HYPERBOLIC DYNAMICAL SYSTEMS AND THE BIRKHOFF-SMALE THEOREM

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Abstract. In this paper, we briefly introduce hyperbolic dynamical systems. Then, we use the example of the Smale horseshoe to explore the properties which can arise on a hyperbolic set. In particular, we show the topological conjugacy between Smale horsehoe map and the shift map on the symbolic space of two symbols. Finally, we prove the Birkhoff-Smale theorem, which shows that the dynamics of the Smale horseshoe map can be found near any transverse homoclinic point of a hyperbolic periodic orbit.

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1. Introduction

Dynamical systems studies the behavior of an iterated map or flow on a space. Of particular interest are the orbits of points, which chart the motion of a single point across the space under both backwards and forwards iterations of a map. In this paper, we will introduce and focus our attention on *hyperbolic dynamical systems*. Hyperbolic dynamical systems involve a "saddle-like" behavior in which, near an orbit or union of orbits, the map expands uniformly in one direction and contracts uniformly in another. Hyperbolicity is important in understanding structural stability (that is, the question of whether the dynamics of a map remain unaffected by small perturbations).

In this paper, we will investigate a striking example of hyperbolic behavior: the Smale horseshoe map. The Smale horseshoe is a structurally stable map with infinitely many periodic orbits all contained in a compact invariant set. Smale saw that the horseshoe can be used to model the dynamics of other complex systems. For example, in his study of celestial mechanics, Poincare noticed the dynamics near a transverse homoclinic point - a point whose orbit forms a loop which begins and ends at a hyperbolic saddle point - were highly complex. Smale tackled this problem by realizing that the dynamics of his horseshoe map could be used to characterize the behavior on a subset which can be found near any transverse homoclinic point. This discovery, the Birkhoff-Smale Theorem, is the culminating result which we will prove in this paper.

In §2, we introduce the basic notions and definitions of dynamical systems. Then, in §3, we will briefly introduce hyperbolic dynamical systems, stating several of the important results of hyperbolic theory, which will be useful for the proof of the Birkhoff-Smale system. In §4, we give a short overview of symbolic dynamics, after [1], which we will use to encode the behavior of the horseshoe map constructed in §5 and in the Birkhoff-Smale Theorem. In §5, we give a geometric construction of the Smale horseshoe as map from S^2 to S^2 and prove the topological conjugacy between the shift map on the symbol space introduced in §5 and the horseshoe map on a compact invariant subset. Finally in §6, we will prove the Birkhoff-Smale Theorem.

2. Basic Definitions

Our initial setting will be that of a compact metric space X and a homeomorphism f.

Definition 2.1. A *dynamical system* is the family $\{f_n\}_{n=-\infty}^{\infty}$, where

$$f^n = \overbrace{f \circ f \circ \cdots \circ f}^{n \text{ times}}, \quad f^0 = id, \text{ and } f^{-n} = (f^n)^{-1}.$$

For a given $x \in X$, the set Orb(x, f) $= \{f^n(x)\}_{n=-\infty}^{\infty}$ is called the *orbit* of x under f. A subset $\Lambda \subset X$ is *invariant* under f if $f(\Lambda) = \Lambda$.

In addition to being interested in the behavior of iterates of the function f on the whole space X, the study of dynamical systems pays close attention to the behavior of f on or near points/subsets of X which have intelligible behavior. One such emphasized notion is that of recurrence, of which there are several forms of varying strength.

The strongest form of recurrence is *periodicity.* A point $x \in X$ is *periodic with period* $n \in \mathbb{N}$ if $f^n(x) = x$, and n is the minimal positive integer which satisfies this equality. A periodic point of period 1 is called a *fixed point.* We can weaken our notion of periodicity by looking at the limit points of subsequences of the orbit of x under either forwards or backwards iterations of f.

A point $y \in X$ is called an ω-*limit point* of $x \in X$ if there exists a subsequence n_i of the natural numbers such that $f^{n_i}(x) \to y$. The ω-limit set of X, denoted by $\omega(x)$, is the set of all ω-limit points of x. A point $x \in X$ is called *positively recurrent* if $x \in \omega(x)$.

Likewise, a point $y \in X$ is called an α-*limit point* of $x \in X$ if there exists a subsequence n_i of the natural numbers such that $f^{-n_i}(x)$ converges to y in X. The α-limit set of X, denoted by $\alpha(x)$ will be the set of α-limit points of x. Finally, a point $x \in X$ is called *negatively recurrent* if $x \in \alpha(x)$.

Remark 2.2. These two forms of recurrence will be most important with regards to the theorems presented in this paper. However, two weaker forms of recurrence exist which are important to the study of dynamics in general. These are the concepts of *nonwandering* and *chain-recurrent* points. A detailed overview may be found in Chapter 1 of [1].

Now, we will discuss *transitivity.*

Definition 2.3. A compact invariant set $\Lambda \subset X$ of f is called *topologically transitive* or *transitive* if there exists $x \in \Lambda$ such that $\omega(x) = \Lambda$.

The following result characterizes the conditions which are equivalent to transitivity.

Theorem 2.4 (Birkhoff)**.** *Let Λ be a compact invariant set of f. The following conditions are equivalent:*

- *(a) Λ is transitive.*
- *(b) For any two open subsets U and V of Λ, there is $n \geq 1$ such that $f^n(U) \cap V \neq \emptyset$.*
- *(c) There is $x \in \Lambda$ whose positive orbit is dense in Λ.*

A full proof may be found in [1].

It is natural for us to seek a method of finding equivalence between dynamical systems. In other words, when are we able to say that the dynamics of two different systems are in some sense "the same"? To answer this question, we introduce the notion of *topological conjugacy.*

Definition 2.5. Two homeomorphisms $f : X \to X$ and $g : X \to X$ are *topologically conjugate* to each other if there exists a homeomorphism $h : X \to X$ such that

$$hf = fg.$$

Such a homeomorphism h is called a *topological conjugacy* from f to g.

A familiar example of topological conjugacy is the concept of matrix similarity from linear algebra; in this case, two square matrices representing the same linear map are conjugate by way of a change of basis matrix. As such, we can think of topological conjugacy as a sort of nonlinear change of coordinates linking two homeomorphisms.

In addition to generally being an equivalence relation on the space of all homeomorphisms from X to X, topological conjugacy ends up being a good choice for a notion of "equivalence" between dynamical systems since it is invariant with respect to iterates (i.e. $hf^n = g^n h$ if $hf = gh$), and therefore preserves orbits for any $x \in X$:

$$h(\mathrm{Orb}(x, f)) = \mathrm{Orb}(h(x), g).$$

Similarly, topological conjugacy also preserves the set of periodic points, the ω-limit set, the nonwandering set, and the chain-recurrent set. By giving us a way to determine if two dynamical systems are equivalent, topological conjugacy will allow us to define the closely related idea of C^r structural stability.

To do so, we transfer settings to a compact, smooth manifold without boundary M. Denote by $\mathrm{Diff}^r(M)$ the set of C^r diffeomorphsisms of M. We endow $\mathrm{Diff}^r(M)$ with the C^r *topology* using a metric. Fix a finite cover of charts $\{(U_i, \varphi_i)\}_{i=1}^n$ such that for each (U_i, φ_i) there exists another chart (V_i, ψ) so that $\overline{U_i} \subset V_i$ and $\psi|_U = \varphi$. We define the C^r*-distance* between two C^r diffeomorphisms f and g to be

$$d^r(f, g) = \sup_{x \in \varphi_i(U_i), 1 \leq i,j \leq n} \{|\varphi_j f \varphi_i^{-1}(x) - \varphi_j f \varphi_i^{-1}(x)|, \ldots, |D^k(\varphi_j f \varphi_i^{-1})(x) - D^k(\varphi_j g \varphi_i^{-1})(x)|\}.$$

Note that the choice of cover of charts does not affect the topology. Now, we say that $f \in \mathrm{Diff}^r(M)$ is *structurally stable* if there exists a neighborhood $\mathcal{U} \subset \mathrm{Diff}^r(M)$

in the C^r topology such that if g is in $\mathcal{U}$, f is topologically conjugate to g. In other words, a C^r diffeomorphism will be C^r structural stable if its dynamics are topologically equivalent to maps C^r-close to itself.

3. Background in Hyperbolic Dynamical Systems

We now give an introduction to hyperbolic dynamical systems, which will be our central objects of study.

Definition 3.1. Let $(E, |\cdot|)$ be a finite-dimensional normed vector space. A linear isomorphism $A : E \to E$ is called *hyperbolic* if E splits into a direct sum

$$E = E^s \oplus E^u$$

which is invariant in the sense that

$$A(E^s) = E^s,\ A(E^u) = E^u,$$

and for which there exist two constants $C \geq 1$ and $0 < \lambda < 1$ such that

$$|A^n v| \leq C\lambda^n v,\ \text{for all } v \in E^s,\ n \geq 0,\ \text{and}$$
$$|A^{-n} v| \leq C\lambda^n v,\ \text{for all } v \in E^u,\ n \geq 0.$$

Given a hyperbolic splitting $E^s \oplus E^u$ of A, we call E^s the *contracting space* of A, and we call E^u the *expanding space* of A. See Figure 1.

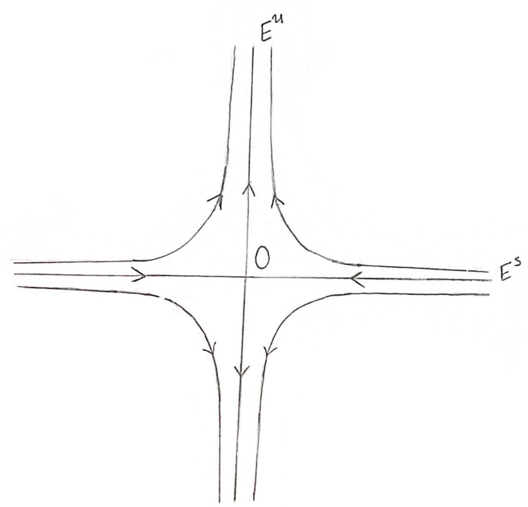

Figure 1. Splitting of E with respect to a hyperbolic linear isomorphism.

Let M be a compact, smooth Riemannian manifold and let $f : M \to M$ be a C^r diffeomorphism. The idea of a hyperbolic set arises when we try to analyze hyperbolic behavior of the tangent map Tf on some invariant set of f. The easiest hyperbolic set to understand is that of a *hyperbolic fixed point*. A fixed point $p \in M$ of f is called *hyperbolic* if the tangent map $Tf : T_pM \to T_pM$ is a hyperbolic linear isomorphism.

For an invariant subset $\Lambda \subset M$, recall that the tangent bundle of Λ is the set

$$T_\Lambda M = \bigsqcup_{x \in \Lambda} T_x M.$$

By looking at the tangent map $Tf : T_{\Lambda M} \to T_\Lambda M$ we may, instead of simply looking at a hyperbolic splitting $E^s \oplus E^u$ induced by the derivative at a fixed point x, consider a family of splittings $E^s_x \oplus E^u_x = T_x M$, for all $x \in \Lambda$, which will be invariant under the tangent map Tf and on which some uniform hyperbolic estimates on the iterates of Tf will hold.

Notation 3.2. We will use $|\cdot|$ to denote the norm induced by the Riemannian metric of M and drop base points from the notation. Then, if π is the bundle projection, we will write $|u| = |u|_{\pi u}$.

Now, we are ready to define a hyperbolic set.

Definition 3.3. Let M a compact smooth Riemannian manifold, $f : M \to M$ a diffeomorphism. An invariant set $\Lambda \subset M$ of f is called *hyperbolic* if, for all $x \in \Lambda$, the tangent space $T_x M$ admits a splitting into a direct sum

$$T_x M = E^s_x \oplus E^u_x,$$

which is invariant under the tangent map Tf, in the sense that

$$Tf(E^s_x) = E^s_{f(x)}, \quad Tf(E^u_x) = E^u_{f(x)},$$

and if in addition, there exist constants $C \geq 1$ and $0 < \lambda < 1$,

$$|Tf^n(v)| \leq C\lambda^n |v|, \text{ for all } x \in \Lambda,\ v \in E^s_x,\ n \geq 0, \text{ and}$$
$$|Tf^{-n}(v)| \leq C\lambda^n |v|, \text{ for all } x \in \Lambda,\ v \in E^u_x,\ n \geq 0.$$

We have already noted that a hyperbolic fixed point is a hyperbolic set. However, it is also completely possible for the entire manifold M to form a hyperbolic set of a diffeomorphism f. Such a diffeomorphism will be called an *Anosov diffeomorphism.*

Example 3.4 (Anosov toral automorphism)**.** Consider a hyperbolic linear isomorphism $A : \mathbb{R}^2 \to \mathbb{R}^2$ with integer entries and determinant ± 1. Such a map A is called an *Anosov auotomorphism.* Because A has integer entries, it maps $\mathbb{Z}^2$ to itself bijectively. Then, A induces a quotient map on the torus:

$$f : \mathbb{T}^2 \to \mathbb{T}^2$$

such that $\pi A = f\pi$, where $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ is the projection which takes each component modulo 1. Since A is invertible and has determinant ± 1, A^{-1} also has integer entries and similarly induces a quotient map on $\mathbb{T}^2$, which is clearly f^{-1}. We can check that both f and f^{-1} are smooth maps, and therefore are diffeomorphisms.

Since the tangent space at $x \in \mathbb{R}^2$ is simply the whole space $\mathbb{R}^2$, we can define a hyperbolic splitting at each point x which is simply the hyperbolic splitting of A. Then, we take the hyperbolic splitting at $a \in \mathbb{T}^2$ to be the same as the splitting at any $x = \pi^{-1}(a)$. Such a map f is called a *Anosov toral automorphism.*

If $\Lambda \subset M$ is a hyperbolic set, then the subspaces E^s_x and E^u_x vary continuously in $x \in \Lambda$ (see [1]). In this way, we obtain two continuous subbundles of $T_\Lambda M$,

$$E^s = \bigsqcup_{x \in \Lambda} E^s_x \text{ and } E^u_x = \bigsqcup_{x \in \Lambda} E^u_x,$$

such that for each $x \in \Lambda$, $E^s_x \oplus E^u = T_xM$. Thus, we write

$$T_\Lambda M = E^s \oplus E^u.$$

Moreover, we call E^s the *stable bundle* and E^u the *unstable bundle*.

The following result states that if a map g is C^1-close to f and Λ is a hyperbolic set of f, then $g-$invariant sets close to Λ will be hyperbolic sets of g. This result will be useful in the proof of the Birkhoff-Smale Theorem in §6.

Proposition 3.5 (Persistence of hyperbolicity for an invariant set)**.** *Let $\Lambda \subset M$ be a compact hyperbolic set of f. Then, there is a C^1 neighborhood $\mathcal{U}_0$ of f in $\mathrm{Diff}^1(M)$ and $a > 0$ such that for all $g \in \mathcal{U}_0$, every compact g-invariant set Δ contained in the ball of radius a around Λ is hyperbolic. In addition, as g approaches f in the C^1 topology and $x \in \Delta$ approaches $y \in \Lambda$, we have that $E^s(x, g)$ also approaches $E^s(y, f)$. The same holds for $E^u(x, g)$ and $E^u(y, f)$.*

The details of the proof can be a found in [1].

Now, we state the Stable Manifold Theorem and define the stable and unstable manifolds. Briefly, the global stable manifold consists of the points which are attracted to a point p of a hyperbolic set under forwards iteration of a diffeomorphism f, while the global unstable manifold is the analogous set of points for backwards iteration. The Stable Manifold Theorem claims that E^s_p can be embedded into M as a C^k disk of dimension $\dim(E^s_p)$ which is tangent to p. In addition, points in the embedded disk are attracted to the point p under forwards iteration. The analogous statement will also be true for E^u_p. We will use the Stable Manifold Theorem throughout the remainder of this section as well as in §6.

Theorem 3.6 (Stable Manifold Theorem)**.** *Let $f : M \to M$ be a C^k diffeomorphism, Λ a hyperbolic set of f with splitting $T_\Lambda M = E^s \oplus E^u$ and hyperbolic constants $0 < \lambda < 1$ and $C \geq 1$. Then there is $r > 0$ such that there are two C^k embedded disks $W^s_r(p, f)$ and $W^u_r(p, f)$ which are tangent at p to E^s_p and E^u_p, respectively.*

Furthermore, there is a C^k map $\sigma^s_p : E^s_p(r) \to E^u_p(r)$ whose derivatives up to order k vary continuously with p, $\sigma^s_p(0_p) = 0_p$, and $D(\sigma^s_p)(0_p) = 0$, such that $W^s_r(x, f)$ is an embedded copy of the graph of σ^s_p. We will also be able to find $\sigma^p_u : E^u(r) \to E^s(r)$ with similar properties so that $W^u_r(x, f)$ is an embedded copy of the graph of σ^u_p.

In addition, for r small enough,

$$\begin{aligned} W^s_r(x, f) &= \{y \in M \mid d(f^n y, f^n x) \leq r, \forall n \geq 0\} \\ &= \{y \in M \mid d(f^n y, f^n x) \leq r \text{ and } d(f^n y, f^n x) \leq C\lambda^n d(y, x), \forall n \geq 0\}. \end{aligned}$$

Likewise,

$$\begin{aligned} W^u_r(x, f) &= \{y \in M \mid d(f^{-n} y, f^{-n} x) \leq r, \forall n \geq 0\} \\ &= \{y \in M \mid d(f^{-n} y, f^{-n} x) \leq r \text{ and } d(f^{-n} y, f^{-n} x) \leq C\lambda^n d(y, x),\ \forall n \geq 0\}. \end{aligned}$$

We call $W^s_r(x, f)$ the *local stable manifold* and $W^u_r(x, f)$ the *local unstable manifold* of x with size r with respect to f.

Remark 3.7. In general, it is possible to formulate the Stable Manifold Theorem on the entire hyperbolic set at once using families of graphs σ^s and σ^u in order to generate a self-coherent family of submanifolds $\{W^s_r(p)\}_{p\in\Lambda}$. Furthermore, note that $r > 0$ may not be the unique r satisfying the properties of the Stable Manifold Theorem and so the local stable and unstable manifolds might likewise not be unique. Going forward, we let $W_r^s(x, f)$ and $W_r^u(x, f)$ refer to some local stable or unstable manifold of size r.

The proof of the Stable Manifold Theorem is quite involved, and so will not be included here. However, a full proof can be found in [1] or [3].

The *global stable manifold* of a point $p \in \Lambda$ is the set

$$W^s(p, f) = \{q \in M \mid \lim_{n\to\infty} d(f^n p, f^n q) = 0\}$$

Likewise, the *global unstable manifold* is the set

$$W^u(p, f) = \{q \in M \mid \lim_{n\to\infty} d(f^{-n} p, f^{-n} q) = 0\}$$

Let $r > 0$ such that $W_r^s(p, f)$ is an embedded C^k submanifold. Then we can obtain the global stable manifold from the local stable manifold of size r by considering the union

$$W^s(p, f) = \bigcup_{n\geq 0} f^{-n} W_r^s(f^n p).$$

Likewise,

$$W^u(p, f) = \bigcup_{n\geq 0} f^{n} W_r^u(f^n p).$$

Therefore, $W^s(p, f)$ and $W^u(p, f)$ are monotone unions of embedded C^k submanifolds and therefore are immersed submanifolds of f. Note that since the global stable and unstable manifolds are immersed, but not necessarily embedded in M, they may have a more complicated structure than that of a C^1 disk. As a result, the local stable or unstable manifold is not simply a truncated version of the global stable or unstable manifold. In other words, if we intersect the global stable or unstable manifold with a ball of radius r around p, we do not necessarily obtain the local stable or unstable manifold of size r. In fact, it will be possible for the global stable or unstable manifold of f to be dense in M (and therefore dense in any open ball of M).

Example 3.8. Recall the example of the Anosov toral automorphism given in Example 3.4.

Let f be an Anosov toral automorphism induced by the Anosov automorphism $A : \mathbb{R}^2 \to \mathbb{R}^2$ with splitting $E^s \oplus E^u$. Then, for any $x \in \mathbb{T}^2$, both the global stable manifold $W^s(x, f)$ and the global unstable manifold $W^u(x, f)$ are dense in $\mathbb{T}^2$. The reason for this is relatively simple. First, it can be shown (as in Theorem 3.8 of [1]) that for any $x \in \mathbb{T}^2$, $W^s(x, f) = \pi(W^s(a, A))$, where a is any point in the preimage $\pi^{-1}(x)$. Recall that in Example 3.4 we mentioned that at $a \in \mathbb{R}^2$, there will be the hyperbolic splitting $E^s \oplus E^u$. It is quick to check that $W^s(a, A) = a + E^s$. Moreover, it can be shown that both E^s and E^u are lines with irrational slope. Thus, E^s will cut through each vertical line through a point in $\mathbb{Z}^2$ at a sequence of irrational points $\{nb\}_{n\in\mathbb{Z}}$. Therefore, modulo integers, E^s intersects the vertical boundary of the unit square - which is the latitude circle S^1 - at a dense subset. Therefore, $W^s(x, f)$ will be dense in $\mathbb{T}^2$. See Figure 2.

It will be convenient to, instead of working with two hyperbolic constants $C \geq 1$ and $0 < \lambda < 1$, find a way to create an immediate contraction and expansion on E^s and E^u.

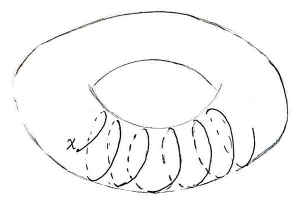

FIGURE 2. The stable manifold $W^s(x, f)$ wrapping around $\mathbb{T}^2$.

Proposition 3.9. *Let $\Lambda \subset M$ be a hyperbolic set of f with splitting $T_\Lambda M = E^s \oplus E^u$ and hyperbolic constants $C \geq 1$ and $0 < \lambda < 1$. There is a C^∞ Riemannian metric $\langle\langle \cdot, \cdot, \rangle\rangle$ of M and a constant $0 < \tau < 1$ such that, with respect to the induced norm $|| \cdot ||$,*

$$||Tf(v)|| \leq \tau ||v||, \text{ for all } v \in E^s,$$
$$||Tf^{-1} v|| \leq \tau ||v||, \text{ for all } v \in E^u.$$

A proof of this fact can be found in [1]. If $\langle\langle \cdot, \cdot \rangle\rangle$ satisfies the conclusion of Proposition 3.9, then it is called *adapted* to Λ. Furthermore, the number

$$\tau(\Lambda) = \sup_{x \in \Lambda} \{ ||Tf|_{E^s_x}||, ||Tf^{-1}|_{E^u_x}|| \} < 1$$

is called the *skewness* of Λ with respect to the norm $|| \cdot ||$.

Now, we will state the Inclination Lemma, a result which will be important to the proof of the Birkhoff-Smale theorem. Briefly, the Inclination Lemma states that a disc transverse to the unstable manifold will eventually accumulate on the unstable manifold. Before doing so, however, we must introduce another concept.

Definition 3.10. Let $p \in \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$, and let $|| \cdot ||$ denote the Euclidean norm on $\mathbb{R}^n$. The *standard horizontal γ-cone* at p is the set

$$H_p^\gamma = \{ (x, y) \in T_p \mathbb{R}^n \mid ||v|| \leq \gamma ||u|| \}.$$

Likewise, the *standard vertical γ-cone* at p is the set

$$V_p^\gamma = \{ (x, y) \in T_p \mathbb{R}^n \mid ||u|| \leq \gamma ||v|| \}.$$

Now, we are able to state the lemma.

Lemma 3.11 (Inclination Lemma)**.** *Let $p \in M$ be a hyperbolic fixed point of $f : M \to M$. If $n = \dim M$, let $k = \dim W^u(p, f)$ and $n - k = \dim W^s(p, f)$. Then, for any k-disc $B \subset W^u(p, f)$, any point $x \in W^s(p)$, any k-disc D transverse to $W^s(p, f)$ at x, and any $\varepsilon > 0$, there is $N > 0$ such that if $n > N$, $f^n(D)$ contains an $(n-k)$-disc that is C^1 ε-close to B. An analogous statement holds true for $(n-k)$-discs transverse to $W^u(p, f)$.*

Remark 3.12. Since we previously defined C^r closeness for a space of diffeomorphisms rather than a manifold, the statement of the Inclination Lemma might cause some confusion. To address this, we will say that two disks A and B parametrized as the graphs of two C^1 diffeomorphisms σ_B and σ_A are C_1-ε close if $d^1(\sigma_B, \sigma_A) < \varepsilon$.

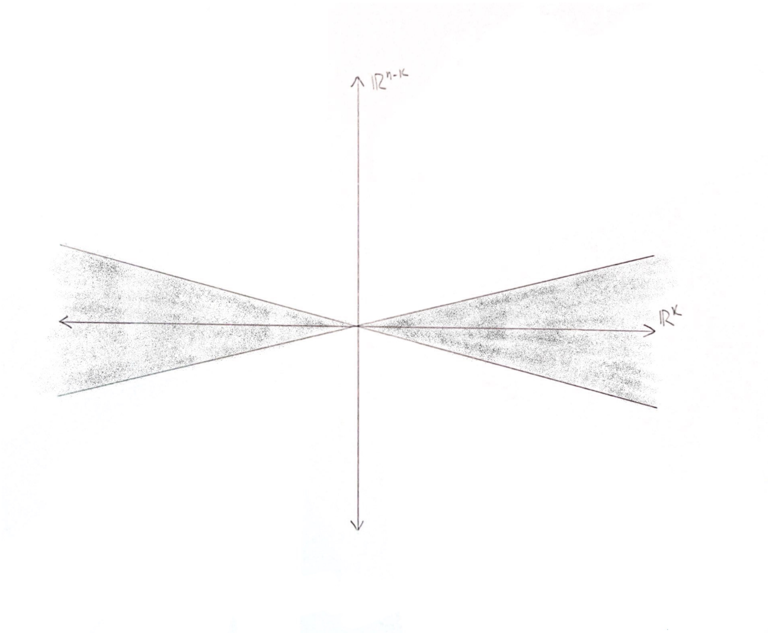

FIGURE 3. A horizontal γ-cone. The closure of the complement of the horizontal cone also forms a vertical $1/\gamma$-cone.

See Figure 4 for a visualization of the statement of the Inclination Lemma.

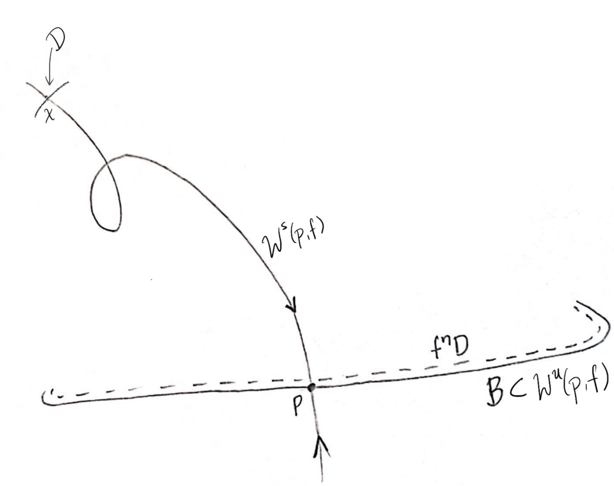

FIGURE 4. Statement of the Inclination Lemma.

Proof. It will be enough to prove a version of the Inclination Lemma in local coordinates. A proof of the local version can be found as Theorem 6.2.23 in [3] or as Theorem 7.1 in [4]. Let $\pi_1 : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ be the the projection map to the

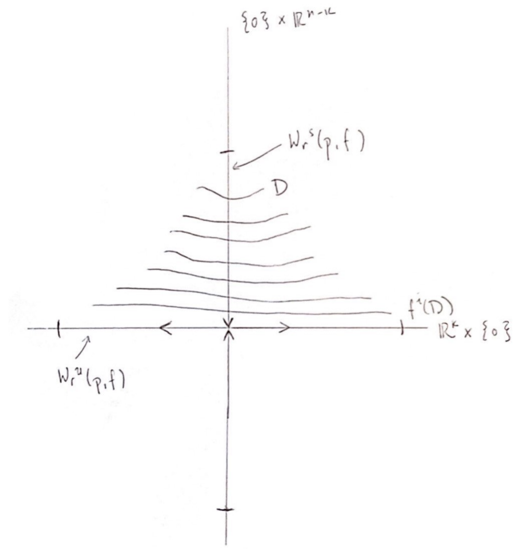

FIGURE 5. Local statement of the Inclination Lemma.

first coordinate. We can take local coordinates around a neighborhood U of p so that $W^u_r(p, f) \subset \mathbb{R}^k \times \{0\}$ and $W^u_s(p, f) \subset \mathbb{R}^{n-k} \times \{0\}$. Then, E^u_p is tangent to $\mathbb{R}^k$ and E^s_p is tangent to $\mathbb{R}^{n-k}$.

Set $K, \eta, \varepsilon > 0$. We will show that when D is is a C^1 disk containing $q \in W^s_p \cap U$ such that for all $x \in D$, $T_x D$ is contained in K-cones about E^u_p and so that $\pi_1(D)$ contains an η-ball around $0 \in \mathbb{R}^k \oplus \{0\}$, then there exists $N \in \mathbb{N}$ such that when $n \geq N$, $\pi_1(f^n(D)) = W^u_r(p, f) \cap U$ and $T_z f^n(D)$ is contained in a ε-cone about E^u for all $z \in f^n(D)$. See Figure 5.

First, notice that since $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ are f-invariant and f is C^1 (so its derivatives vary continuously), at $(x, y) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ we can put $Df(x, y)$ in the form of a block matrix:

$$Df(x, y) = \begin{pmatrix} A_{uu} & A_{su} \\ A_{us} & A_{ss} \end{pmatrix},$$

where

$$\begin{aligned} &||A^{-1}_{uu}||,\ ||A_{ss}|| \leq \tau + \delta. \\ &||A_{su}|| = o(||x||),\ ||A_{us}|| = o(||y||). \end{aligned} \tag{3.13}$$

Here, τ is an (adapted) contraction/expansion rate of $Df(p)$ and $Df^{-1}(p)$, and $\delta > 0$ is taken to be arbitrarily small by shrinking U and possibly replacing D with some iterate $f^n(D)$. In other words, near the origin $(0, 0)$, $Df(x, y)$ approximates the hyperbolic behavior of $Df(p)$.

Now, we consider tangent planes in horizontal $\gamma-$cones as graphs of linear maps whose operator norm $|| \cdot ||$ is bounded by γ. After shrinking D, we can assume that $D \cap (\{0\} \oplus \mathbb{R}^{n-k})$ is a single point $\{z\}$. We want to find $n \in \mathbb{N}$ such that the tangent space $T_{f^n(z)} f^n(D)$ is contained in some ε-cone. To do so, we will consider a linear map $E_z : \mathbb{R}^k \to \mathbb{R}^{n-k}$ parametrizing the tangent space $T_z D$. The graph $gr(E_z) = \{(x, E_z(x) \mid x \in \mathbb{R}^k\} \subset \mathbb{R}^n \times \mathbb{R}^{n-k}$ can be written as the image of the linear map

$$\begin{pmatrix} I_k \\ E_z \end{pmatrix} : R^k \to \mathbb{R}^k \times R^{n-k}$$

where I_k is the identity map on $\mathbb{R}^k$. We can compose this map with $Df(z) = Df(x,y)$, giving us

$$\begin{pmatrix} A_{uu,z} & A_{su,z} \\ A_{us,z} & A_{ss,z} \end{pmatrix} \begin{pmatrix} I_k \\ E_z \end{pmatrix}.$$

Note that that we can calculate $Df(p)(E_p)$ the same way for any point $p \in D$ and any linear map E_p parametrizing the tangent plane at p. Since in this case we are looking at $z = (0, y)$, then we have $A_{su,z} = 0$. In addition, since $A_{uu} \in M_{k\times k}$ is invertible, we know that the image of A_{uu}^{-1} is the whole subspace $\mathbb{R}^k$. As a result, the image of $\mathbb{R}^k \oplus \{0\}$ under $\begin{pmatrix} A_{uu,z} \\ A_{us,z} + A_{ss,z}E_z \end{pmatrix}$ is the same as its image under precomposition by $A_{uu,z}^{-1}$:

$$\begin{pmatrix} A_{uu,z} \\ A_{us,z} + A_{ss,z}E_z \end{pmatrix} (A_{uu,z}^{-1}) = \begin{pmatrix} I_k \\ A_{us,z}A_{uu,z}^{-1} + A_{ss,z}E_z(A_{uu,z})^{-1} \end{pmatrix}.$$

Therefore, we set

$$E_{f(z)} = A_{us,z}(A_{uu,z})^{-1} + A_{ss,z}E_z(A_{uu,z})^{-1}.$$

Then, using the bounds stated in (3.13),

$$||E_{f(z)}|| \le (\tau + \delta)(||A_{us,z}|| + ||E_z||),$$

which implies that

$$||E_{f^n(z)}|| \le \sum_{i=0}^{n-1} (\tau+\delta)^{n-i} ||A_{us,f^i(z))}|| + (\tau+\delta)^n ||E_z||. \tag{3.14}$$

Now, by iterating $gr(E_z)$ under $Df(z)$, we have found a sequence of linear maps $E_{f^n(z)}$, the graphs of which parametrize each tangent plane $T_{f^n(z)}D$. If we show that $||E_{f^n}(z)||$ eventually becomes small, we will succeed in showing that $T_{f^n(z)}D$ is contained in an ε-cone.

To this end, we want to shrink the sum in (3.14). First, since $||E_z||$ is constant and $\tau + \delta < 1$, there exists $N \in \mathbb{N}$ such that $(\tau+\delta)^n ||E_z|| < \varepsilon$ when $n \ge N$. Similarly, since $z \in W^s(p)$, we know that forwards iterates of z under f approach the origin. Therefore, since $||A_{us,f^i(z)}|| = o(||f_2^i(y)||)$, where $f^i(z) = (f_1^i(x), f_2^i(y))$, we have that $||A_{us,f^i(z)}||$ decreases to 0 as i gets large. Using this fact, we can make the last terms of the sum in (3.14) small. That is, there exists $N' \in \mathbb{N}$ large enough so that when $n \ge N'$,

$$\sum_{i=N}^{n-1} (\tau+\delta)^{n-i} ||A_{us,f^i(z)}|| < \varepsilon.$$

Likewise, there is $N'' \in \mathbb{N}$ such that when $n \ge N''$, the first $N-1$ terms of the sum are small. Finally, letting $N_0 = \max\{N, N', N''\}$, we have that $||E_{f^n(z)}|| < \varepsilon$ when $n \ge N$. By taking N_0 perhaps even larger, we may assume that $||A_{us,(x,y)}|| < (1-\tau-\delta)\varepsilon$, when $|y| < ||f^{N_0}(z)||$. In addition, the linear map $E_{f^i(p)}$ parametrizing the tangent space of $f^i(p) \in f^i(D)$ varies continuously with the base point p, since f is a C^k diffeomorphism and D is a C^1 disk. So, by shrinking D, we may assume that all tangent planes of points in $f^{N_0}(D)$ are contained within ε-cones and that $||A_{us,p}|| < (1-\tau-\delta)\varepsilon$, for all $p \in \bigcup_{i>N_0} f^i(p)$. Then, if $\varepsilon > 0$ is small enough, it

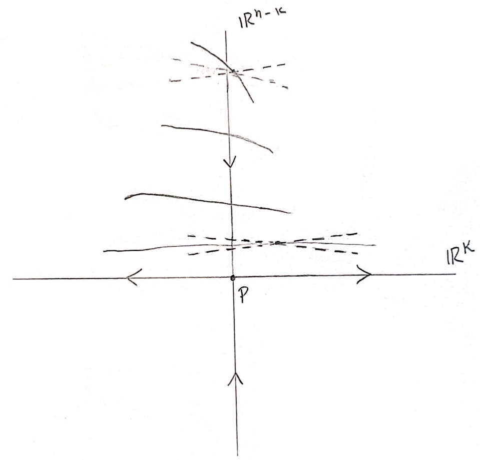

FIGURE 6. Each tangent plane to D is eventually contained in ε-cone.

will be true that when $||E|| < \varepsilon$,

$$||(A_{uu} + A_{su}E)^{-1}|| < 1.$$

Thus, for any $p \in D$, $||E_{f^i(p)}|| < \varepsilon$, we have

$$\begin{aligned} ||E_{f^{i+1}(p)}|| &= ||(A_{us,f^i(p)} + A_{ss}E_{f^i(p)})(A_{uu,f^i(p)} + A_{su,f^i(p)}E_{f^i(p)})^{-1}|| \\ &\leq ||A_{us,f^i(p)} + A_{ss}E_{f^i(p)}|| \cdot ||(A_{uu,f^i(p)} + A_{su,f^i(p)}E_{f^i(p)})^{-1}|| \\ &\leq ||A_{us,f^i(p)}|| + ||A_{ss,f^i(p)}|| < (1-\tau-\delta)\varepsilon + (\tau+\delta)\varepsilon = \varepsilon. \end{aligned}$$

In other words, if $T_{f^i(p)}f^i(D)$ is contained within an $\varepsilon-$cone, the tangent space $T_{f^{i+1}(p)}f^{i+1}(D)$ at the next iterate will be contained within an $\varepsilon-$cone. Therefore, when $n \geq N_0$, the tangent plane $T_p f^n(D)$ is contained within an $\varepsilon-$cone for all $p \in f^n(D)$.

It remains to show that $\pi_1(f^n(D)) = W_p^u \cap U$ when $n \geq N_0$. Let v_n be a nonzero tangent vector to $p \in f^n(D)$ with image v_{n+1} under $Df^n(p)$. Then, since $||A_{uu,f(x)}|| > 1$, we have that

$$\frac{||v_{n+1}||}{||v_n||} \geq ||Df(p)|| > 1.$$

Therefore, the diameter of $f^n(D)$ increases at constant rate greater than 1, implying that for n large enough, the image of $f^n(D)$ under the projection map π_1 covers the local unstable manifold $W_r^u(p,f) \cap U$.

We now see for all n large enough, $f^n(D)$ will eventually contain a disk which can be represented as a graph of a C^1 function $\sigma_n : W_r^u(p,f) \cap U \to \mathbb{R}^{n-k}$. Since $f^n(q) \in W_r^s(p,f)$ approaches $(0,0)$, by continuity points in $f^n(D)$ near $f^n(q)$ become ε-close to $\mathbb{R}^k \times \{0\}$. Moreover, since, as shown previously, all tangent spaces of $f^n(D)$ are contained within ε-cones, the norm $||D\sigma||$ will be less than ε. Thus, σ_n will be ε-close to $0 : W_r^u(p,f) \to \mathbb{R}^{n-k}$ in the C^1 topology.

Returning to our global setting, we see that we have shown that $f^n(D)$ contains a k-disk that is C^1-ε close to a k-disk contained in B (i.e. we have shown that it is

C_1 ε-close to a disk in the local unstable manifold). This is enough, however, since successive applications of f will eventually cover the entire disk B. $\square$

4. Symbolic Dynamics

In this section, we will give a quick overview on symbolic dynamics using two symbols. Symbolic dynamics is an extremely useful tool in understanding the dynamics of the Smale horseshoe, since we will construct a topological conjugacy between a homeomorphism in symbolic dynamics and an invariant set of the horseshoe. Remarkably, we will be able to encode the entire dynamics of the horseshoe using a simple "shift" mechanism on strings of 0's and 1's. We will now begin by giving some basic definitions.

Definition 4.1. The *symbolic space* Σ_2 of two symbols is defined to be the bi-infinite product

$$\Sigma_2 = \{0, 1\}^{\mathbb{Z}}$$

endowed with the product topology.

Then, we can denote a point $a \in \Sigma_2$ by a bi-infinite sequence $\{a_n\}_{n\in\mathbb{Z}}$ where each a_n is either 0 or 1.

Note that we will always treat the bi-infinite sequences as if they are centered around 0. For example, denoting by a hat the 0-th place of a sequence in Σ_2, the sequence $\ldots 00\hat{1}00 \ldots$ differs from $\ldots 01\hat{0}00 \ldots$ by a shift to the left of one place.

We define a basis of open neighborhoods of $a \in \Sigma_2$ using sets of the form

$$C_j(a) = \{b \in \Sigma_2 \mid b_n = a_n, \forall -j \leq n \leq j\}.$$

Therefore, we consider b to be close to a if for a large j, b agrees with a on the interval $[-j, j]$ centered at 0. This topology on Σ_2 is metrizable by the metric

$$d(a, b) = \sum_{n\in\mathbb{Z}} \frac{|a_n - b_n|}{2^{|n|}}.$$

Once again, we see that a and b are close if a_n and b_n agree on a long string centered about 0.

In §5 and §6, it will become relevant that Σ_2 has the structure of a Cantor set. Therefore, we will quickly define Cantor sets and prove this fact here.

Definition 4.2. A *Cantor set* is a topological space which is compact, perfect, totally disconnected, and metrizable. All such sets are homeomorphic to the canonical *middle-thirds Cantor set* on the real line.

Proposition 4.3. *Σ_2 is a Cantor set.*

Proof. We have already mentioned that Σ_2 is metrizable. By definition, the topology on Σ_2 is the infinite product of the compact space $\{0, 1\}$, endowed with the product topology. Since the product of compact topological spaces is compact (with respect to the product topology), we immediately have that Σ_2 is compact.

To show that Σ_2 is totally disconnected, we first let $a = (\ldots a_{-1}\ a_0\ a_1 \ldots)$ and $b = (\ldots\ b_{-1}\ b_0\ b_1 \ldots)$ be distinct elements of Σ_2. Since $b \neq a$, there exists $m \in \mathbb{Z}$ such that $a_m \neq b_m$. Let

$$V(a) = \{c \in \Sigma_2 \mid c_m = a_m\}.$$

$V(a)$, which contains a, is open, and $\Sigma_2 \setminus V(a)$, which contains b, is also open. Therefore, a and b must lie in different connected components. Finally, we see that

Σ_2 is perfect. Let $a \in \Sigma^2$. For any $j \geq 1$, a is not the unique element in $C_j(a)$. Therefore, any open neighborhood of a will have some element $b \neq a$. So, a is not an isolated point, and $a \in \Sigma_2$. □

Definition 4.4. We define the *shift map*

$$\sigma : \Sigma_2 \to \Sigma_2$$

by

$$(\sigma(a))_n = a_{n+1}, \quad \text{for all } n \in \mathbb{Z}.$$

That is, σ takes the bi-infinite sequence $a \in \Sigma_2$ to the bi-infinite sequence $\sigma(a) \in \Sigma_2$ which is a shifted one unit to the left.

The following proposition characterizes some of the properties of the shift map σ.

Proposition 4.5. *Periodic points of σ are dense in Σ_2, and σ is transitive on Σ_2.*

Proof. Let $a \in \Sigma_2$ and $j \geq 1$. Let $b \in \Sigma_2$ be sequence which repeats the $(2j+1)$-tuple $a_{-j} \dots a_j$ in both directions. Then b is periodic and $b \in C_j(a)$. Since we can find such a sequence b for any $j \geq 1$, then we can find b arbitrarily close to a. Therefore, periodic points are dense in Σ_2.

By Theorem 2.4, finding $x \in \Sigma_2$ whose positive orbit is dense in Σ_2 will be enough to show that σ is transitive. Let $x_n = 0$ for all $n \geq -1$. Since there are only countably many possible finite tuples of $0, 1$, we may place them in some order. Therefore, starting with $n = 0$, we set $\{x_n\}_{n \geq 0}$ to successively iterate through all possible finite tuples of $0, 1$. Now, let $a \in \Sigma_2$, $j \in \mathbb{N}$. The string $a_{-j} \dots a_j$ is present somewhere in the positive part of x_n. Therefore, there exists $m \in \mathbb{N}$ such that $\sigma^m(x)$ agrees with a on the interval $[-j, j]$. Since we can do this for any integer $j \geq 0$, the positive orbit of x will be arbitrarily close to a. Thus, σ is transitive on Σ_2. □

Now, we are ready to describe the Smale horseshoe.

5. Description of the Smale Horseshoe as a map from S^2 to S^2

The goal of this section will be to define the Smale horseshoe geometrically, as a map S^2 to S^2. Then, we will prove the topological conjugacy between the Smale horseshoe and the shift map $\sigma : \Sigma_2 \to \Sigma_2$.

Let $Q \subset \mathbb{R}^2$ be a unit square. Let f be a diffeomorphism which contracts Q horizontally, expands it vertically, and then folds it into a horseshoe shape. Furthermore, we require f be defined so that the horseshoe is put back onto the original square Q and $f^n(Q) \cap Q$ consists of two vertical rectangles V_0 and V_1 crossing the whole height of Q. By construction, the preimages of V_0 and V_1 will be the horizontal rectangles H_1 and H_2, which also cross the whole length of Q. That is, $Q \cap f^{-1}(Q) = H_1 \cup H_2$. See Figure 7. Since f does not map Q into itself, some of the points of Q do not have second iterates. To fix this, we extend the map to a global diffeomorphism on S^2 which is affine on H_i, has a contraction rate of $0 < \mu < 1$ and an expansion rate of $1/\mu$. Let

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(Q).$$

Notice that any $f-$invariant set contained within Q must, by necessity, be contained in the intersection of all positive and negative iterates of Q by f. However, this

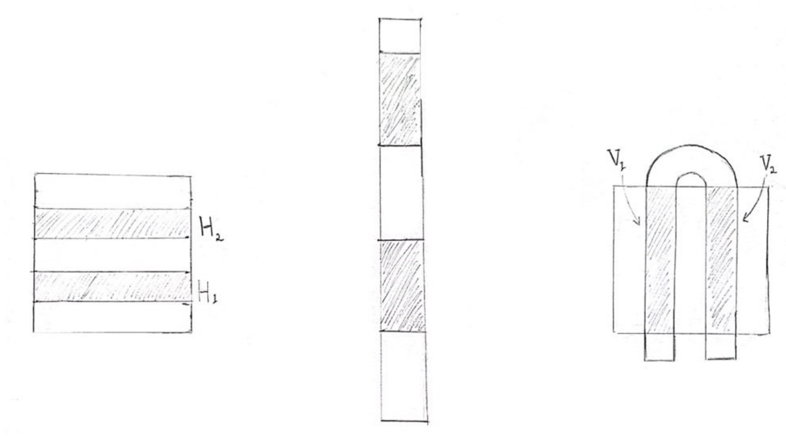

FIGURE 7. Images of H_1 and H_2 under the horseshoe map.

intersection is, by definition, exactly Λ, so Λ must be the maximal f-invariant set inside of Q. Going forward, we will call Λ the *horseshoe set*.

The following theorem establishes the topological conjugacy between the shift map σ and the horseshoe map f restricted to the horseshoe set Λ. The proof of the Birkhoff-Smale Theorem (Theorem 6.2) in §6 will in many ways mirror the proof given here.

Theorem 5.1. *The horseshoe map $f : \Lambda \to \Lambda$ is topologically conjugate to $\sigma : \Sigma_2 \to \Sigma_2$.*

Proof. To show that there exists topological conjugacy between f and σ, we will construct a homeomorphism associating points in Λ to sequences in Σ_2 which encode their orbits. Then, the shift σ will naturally align with a shift in the orbit of x to the orbit of fx.

Suppose that $x \notin H_1 \cup H_2$. Then $f(x) \notin Q$. Therefore,

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(H_1 \cup H_2).$$

Since H_1 and H_2 do not intersect, for each $n \in \mathbb{Z}$, $x \in \Lambda$, the iterate $f^n(x)$ will either be in H_0 or H_1, but never both. This lends a natural encoding of the orbit of x by bi-infinite strings of 0's and 1's. Indeed, for any $x \in \Lambda$, there is a unique sequence $a \in \Sigma_2$ such that for all $n \in \mathbb{Z}$

$$f^n(x) \in H_{a_n}.$$

We call a the *itinerary sequence* of $x \in \Lambda$. Consider the mapping $h : \Lambda \to \Sigma_2$ which sends $x \in \Lambda$ to its itinerary sequence. Notice that the itinerary sequence of $f(x)$ is simply the shift of the itinerary sequence of x by σ, giving us the relation $hf = \sigma h$.

To complete the proof that h is a topological conjugacy between f and σ, we must prove that it is a homeomorphism. To show that h is continuous, we see that for any $j \geq 1$, if $x, y \in \Lambda$ are sufficiently close, then their iterates $f^n(x)$ and $f^n(y)$ will be within $\frac{1}{2}\mu$ of each other for all $-j \leq n \leq j$. As a result, for all iterates in the interval $[-j, j]$, the orbits of x and y will pass through the same series of horizontal strips H_0 or H_1. Thus, their itinerary sequences will agree on $[-j, j]$. Since we can do this for any $j \geq 1$, h must be continuous. Then, because Λ is compact, showing that h is bijective will be enough to show that it is a homeomorphism. This follows from the fact that continuous bijections between compact spaces are homeomorphisms.

To show that h is a bijection, we need to see that for any $a \in \Sigma_2$, the intersection

$$\bigcap_{n\in\mathbb{Z}} f^{-n}(H_{a_n})$$

is a single point $\{x\}$. That is, one and only one $x \in \Lambda$ travels through each possible sequence of rectangles H_0 and H_1 under forwards and backwards iterations of f.

First, we define two families of sets $\{I_i\}_{i\in\mathbb{N}}$ and $\{J_i\}_{n\in\mathbb{N}}$:

$$\begin{aligned} &\cdots \cap f^2 H_{a_{-2}} \cap f H_{a_{-1}} \cap H_{a_0} \cap f^{-1} H_{a_1} \cap f_{-2} H_{a_2} \cap \ldots . \\ &= \underbrace{\ldots f^2 V_{a_{-3}} \cap \underbrace{f V_{a_{-2}} \cap \underbrace{V_{a_{-1}}}_{J_0}}_{J_1}}_{J_2} \cap \underbrace{\underbrace{\underbrace{H_{a_0}}_{I_0} \cap f^{-1} H_{a_1}}_{I_1} \cap f^{-2} H_{a_2} \cap \ldots}_{I_2} . \end{aligned} \tag{5.2}$$

Because the families of sets J_i and I_i are built off of successive intersections, we have that each of these families is nested:

$$I_{n+1} \subset I_n \text{ and } J_{n+1} \subset J_n.$$

Furthermore, notice that for any vertical strip V, the intersection $fV \cap V_i$ is a vertical strip with width contracted by μ. Likewise, for any horizontal strip H, $f^{-1}(H) \cap H_i$ is a horizontal strip with height contracted by μ. This is so since

$$f^{-1} H \cap H_i = f^{-1}(H \cap f H_0) = f^{-1}(H \cap V_i).$$

Then, $H \cap V_i$ is a rectangle which crosses V_i horizontally, so its preimage will be a contracted horizontal strip. Using these facts, we see that every I_n will be a horizontal strip with height less than μ^n, and every J_n will be a vertical strip with width $\leq \mu^n$. Since J_0 is a vertical strip, we see that

$$J_1 = f(V_{a_{-2}}) \cap V_{a_{-1}}$$

is a vertical strip. Likewise,

$$J_2 = f(f(V_{a_{-3}}) \cap V_{a_{-2}})) \cap V_{a_{-1}}$$

must be a vertical strip. The same will be true for the family of sets I_n. Then

$$\bigcap_{n\in\mathbb{Z}} f^{-n}(H_{a_n})$$

is the intersection of a horizontal interval and a vertical interval, so it must be a single point, which is what we wanted. □

Corollary 5.3. *The horseshoe set Λ is a Cantor set. In addition the horsehoe map f is transitive on Λ, and periodic points are dense in f.*

Proof. The fact that Λ is a Cantor set follows from the fact it is homeomorphic to Σ_2, which is a Cantor set. Transitivity and the density of periodic points follows from the conjugacy between σ and f. □

6. Horseshoes and Transverse Homoclinic Points on Manifolds

Now we will introduce the concept of transverse homoclinic points and study the dynamical behavior which occurs near them, which will resemble the behavior of the Smale horseshoe.

Definition 6.1. Let $p \in M$ be a hyperbolic periodic point of a diffeomorphism $f : M \to M$. We call some $q \in M$ a *homoclinic point* of p if

$$q \in W^s(p, f) \cap W^u(p, f) \setminus \{p\}.$$

The point q will be a *transverse homoclinic point* if the intersection of $W^s(p, f)$ and $W^u(p, f)$ is transverse at q.

Since the stable and unstable manifolds are invariant, iterates of transverse homoclinic points must also be transverse homoclinic points. Therefore, the global stable and unstable manifolds will have to intersect transversely at every point in the orbit of q. This leads to a great complication in the dynamics of f, as the stable and unstable manifolds will have to oscillate and double back upon themselves in order to keep intersecting transversely at iterates of q. See Figure 8.

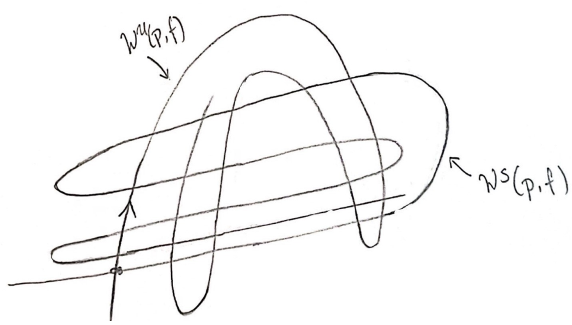

Figure 8. Shape of stable and unstable manifolds at a homoclinic point.

We are finally ready to state and prove the Birkhoff-Smale theorem. In this proof, we will recover the entire behavior of the Smale horseshoe map within the neighborhood of a transverse homoclinic point.

Theorem 6.2 (Birkhoff-Smale)**.** *Let $f : M \to M$ be a diffeomorphism, p be a hyperbolic periodic point, and q be a transverse homoclinic point for p. Then for any neighborhood U of $\{p, q\}$, there exists $n \geq 0$ such that f^n has a hyperbolic invariant set $\Lambda \subset U$ such that $p, q \in \Lambda$ and on which f^n is topologically conjugate to the shift map $\sigma : \Sigma_2 \to \Sigma_2$.*

Proof. The following proof is modeled after Theorem 4.5 in [2]. To prove the main result, we will find a way to construct sets $\tilde{V_0}$ and $\tilde{V_1}$ in a neighborhood U of $\{p, q\}$ which will play an analogous role to the rectangles V_0 and V_1 in the proof of Theorem 5.1. Using $\tilde{V_0}$ and $\tilde{V_1}$, we will then encode the orbits of points in some maximal invariant subset of U and apply the same technique of itinerary sequences that we did in Theorem 5.1.

Let $\Lambda_q = \mathrm{Orb(p)} \cup \mathrm{Orb}(q)$. First, we will show that Λ_q is a hyperbolic set. Then, we will show that if V is a small enough neighborhood of Λ_q, the maximal

f-invariant subset $\bigcap_{n\in\mathbb{Z}} f^n(V)$ is also a hyperbolic set. Finally, by looking at the local behavior of the unstable and stable manifolds in a neighborhood U contained within V, we may construct boxes $\tilde{V_0}$ and $\tilde{V_1}$ and find some iterate $n \in \mathbb{N}$ such that a topological conjugacy between $\sigma : \Sigma_2 \to \Sigma_2$ and $f^n : \Lambda_U \to \Lambda_U$ becomes apparent.

Let $\Lambda_q = \mathrm{Orb}(p) \cup \mathrm{Orb}(q)$. For this proof, we will assume that p is hyperbolic fixed point with splitting $E^s_p \oplus E^u_p$. However, adaptations can be made to allow for the case that $\mathrm{Orb}(p)$ is a hyperbolic periodic orbit. To find a hyperbolic splitting on Λ_q, we consider the direct sum

$$T_qW^u(p) \oplus T_qW^s(p) = T_qM. \tag{6.3}$$

Since f is a diffeomorphism, the derivative $Df(x) : T_xM \to T_{f(x)}M$ is linear isomorphism for each $x \in M$. Therefore, we may iterate $Df(x)$ on (6.3) in order to obtain the invariant splitting

$$\begin{aligned} Df^m(q)(T_qW^u(p) \oplus T_qW^s(p)) &= Df^m(q)(T_qW^u(p)) \oplus Df^m(q)(T_qW^s(p)) \\ &= T_{f^m(q)}W^u(p) \oplus T_{f^m(q)}W^s(p) \end{aligned}$$

on $\mathrm{Orb}(q)$. At each point $f^m(q)$ in the orbit of q, set $E^s_{f^m(q)} = T_{f^m(q)}W^s(p)$ and $E^u_{f^m(q)} = T_{f^m(q)}W^u(p)$. We claim that the splitting

$$E^s \oplus E^u = \bigcup_{x\in\Lambda_q} E^s_x \oplus E^u_x$$

is hyperbolic. To this end, we first ensure that the splitting $E^s \oplus E^u$ on the orbit of q continuously approaches $E^s_p \oplus E^u_p$. Since the stable manifold is locally an embedded C_1 disk which is tangent to p, we must have that $T_{f^m(q)}W^s(p)$ converges to E^s_p as $f^m(q)$ converges to p. On the other hand, since $W^s(p)$ and $W^u(p)$ are transverse at every point in the orbit of q, by the Inclination Lemma (Lemma 3.11), any disk in $W^u(p)$ containing q eventually C_1-converges to the local unstable manifold $W^u_r(p)$. Therefore, under forwards iteration, the tangent space $T_qW^u(p) = E^u(q)$ converges to $E^u_p = T_pW^u(p)$. An analogous proof shows that $E^s_{f^m(q)} \oplus E^u_{f^m(q)}$ converges to $E^s_p \oplus E^u_p$ as m approaches $-\infty$. Up until now, we have verified that $E^s \oplus E^u$ is a continuous, invariant splitting on Λ_q. Now we must show that it is hyperbolic.

Indeed, we want to show there exist C, λ such that for all $v \in E^s_x$,

$$|Df^n(x)(v)| \leq C\lambda^n|v|.$$

Let $||\cdot||$ be an adapted norm on $E^u_p \oplus E^s_p$. Set λ to be the skewness of $||\cdot||$. By the continuity of the splitting $E^s \oplus E^u$, we may find a neighborhood O of p such that $||Df(x)(E^s_x)|| < \lambda$ when $x \in \mathrm{Orb}(q) \cap O$. Because q is a homoclinic point, it will approach p under both positive and negative iterates of x. That is, there is $M \in \mathbb{N}$ such that when $n \geq M$ or $n \leq -M$, $f^n(q) \in O$. Therefore, the neighborhood O of p will necessarily contain all but finitely many elements of $\mathrm{Orb}(q)$. Therefore, we may find $C \geq 1$ such that if $x \in \mathrm{Orb}(q)$ such that $f^i(x) \notin O$ for $0 \leq i < k$, then $||Df^i(x)(E^s_x)|| \leq C\lambda^i$. Furthermore, for any $x \in \mathrm{Orb(q)}$, there is at most one such string of positive iterates i, for $0 \leq i < k$, of x such that $f^i(x) \notin O$. Thus, we find

$$||Df^i(x)(E^s_x)|| \leq C\lambda^i$$

for all positive iterates i. We can carry out a similar process by exchanging f and f^{-1} to obtain to an appropriate C and λ for both Tf and Tf^{-1}. This completes the proof that Λ_q has a hyperbolic structure.

The next step in the proof is to show that if V is a small enough neighborhood of Λ_q, the maximal $f-$invariant set

$$\Lambda_V = \bigcap_{i\in\mathbb{Z}} f^i(V)$$

also has a hyperbolic structure. But this is immediate from *Proposition* 3.5.

Now, we construct the sets $\tilde{V_0}$ and $\tilde{V_1}$. First, as in the proof of Lemma 3.11 we can take local coordinates so that E^s_p becomes identified with $\mathbb{R}^k$, E^u is identified with $\mathbb{R}^{n-k}$ and a neighborhood near p is a subset of $E^s \times E^u$. Furthermore, we identify the local stable manifold $W^s_r(p)$ with the disk $E^s(r)\times\{0\}$ and the local unstable manifold $W^u_r(p)$ is given by the disk $E^u(r)\times\{0\}$. For $\varepsilon_s, \varepsilon_u > 0$, we denote $D^s = W^s_{\varepsilon_s}(p)$ and $D^u = W^u_{\varepsilon_u}(p)$. Since we have identified $W^s_r(p)$ with $E^s(r)$ and $W^u_r(p)$ with E^u_p, we similarly let $D^s = E^s(\varepsilon_s)$ and $D^u = E^u(\varepsilon_u)$. Then we take $\varepsilon_s, \varepsilon_u > 0$ and $k \geq 0$ such that

$$q \in \text{int}(f^{-k}(D^s) \setminus (f^{-(k-1)}(D^s))$$
$$q \in \text{int}(f^{k}(D^u) \setminus (f^{(k-1)}(D^u)),$$

adjusting ε_s or ε_u so that they work the same k in forwards and backwards iterations. Because p is a fixed point, it will be contained in both the image $f^k(D^u)$ and the preimage $f^{-k}(D^s)$. See Figure 9.

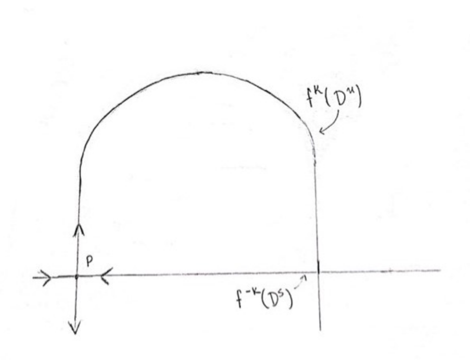

FIGURE 9. The intersection of $f^k(D^s)$ and $f^k(D^u)$ at p and q.

Take $j_0 \geq 0$ such that for all $j \geq j_0$, the disk $f^k(D^u)$ crosses $f^{-k}(D^s \times f^{-j}(D^u))$ transversally in the connected component of the intersection

$$f^{-k}(D^s \times f^{-j}(D^u)) \cap f^k(D^u) \tag{6.4}$$

containing either p or q. That is, the disk $f^k(D^u)$ intersects $f^{-k}(D^s \times \{y\})$ at a single point in each component, and this intersection is transverse. By choosing j large, we ensure that the image of the rectangle $D^s \times f^{-j}(D^u) \subset D^s \times D^u$ under f^{-k} is "thin" enough so that $f^k(D^u)$ crosses through it vertically. See Figure 10. Thus, (6.4) is a vertical disk through p in one component and a vertical disk through q in another.

Now, we find a neighborhood of $f^k(D^u)$ which acts as a vertical strip through q and p. For $j \geq j_0$, consider the set

$$f^{2k+j} \circ f^{-k}(D^s \times f^{-j}(D^u)) = f^{k+j}(D^s \times f^{-j}D^u)). \tag{6.5}$$

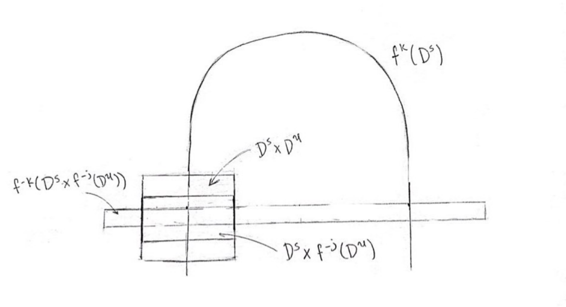

FIGURE 10. Intersection of the image of D^s with $D^s \times f^{-j}(D^u)$.

By Lemma 3.11, we can guarantee that $f^{k+j}(D^s \times f^{-j}D^u)$ forms a "thin" strip around $f^k(D^u)$ when j is large enough. Therefore, for large j, we find that $f^{k+j}(D^s \times f^{-j}(D^u))$ crosses $f^{-k}(D^s \times f^{-j}(D^u)$ transversally in the components of the intersection

$$f^{k+j}(D^s \times f^{-j}D^u)) \cap f^{-k}(D^s \times f^{-j}D^u)$$

containing p and q. In this case, we see that the intersection of $f^{k+j}(\{x\} \times f^{-j}D^u)$ and $f^k(D^s \times \{y\})$ is exactly one point and the intersection is transverse, for any $x \in D^s$, $y \in f^{-j}(D^u)$.

Then, for a fixed j large enough to satisfy the previous condition, set $n = 2k + j$ and $\mathcal{D} = f^{-k}(D^s \times f^{-j}D^u))$. Notice that the "vertical" intersection of $f^n(\mathcal{D})$ with $\mathcal{D}$ is similar to the intersection $f^n(Q) \cap Q$ in our construction of the Smale horseshoe map in the previous section. Therefore, the following give good candidates for $\tilde{V_0}$ and $\tilde{V_1}$ in our present construction.

Let $\tilde{V_0}$ be the connected component of $f^n(\mathcal{D}) \cap \mathcal{D}$ which contains p. Likewise, let $\tilde{V_1}$ be the connected component of $f^n(\mathcal{D}) \cap \mathcal{D}$ which contains q. Let U be an open neighborhood of p and q which is contained in V. Let

$$\Lambda = \bigcap_{i \in \mathbb{Z}} f^{in}(\tilde{V_0} \cup \tilde{V_1}).$$

By construction, $\tilde{V_0}$ and $\tilde{V_1}$ are disjoint. Therefore, a point $x \in \Lambda$ must be in either $\tilde{V_0}$ or $\tilde{V_1}$, but never both. In addition, by our choices of $\tilde{V_0}$ and $\tilde{V_1}$, the preimage

$$f^{-n}(\tilde{V_0} \cup \tilde{V_1})$$

has two connected components which cross $\mathcal{D}$ horizontally, transverse to $f^{k+j}(\{x\} \times f^{-j}(D^u))$, for all $x \in D^s$. Since Λ_V is hyperbolic, we know that $\Lambda \subset \Lambda_V$ is hyperbolic. Let $h : \Lambda \to \Sigma_2$ be the map sending $x \in \Lambda$ to its itinerary sequence. The contraction and expansion rates of Λ as well as the properties of $\tilde{V_0}$ and $\tilde{V_1}$ are such that the arguments from the proof of Theorem 5.1 apply. Thus, h is a topological conjugacy between Λ and Σ_2. This ends the proof of the Birkhoff-Smale theorem. $\square$

Corollary 6.6. *The map f_Λ is topologically conjugate to the Smale horseshoe map restricted to the maximal invariant set on Q. Therefore, Λ is a Cantor set, periodic points are dense in Λ, and $f|_\Lambda$ is transitive.*

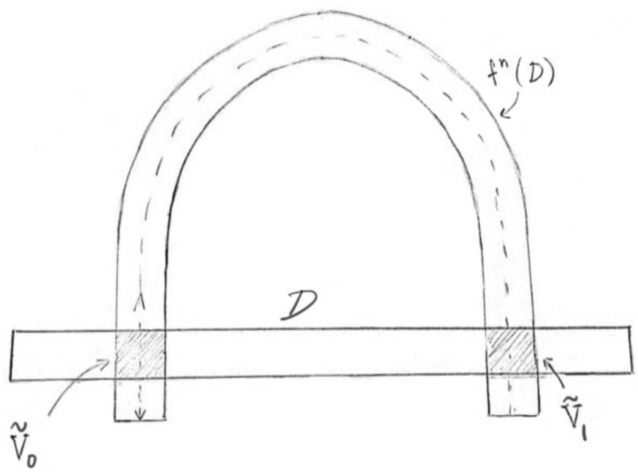

FIGURE 11. Choice of $\tilde{V_0}$ and $\tilde{V_1}$.

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