $j = 0, 1, \ldots, n$ such that for $t \in [t_j, t_{j+1})$ $X_t = Y_j$.

Definition 2.8. Let $X_t$ be a simple process as in the above definition. We then define

$$\int_0^t X_s dB_s = \sum_{i=0}^{n-1} Y_i (B_{t_{i+1}} - B_{t_i})$$

Theorem 2.9. Suppose $X_t$ is a continuous, bounded process. Then there exists a sequence of simple, bounded processes $X^{(n)}_t$ such that

$$\lim_{n \to \infty} \int_0^t E(\|X_s - X^{(n)}_s\|^2) ds = 0$$

for all $t$.

Remark 2.10. From the above theorem, we can expand the stochastic integral from simple processes to continuous bounded processes. For a continuous, bounded process $X_t$, we choose a sequence of simple, bounded processes $X^{(n)}_t$ satisfying the above theorem and then define

$$\int_0^t X_s dB_s = \lim_{n \to \infty} \int_0^t X^{(n)}_s dB_s$$

Theorem 2.11. Ito’s Formula: Let $B_t$ be a standard Brownian motion and $f$ be a $C^2$ function. Then

$$f(B_t) = f(B_0) + \int_0^t f'(B_s) dB_s + \frac{1}{2} \int_0^t f''(B_s) ds$$
Remark 2.12. We write Ito’s formula in terms of stochastic integrals because these are better defined (Brownian motions are nowhere differentiable) compared to the shorthand differential form $df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt$. We do however, use the shorthand differential form throughout the paper in place of the well defined stochastic integrals.

Theorem 2.13 (Multivariate Ito’s Formula). Let $X, Y$ be two stochastic processes and $f$ be a $C^2$ function. Then,

$$df(X, Y) = \frac{\partial f(X, Y)}{\partial X}dX + \frac{\partial f(X, Y)}{\partial Y}dY + \frac{1}{2} \left( \frac{\partial^2 f(X, Y)}{\partial X^2}d[X, X] + 2\frac{\partial^2 f(X, Y)}{\partial X\partial Y}d[X, Y] + \frac{\partial^2 f(X, Y)}{\partial Y^2}d[Y, Y] \right)$$

Remark 2.14. Let $X, Y$ be two stochastic processes. Then $d(XY) = XdY + (dX)Y + d[X, Y]$. This follows from the above theorem. This order is important when dealing with matrix stochastic processes as we will be later in this paper. When writing $dXdY$ we in fact mean $d[X, Y]$. Similarly, $dXYdZ$ for matrices means $dXYdZ_{ij} = \sum_{k,l}dX_{ik}Y_{kl}dZ_{lj} = \sum_{k,l}Y_{kl}dX_{ik}dZ_{lj} = \sum_{k,l}Y_{kl}[dX_{ik}Z_{lj}]$

3. Wishart Processes

Definition 3.1. A Brownian matrix of dimension $n \times p$ is a $n \times p$ matrix process whose entries are independent Brownian motions.

Definition 3.2. $X_t$ is a Wishart process of dimension $p$, index $n$, and initial state $X_0$ if $X_t = B_t^\top B_t$ where $B_t$ is a Brownian matrix of dimension $n \times p$ and $X_0 = D^\top D$ for some $n \times p$ matrix $D$.

Theorem 3.3. Let $X_t$ be a Wishart process of dimension $p$ and index $n$. Then $X_t$ satisfies the stochastic differential equation

$$dX_t = \sqrt{X_t}dW_t + dW_t^\top \sqrt{X_t} + nI_p dt$$

where $W_t$ is a $p \times p$ Brownian matrix and $I_p$ is the $p \times p$ identity matrix. Furthermore, $\sqrt{X_t}$ is the symmetric matrix $C$ such that $C^2 = X$.

Proof. We can diagonalize $X = QDQ^{-1}$ where $Q$ is the matrix whose columns are the eigenvectors of $X$ and $D$ is the diagonal matrix whose diagonal entries are the corresponding eigenvalues of $X$. We then have $\sqrt{X} = QD^{1/2}Q^{-1}$ and consequently $\sqrt{X^{-1}} = QD^{-1/2}Q^{-1}$. Thus

$$\sqrt{X^{-1}}B^\top dB = (QD^{-1/2}Q^{-1})(XB^{-1})dB = QD^{-1/2}Q^{-1}QDQ^{-1}B^{-1}dB = \sqrt{X}B^{-1}dB = dB$$

So we have $B^\top dB = \sqrt{X}dW$ and $(dB)^\top B = (B^\top dB)^\top = (\sqrt{X}dW)^\top = dW^\top \sqrt{X} = dW^\top \sqrt{X}$ since $\sqrt{X}$ is symmetric. We then have by Remark 2.14

$$dX = d(B^\top B) = B^\top dB + dB^\top B + d[B^\top, B] = \sqrt{X}dW + dW^\top \sqrt{X} + d[B^\top, B]$$

Once we show $[B^\top, B]_t = ntI_p$ we are done. We have
We now introduce the following matrices $d$:

$$[B^T, B]_t = \lim_{\|P\| \to 0} \sum_{i=1}^n (B_{ti}^T - B_{ti-1}^T)(B_{ti} - B_{ti-1})$$

which is a $p \times p$ matrix whose entry in the $i$ row and $j$ column is

$$\sum_{k=1}^n \lim_{\|P\| \to 0} \sum_{i=1}^N (b_{ki}(t_i) - b_{ki}(t_{i-1}))(b_{kj}(t_i) - b_{kj}(t_{i-1})) = \sum_{k=1}^n [b_{ki}, b_{kj}]_t$$

where $b_{ki}(t)$ is the entry in the $k$ row and $i$ column of $B(t)$. Since $[b_{ki}, b_{kj}]_t = t$ only if $i = j$ and 0 otherwise (by Lemmas 2.5 and 2.6), $d[B^T, B]_t$ is a $p \times p$ matrix with nonzero entries only along the diagonal which are $n t$ so $[B^T, B]_t = n t I_p$. 

**Remark 3.4.** A Wishart process $X_t$ of dimension $p = 1$ and index $n$ satisfies the stochastic differential equation

$$dX_t = 2\sqrt{X_t}dW_t + ndt$$

which is a squared Bessel process with index $n$.

**Theorem 3.5.** Let $X_t$ be a Wishart process of dimension $p$ and index $n$. Let $\lambda_i(t)$ be the $p$ eigenvalues of $X_t$. Suppose the $p \lambda_i(0)$'s are distinct. Then the $\lambda_i(t)$'s satisfy the stochastic differential equation

$$d\lambda_i(t) = 2\sqrt{\lambda_i(t)}dW_i(t) + ndt + \sum_{k \neq i} \frac{\lambda_i(t) + \lambda_k(t)}{\lambda_i(t) - \lambda_k(t)} dt$$

where the $W_i$ are independent Brownian motions.

**Proof.** We follow the proof in [2]. We can choose orthogonal $Q_t$ to diagonalize $X_t$ such that $Q_t^T X_t Q_t = \text{diag}(\lambda_i(t)) = \Lambda_t$ where $\lambda_i(t)$ are the eigenvalues of $X_t$ and the columns of $Q_t$ are the corresponding eigenvectors. Thus by Remark 2.14,

$$d\Lambda = Q^TXdQ + d(Q^TX)Q + d[Q^TX, Q] = Q^TXdQ + d(Q^TX)Q + [d(Q^TX), dQ]$$

Since $d(Q^TX) = Q^TdX + dQ^TX + d[Q^T, X]$$d\Lambda = Q^TXdQ + Q^TdXQ + dQ^TXQ + dQ^TXdQ + Q^TdXQ + Q^TXdQ$$d\Lambda = Q^TXdQ + Q^TdXQ + dQ^TXQ + dQ^TXdQ + Q^TdXQ + Q^TXdQ$.

We now introduce the following matrices

$$dA = Q^TdQ + \frac{1}{2}dQ^TdQ, \quad d\Gamma = \frac{1}{2}(dA)(dA)$$

$$d\Phi = Q^TdxQdA, \quad \text{and} \quad d\mu = (dA)^\top \Lambda(dA)$$

and see that

$$d\Phi = (QdA)^\top (Q^TdX)^\top = dA^TQ^TdX^\top Q = dA^TQ^TdXQ.$$ 

Note that we can write $dQ = QdA + \frac{1}{2}QdAdA$ and thus $dQ = Q(dA + d\Gamma)$. Note for $d\Phi^\top$ we used the fact that since $X$ is always symmetric, $dX$ is always symmetric.
and thus \((dX)^{\top} = dX\). We have that \(A\) is always skew symmetric. To show this, since \(A_0 = 0\), we merely need to show \(dA\) is always skew symmetric.

\[
dA_{ij} + dA_{ji} = \sum_{k=1}^{p} Q_{ki} dQ_{kj} + \frac{1}{2} dQ_{ki} dQ_{kj} + Q_{kj} dQ_{ki} + \frac{1}{2} dQ_{kj} dQ_{ki}
\]

\[
= \sum_{k=1}^{p} (Q_{ki} + dQ_{ki})(Q_{kj} + dQ_{kj}) - \sum_{k=1}^{p} Q_{ki} Q_{kj} = 0
\]

since \(Q\) is always orthogonal (so \(Q_t + dQ_t\) is also orthogonal) and any two columns in an orthogonal matrix are orthogonal. Since \(dA\) is always skew symmetric, \(d\Gamma\) is always symmetric. We rewrite \(d\Lambda\) as

\[
d\Lambda = Q^{\top} X Q (dA + d\Gamma) + Q^{\top} dX Q + (dA^{\top} + d\Gamma^{\top}) Q^{\top} X Q + (dA^{\top} + d\Gamma^{\top}) Q^{\top} dX Q
\]

\[
+ Q^{\top} dX Q (dA + d\Gamma) + (dA^{\top} + d\Gamma^{\top}) Q^{\top} X Q (dA + d\Gamma)
\]

Since \(Q^{\top} X Q = \Lambda\)

\[
d\Lambda = \Lambda dA + \Lambda d\Gamma + Q^{\top} dX Q + dA^{\top} \Lambda + d\Gamma^{\top} \Lambda + dA^{\top} Q^{\top} dX Q
\]

\[
+ Q^{\top} dX Q dA + dA^{\top} \Lambda dA
\]

\[
= dA^{\top} \Lambda + (d\Gamma^{\top} \Lambda + \Lambda d\Gamma) + d\Phi^{\top} + d\Phi + d\mu
\]

Let \(q_{ij}, da_{ij}, d\gamma_{ij}, d\phi_{ij}, d\mu_{ij}, \lambda_{ij}\) denote the entry in the \(i\) row and \(j\) column in \(Q, dA, d\Gamma, d\Phi, d\mu, \Lambda\), respectively and as in the theorem statement \(\lambda_i\) are the eigenvalues of \(X\). Since \(\Lambda\) is diagonal with \(\Lambda_{ii} = \lambda_i\) we have \(d\Lambda_{ii} = d\lambda_i\). Looking at diagonal entries

\[
d\Lambda_{ii} = \sum_{k=1}^{p} \sum_{l=1}^{p} q_{kl} dx_{kl} q_{li} + \sum_{k=1}^{p} da_{ki} \lambda_{ki} + \sum_{k=1}^{p} \lambda_{ki} da_{ki} + \sum_{k=1}^{p} d\gamma_{ki} \lambda_{ki} + \sum_{k=1}^{p} \lambda_{ki} d\gamma_{ki}
\]

\[
+ d\phi_{ii} + d\phi_{ii} + d\mu_{ii}
\]

\[
= \sum_{k=1}^{p} \sum_{l=1}^{p} q_{kl} dx_{kl} q_{li} + da_{ii} \lambda_{ii} + \lambda_{ii} da_{ii} + d\gamma_{ii} \lambda_{ii} + \lambda_{ii} d\gamma_{ii} + 2 d\phi_{ii} + d\mu_{ii}
\]

Since \(dA\) is skew symmetric, \(da_{ii} = 0\) so

\[
d\lambda_i = \sum_{k=1}^{p} \sum_{l=1}^{p} q_{kl} dx_{kl} q_{li} + 2 \lambda_i d\gamma_{ii} + 2 d\phi_{ii} + d\phi_{ii}
\]

\[
= 2 \sqrt{\lambda_i} dW_i + ndt + 2 \lambda_i d\gamma_{ii} + 2 d\phi_{ii} + d\phi_{ii}
\]

Looking at the off-diagonal entries,
Thus, 

\[ d\Lambda_{ij} = 0 = \sum_{k=1}^{p} \sum_{l=1}^{p} q_{ki} dx_{kl} q_{lj} + \sum_{k=1}^{p} da_{ki} \lambda_{kj} + \sum_{k=1}^{p} \lambda_{ik} da_{kj} + \sum_{k=1}^{p} d\gamma_{ki} \lambda_{kj} \]

\[ + \sum_{k=1}^{p} \lambda_{ik} d\gamma_{kj} + d\phi_{ji} + d\phi_{ij} + d\mu_{ij} \]

\[ = \sum_{k=1}^{p} \sum_{l=1}^{p} q_{ki} dx_{kl} q_{lj} + da_{ij} \lambda_{jj} + \lambda_{ii} da_{ij} \]

\[ + d\gamma_{ji} \lambda_{jj} + \lambda_{ii} d\gamma_{ij} + d\phi_{ji} + d\phi_{ij} + d\mu_{ij} \]

Since \( A \) is skew symmetric and \( d\Gamma \) is symmetric, \( da_{ji} = -da_{ij} \) and \( d\gamma_{ij} = d\gamma_{ji} \) so

\[ 0 = \sum_{k=1}^{p} \sum_{l=1}^{p} q_{ki} dx_{kl} q_{lj} + (\lambda_{i} - \lambda_{j}) da_{ij} + (\lambda_{i} + \lambda_{j}) d\gamma_{ij} + d\phi_{ij} + d\phi_{ji} + d\mu_{ij} \]

and thus,

\[ (\lambda_{j} - \lambda_{i}) da_{ij} = \sum_{k=1}^{p} \sum_{l=1}^{p} q_{ki} dx_{kl} q_{lj} + (\lambda_{i} + \lambda_{j}) d\gamma_{ij} + d\phi_{ij} + d\phi_{ji} + d\mu_{ij} \]

which gives

\[ d[a_{ij}, a_{ij}] = \lambda_{i} + \lambda_{j} \]

\[ \frac{(\lambda_{j} - \lambda_{i})^{2}}{dt} , \quad \text{and} \quad da_{ij} = \sqrt{\frac{\lambda_{i} + \lambda_{j}}{\lambda_{j} - \lambda_{i}}} \]

Thus,

\[ d\gamma_{ii} = \frac{1}{2} \sum_{k=1}^{p} d[a_{ik}, a_{ki}] = -\frac{1}{2} \sum_{k=1}^{p} d[a_{ik}, a_{ki}] = -\frac{1}{2} \sum_{k \neq i} \frac{\lambda_{i} + \lambda_{k}}{(\lambda_{k} - \lambda_{i})^{2}} dt \]

\[ d\mu_{ii} = \sum_{k=1}^{p} \sum_{j=1}^{p} \lambda_{jk} d[a_{ji}, a_{ki}] = \sum_{k=1}^{p} [a_{ki}, a_{ki}] \lambda_{k} = \sum_{k \neq i} \frac{\lambda_{i} \lambda_{k} + \lambda^{2}_{k}}{(\lambda_{k} - \lambda_{i})^{2}} dt \]

We also have from the equation derived from taking off-diagonal entries that

\[ (\lambda_{i} - \lambda_{k}) d[a_{ki}, x_{kl}] = (\lambda_{i} + \lambda_{j})(q_{ki} q_{lj} + q_{ki} q_{kj}) dt \]

and

\[ d\phi_{ii} = \sum_{k,l,r=1}^{p} q_{ki} q_{lj} d[x_{kl}, a_{ri}] = \sum_{k \neq i} \frac{\lambda_{i} + \lambda_{k}}{\lambda_{i} - \lambda_{k}} dt. \]

Thus,

\[ 2\lambda_{i} d\gamma_{ii} + 2d\phi_{ii} = \sum_{k \neq i} \frac{\lambda^{2}_{i} + \lambda_{i} \lambda_{k}}{(\lambda_{k} - \lambda_{i})^{2}} dt + \sum_{k \neq i} \frac{2\lambda_{i} + 2\lambda_{k}}{\lambda_{i} - \lambda_{k}} dt + \sum_{k \neq i} \frac{\lambda_{i} \lambda_{k} + \lambda^{2}_{k}}{(\lambda_{k} - \lambda_{i})^{2}} dt \]

\[ = \sum_{k \neq i} \frac{\lambda^{2}_{i} - \lambda^{2}_{k}}{(\lambda_{i} - \lambda_{k})^{2}} dt \]

\[ = \sum_{k \neq i} \frac{\lambda_{i} + \lambda_{k}}{\lambda_{i} - \lambda_{k}} dt \]
which finally gives

\[ d\lambda_i = 2\sqrt{\lambda_i}dW_i + ndt + \sum_{k \neq i} \frac{\lambda_i + \lambda_k}{\lambda_i - \lambda_k} dt \]

\[ \square \]

Remark 3.6. Note that the matrices \( d\Gamma, d\Phi, d\mu \) are all diagonal. Furthermore, the eigenvalues behave like squared Bessel processes of index \( n \) with the additional repulsion force displayed in the last \( \sum_{k \neq i} \frac{\lambda_i(t)+\lambda_k(t)}{\lambda_i(t)-\lambda_k(t)} dt \) term. In fact, the eigenvalues never collide in finite time [2]. We have that since \( X_t = B_t^\top B_t \), \( X_t \) is always positive semidefinite and thus has nonnegative eigenvalues at all times. Furthermore, the eigenvalues remain positive [3].

Theorem 3.7. Let \( X(t) \) be a Wishart process of dimension \( p \) and index \( n \) and the \( p \lambda_i(0) \)’s are distinct. Let \( Q(t) \) be the matrix with columns the eigenvectors of \( X(t) \). Let \( q_{ij}(t) \) denote the entry in the \( i \) row and \( j \) column of \( Q(t) \). Then \( q_{ij}(t) \) satisfies the stochastic differential equation

\[ dq_{ij}(t) = \sum_{k \neq j} q_{ik}(t) \sqrt{\frac{\lambda_k(t) + \lambda_j(t)}{\lambda_k(t) - \lambda_j(t)}} dW_{kj}(t) - \frac{1}{2} q_{ij}(t) \sum_{k \neq j} \frac{\lambda_k(t) + \lambda_j(t)}{(\lambda_k(t) - \lambda_j(t))^2} dt \]

where the \( dW_{kj}(t) \)'s are independent Brownian motions.

Proof. As in Theorem 3.5, we can diagonalize \( X_t = Q_t \Lambda_t Q_t^\top \) where \( \Lambda_t \) is diagonal with entries the eigenvalues of \( X_t \) and \( Q_t \) orthogonal with columns the corresponding eigenvectors. We can write \( \Lambda_t = Q_t^\top X_t Q_t \). From the definition of \( dA \) and the expressions for \( da_{ij}^2 \) and \( da_{ij} \) in Theorem 3.5’s proof we have

\[ dq_{ij} = \sum_{k=1}^p q_{ik} da_{kj} + \sum_{k=1}^p \sum_{l=1}^p q_{jl} da_{ik} \]

\[ = \sum_{k \neq j} q_{ik} da_{kj} - \frac{1}{2} \sum_{k \neq j} q_{ik} da_{kj}^2 \]

\[ = \sum_{k \neq j} q_{ik} \sqrt{\frac{\lambda_k + \lambda_j}{(\lambda_k - \lambda_j)^2}} dW_{kj} - \frac{1}{2} q_{ij} \sum_{k \neq j} \frac{\lambda_k + \lambda_j}{(\lambda_k - \lambda_j)^2} dt \]

\[ \square \]

4. Simulations

Using the stochastic differential equation in Theorem 3.5, we simulate in the appendix the eigenvalues of a Wishart process of dimension \( p = 10 \) and index \( n = 20 \) from \( t = 0 \) to \( t = 100 \). We clearly have to discretize this process and choose time steps \( \frac{100-0}{N} = \frac{100}{N} \) for various values of \( N \). Thus we generate \( dW_i \sim N(0, 100/N) \) at each step. We use R to run these simulations and plot the processes. We use \( N=20000 \) for the simulations below. The first two simulations are the eigenvalue processes with the initial \( B_0 \) matrix having iid entries drawn from a normal distribution with mean 10 and standard deviation 10 (random choice to have initial eigenvalues spaced out from one another). The next two simulations are the eigenvalue processes with the initial \( B_0 \) matrix having iid entries drawn from a normal distribution with mean 0
and standard deviation 1 (initial eigenvalues are very close to each other). The last two simulations are the eigenvalue processes again with the initial $B_0$ matrix having iid entries drawn from a normal distribution with mean 0 and standard deviation 1 (initial eigenvalues are very close to each other). However, the plots show the eigenvalue processes from $t=(100/20000)*300$ to $t=(100/20000)*600$. This is not only presented to show a close-up of the processes during a certain time window when it is hard to distinguish them in the 3rd and 4th plots but also to again show the eigenvalue processes for a matrix with eigenvalues spaced apart. Clearly the actual eigenvalue processes will never collide since when getting the eigenvalues for $X(t)$ at each time step, R returns the eigenvalues in order. However, when simulating these eigenvalue processes, there is a chance of collision due to the discretization of this process. Decreasing the time step greatly reduces the chance of this happening and large jumps when two eigenvalues come very close to each other. Even with extremely small time steps there is still a chance of collision due to the random $dW$ term which is different for each eigenvalue and R rounding errors. When N was increased to 50000 and 100000 (simulations not pictured) there were no visible collisions. In the last two plots, we can also see much more clearly the non-colliding behavior of the eigenvalues.

5. Financial Applications

The structure of Wishart processes (the transpose of a Brownian matrix multiplied by the same Brownian matrix) result in the natural investigation at attempting to model stochastic covariance matrices with Wishart processes. Many processes in finance are frequently modeled by stochastic processes due to the randomness of the market. While the famous Black-Scholes model for pricing options does assume the underlying stock follows a geometric Brownian motion, it also assumes the volatility of the underlying stock is constant which has repeatedly been shown to not be the case—leading to attempts to model volatilities stochastically. In [4], Fonseca attempts to model the $n \times 1$ vector asset $S_t$ with the stochastic differential equation below

$$dS_t = \text{diag}(S_t)(r \mathbf{1}_t dt + \sqrt{X_t} dW_t)$$

where $X_t$ is a covariance matrix following a Wishart process and $\mathbf{1}$ is the $n \times 1$ vector consisting of only 1s. This model is better at pricing options than Black Scholes as it reproduces some of the nonconstant volatility observed in real market conditions such as the volatility skews and smiles. For the single asset, Fonseca models the asset $S_t$ with the below stochastic differential equation

$$dS_t = S_t(r dt + Tr(\sqrt{X_t} dW_t))$$

where $X_t$ is again Wishart compared to the traditional geometric Brownian motion $dS_t = S_t(\mu dt + \sigma dW_t)$ (where the volatility is constant) used to model assets.

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References


6. Appendix