# PAPPUS' THEOREM AND 7 PROOFS

ATHENA ROSCOE

Abstract. Pappus' (Hexagon) Theorem is a degenerate case of Pascal's Theorem, where a conic is instead flattened into two lines. We are interested in the degenerate cases for the theorem, and so we discuss how Pappus' theorem is proven over the course of seven proofs, first within Euclidean geometry, then in projective space under basic transformations, and finally by virtue of Pascal's theorem.

Contents

1. Introduction 1
2. Euclidean Proofs 3
3. Projective Proofs 6
4. Conic Proofs 11
Acknowledgments 13
References 14

## 1. Introduction

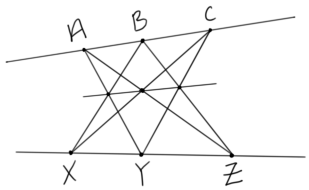
Throughout this paper we look at seven different proofs for Pappus' Theorem. The proofs primarily follow from [2]. The theorem itself is interesting because it allows us to generalize conditions to create a colinear set of points. To be more precise, Pappus' Theorem can be stated as follows:

**Theorem 1.1** (Pappus' Theorem)**.** *Let $A, B, C$ be three points on a straight line, and let $X, Y, Z$ be three points on another line. If the lines $\overline{AY}, \overline{BZ}, \overline{CX}$ intersect lines $\overline{BX}, \overline{CY}, \overline{AZ}$ respectively, then the three points of intersection are colinear.*

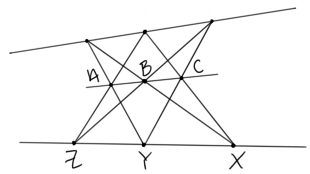
Although the theorem exists in the Euclidean plane, the proofs use properties of projective space, which allow for a generalization of Pappus' theorem in the case of conics. This paper includes seven proofs of Pappus' theorem in increasing generality. The first two will use Euclidean space and the line at infinity. The next three will occur in projective space, and the last two will generalize the theorem to conics. Figure 1 includes examples of Pappus' Theorem.

**Definition 1.2** (Projective Spaces)**.** $\mathbb{P}^2$ is the quotient $\mathbb{R}^3 - \{(0,0,0)\}/\sim$, where $(x_1, x_2, x_3) \sim (y_1, y_2, y_3)$ if there exists $\lambda \in \mathbb{R}^x$ such that $x_1 = \lambda y_1, x_2 = \lambda y_2, x_3 = \lambda y_3$. In other words, $\mathbb{P}^2$ is the collection of lines in $\mathbb{R}^3$ passing through the origin.

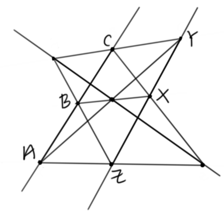
*Date*: December 24, 2021.




(A)




(B)




(C)

FIGURE 1. Pappus' Theorem Examples

Less formally, projective space can be considered as a space that is invariant under all general linear homogenous transformations within the space, and also invariant under any linear combinations consisting only of those transformations. Definitively, a projective plane obeys the following three axioms.

(1) For any two distinct points, there is exactly one line incident with both of them.
(2) For any two distinct lines, there is exactly one point incident with both of them.
(3) There are four distinct points such that no line is incident with more than two of them.

Throughout this set of proofs, it will become relevant to discuss the nature of projective geometry and its use of projective spaces, as opposed to classic Euclidean geometry. Generally, we can think of projective geometry as an extension of Euclidean geometry; it is designed to account for some of the logical extent that Euclidean geometry can't address, and that specifically affects the interpretation and use of infinite spaces. Projective geometry concerns itself with infinity, and adds or added points at infinity.

With relevance to the following proofs, effectively projective space allows for parallel lines to meet - it accounts for the exception to the statement "all lines intersect" (except when parallel), and instead sends the intersection of parallel lines to a point at infinity. With many different sets of parallel lines, there are multiple points at infinity, which, together, form the line at infinity $l_\infty$. This extension is valuable in clarifying degenerate cases of any theorem or proof that involves the intersection of lines, parallel lines, slope of lines, etc, and projective geometry will become valuable in this paper from proof 2 onwards.

**Definition 1.3** (Projective Transformations)**.** A projective transformation is a linear transformation on homogenous 3-vectors represented by an invertible $3 \times 3$ matrix.

Functionally, projective transformations can move objects, deform objects, or both. It can be shown that projective transformations have the following properties.

(1) Projective transformations preserve colinearity.
(2) Projective transformations preserve incidences of lines and points.

Throughout our proofs, the preserved colinearity and incidence of points of projective transformations allow us to make assumptions about the locations or relationships of points without loss of generality. Because one such transformation is possible in every relevant context, a proof can be implied in full generality after transformation.

Finally, there are a couple of assumptions about any full proof of Pappus' Theorem that we should be able to make here, which become relevant later.
First, we have the nondegeneracy condition - this theorem applies under the condition that there must be no identical points, and no identical lines.
Second, we have that the order of points $A, B, C$ and $X, Y, Z$ should not be relevant (each proof should hold despite reordering).

## 2. Euclidean Proofs

In this section, we give two proofs of Pappus' theorem in the Euclidean Space. Both proofs rely on sending the line of intersections to the line at infinity in projective space. More specifically, this line is defined as $l_\infty = \{(x : y : 0)|x, y, \neq 0\}$, or where parallel lines in the Euclidean plane meet at infinity.

**Lemma 2.1.** Let $P$ represent the intersection of two lines. Let $\overline{v}//\overline{w}$ imply that lines $\overline{v}$ and $\overline{w}$ are colinear. Now let $a, b$ be two points on one line, not separated by point $P$, and let $x, y$ be two points on the second line, also not separated by point $P$. Then

$$\frac{|Pa|}{|Pb|} = \frac{|Px|}{|Py|} \Rightarrow \overline{ax}//\overline{by}$$

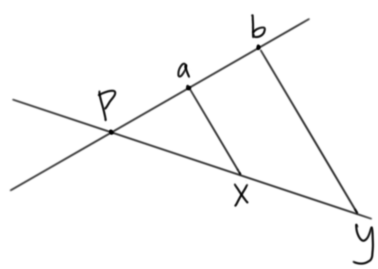



Figure 2. Lemma 2.1

*Pappus' Theorem Proof 1.* First note that three points on the line at infinity are considered to be colinear. Also recall that in a projective space, two colinear lines intersect at the line at infinity. Then if we assume that $A, B, C$ and $X, Y, Z$ are not separated by the intersection of their respective lines, we can transform all relevant instances of $A, B, C$ and $X, Y, Z$ so that for an intersection P of the two lines, $\frac{|Pa|}{|Pb|} = \frac{|Py|}{|Px|}$ and $\frac{|Pb|}{|Pc|} = \frac{|Pz|}{|Py|}$. Then, using Lemma 2.1,

$$\frac{|PA|}{|PB|} = \frac{|PY|}{|PX|} \text{ and } \frac{|PB|}{|PC|} = \frac{|PZ|}{|PY|} \text{ implies } \overline{AY}//\overline{BZ} \text{ and } \overline{BX}//\overline{CY}.$$

In a similar vein, we can also multiply our two equalities to show that

$$\frac{|PA|}{|PB|} = \frac{|PY|}{|PZ|} \text{ and } \frac{|PB|}{|PC|} = \frac{|PX|}{|PY|}$$
$$\Rightarrow \frac{|PA|}{|PC|} = \frac{|PX|}{|PZ|}$$
$$\Rightarrow \overline{AX}//\overline{CZ}.$$

Now note that because this is a projective space, $\overline{AY}//\overline{BZ}$ implies $\overline{AY}$ and $\overline{BZ}$ have an intersection $\alpha$ on the line at infinity. Similarly, $\overline{BX}//\overline{CY}$ implies $\overline{BX}$ and $\overline{CY}$ have an intersection $\beta$ on the line at infinity.

So therefore, if we have any $A, B, C$ and $X, Y, Z$ on two lines, where the points $A, B, C$ and $X, Y, Z$ are not separated by the the intersection of the two lines, then they can be transformed so that $\overline{AY}$ and $\overline{BZ}$, and $\overline{BX}$ and $\overline{CY}$, intersect at $\alpha$ and $\beta$ on the line at infinity. This same transformation also implies that

$$\overline{AX}//\overline{CZ}$$

$$\overline{AX} \text{ and } \overline{CZ} \text{ also intersect at infinity, at } \gamma$$

Then if $\alpha, \beta, \gamma$ are all on the line at infinity, then they are colinear. And since colinearity is preserved through transformation, this relationship holds, and therefore Pappus' Theorem holds.

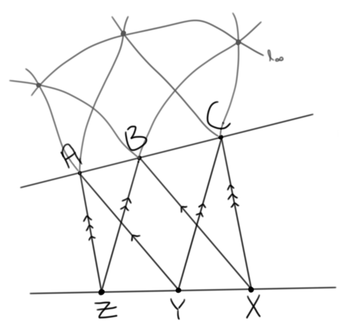



FIGURE 3. Euclidean Parallelism

□

**Definition 2.1** (Triangle Orientation)**.** Triangle orientation is relevant when considering area. Let $A, B, C$ denote the vertices of a triangle. We can assign signs to the area of this triangle by stating that triangle (A,B,C) is positive if the sequence of points is counterclockwise, and negative if the sequence of points is clockwise.

**Lemma 2.2.** Let A,B,C,D denote the four vertices of a quadrilateral as shown below. If we have $area(BCA) + area(CBD) = 0$, then the quadrilateral is a trapezoid, i.e., $AD//BC$. Likewise, $AD//BC$ implies $area(BCA) + area(CBD) = 0$.

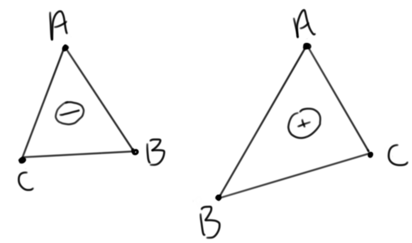



FIGURE 4. Triangle Orientation

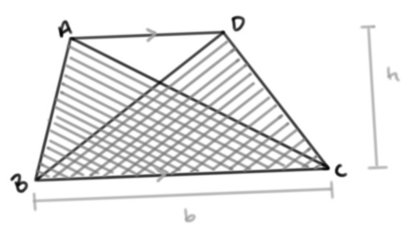



FIGURE 5. Equal Area Implies Parallelism

*Proof.* Equivalent areas under the diagonals implies parallel lines. Specifically, if there is a given quadrilateral $A, B, C, D$, we are concerned with triangles $BCA$ and $CBD$. Note that the sequence of points runs in opposite directions, so if the triangles are equal in area, they will cancel out. Then, since the two triangles share an edge $\overline{BC}$, and the heights will only ever be equal if $\overline{AD}$ and $\overline{BC}$ are parallel,

$$area(BCA) + area(CBD) = 0 \Rightarrow \overline{AD}//\overline{BC}$$

□

*Pappus' Theorem Proof 2 (Area Method).* Consider $A, B, C, X, Y, Z$ arranged as follows.

Note that $ACB$ and $XYZ$ form a triangle, and that $BYXA$, $CZYB$, and $CZXA$ form quadrilaterals. Let

$$\begin{aligned}\beta &= area(ACB) + area(XYZ) + area(BYXA) + area(CZYB) - area(CZXA)\\ &= 0.\end{aligned}$$

Therefore, since $\beta$ is 0 regardless of the placement of $A, B, C, X, Y, Z$, $\beta$ must be a zero polynomial. Now we consider $A, B, C$ and $X, Y, Z$ such that each set exists on two respective lines. We also assume $\overline{AY}//\overline{BZ}$ and $\overline{BX}//\overline{CY}$ after transformation. As with Proof 1, we seek to use this assumption to prove that $\overline{AX}//\overline{CZ}$.

Then, in the new configuration,

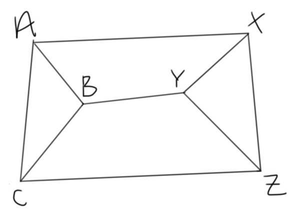



FIGURE 6. Approximate Arrangement

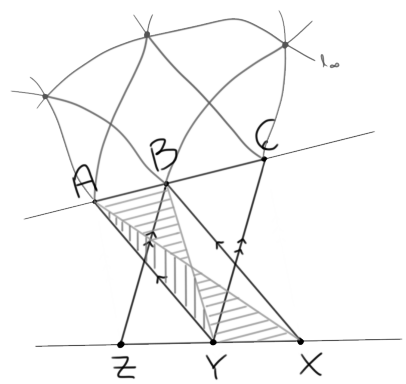



FIGURE 7. Parallelism Implies Area is 0

$$area(ABC) = 0 = area(XYZ)$$

And, using Theorem 2.2 we can assert that because $\overline{AY}//\overline{BZ}$ and $\overline{BX}//\overline{CY}$, then

$$area(B, Y, X, A) = 0 = area(C, Z, Y, B).$$

Then

$$area(CZXA) = area(ACB) + area(XYZ) + area(BYXA) + area(CZYB) = 0.$$

That, in turn, proves that $\overline{AX}//\overline{CZ}$, so that our intersections are pushed to $l_\infty$, and are therefore colinear. $\square$

## 3. Projective Proofs

In this section, we give three proofs of Pappus' theorem in projective space, all of which make use of projective transformations. Projective transformation allows proofs to assume certain determinant values for triples of points, as well as Ceva's thorem.

**Lemma 3.1** (colinearity of Points)**.** For a given $A = (a_1, a_2, a_3), B = (b_1, b_2, b_3)$, and $C = (c_1, c_2, c_3)$, the

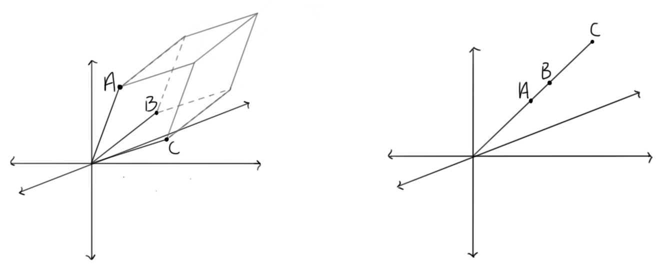
$$\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

is equivalent to the volume of a parallelopiped formed by $A, B, C$.

Therefore, if $A, B, C$ are colinear, there will be no volume, and

$$\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = 0.$$

This is displayed in Figure 8.




(A) classic example (B) colinear A,B,C

FIGURE 8. Determinant Parallelopiped Examples

*Pappus' Theorem Proof 3.* Before we begin, we will first rename our various points and intersections to align roughly with the configuration in Figure 9.

Then note that under the nondegeneracy condition discussed earlier (under Pappus' Theorem assumptions), (1,4,7) cannot be colinear without combining several lines. We can also assign points 1 through 9 to a three dimensional plane formed by the two lines holding A,B,C and X,Y,Z respectively. Then, after an affine transformation we can assume without loss of generality that (1,4,7) forms an equilateral triangle.

Then we can use the fact that for colinear points $a, b, c$, the determinant $[a, b, c] = 0$. Note that moving forward, we will use the notation that $[x, y, z]$ refers to the determinant of points $x, y, z$. In Pappus' Theorem (Theorem 1.1), we have eight lines that we can use to prove the colinearity of a ninth line. Then

$$[1, 2, 3] = 0 \Rightarrow ce = bf,$$
$$[4, 5, 6] = 0 \Rightarrow gl = ij,$$
$$[1, 9, 5] = 0 \Rightarrow iq = hr,$$
$$[1, 8, 6] = 0 \Rightarrow ko = ln,$$
$$[2, 9, 4] = 0 \Rightarrow ar = cp,$$
$$[2, 7, 6] = 0 \Rightarrow bj = ak,$$
$$[3, 8, 4] = 0 \Rightarrow fm = do,$$
$$[3, 7, 5] = 0 \Rightarrow dh = eg.$$

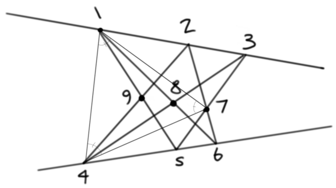



Figure 9. Pappus Diagram

| *point* | $x$ | $y$ | $z$ |
|---|---|---|---|
| 1 | 1 | 0 | 0 |
| 2 | $a$ | $b$ | $c$ |
| 3 | $d$ | $e$ | $f$ |
| 4 | 0 | 1 | 0 |
| 5 | $g$ | $h$ | $i$ |
| 6 | $j$ | $k$ | $l$ |
| 7 | 0 | 0 | 1 |
| 8 | $m$ | $n$ | $o$ |
| 9 | $p$ | $q$ | $r$ |

Figure 10. Coordinate Table

If we multiply all sides and cancel out terms, this gives us

$$mq = np$$

$$[7, 8, 9] = 0$$

which is what we want. Note that multiplying and cancelling out terms is possible because (1,4,7) includes coordinates (1,0,0), (0,0,1), and (0,1,0) - nondegeneracy conditions prevents variables from being equivalent to 0. □

**Lemma 3.2** (Grassman-Plücker Relation)**.** The full version of the Grassman-Plücker relation states that for any $A, B, C, D, E$ in $\mathbb{R}^3$,

$$[A, B, C][A, D, E] - [A, B, D][A, C, E] + [A, B, E][A, C, D] = 0.$$

*Proof.* First, we will prove a variation on our target Grassman-Plücker Relation. That states that for any $a, b, c, d$ in $\mathbb{R}^2$,

$$[a, b][c, d] - [a, c][b, d] + [a, d][b, c] = 0$$

We do this by projective transformation, which allows us to assume generally that $a = (1, 0)$ and $b = (0, 1)$. Then we have

$$[c, d] + c_2 d_1 - d_2 c_1 = 0$$

which holds. Then we can consider the full version. After a suitable affine transformation, we can assume without loss of generality that E = (1,0,0) which means that given the first variation, the second holds. □

*Pappus' Theorem Proof 4.* We use similar assumptions to proof 3 - specifically, we assume that (1,4,7) is not colinear under the nondegeneracy condition. More importantly, 1, 4, and 7 cannot fall on any of the same lines under the nondegeneracy conditions, which means that each of the eight lines intersects with one and only one of the three points. Then for each of the eight relevant lines in the theorem (including the line (1,2,3)), we can assume that, first, the determinant is 0 ($[1, 2, 3] = 0$) and then apply the Grassman-Plücker (Theorem 3.2) relation so that

$$[1, 4, 7][1, 2, 3] - [1, 4, 2][1, 7, 3] + [1, 4, 3][1, 7, 2] = 0$$

$$0 + [1, 4, 3][1, 7, 2] = [1, 4, 2][1, 7, 3]$$

Then, if we observe our determinants, we can see that both (1,4,3) and (1,7,2) cannot be colinear due to the nondegeneracy condition. Now, if we apply the same process to the other seven lines, we end up with

$$[1,4,7][1,8,6] - [1,4,8][1,7,6] + [1,4,6][1,7,8]$$
$$[1,4,6][1,7,8] = [1,4,8][1,7,6]$$
$$[1,4,7][1,9,5] - [1,4,9][1,7,5] + [1,4,5][1,7,9]$$
$$[1,4,9][1,7,5] = [1,7,9][1,4,5]$$
$$[4,1,7][4,9,2] - [4,1,9][4,7,2] + [4,1,2][4,7,9]$$
$$[4,1,2][4,7,9] = [4,1,9][4,7,2]$$
$$[4,1,7][4,8,3] - [4,1,8][4,7,3] + [4,1,3][4,7,8]$$
$$[4,1,8][4,7,3] = [4,1,3][4,7,8]$$
$$[4,1,7][4,5,6] - [4,1,5][4,7,6] + [4,1,6][4,7,5]$$
$$[4,1,5][4,7,6] = [4,1,6][4,7,5]$$
$$[7,1,4][7,2,6] - [7,1,2][7,4,6] + [7,1,6][7,4,2]$$
$$[7,1,6][7,4,2] = [7,1,2][7,4,6]$$
$$[7,1,4][7,3,5] - [7,1,3][7,4,5] + [7,1,5][7,4,3]$$
$$[7,1,5][7,4,3] = [7,1,3][7,4,5].$$

The application of Grassman-Plücker (Theorem 3.2) to (1,4,7) and all lines is made possible by the noncolinearity of (1,4,7). Similarly to the first case, we can show that each determinant is noncolinear and therefore does not equal zero, because each of them includes at least two out of (1,4,7). This allows us to multiply all the equations together, which gives us

$$[7,1,8][7,4,9] = [7,1,9][7,4,8]$$

Then if we consider the Grassman-Plücker relation (Theorem 3.2) of the points (7,1,4,8,9), we get that

$$[7,1,4][7,8,9] - [7,1,8][7,4,9] + [7,1,9][7,4,8] = 0$$

then since $[7,1,8][7,4,9] = [7,1,9][7,4,8]$,

$$[7,1,4][7,8,9] = 0.$$

Finally, because [7,1,4] is not colinear and cannot equal 0, [7,8,9] = 0. Therefore, [7,8,9] is colinear. □

For our final projective proof, we need to use Ceva's theorem. Ceva's theorem concerns the ratio of the border of a triangle when there are intersecting lines drawn from vertices. We can prove the relation holds when all but one line is drawn, and all but one triangle is drawn (in a structure formed of multiple triangles). In the proof, we can layer triangles until we roughly form a Pappus' configuration, and then prove that the final line holds.

**Theorem 3.1** (Ceva's Theorem)**.** *If a triangle $A, B, C$ has three lines which all intersect at one center, each running through one vertice and intersecting the opposite line at points $X, Y, Z$ respectively, then*

$$\frac{\overline{BX}}{\overline{XC}}\frac{\overline{CY}}{\overline{YA}}\frac{\overline{AZ}}{\overline{ZB}} = 1$$

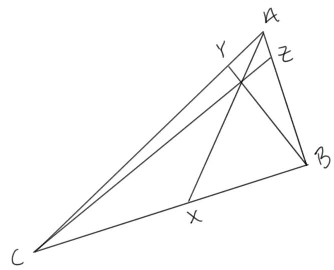



FIGURE 11. Ceva Theorem 1

Furthermore, this relationship continues to hold true for any shape that can be composed of Ceva triangles - the ratio will instead hold for the border, while any internal borders will be canceled out. We can prove this for the example of a quadrilateral $ABCD$ formed from two triangles $ABC$ and $BCD$, intersected by $VWXYZ$. Then we end up with

$$\frac{\overline{AV}\,\overline{BX}\,\overline{CW}}{\overline{VB}\,\overline{XC}\,\overline{WA}} = 1$$

and

$$\frac{\overline{BY}\,\overline{DZ}\,\overline{CX}}{\overline{XB}\,\overline{YD}\,\overline{ZC}} = 1$$

so that if we multiply the two and cancel, out, we end up with

$$\frac{\overline{AV}\,\overline{BY}\,\overline{DZ}\,\overline{CW}}{\overline{VB}\,\overline{YD}\,\overline{ZC}\,\overline{WA}} = 1.$$

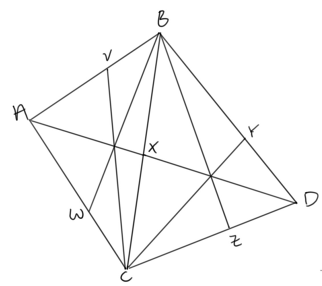
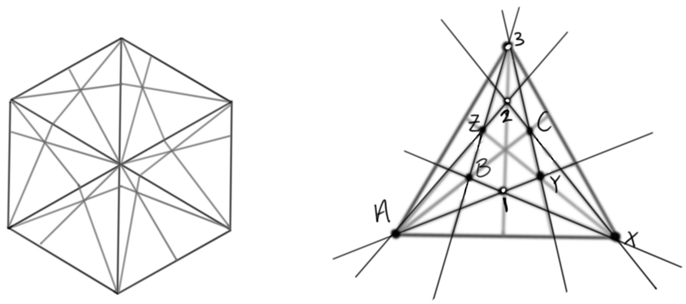



FIGURE 12. Ceva Theorem 2

Then this fraction represents the application of Ceva's Theorem to the outline of the quadrilateral. In general, we can also show that for any shape formed of triangles, if the Ceva conditions are met for every triangle except for one, then the Ceva configuration will automatically hold for the last one as well. This can be shown by considering a shape with a triangular hole in the middle - the Ceva configuration is already upheld, and inserting a triangle will not change it.

*Pappus' Theorem Proof 5.* Consider a hexagon formed of six Ceva triangles. If we fold all six configurations down into one triangle, we will have one triangle with six transplanted centers, and one created center. We can then assign points as below, so that the folded Ceva triangle fulfills the Pappus' Theorem's conditions.




(A) Hexagon Ceva (B) After Folding

FIGURE 13. Ceva Theorem Pappus Proof

Then we can apply Ceva's Theorem (Figure 11) again to state that if five out of six triangles hold Ceva conditions, so will the last - then (7,8,9) is implied by the other lines, and we have shown Pappus' Theorem.

□

## 4. CONIC PROOFS

We provide two conic proofs in this paper. Both of these proofs primarily prove Pascal's theorem, which is a conic general case of Pappus' theorem. Papcal's theorem is as follows.

**Theorem 4.1** (Pascal's Theorem)**.** *Let $A, B, C, X, Y, Z$ be six points on a conic. If the lines $\overline{AY}, \overline{BZ}, \overline{CX}$ intersect lines $\overline{BX}, \overline{CY}, \overline{AZ}$ respectively, then the 3 points of intersection are colinear. Look to Figure 14 for an example of Pascal's theorem.*

We actually consider Pappus' theorem to be a degenerate case of Pascal's theorem, because Pappus' theorem can be created by flattening the slope of the conic in Pascal's theorem. Therefore, by proving Pascal's theorem, we prove Pappus' theorem.

**Lemma 4.1.** For any six points $A, B, C, X, Y, Z$ on a conic,

$$[A, B, C][A, Y, Z][X, B, Z][X, Y, C] = [X, Y, Z][X, B, C][A, Y, C][A, B, Z]$$

*Pappus' Theorem Proof 6.* We will prove Pascal's Theorem (and therefore Pappus' Theorem), by first using the available Grassman-Plücker relations, and then applying Lemma 4.1. Since we are trying to prove Pascal's Theorem instead, there are only 6 colinear lines to consider for the Grassman-Plücker relations so that we can write

$$[1, 5, 7][2, 5, 9] = [1, 2, 5][5, 9, 7]$$

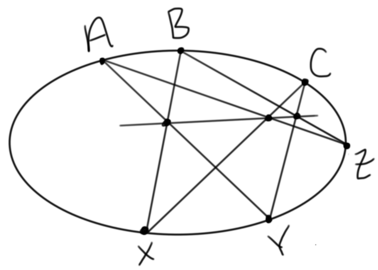



FIGURE 14. Pascal's Theorem On Ellipse

$$[1,2,6][3,6,8] = [1,3,6][2,6,8]$$
$$[2,4,5][2,9,7] = [2,4,7][2,5,9]$$
$$[2,4,7][2,6,8] = [2,4,6][2,8,7]$$
$$[3,4,6][3,5,8] = [3,4,5][3,6,8]$$
$$[1,3,5][5,8,7] = [1,5,7][3,5,8]$$

so that if we multiply all the equations and cancel out terms, we get

$$[1,2,6][2,4,5][2,9,7][3,4,6][1,3,5][5,8,7] = [1,2,5][5,9,7][1,3,6][2,4,6][2,8,7].$$

Then we can apply Lemma 4.1 to the six points on the conic, so that

$$[1,2,5][1,3,6][2,4,6][3,4,5] = [1,2,6][1,3,5][2,4,5][3,4,6]$$

Then if we multiply this equation against the Grassman-Plücker relation results, we end up with

$$[2,8,7][5,9,7] = [2,9,7][5,8,7] \Rightarrow [9,8,7] = 0$$

which therefore proves that the intersections (7,8,9) would be colinear, and Pascal's Theorem holds. □

**Theorem 4.2** (Cayley-Bacharach-Chasles Theorem (CBC))**.** *Let $A$ and $B$ be two curves of degree three intersecting in nine proper points. If six of these points are on a conic, the remaining three points are colinear.*

To prove this theorem, we need a result in algebraic geometry that counts the number of intersection points of two algebraic curves. An algebraic curve $C$ is the zero-set of a polynomial $f$ in two variables. If $f$ has degree $d$ then we say that $C$ has degree $d$.

**Theorem 4.3** (Bezout's Theorem)**.** *Let $A$ and $B$ be algebraic curves of degree $a$ and $b$ respectively. Then $A$ and $B$ either intersect at $ab$ points, counting multiplicity, or the polynomials defining them have some shared component.*

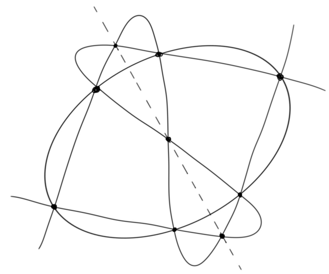


FIGURE 15. Cayley-Bacharach-Chasles Theorem On Ellipse

*Pappus' Theorem Proof 7.* Let $A(x, y, z)$ and $B(x, y, z)$ represent the polynomials for two algebraic curves of degree 3. According to Bezout's Theorem, then $A(x, y, z)$ and $B(x, y, z)$ can only have, at maximum, $3x3 = 9$ intersection points, unless the number of intersections between $A$ and $B$ is infinite. Then we can characterize intersection points uniquely as being a point $P = (x, y, z)$ at which both polynomials $A$ and $B$ are equivalent to 0. Now assume that there exists a conic $C$ such that six of those intersection points fall along the conic (aligning with the points 1,...,6 in earlier arrangements). Then, we can create a polynomial $D(x, y, z)$ such that

$$D(x, y, z) = A(x, y, z) + kB(x, y, z)$$

for some k. It's most important to note at each intersection of A and B,

$$D(P) = 0 + 0 = 0$$

$C$ will also intersect with both of them. Therefore, we know that there are at least six intersection points between $C$ and $D$, regardless of $k$. Then, consider a point S (seventh point) on C. We can choose some k such that

$$D(S) = A(S) + kB(S) = 0$$

Therefore, we choose an identity for $D$ such that $D$ has seven intersections with $C$. Since a conic function is degree 2 and function D is degree 3, the number of intersections shouldn't exceed 6 unless the intersections are infinite. According to Bezout's Theorem (Theorem 4.3), this implies that $D$ has a component C. Therefore

$$D(x, y, z) = CL$$

where L is some function. However, since a conic function is degree 2 and D is degree 3, L must be linear. Therefore, for 7,8,9 which exist as intersection points between $A, B, D$ which are not on $C$, $7, 8, 9$ must exist on $L$. This shows that six points on a conic, which can be described as the intersection of two degree three polynomials, imply the existence of colinear intersection points. □

## ACKNOWLEDGMENTS

I would like to thank my mentor, Livia Xu, for helping me so much with selecting and understanding my topic. I would also like to thank Professor Rudenko, for the wonderful lectures, Professor May for organizing the REU, as well as all the mathematicians who were part of the program and made it into a reality.

## References

[1] Weisstein, Eric W. *Projective Space.* Mathworld.
[2] Richter-Gebert, Jurgen. *Perspectives on Projective Geometry* Springer-Verlag, Berlin, Heidelberg, 2011.